Coherency and purity for monoids

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Coherency for Monoids: a finitary condition

Throughout, S will denote a monoid.

Finitary condition

A condition satisfied by all finite monoids.

Example

Every element of S has an idempotent power.

Finitary conditions were introduced by **Noether** and **Artin** in the early 20th Century to study rings; they changed the course of algebra entirely.

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Example

Every right congruence on S is finitely generated, i.e. S is **right Noetherian**.
Every right ideal of S is finitely generated, i.e. S is **weakly right Noetherian**.

Coherency for Monoids: the definition

Coherency

This is the (first) finitary condition of importance to us today

Definition

S is right coherent if every finitely generated S-subact of every finitely presented right S-act is finitely presented.[a]

[a] This definition comes from Wheeler (1976)

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Left coherency is defined dually: S is coherent if it is left and right coherent.

Acts over monoids: S-acts
Representation of monoid S by mappings of sets

A **(right)** S-**act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Beware: an S-act is also called an S-set, S-system, S-action, S-operand, or S-polygon.

Let Act-S denote the class of all S-acts.

Acts over monoids: S-acts
Standard definitions/Elementary observations

- S-acts form a variety of universal algebras, to which we may apply the usual notions of subalgebra (S-**subact**), morphism (S-**morphism**), congruence, etc.
- S-acts and S-morphisms form a category, **Act-S**.
- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- Free S-acts are **disjoint unions of copies of** S.

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- We have usual definitions of **free, projective, injective**, etc. including variations on **flat**.
- Free S-acts are **disjoint unions of copies of** S.
- A is **finitely presented** if

$$A \cong F_S(X)/\rho$$

for some finitely generated free S-act $F_S(X)$ and finitely generated congruence ρ.

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Example: Fountain (92)

There is a monoid S which is weakly right noetherian but which is not right coherent.

Let us call the example above the **Fountain monoid**: it is made up of a group and a 4 element nilpotent semigroup.

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- Certain nice classes of monoids are coherent (more later).
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- Coherency is related to **purity** (more later).

Why is coherency interesting?

Theorem: Wheeler (1976); G (1986), Ivanov (1992)

The following are equivalent for a monoid S:

1. S is right coherent;
2. the existentially closed S-acts form an axiomatisable class;
3. the first-order theory of S-acts has a model companion.

Equations over S-acts

Let A be an S-act. An **equation** over A has the form

$$xs = xt, \; xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

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A set of equations is **consistent** if it has a solution in some S-act $B \supseteq A$.

Absolutely pure and almost pure

A is **absolutely pure (almost pure)** if every finite consistent set of equations over A (in 1 variable) has a solution in A.

absolutely pure = algebraically closed
almost pure = 1-algebraically closed

Equations and inequations over S-acts

Let A be an S-act. An **inequation** over A has the form

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Existentially closed and 1-existentially closed

A is **existentially closed (1-existentially closed)** if every finite consistent set of equations and inequations over A (in 1 variable) has a solution in A.

Why is coherency interesting?

Theorem: G (1986)

The following are equivalent for a monoid S:

1. S is right coherent;
2. the existentially closed S-acts $\mathcal{E}$ form an axiomatisable class;
3. the first-order theory of S-acts has a model companion;
4. the 1-existentially closed S-acts $\mathcal{E}_1$ form an axiomatisable class;
5. the absolutely pure S-acts $\mathcal{A}$ form an axiomatisable class;
6. the almost pure S-acts $\mathcal{A}_1$ form an axiomatisable class.

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- the free left restriction monoid on X.

Theorem:

The free inverse monoid on X for $|X| \geq 2$ is not right coherent.

Completely right pure monoids

Definition: A monoid is completely right pure

if every S-act is absolutely pure.

Clearly

$$\mathcal{A} \subseteq \mathcal{A}_1 \subseteq Act\text{-}S.$$

Theorem: G (1991)

A monoid S is completely right pure if and only if all S-acts are almost pure, i.e.

$$Act\text{-}S = \mathcal{A}_1 \iff Act\text{-}S = \mathcal{A}.$$

Completely right pure monoids

The fact $Act\text{-}S = \mathcal{A}_1 \iff Act\text{-}S = \mathcal{A}_1$ enabled me to characterise **completely right pure monoids (1991)** in a way analogous to that of Skornjakov (1979), and Fountain (1974) and Isbell (1972) for **completely right injective monoids** .

A Question

Does there exist a monoid S and an S-act A such that A is almost pure but not absolutely pure?????

Purity: Absolute purity vs almost purity
The Question: does $\mathcal{A} = \mathcal{A}_1$ for every monoid S?

Theorem: G, Yang Dandan, Salma Shaheen (2016)

Let S be a finite monoid and let A be an almost pure S-act. Then A is absolutely pure.

Consequently: $\mathcal{A} = \mathcal{A}_1$ is a finitary condition.

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Theorem: G, Yang Dandan (2016/7)

Let S be a right coherent monoid and let A be an almost pure S-act. Then A is absolutely pure.

That is,

$$S \text{ right coherent} \Longrightarrow \mathcal{A} = \mathcal{A}_1$$

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Counterexample G, Yang Dandan (2017)

No! The Fountain Monoid is an example of a non-coherent monoid such that $\mathcal{A} = \mathcal{A}_1$.

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For an S-act A we can build canonical absolutely pure (almost pure) extensions $A(\aleph_0)$ ($A(1)$).

Proposition G: 2017

The following are equivalent for a monoid S:

1. every almost pure S-act is absolutely pure;
2. for every **finitely generated subact of every finitely presented S-act** A, we have $A(1)$ is a retract of $A(\aleph_0)$.

Questions, Questions...!

- Does there exist a monoid S and an S-act A such that A is almost pure but not absolutely pure? Use the last result to write down a condition on chains of right congruences such that every almost pure S-act is absolutely pure; **now find an S not satisfying this condition.**

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- Determine exact connections of right coherency with products/ultraproducts of flat **left** S-acts.

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- Is $\mathcal{I}_X$ coherent? Is $\mathcal{T}_X$ right (or left) coherent?
- Determine exact connections of right coherency with products/ultraproducts of flat **left** S-acts.
- Other finitary conditions arise from model theoretic considerations of S-acts; many open questions remain!

Why am I interested in coherency?
Purity: absolute purity vs almost purity

Purity properties may be reformulated as weak injectivity properties. Injectivity may be reformulated as a stronger purity property.

Definition: A monoid is completely right injective (completely right pure)

if every S-act is injective (absolutely pure).

Fountain (1974), Isbell (1972) (following work of **Skornjakov (69)** and others)

Characterised completely right injective monoids in terms of right ideals and elements.

Completely right injective monoids

Theorem: Skornjakov (1969)

A monoid S is completely right injective if S has a left zero and S satisfies (*) for any right ideal I of S and right congruence ρ on S, there is an $s \in I$ such that for all $u, v \in S, w \in I$, $sw\,\rho\,w$ and if $u\,\rho\,v$ then $su\,\rho\,sv$.

Theorem: Fountain (1974)

A monoid S is completely right injective if and only if S has a zero, and each right ideal I has an idempotent generator e such that, for each pair of elements $a, b \in S \setminus I$, we have $a'ea\,\mathcal{R}\,b'eb$ for all $a' \in V(a), b' \in V(b)$ implies that $a'ea = b'eb$ for all $a' \in V(a), b' \in V(b)$.