Decomposing 4-connected planar triangulations into two trees and one path

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Abstract

Refining a classical proof of Whitney, we show that any 4-connected planar triangulation can be decomposed into a Hamiltonian path and two trees. Therefore, every 4-connected planar graph decomposes into three forests, one having maximum degree at most 2. We use this result to show that any Hamiltonian planar triangulation can be decomposed into two trees and one spanning tree of maximum degree at most 3. These decompositions improve the result of Gonçalves [Covering planar graphs with forests, one having bounded maximum degree. *J. Comb. Theory, Ser. B*, 100(6):729–739, 2010] that every planar graph can be decomposed into three forests, one of maximum degree at most 4. We also show that our results are best-possible.

1 Introduction

All graphs considered here are finite, undirected, and simple, i.e., contain no loops nor multiple edges. The *fractional arboricity* $a_f(G) := \max_{S \subseteq V; |S| \geq 2} \frac{|E(S)|}{|S|-1}$ of a graph $G = (V, E)$ (sometimes also denoted by $\Upsilon_f(G)$) is a classical measure of density, introduced by Payan [13]. The famous Nash-Williams Theorem [12] says that the edges of any graph G can be decomposed into $\lceil a_f(G) \rceil$ forests. While the number of forests cannot be reduced, it might still be possible to improve the decomposition by imposing a low maximum degree on one forest.

For positive integers k, d, a graph is called $(k,d)^\star$*-decomposable* if its edges decompose into $k+1$ forests, one of maximum degree at most d. A weaker notion than $(k,d)^\star$-decomposable is the following: A graph is called (k,d)*-decomposable* if its edges decompose into k forests and one subgraph of maximum degree at most d. Let us also define the following notion that is stronger than $(k,d)^\star$-decomposability: A graph is called $[k,d]^\star$*-decomposable* if its edges decompose into k forests and one tree of maximum degree at most d.

Affirming a conjecture of Balogh *et al.* [1], Gonçalves [5] proves that every planar graph is $(2,4)^\star$-decomposable, while large enough planar 3-trees are not $(2,3)$-decomposable [1].

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Our Results. In this paper we consider $[k,d]^\star$-decompositions of maximally planar graphs, which we will call planar triangulations, or just triangulations for short. First, we improve the classical result of Whitney [16] that every 4-connected triangulation is Hamiltonian. Recall that a graph $G = (V, E)$ is 4-connected if $|V| \geq 5$ and for any triple S of vertices the graph $G-S$ is connected. Moreover, a Hamiltonian path in G is a simple path in G that contains all vertices of G, and G is Hamiltonian if it contains at least one Hamiltonian path.

Theorem 1.1. *Every* 4*-connected planar triangulation* G *decomposes into two trees and one Hamiltonian path. In particular,* G *is* $[2,2]^\star$*-decomposable.*

Combining this result with recursive decompositions of non-4-connected triangulations, we obtain decompositions of Hamiltonian triangulations.

Theorem 1.2. *Every Hamiltonian planar triangulation* G *decomposes into two trees and a spanning tree of maximum degree* 3*. In particular,* G *is* $[2,3]^\star$*-decomposable.*

Furthermore, our methods give a new proof of (a slight strengthening of) Gonçalves' result.

Theorem 1.3. *Every planar triangulation* G *decomposes into two trees and a spanning tree of maximum degree* 4*. In particular,* G *is* $[2,4]^\star$*-decomposable.*

Finally, we show that all our results are best-possible, where in the last case we provide a family of examples that is richer than just planar 3-trees, as given in [1].

Theorem 1.4. *Each of the following holds.*

(i) Some 4*-connected planar triangulations are not* $(2,1)$*-decomposable.*

(ii) Some Hamiltonian planar triangulations are not $(2,2)$*-decomposable.*

(iii) Some planar triangulations are not $(2,3)$*-decomposable.*

Related Work. While the present paper is focused on planar triangulations and the cases $k = 2$, $d \in \{2,3\}$, let us give an account of the history of $(k,d)^\star$-decomposability and (k,d)-decomposability for general graphs and general integers $k, d \geq 1$.

One motivation for $(k,d)^\star$-decomposability are applications to bounding the (incidence) game-chromatic number [2, 6, 11] and the spectral radius of a graph [4]. However, most of the research in this field was inspired by the famous Nine Dragon Tree Conjecture of Montassier, Ossona de Mendez, Raspaud, and Zhu [10], which states that if the difference between $\lceil a_f(G) \rceil$ and $a_f(G)$ is large, then G decomposes into $\lceil a_f(G) \rceil$ forests, where the maximum degree of one forest can be bounded. More precisely, if $a_f(G) \leq k + \frac{d}{k+d+1}$ for positive integers k, d, then G is $(k,d)^\star$-decomposable. The Nine Dragon Tree Conjecture was proved for several special cases [3, 8–10] before it was confirmed in full generality by Jiang and Wang in 2016 [7].

The original motivation for the Nine Dragon Tree Conjecture in [10] comes as a generalization of decomposition results in sparse planar graphs: Planar graphs are known to be $(1,1)^\star$-decomposable, i.e., decompose into a forest and a matching, when they have girth at least 8 [10, 15], while some planar graphs of girth 7 are not $(1,1)$-decomposable [10]. For $d = 2$, He *et al.* [6] show that planar graphs of girth at least 7 are $(1,2)$-decomposable, which was improved to $(1,2)^\star$-decomposability by Gonçalves [5], while some planar graphs of girth 5 are not $(1,2)$-decomposable [10]. In [6] it is further shown that planar graphs of girth at least 5 are $(1,4)$-decomposable. All these decomposition results follow immediately from the Nine Dragon Tree Theorem [7] as for every planar graph G of girth at least g we have $a_f(G) \leq \frac{g}{g-2}$, and thus these decompositions rely purely on the low fractional arboricity of graphs.

