A first introduction to graph reconstruction

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Abstract. In the present paper I give a brief introduction to the reconstruction conjecture for finite simple graphs.

1. Introduction

In this paper I only consider undirected, finite graphs without loops and multiple edges. I introduce to the reconstruction conjecture which is one of the longer standing open problems in graph theory. Bondy lists it as the first one in his list of beautiful conjectures in graph theory [**4**]. It dates back to Ulam's and Kelly's work in the 1940s [**5**] . Early related surveys were published by Harary, Bondy, Bondy and Hemminger and ONeil [**11, 12, 3, 5, 17**]. More recent results are contained in Lauri's chapter in [**10**]. Also [**21**] and [**18**] provide insight.

2. Basic concepts

I denote the cardinality of a finite set S by $|S|$. For each natural number n, I denote the set $\{1, 2, 3, \ldots, n\}$ by $\mathbb{N}_n$. For a mapping $\phi : D \to R$, I denote the set $\{\phi(d) \mid d \in D\}$ by $im(\phi)$. A graph G is a pair $G = (VG, EG)$ of a finite set VG of so-called vertices and a set EG of so-called edges, i. e. two-element-subsets of VG. Let $G = (VG, EG)$ be a graph. Instead of $\{g, g'\} \in EG$, I am going to write $gg' \in EG$. Let $K = (VK, EK)$ be a graph. Then a mapping $\phi : VG \to VK$ is called preserving, reflective or weakly reflective if $gg' \in EG$ implies $\phi(g)\phi(g') \in EK$, if $\phi(g)\phi(g') \in EK$ implies $gg' \in EG$, or if $\phi(g)\phi(g') \in EK$ implies the existence of $\gamma \in \phi^{-1}(\phi(g))$ and $\gamma' \in \phi^{-1}(\phi(g'))$, such that $\gamma\gamma' \in EG$, respectively. More commonly, weakly reflective mappings are being referred to as being strong [**15**]. A mapping $\phi : VG \to VK$ is called a morphism from G to K, if it is preserving. To denote that ϕ is a morphism, I write $\phi : G \to K$. A morphism $\phi : G \to K$ is called an isomorphism if it is invertible and its inverse is a morphism. Obviously ϕ is an isomorphism if and only if it is bijective, preserving and reflective. If there is an isomorphism from G to K then G and K are called isomorphic to each other. This then is denoted by $G \approx K$.

A graph $H = (VH, EH)$ is called a subgraph of G if $VH \subseteq VG$ and $EH \subseteq EG$. For any vertex g of the graph G the vertex-reduced subgraph G_g of G is the graph $G_g = (VG \setminus \{g\}, \{\gamma\gamma' \in EG \mid \gamma \neq g \neq \gamma'\})$. For any edge $gg' \in EG$ one says that g and g' are adjacent to one another. One also says that the edge gg' is incident with g as well as g'. The degree $\delta(g)$ of a vertex $g \in VG$ is the number of G's vertices that are adjacent to g. G is called regular, if all its vertices have the same degree. A vertex of degree 0, 1 or $|VG|-1$ is called an isolated vertex, end vertex or central vertex, respectively. A path P in G from $\alpha \in VG$ to $\omega \in VG$ is a sequence $P = g_0, g_1, g_2, \dots, g_n$ of vertices of G, such that $g_0 = \alpha$, $g_n = \omega$ and $g_i g_{i+1} \in EG$, for each $i \in \mathbb{N}_{n-1} \cup \{0\}$. The number n of edges of P is called that path's length. The distance $d(\alpha, \omega)$ between α and ω is the length of a shortest path in G between these vertices.

If there is a path from α to ω then these vertices are called reachable from one another. Reachability via a path is an equivalence relation on the set VG. The related equivalence classes are called G's connected components. The graph G is called connected if a path in G exists between any two vertices of it. G is called disconnected, if it is not connected. A vertex g of G is called a cut vertex of G if, removing it from G along with all the edges it is incident with, produces a graph with more connected components than G has. The graph $\overline{G} = (VG, \{\{g, g'\} \subseteq VG \mid gg' \notin EG\})$ is called the complement of G.