However, for planar triangulations the fractional arboricity tends to 3 as the number of vertices tends to infinity. Thus the Nine Dragon Tree Theorem does not give $(2,d)^\star$-decomposability of all planar graphs for any fixed d. Hence, for the following results (as well as our results in the present paper) the structure of planar graphs had to be exploited on a different level: In [6] it is shown that planar graphs are $(2,8)$-decomposable, which is strengthened to $(2,8)^\star$-decomposability in [1]. Moreover, Balogh *et al.* [1] show that Hamiltonian and consequently 4-connected planar graphs are $(2,6)$-decomposable. Finally Gonçalves [5] improved these results to $(2,4)^\star$-decomposability of all planar graphs, which is best-possible [1].

Organization of the Paper. In Section 2 we prove our key lemma, Lemma 2.2, which is crucial for all our decomposition results. In Section 3 we decompose any planar triangulation along its separating triangles and introduce triangle assignments. Here we also combine Lemma 2.2 to obtain $[2,k+2]^\star$-decompositions for $k \in \{0,1,2\}$ of planar triangulations admitting so-called k-assignments, c.f. Proposition 3.2. In Section 4 we prove our main decomposition results, namely Theorems 1.1–1.3. To this end, we show that 4-connected (respectively Hamiltonian and general) triangulations admit 0-assignments (respectively 1-assignments and 2-assignments) and use the decompositions given by Proposition 3.2 in Section 3. In Section 5 we show that our results are best-possible by constructing 4-connected (respectively Hamiltonian and general) triangulations that are not $(2,1)$-decomposable (respectively $(2,2)$-decomposable and $(2,3)$-decomposable); In other words, we prove Theorem 1.4. Finally, we conclude the paper in Section 6.

2 The Key Lemma

This section is devoted to the proof of Lemma 2.2, which is a central element of the proofs of all our Theorems.

Let G be a plane embedded graph with a simple outer cycle C. Moreover, let G be inner triangulated, that is, every inner face of G is a triangle. For two outer vertices u, v of G we denote by P_{uv} the path from u to v along the outer cycle in counterclockwise direction, and define $P^\circ_{uv} = P_{uv} \setminus \{u,v\}$. If $u = v$ then P_{uv} consists of only one vertex and $P^\circ_{uv} = \emptyset$. A *filled triangle* in G is a triple of pairwise adjacent vertices, such that at least one vertex of G lies inside this triangle. A *separating triangle* in G is a triple of pairwise adjacent vertices, such that at least one vertex of G lies inside this triangle and at least one vertex lies outside this triangle. It is well-known and easy to see that a planar triangulation is 4-connected if and only if it does not have[1] any separating triangles.

Definition 2.1. *A plane inner triangulated graph $G = (V, E)$ with simple outer cycle C is a* Whitney graph with respect to (x, y, z) *if x, y, z are outer vertices of G with $z \neq x, y$, such that:*

- *G contains no filled triangle.*
- *x, y, z appear in this counterclockwise order around C.*
- *P_{xy}, P_{yz}, P_{zx} are induced paths in G.*
- *If $x = y$, then zx is not an edge of G.*

Note that the outer face of a Whitney graph with at least four vertices cannot be a triangle. Moreover, any inner triangulated 4-connected graph with some outer vertices x, y, z in this counterclockwise order is a Whitney graph with respect to (x, y, z).

Lemma 2.2. *If $G = (V, E)$ is a Whitney graph with respect to (x, y, z), then the edges of G can be oriented and colored black, red, and blue, such that each of the following holds.*

(1) *The black edges form a directed Hamiltonian path from x to z in G.*

(2) *Every inner vertex has precisely one outgoing red edge and one outgoing blue edge.*

(3) *Every vertex on P°_{xy} has precisely one outgoing blue edge and no outgoing red edge.*

(4) *Every vertex on P°_{yz} has precisely one outgoing red edge and no outgoing blue edge.*

(5) *Every vertex on P°_{zx} has precisely one outgoing blue edge and no outgoing red edge.*

(6) *Neither y nor z has outgoing red edges nor outgoing blue edges.*

(7) *If $x \neq y$, then vertex x has precisely one outgoing blue edge which is xz if it exists and no outgoing red edge. If $x = y$, then x has no outgoing blue and no outgoing red edge.*

(8) *There is no monochromatic directed cycle in G.*

Proof. Recall that our proof is based on the decomposition of planar 4-connected triangulations by Whitney [16]. We do induction on the number $|V|$ of vertices of G. If $|V| = 3$ then G is a triangle with vertices x, y and z. We define P to be the path $x - y - z$ and orient xz from x to z and color it blue. It is easy to see that (1)–(8) are satisfied. If $|V| > 3$ we distinguish seven cases, which we go through in this order, i.e., when considering Case i we sometimes make use of the fact that Case j does not apply for $j < i$.

Case 1: $x = y$.
Let x' and y' be the neighbor of x on P_{zx} and P_{yz}, respectively. Then $G' = G \setminus \{x\}$ is a Whitney graph with respect to (x', y', z). Indeed the

[1]Using planar triangulation instead of plane triangulation is justified as the existence of separating triangles is independent of the chosen plane embedding.

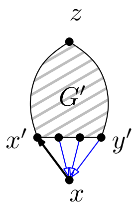


Figure 1: **Case 1:** $x = y$

vertices on $P_{x'y'}$ are neighbors of x and hence a chord $\{u, v\}$ in $P_{x'y'}$ would give a separating triangle $\{x, u, v\}$ in G. Hence $P_{x'y'}$ is an induced path in G'. Moreover, $P_{y'z}$ and $P_{z,x'}$ are subsets of the induced paths P_{yz} and P_{zx}, respectively, and thus induced, too.

By induction there is a Hamiltonian path P' from x' to z in G' and an orientation and coloring of the edges in $E(G') \setminus E(P)$, such that (2)–(8) are satisfied. We extend P' by the edge xx', i.e., $P = \{xx'\} \cup P'$, and the coloring/orientation by orienting all incident edges at x (except xx') towards x and coloring them red. See Figure 1 for an illustration.