A hypomorphism $\sigma : G \to K$ from the graph G to the graph K is a bijection $\sigma : VG \to VK$, such that for each vertex g of G the vertex reduced subgraph G_g is isomorphic to the vertex reduced subgraph $K_{\sigma(g)}$. If there is a hypomorphism $\sigma : G \to K$, then K is called hypomorphic image or reconstruction of G. I am going to denote the set of hypomorphic images and the set of hypomorphisms from G to K by $Hyp(G)$ and $Hyp(G, K)$, respectively. A graph is called reconstructible, if it is isomorphic to each of its hypomorphic images. The reconstruction conjecture is the assertion, that each graph on at least three vertices is reconstructible. It is obvious that graphs with exactly two vertices have to be excluded from the reconstruction conjecture, since the path on two vertices and the edgeless graph on two vertices have isomorphic vertex reduced subgraphs.

Hypomorphisms induce a relation, hypomorphy, on the class of all graphs. As it is obviously reflexive, symmetric and transitive, it is an equivalence relation. One motive for studying graph reconstruction is, that this equivalence relation might actually be coarser than graph isomorphy and thus might give a graph classification that occasionally might be more helpful than graph isomorphy. Obviously, graph reconstruction research has gained some momentum due to the huge amount of available research.

As is well-known the key modeling language in computer science is graphs. For example, entity-relationship-diagrams are frequently used to specify database schemata. At times these schemata can become quite large, so that it is not a trivial matter to find out whether or not two given such diagrams A and B are isomorphic to each other, if considered as graphs. Obviously, one would tend to consider their information content not to be equal if they are not isomorphic to each other. Suppose further, that in the media, such as a sheet of paper or a computer screen, commonly used to represent these diagrams, only all but one of the entities of an entity-relationship-diagram can be represented.[1] Then the natural question arises: Suppose, two such diagrams are in a 1:1 relationship σ so that in each case of removing an entity a from diagram A and removing the corresponding entity $\sigma(a)$ from diagram B one ends up with isomorphic diagrams $A_a \approx B_{\sigma(a)}$. Is is then $A \approx B$? A final answer to that question could help to decide whether or not

[1]Note that each of these entities essentially represents a root of persistency, such as a table, for the data the database under design is going to hold.

two given schemas are equivalent with regard to their information content. Thus to some extent I would even argue that solving the reconstruction conjecture can have some practical impact.

3. Reconstruction folk knowledge

Let, in what follows, be $G = (VG, EG)$ a graph, $|VG| \geq 3$, $K = (VK, EK) \in Hyp(G)$ and $\sigma \in Hyp(G, K)$. Strangely, as it might look at the first glance, hypomorphisms preserve quite a number of interesting graph characteristics. This starts with the edge count:

Proposition 1.

$|EG| = |EK|$ and $\delta(g) = \delta(\sigma(g))$, for each $g \in VG$.

Proof. By definition, $G_g \approx K_{\sigma(g)}$, for each $g \in VG$. It follows

$$\sum_{g \in VG} |EG_g| = \sum_{g \in VG} |EK_{\sigma(g)}|$$

Since $|EG_g| = |EG| - \delta(g)$ and $|EK_{\sigma(g)}| = |EK| - \delta(\sigma(g))$ it follows $\sum_{g \in VG} |EG_g| = \sum_{g \in VG} |EG| - \sum_{g \in VG} \delta(g)$ and similarly for K. Since the sum of the vertex degrees is twice the edge count, it follows with $m = |VG|$, that

$$(m-2)|EG| = (m-2)|EK|$$

and thus $|EG| = |EK|$. Because of that also $\delta(g) = \delta(\sigma(g))$, for each $g \in VG$. □

Proposition 2.

If G is regular then it is reconstructible.