We need to argue that P together with the coloring/orientation satisfies (2)–(8). Indeed (2) follows from (2), (3) with respect to G' and the coloring/orientation of the edges incident to x, for (3) there is nothing to show, (4)–(6) follow from (4)–(6) with respect to G', and (7) follows again from the coloring/orientation of the edges incident to x. Finally, we need to show (8), namely that there is no directed monochromatic cycle in G. By (8) with respect to G' such a cycle would contain x, which has not incoming and outgoing edges of the same color.

Case 2: There is an edge xu with $u \in P^{\circ}_{yz}$.

We choose $u \in P^{\circ}_{yz}$ to be the vertex that is a neighbor of x and is closest to z on P°_{yz}. The three illustrations in Figure 2 display the three subcases that we sometimes have to treat differently along the construction: $zx \notin E$, $zx \in E$ and $uz \notin E$, and $zx, uz \in E$.

Define G_1 to be the inner triangulated subgraph of G with outer cycle $P_{xy} \cup P_{yu} \cup \{ux\}$. Then G_1 is a Whitney graph with respect to (x, y, u). We define G_2 to be the graph $G \setminus (G_1 \setminus \{u\})$. If both, xz and uz, are edges in G, then G_2 is just a single edge, which we put into the Hamiltonian path. Otherwise G_2 is a Whitney graph with respect to (u, y', z) (or rather (u, u, z) in case $xz \in E$), where y' is the neighbor of x on P_{zx} if $xz \notin E$. However, if $xz \notin E$ the embedding of G_2 needs to be flipped so that u, y', z appear in counterclockwise order. Indeed $P_{y'u}$ is induced since it consists solely of neighbors of x, and similarly P_{uz} is induced in case $xz \in E$. We apply induction to both, G_1 and G_2 (if G_2 is not just an edge), and concatenate the obtained Hamiltonian paths in G_1 and G_2 to a Hamiltonian path from x to z in G.

It remains to color and orient the edges incident to x, but not contained in G_1. Additionally, we also define a color and orientation for the edge ux, disregarding its color and orientation given by induction on G_1. (Note that ux is certainly not in the Hamiltonian path of G_1.) If $xz \in E$ we

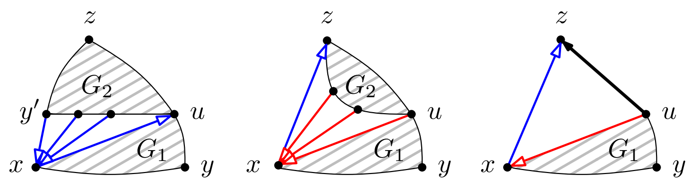


Figure 2: Considered subcases of **Case 2:** There is an edge xu with $u \in P^\circ_{yz}$.

orient these edges, as well as the edge ux, towards x and color them red, except for xz which we color blue and orient to z. On the other hand if $xz \notin E$ we swap the colors red and blue in the coloring for G_2, but keep the orientation the same. Moreover, we color the edges incident to x blue and orient them towards x, except for xu which is oriented towards u. See Figure 2 for an illustration.

It is straightforward to check that (2)–(7) follow from (2)–(7) with respect to G_1 and G_2 and the coloring/orientation of the edges incident to x. Moreover, by (8) with respect to G_1 and G_2, every directed monochromatic cycle has to contain x. The only case in which x has incoming and outgoing edges of the same color is when $xz \notin E$. There the only cycle would have to be blue and go through u which has no blue outgoing edges by (7) for G_2 and since red and blue were swapped in G_2. Thus there is no such directed monochromatic cycle in G, i.e., (8) is satisfied.

Case 3: There is an edge uv with $u \in P^\circ_{xy}$ and $v \in P_{zx}$.

We choose $u \in P^\circ_{xy}$ to be the vertex that has a neighbor on P_{zx} and is closest to x on P_{xy}, and v to be the neighbor of u on P_{zx} that is closest to z on P_{zx}. Note that $u, v \neq x, y$, but possibly $v = z$. The three illustrations in Figure 2 display the three sub-cases that we sometimes have to treat differently along the construction: $xv \notin E$, $xv \in E$ and $xu \notin E$, and $xv, xu \in E$. We define G_2 to be the inner triangulated subgraph of G with outer cycle $P_{uy} \cup P_{yz} \cup P_{zv} \cup \{vu\}$. Then G_2 is a Whitney graph with respect to (u, y, z). Indeed, P_{zu} in G_2 is induced since P_{zx} in G is and by the choice of v there is no edge between u and $P_{zv} \setminus \{v\}$. Next consider $G_1 = G \setminus (G_2 \setminus \{u\})$ and the neighbor y' of v on P_{vx}. If both, xu and xv, are edges in G, then G_1 is just the edge xu and we put this edge into the Hamiltonian path. Otherwise G_1 is a Whitney graph with respect to (x, y', u), since, by the choice of u, there is no edge from v to a vertex on P_{xy} between x and u. However, the embedding of G_1 needs to be flipped so that x, y', u appear in counterclockwise order.

We apply induction to both, G_1 (if G_1 is not just an edge) and G_2, and concatenate the obtained Hamiltonian paths in G_1 and G_2 to a Hamiltonian path from x to z in G. We orient the edges incident to v towards v and color them blue. Note that if $y' = x$ then the vertices on P_{xy} that are contained in G_1 have no outgoing *red* edge, since we flipped the embedding of G_1. Similarly the neighbors of v in G_1 have no outgoing blue edge within G_1.

It is again straightforward to check that (2)–(7) follow from (2)–(7) with respect to G_1 and G_2 and the coloring/orientation of the edges incident

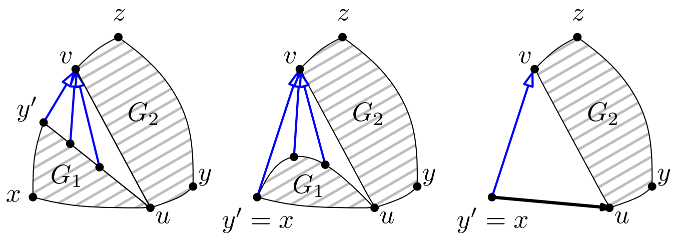


Figure 3: Considered subcases of **Case 3:** There is an edge uv with $u \in P^{\circ}_{xy}$ and $v \in P_{zx}$.