Proof. Let G be regular of the degree d. Then also K is regular of the degree d. Now, consider a vertex $g \in VG$ and an isomorphism $\phi : G_g \to K_{\sigma(g)}$. Then $|VG| \cdot d \div 2 = |EG| = |EK|$. The subgraph of K induced by $im(\phi)$, however, in comparison to K is short of d edges. It thus consequently has d vertices with the degree $d-1$ in $K_{\sigma(g)}$. Since on the other hand $\sigma(d)$ has the degree d in K, each of the former vertices is adjacent to $\sigma(d)$. Finally, the vertices of degree $d-1$ exactly are the images of vertices of degree $d-1$ in G_g under ϕ. Consequently, $G \approx K$. □

It is well-known that disconnected graphs are reconstructible. To show this, the following Lemma is helpful:

Lemma 3.

If G is connected then at least one of its vertices is not a cut vertex.

Proof. To show this, let G be connected. Consider two vertices α and ω of G such that their distance $M = d(\alpha, \omega)$ is the maximal distance in G. Let $x_1, x_2, x_3, \dots x_n$ be all the neighbors of ω. Assume, that ω is a cut vertex. Then there exist vertices p, q of G such that each path between p and q includes ω. Then, without loss of generality, each such path contains the two-path $x_1 - \omega - x_2$. However, for each $i \in \mathbb{N}_n$, the distance $d(x_i, \alpha) < M$ and hence a path from x_i to α exists that does not include ω. Considering now a path of shortest length between p and q reveals that it has exactly one occurrence of ω in it. Now, since there are paths between x_1 and α and x_2 and α, that do not include ω, contrary to the implication from the assumption, there is a path between p and q, that does not include ω. Thus the assumption is wrong and ω is not a cut vertex. Obviously, the same reasoning applies to α. $\square$

The Lemma can now be used to establish that hypomorphisms preserve connectedness.

PROPOSITION 4.

If G is connected then K is connected.

PROOF. The Lemma implies that there is a non-cut-vertex g in G. Consequently G_g and thus $K_{\sigma(g)}$ is connected. Assume K is disconnected. Since σ preserves the vertex degree neither G nor K has an isolated vertex. Therefore, contrary to $K_{\sigma(g)}$ being connected, each vertex reduced subgraph of K is disconnected. Thus K is connected. $\square$

PROPOSITION 5.

If G is disconnected then G is reconstructible.

PROOF. If G has an isolated vertex g, then also $\sigma(g)$ is isolated. Since $G_g \approx K_{\sigma(g)}$ this implies that $G \approx K$. I may thus now suppose that neither G nor K has an isolated vertex. Each connected component of G thus at least has two elements. The Lemma implies that the number of connected components of G is the same as the one of K. Let that number be n. Then G has exactly the connected components $G_1, G_2, G_3, \ldots, G_n$. Choose non-cut-vertices g,h from G_1 and G_2, respectively. Then $K_{\sigma(g)}$ has the connected components $K_1', K_2, K_3, \ldots, K_n$ and $K_{\sigma(h)}$ has the connected components $K_1, K_2', K_3, \ldots, K_n$ and without loss of generality I may suppose that (1) $\sigma_g(G_3, \ldots, G_n) = \{K_3, \ldots, K_n\} = \sigma_h(G_3, \ldots, G_n)$, (2) K_1' and K_2' is a subgraph of K_1 and K_2, respectively and (3) $K_i \approx G_i$, for each $i \in \mathbb{N}_n$. Therefore $G \approx K$. $\square$

PROPOSITION 6.

(i) G is reconstructible if and only if its complement is reconstructible.
(ii) G is reconstructible if it has a central vertex.

PROOF. As the assertion (i) holds trivially it suffices to focus on the assertion (ii). It, however, follows from the assertion (i), the Proposition 5 and the observation that a vertex is central in a graph if and only if it is isolated in that graph's complement. $\square$

It was the first major result in graph reconstruction, that trees on at least three vertices are reconstructible [**13**]. Graphs with less than twelve vertices are known to be reconstructible [**16**]. Finally, almost all graphs are reconstructible [**1**]. The reconstruction conjecture is not true for tournaments and for infinite graphs [**9, 19**].

Following Harary one calls each isomorphy class of a vertex reduced subgraph of G a card of G and the multi set of all its cards the deck DG of G. The reconstruction conjecture, RC, can then, using that terminology, be rephrased as, HC, i. e. that G up to isomorphism can be recovered from its deck.