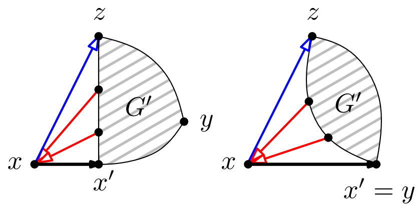


Figure 4: Considered subcases of **Case 4:** $xz \in E$

to v and u. It remains to show that (8) is satisfied, i.e., there is no monochromatic directed cycle in G. By (8) with respect to G_1 and G_2, such a cycle would contain edges from G_1 to v and pass through u, but u has no outgoing edges towards G_1.

Case 4: $xz \in E$**.**

Let x' be the neighbor of x in P_{xy}. Note that possibly $x' = y$, which is illustrated second in Figure 4. Moreover $x'z$ is not an edge since Case 3 does not apply. Now $G' = G \setminus \{x\}$ is a Whitney graph with respect to (x', y, z), since the vertices in $P_{zx'}$ are neighbors of x and hence $P_{zx'}$ is induced. We apply induction to G' and extend the obtained Hamiltonian path P' by the edge xx' and orient xz from x to z and color it blue. We orient the remaining edges incident to x towards x and color them red. See Figure 4 for an illustration.

Now (2) follows from (2), (5) with respect to G' and the coloring/orientation of the edges incident to x, (3), (4) and (6) follow from (3), (4) and (6) (and (7) in case $x' = y$) with respect to G', and (5) and (7) follow again from the orientation/coloring of the edges at x. Finally, (8) follows from (8) with respect to G' and the fact that x has no outgoing and incoming edges of the same color.

Case 5: $yz \in E$**.**

Let y' be the neighbor of z in P_{zx}. Note that $y' \neq x$ since Case 4 does not apply. Now $G' = G \setminus \{z\}$ is a Whitney graph with respect to (x, y', y). However, the embedding of G' needs to be flipped so that x, y', y appear in counterclockwise order. Indeed $P_{yy'}$ is an induced path since all its vertices are neighbors of z. We apply induction to G' and extend the obtained Hamiltonian path P' by the edge yz. We orient the remaining edges incident to z towards z and color them blue. See Figure 5 for an

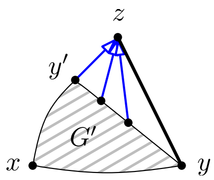


Figure 5: **Case 5:** $yz \in E$

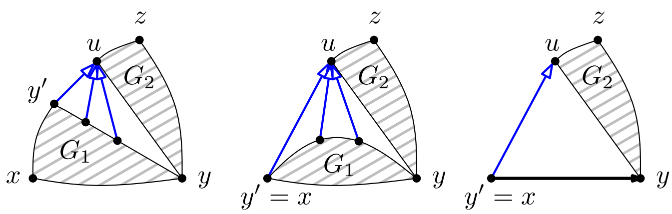


Figure 6: Considered subcases of **Case 6:** There is an edge yu with $u \in P^\circ_{zx}$.

illustration.

Now (2) follows from (2), (4) with respect to G' and the coloring/orientation of the edges incident to z, (3) and (5)–(7) follow from (3) and (5)–(7) with respect to G', and for (4) there is nothing to show. Finally, (8) follows from (8) with respect to G' and the fact that z has no outgoing and incoming edges of the same color.

Case 6: There is an edge yu with $u \in P^\circ_{zx}$.

We choose u to be any neighbor of y on P_{zx}. Note that $u \neq z$ since Case 5 does not apply. We define G_2 to be the inner triangulated subgraph of G with outer cycle $P_{yz} \cup P_{zu} \cup \{uy\}$. Then G_2 is a Whitney graph with respect to (y, u, z). However, the embedding of G_2 needs to be flipped so that y, u, z appear in counterclockwise order. Indeed, P_{zy} in G_2 is induced since P_{zu} in G is so and by the choice of u there is no edge between y and $P_{zu} \setminus \{u\}$. Next consider $G_1 = G \setminus (G_2 \setminus \{y\})$ and the neighbor y' of u on P_{ux}. The three illustrations in Figure 6 display the three subcases that we sometimes have to treat differently along the construction: $xu \notin E$, $xu \in E$ and $xy \notin E$, and $xu, xy \in E$. If both, ux and xy, are edges in G, then G_1 is just the edge xy and we put this edge into the Hamiltonian path. Otherwise G_1 is a Whitney graph with respect to (x, y', y). However, the embedding of G_1 needs to be flipped so that x, y', y appear in counterclockwise order.

We apply induction to both, G_1 (if G_1 is not just an edge) and G_2, and concatenate the obtained Hamiltonian paths in G_1 and G_2 to a Hamiltonian path from x to z in G. We swap the colors red and blue in the coloring for G_2, but keep the orientation the same. We orient the edges incident to u towards u and color them blue. Note that if $y' = x$ then the vertices on P_{xy} that are contained in G_1 have no outgoing *red* edge, since we flipped the embedding of G_1. Similarly the neighbors of u in G_1 have no outgoing blue edge within G_1.

It is again straightforward to check that (2)–(7) follow from (2)–(7) with respect to G_1 and G_2 and the coloring/orientation of the edges incident to u and y. By (6) there is no outgoing edge at y towards G_1. Thus no directed cycle contains y and hence from (8) with respect to G_1 and G_2 follows (8) for G, i.e., there is no directed monochromatic cycle in G.

Case 7: None of Case 1 – Case 6 applies.

Let uv be the edge between a vertex $u \in P^\circ_{xy}$ and a vertex $v \in P^\circ_{yz}$ farthest away from y. If no such edge exists set $u = y$ and let v be the neighbor of y and on P_{yz}. Then $u \neq x$ and $v \neq z$, since otherwise an earlier case would apply.

Let w be the neighbor of x on P_{zx}. Since earlier cases do not apply, we have $w \neq z$. Consider the subgraph $\tilde{G}$ of G induced by all vertices that are not on P_{xy} but have at least one neighbor in P_{xy}. Since none of Case 2, Case 3 and Case 6 applies, v and w are the only outer vertices of G that are contained in $\tilde{G}$. Let P be a shortest vw-path in $\tilde{G}$. (Clearly $\tilde{G}$ is connected since G is inner triangulated.) We define G' to be the inner triangulated subgraph of G with outer cycle $P \cup P_{vz} \cup P_{zw}$. Then G' is a Whitney graph with respect to (v, w, z). However, the embedding of G' needs to be flipped so that v, w, z appear in counterclockwise order. Indeed, the path P is induced since it is a shortest path.

Moreover, if $u \neq y$ define G'' to be the inner triangulated subgraph of G with outer cycle $P_{uy} \cup P_{yv} \cup vu$, which clearly is a Whitney graph with respect to (u, y, v). If $u = y$, then v is the neighbor of y on P_{yz} and we set G'' to be the edge uv.

Furthermore we define $G_1, \ldots, G_k$ to be the blocks of $G \setminus (G' \cup G'' \setminus \{u\})$, where the numbering is according to their appearance along P_{xy}. For every $i = 1, \ldots, k$ the graph G_i contains two vertices x_i and z_i that lie on P_{xy} and have a neighbor on P. Let x_i be the one that is closer to x on P_{xy}. If x_i and z_i have a common neighbor on P, then G_i is either just an edge or a Whitney graph with respect to (x_i, x_i, z_i). Otherwise there is a unique vertex y_i in G_i that has two neighbors on P, and G_i is a Whitney graph with respect to (x_i, y_i, z_i). Whenever G_i is not just an edge, we flip the embedding of G_i so that (x_i, y_i, z_i) appear in counterclockwise order.

We color and orient all the Whitney graphs $G', G'', G_1, \ldots, G_k$ by induction, where if $G'' = uv$, we put uv directed from u to v into the black path, and whenever $G_i = x_i z_i$, we put $x_i z_i$ directed from x_i to z_i into the black path. In G' we swap the colors red and blue.

The remaining edges emanating from P outside G' are oriented as follows. All such edges at a vertex on P are blue and oriented towards it until the last edge from P_{xy}, which is blue but oriented towards P_{xy}. If there are more edges to the vertex they are red and oriented towards it. See Figure 7 for an illustration.

It is maybe a bit tedious but again straightforward to check that (2)–(7) follow from (2)–(7) with respect to $G', G'', G_1, \ldots, G_k$ after swapping red and blue in G'.

In order to see (8), first note that no red edges leave G' and thus any monochromatic cycle would have to be blue. Moreover, vertices on P have no blue outgoing edges towards G', i.e., a blue cycle has to use $G_1, \ldots, G_k$

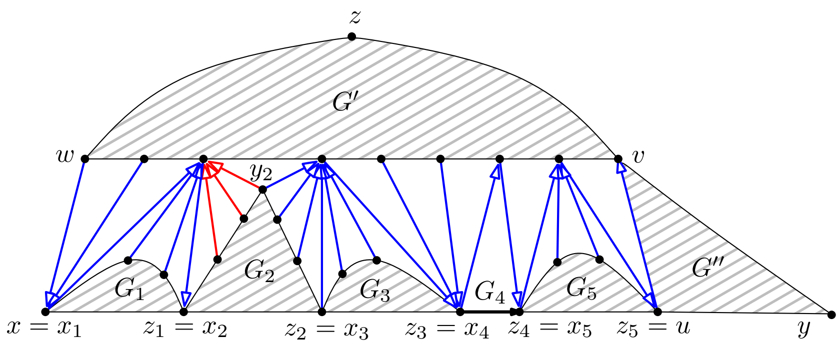


Figure 7: **Case 7:** None of Case 1 – Case 6 applies.

and the explicitly colored edges. The latter form a set of rooted trees, where the only vertices with ingoing blue edges in $G_1, \ldots, G_k$ are the vertices $x_1, z_1, \ldots, z_k$, since these have no outgoing blue edges within the respective G_i any directed blue path continues to the right until eventually reaching u and then possibly v, which has no blue outgoing edge. This gives (8) and concludes the proof. □

3 Triangle Assignments

In this section we will establish the notion of triangle assignments and under which conditions they are sufficient to apply Lemma 2.2 to recursive decompositions of planar triangulations.

Let G be a plane triangulation and let $X = X(G)$ and $Y = Y(G)$ denote the set of filled triangles in G and inner faces in G, respectively. In particular, $X \dot\cup Y$ is a partition of the triangles in G, we have $|X| = 0$ if and only if G is just a triangle, and we have $|X| = 1$ if and only if G is 4-connected. The set X is naturally endowed with a partial order $\prec$, where $\Delta \prec \Delta'$ whenever the interior of Δ is strictly contained in the interior of Δ'. Since any two filled triangles Δ_1, Δ_2 that both contain a third filled triangle Δ_3 (i.e., $\Delta_3 \prec \Delta_1$ and $\Delta_3 \prec \Delta_2$) are necessarily contained in each other ($\Delta_1 \prec \Delta_2$ or $\Delta_2 \prec \Delta_1$), we have that the partially ordered set $(X, \prec)$ has the structure of a rooted tree T_G whose root is the outer triangle Δ_{out} and where $\Delta \prec \Delta'$ if and only if Δ, Δ' lie on a root-to-leaf path in T with Δ' being closer to the root Δ_{out} than Δ. Tree T_G is sometimes called the *separation tree* of G, see [14] for an early appearance of the term.

For each triangle $\Delta \in X$, let G_Δ denote the subgraph of G induced by Δ and all vertices inside Δ. We further define ∂G to be the plane triangulated graph obtained from G by removing for each $\Delta \in X(G) - \Delta_{\text{out}}$ all inner vertices of G_Δ. Note that $\partial G = G$ if and only if G is 4-connected. For the graphs G_Δ and ∂G_Δ with $\Delta \in X$ we have that

(P1) ∂G_Δ is a 4-connected triangulation for every $\Delta \in X(G)$,

(P2) every separating triangle of G is an inner face of ∂G_Δ for exactly one $\Delta \in X(G)$,

(P3) the graphs $\{\partial G_\Delta - E(\Delta) \mid \Delta \in X(G)\}$ form an edge-partition of G.