PROPOSITION 7.

RC is true if and only if HC is true.

PROOF. Let X, Y be graphs with at least three vertices. Let at first be RC true. Consider the deck DX of X and suppose that DX too is the deck of Y. Then $Y \in Hyp(X)$ and $X \approx Y$, i. e. HC. Let now HC be true. Then X, up to isomorphism, can be recovered from its deck DX. Let now be $Y \in Hyp(X)$. Then $DX = DY$ and X as well as Y, up to isomorphism, can be recovered from DX and DY. Therefore $X \approx Y$ and thus RC is true. □

Harary has also brought up the legitimate deck problem that asks to, for a given multiset of equivalence classes of graphs, decide whether or not it is a deck of a graph. To my knowledge also this problem has resisted a solution so far.

Since it turned out to be hard to settle the reconstruction conjecture, colleagues have turned to two major approaches with which they seek new results. First, one tries to prove that certain classes of graphs are reconstructible. I have discussed elementary examples of that before and the related graph classes were disconnected graphs, regular graphs and trees. Second, one tries to recover graph characteristics such as vertex count, edge count, number of connected components etc. from the deck, or which is sort of the same, one tries to show that hypomorphisms preserve the related graph characteristics. Obviously, the first strategy is about sufficient conditions and the second strategy is about necessary conditions for a graph to be reconstructible. A third possibility would, however, be, to provide necessary and sufficient conditions for the reconstruction conjecture to be true.

Obviously, in any category reconstruction can be considered in which the concept of maximal sub-object is defined.

4. Concerning the truth of the reconstruction conjecture

It seems to me that a graph, for whom the reconstruction conjecture is not true, must have certain structural characteristics that set it apart from the many known reconstructible graphs. It therefore comes to mind that the non-reconstructible graphs can be characterized in terms of forbidden subgraphs. However, things are not that easy since, for any graph G, by adding an isolated vertex to it, reconstructibility is attained and thus no such forbidden subgraphs in the conventional sense exist. My recent result, listed below, indicates that also the possibilities are quite limited to characterize non-reconstructible graphs by substructures that can be extended into a co-retract.

As I like to view the matter, a very smart, swift and virtually immortal creature could, if needed, first create each finite simple graph and ultimately judge as to whether or not it is reconstructible. Thus such creature would be able to decide about the truth of the reconstruction conjecture. In my view thus the problem regarding the reconstruction conjecture is to find a proof or refutation of it and, if that should turn out to be impossible within the language of current finite set theory, add the weakest possible axiom such that the reconstruction conjecture becomes provable or refutable. The possibility of the reconstruction conjecture actually requiring an increase of the strength of the axiomatic base of finite set theory, suggests that it might be particularly promising to work more on necessary and sufficient conditions for the reconstruction conjecture to be true.

Clearly, by applying it recursively, if the reconstruction conjecture is true, then each graph up to isomorphism can be recovered from the multi set of isomorphism classes of its subgraphs on three vertices. For a more profound discussion see

[20]. Schwenk [18] has said, that he does not believe anymore the reconstruction conjecture is true.

5. A new result

Any neighbor of an end vertex is called that vertices bucket. The order of a bucket is the number of end vertices that share the bucket. The runway $\rho(g)$ of an end vertex g of a graph G is the longest path $\rho(g) = g_0 - g_1 - g_2 - \ldots - g_{n-1} - g_n$ such that $g = g_0$ and $\delta(g_1) = 2$, for each $i \in \mathbb{N}_{n-1} \setminus \{0\}$. The length $\lambda(\rho(g))$ of the runway $\rho(g)$ of an end vertex g is the number of its edges.

THEOREM 8. [**14**]

A graph on at least three vertices is reconstructible, if it has a bucket of order at least two or a runway of length at least two.

COROLLARY 9.

Each tree on at least three vertices is reconstructible.

PROOF. Consider a leaf l of a tree T on at least three vertices. If the leaf has a sibling then the Theorem warrants the reconstructibility of T. If the leaf has no sibling, then it has a runway of a length at least two. Again the Theorem steps in. □

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