Properties **(P1)**–**(P3)** will enable us to find $[2, k]^\star$-decompositions of G for small k, based on $[2, 2]^\star$-decompositions of $\partial G_\Delta - E(\Delta)$, $\Delta \in X(G)$, given by the following lemma. Let $\Delta_{\text{out}} = v_0v_1v_2$ be the outer triangle of G. An outer vertex of $\partial G - E(\Delta_{\text{out}})$ different from v_0, v_1, v_2 is called *special vertex* of G, and is denoted by u_i, when it is adjacent to v_{i-1} and v_{i+1} (indices modulo 3). We also say that u_i is the special vertex *opposing* v_i. Note that u_0, u_1, u_2 are pairwise distinct when $|V(\partial G)| \geq 5$, and pairwise coincide when $|V(\partial G)| = 4$, i.e., $\partial G \cong K_4$. An immediate consequence of Lemma 2.2 is the following.

Lemma 3.1. *Let G be a plane triangulation with outer triangle $\Delta_{\text{out}} = v_0v_1v_2$ and corresponding special vertices u_0, u_1, u_2. Then for any $\{x, y, z\} = \{0, 1, 2\}$ the edges of $\partial G - E(\Delta_{\text{out}})$ can be partitioned into three forests F_x, F_y, F_z such that*

- *F_x is a Hamiltonian path of $(\partial G - E(\Delta_{\text{out}})) \setminus \{v_y, v_z\}$ going from v_x to u_x,*
- *F_y is a spanning tree of $(\partial G - E(\Delta_{\text{out}})) \setminus \{v_x, v_z\}$,*
- *F_z is a spanning forest of $(\partial G - E(\Delta_{\text{out}})) \setminus \{v_y\}$ consisting of two trees, one containing v_x and one containing v_z, unless $\partial G \cong K_4$. In this case $F_z = v_zu_z$.*

Proof. If ∂G has only 4 vertices, i.e., $\partial G \cong K_4$, then the decomposition $F_i = v_iu_i$ for all $i \in \{x, y, z\}$ has the desired properties. So assume that ∂G has at least 5 vertices. Since ∂G is 4-connected, $\partial G - V(\Delta_{\text{out}})$ is a Whitney graph with respect to (u_0, u_1, u_2). By rotating and flipping we can assume without loss of generality that $(y, z, x) = (0, 1, 2)$. Then by Lemma 2.2 the edges of $\partial G - V(\Delta_{\text{out}})$ can be oriented and colored black, red and blue with the following properties. The black edges form a Hamiltonian path from u_0 to u_2, which we extend by the edge $u_0v_2 = u_yv_x$ to obtain F_x. Every inner vertex has exactly one outgoing red edge and one outgoing blue edge, every vertex on $P^\circ_{u_2u_0} \cup P^\circ_{u_0u_1} \cup \{u_0\}$ has an outgoing blue but no outgoing red edge, every vertex on $P^\circ_{u_1u_2}$ has an outgoing red but no outgoing blue edge, and neither u_1 nor u_2 have outgoing blue or red edges. Since there are no directed monochromatic cycles. This implies that red and blue edges form a forest each, where the roots of the red trees correspond to the vertices on $P_{u_2u_0} \cup P_{u_0u_1}$ and the roots of the blue trees $P_{u_1u_2}$. Thus, coloring all edges except u_yv_x from $P_{u_2u_0} \cup P_{u_0u_1}$ to v_x or v_z red gives F_z and coloring all edges from $P_{u_1u_2}$ to v_y blue gives F_y. See Figure 8 for an illustration. We have obtained the desired decomposition of the edges of $\partial G - E(\Delta_{\text{out}})$. □

We define for $k \in \{0, 1, 2\}$ a *k-assignment* of a plane triangulation G with outer triangle Δ_{out} to be a map $\phi : X(G) \to V(G)$ which satisfies $\phi(\Delta) \in V(\Delta)$ for every $\Delta \in X$ and

$$|\phi^{-1}(v)| \leq \begin{cases} k+1 & \text{if } v = \phi(\Delta_{\text{out}}) \text{ or there is some } \Delta \in X(G) \text{ such that} \\ & \quad v \text{ is the special vertex in } G_\Delta \text{ opposing } \phi(\Delta), \\ k & \text{otherwise.} \end{cases}$$

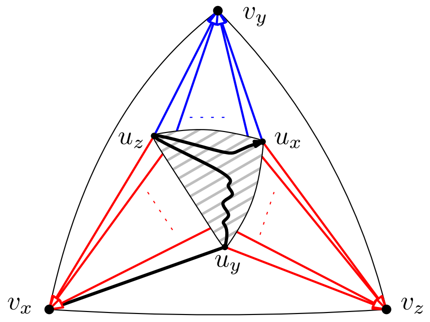


Figure 8: The construction in Lemma 3.1.

Proposition 3.2. *Let G be a plane triangulation with outer triangle $\Delta_{\text{out}} = w_0w_1w_2$ such that w_0w_1 or w_0w_2 is in no separating triangle. If G admits a k-assignment ϕ, $k \in \{0,1,2\}$, such that $\phi(\Delta_{\text{out}}) = w_0$, then G can be decomposed into two trees and one spanning tree with maximum degree at most $k+2$, i.e., G is $[2,k+2]^\star$-decomposable.*

Proof. Without loss of generality assume that w_0w_2 is in no separating triangle.

We will prove that G decomposes into three trees T_0, T_1, T_2, where T_0 is spanning and has maximum degree $k+2$, T_1 spans $G \setminus \{w_0, w_2\}$, and T_2 spans $G \setminus \{w_1\}$. Even stronger, for every vertex v we shall have $\deg_{T_0}(v) = 1 + |\phi^{-1}(v)|$ if $v = w_0$ or v is a special vertex opposing $\phi(\Delta^*)$ for some $\Delta^* \in X(G)$, and $\deg_{T_0}(v) = 2 + |\phi^{-1}(v)|$ otherwise. This will be done by induction along the separation tree of G.

If G is just the triangle $\Delta_{\text{out}} = w_0w_1w_2$ we set $T_0 = \{w_0w_1, w_1w_2\}$, $T_1 = \emptyset$, and $T_2 = w_2w_0$, and we are done.

Let now $\Delta = v_0v_1v_2 \in X(G)$ be an inclusion-minimal filled triangle of G. Hence G_Δ is 4-connected and we have $\partial G_\Delta = G_\Delta$. We apply induction to the graph G' obtained from G by removing all vertices inside Δ, together with the k-assignment ϕ' obtained from ϕ by restricting to $X(G') = X(G) \setminus \{\Delta\}$, and obtain three trees T'_0, T'_1, T'_2 with the desired properties. We have to include the edges of $G_\Delta - E(\Delta)$ into our decomposition.

Note that at most two vertices of Δ also appear in Δ_{out} but by the choice of w_0w_2 these cannot be w_0 and w_2 simultaneously. Without loss of generality let $\phi(\Delta) = v_0$ and v_2 be such that $\{v_0, v_2\} \not\subseteq \{w_0, w_1, w_2\}$. Since w_0w_2 is in no separating triangle, we may also assume that $v_1 \notin \{w_0, w_2\}$. Let F_0, F_1, F_2 be a partition of the edges in $G_\Delta - E(\Delta)$ as given by Lemma 3.1 for $x = 0$, $y = 1$ and $z = 2$. (Recall that $\partial G_\Delta = G_\Delta$.) We include F_0 into T'_0, F_1 into T'_1, and F_2 into T'_2. Clearly, by **(P3)** we obtain a set of trees T_0, T_1, T_2 with the desired properties, except that it remains to bound the degrees of vertices in T_0.

First consider any inner vertex v of G_Δ. If $v = u_0$ is the vertex in G_Δ opposing $v_0 = \phi(\Delta)$, then v has one incident edge in F_0 and no incident edge in T'_0, and thus $\deg_{T_0}(v) = 1 + |\phi^{-1}(v)|$ with $|\phi^{-1}(v)| = 0$. If v is an inner vertex of G_Δ different from u_0, then v has two incident edges in F_0 and no incident edge in T'_0, and thus $\deg_{T_0}(v) = 2 + |\phi^{-1}(v)|$ with $|\phi^{-1}(v)| = 0$.

Now consider any vertex v of G'. Note that v is a special vertex opposing $\phi(\Delta^*)$ for some $\Delta^* \in X(G)$ if and only if that is also the case for ϕ' and G'. If $v = v_0$, then $|\phi^{-1}(v)| = |\phi'^{-1}(v)| + 1$ and $\deg_{T_0}(v) = \deg_{T'_0}(v) + 1$,

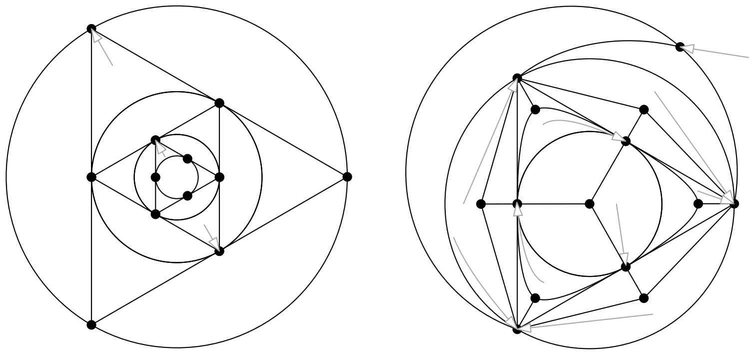


Figure 9: A non-4-connected planar triangulation with 0-assignment qualifying for Proposition 3.2 and a non-Hamiltonian planar triangulation with 1-assignment qualifying for Proposition 3.2.

as v has exactly one incident edge in F_0, which implies the desired degree of v in T_0 for this case. Finally, if v is any vertex of G' different from v_0, then $|\phi^{-1}(v)| = |\phi'^{-1}(v)|$ and $\deg_{T_0}(v) = \deg_{T'_0}(v)$, as v has no incident edge in F_0, which implies the desired degree of v in T_0 also for this case. □

4 Proofs of Decomposition Results

In this section, we show that all the classes of concern admit triangle assignments qualifying for Proposition 3.2 with respect to the claimed parameters. This will suffice to prove the Theorems of this paper.

Now Proposition 3.2 immediately implies Theorem 1.1, namely that every 4-connected triangulation is $[2,2]^\star$-decomposable.

Proof of Theorem 1.1. Clearly, if G is a 4-connected plane triangulation, then $X(G)$ consists only of the outer face $\Delta_{\text{out}} = xyz$ and $\phi(\Delta_{\text{out}}) = x$ is a 0-assignment of G. Since G contains no separating triangles, applying Proposition 3.2 yields the result. □

We remark that some plane triangulations (such as the one in the left of Figure 9) are not 4-connected and still admit a 0-assignment qualifying for Proposition 3.2.

Next, we shall turn our attention to general plane triangulations. Recall that $Y(G)$ denotes the set of all inner faces of a plane triangulation G.

Lemma 4.1. *Let G be a plane triangulation on at least four vertices and u be a special vertex of G. Then there is a map $\psi : Y(G) \to V(G)$ with $\psi(\Delta) \in V(\Delta)$ for every $\Delta \in Y(G)$ and*

$$|\psi^{-1}(v)| = \begin{cases} 0 & \text{if } v \text{ is an outer vertex,} \\ 2 & \text{if } v \text{ is an inner vertex, } v \neq u, \\ 3 & \text{if } v = u. \end{cases}$$

Proof. Let v_1, v_2 be two outer vertices adjacent to the special vertex u and $\Delta^\star \in Y(G)$ be the inner face formed by v_1, v_2 and u. Consider a plane straight-line embedding of G in which the only horizontal edge is v_1v_2. For every inner face $\Delta \in Y(G) - \Delta^\star$, let $\psi(\Delta)$ be the vertex in $V(\Delta)$ with the middle y-coordinate. Moreover, let $\psi(\Delta^\star) = u$. It is easily seen that ψ has the desired properties. □

Lemma 4.2. *Every plane triangulation admits a* 2*-assignment.*

Proof. Let G be any plane triangulation with outer triangle Δ_{out}. By **(P2)** we have that $\{\Delta_{\text{out}}\} \cup \bigcup_{\Delta \in X(G)} Y(\partial G_\Delta) \supset X(G)$. Similarly, every inner vertex of G is an inner vertex of ∂G_Δ for exactly one $\Delta \in X(G)$. By Lemma 4.1 we have maps $\psi_\Delta : Y(\partial G_\Delta) \to V(\partial G_\Delta)$ mapping the inner faces of ∂G_Δ to inner vertices of ∂G_Δ such that every inner vertex is hit twice, except for a fixed special vertex of ∂G_Δ, which is hit three times. From this collection of maps $\{\psi_\Delta \mid \Delta \in X(G)\}$ we obtain a desired 2-assignment $\phi : X(G) \to V(G)$ by setting $\phi(\Delta_{\text{out}}) = v$ for some outer vertex v of G and $\phi(\Delta') = \psi_\Delta(\Delta')$ whenever $\Delta' \in Y(G_\Delta)$. □

We can now give an alternative proof of Gonçalves's result [5] that every planar graph is $(2,4)^\star$-decomposable and indeed a slight strengthening thereof, since in [5] the forests of the decomposition are not necessarily connected. That is, we prove Theorem 1.3 stating that every planar triangulation decomposes into two trees and one spanning tree of maximum degree 4.

Proof of Theorem 1.3. Lemma 4.2 gives a 2-assignment ϕ for a planar triangulation for any prescribed outer triangle Δ_{out}, where moreover the choice of $\phi(\Delta_{\text{out}})$ is arbitrary. Since clearly any planar triangulation contains an edge e, that is not contained in a separating triangle we can choose an outer triangle containing e. Use Lemma 4.2 to get a 2-assignment such that $\phi(\Delta_{\text{out}})$ is a vertex of e and apply Proposition 3.2 to get the desired decomposition. □

Next, we shall turn our attention to Hamiltonian planar triangulations.

Lemma 4.3. *Let G be a plane triangulation with outer triangle $\Delta_{\text{out}} = v_0v_1v_2$.*

(i) If G admits a Hamiltonian v_0-v_2-path, then G admits a 1*-assignment ϕ with $\phi(\Delta_{\text{out}}) = v_1$ and*

$$|\phi^{-1}(v)| \leq \begin{cases} 1 & \text{if } v = v_1, v_2, \\ 0 & \text{if } v = v_0. \end{cases}$$

(ii) If G admits a Hamiltonian v_0-v_2-path, then G admits a 1*-assignment ϕ with $\phi(\Delta_{\text{out}}) = v_1$ and*

$$|\phi^{-1}(v)| \leq \begin{cases} 2 & \text{if } v = v_1, \\ 0 & \text{if } v = v_0, v_2. \end{cases}$$

(iii) If $G - v_1$ admits a Hamiltonian v_0-v_2-path, then G admits a 1-assignment ϕ with $\phi(\Delta_{\text{out}}) = v_2$ and

$$|\phi^{-1}(v)| \leq \begin{cases} 1 & \text{if } v = v_2, \\ 0 & \text{if } v = v_0, v_1. \end{cases}$$

Proof. Let G be a plane triangulation with outer triangle $\Delta_{\text{out}} = v_0v_1v_2$ and let P be a fixed Hamiltonian v_0-v_2-path in G for items (i),(ii) or in $G - v_1$ for item (iii). We shall prove all three items by induction on $|X(G)|$, the number of filled triangles in G. If $|X(G)| = 1$, i.e., G is 4-connected, then a desired 1-assignment ϕ is given by $\phi(\Delta_{\text{out}}) = v_1$ for (i),(ii) and $\phi(\Delta_{\text{out}}) = v_2$ for (iii), respectively.

So assume that $|X(G)| \geq 2$, i.e., G has at least one separating triangle. Let $Z \subseteq X(G) - \Delta_{\text{out}}$ denote the set of all inclusion-maximal separating triangles in G. For each $\Delta \in Z$, let $P_\Delta \subset P$ denote the inclusion-maximal subpath of P consisting only of edges in $G_\Delta - E(\Delta)$. Then P_Δ has distinct endpoints, which both lie on Δ. Note that P_Δ is a Hamiltonian path either in G_Δ, or in $G_\Delta - v$ for one $v \in \Delta$. Moreover, $E(P_\Delta) \subset E(G_\Delta) - E(\Delta)$ for each $\Delta \in Z$ and thus $E(P_\Delta) \cap E(P_{\Delta'}) = \emptyset$ for $\Delta \neq \Delta' \in Z$, by **(P3)**.

Let u denote the special vertex of G opposing v_1 for items (i),(ii) and opposing v_0 for item (iii). We endow the path P in G with an orientation by orienting every edge towards u. This implies an orientation of P_Δ for each $\Delta \in Z$. Now depending on P_Δ and its orientation, we consider a 1-assignment ϕ_Δ of G_Δ for each $\Delta \in Z$, which we obtain by induction.

Case 1: u is an inner vertex of P_Δ.
Denote by $v_0(\Delta)$ and $v_2(\Delta)$ be the endpoints of P_Δ. Since u is a special vertex of G, we have $\Delta = v_0(\Delta)uv_2(\Delta)$. (Recall that, by definition, special vertices are not contained in any separating triangle of G.) In particular P_Δ is a Hamiltonian path in G_Δ. Then by induction (item (ii)) there exists a 1-assignment ϕ_Δ of G_Δ with $|\phi_\Delta^{-1}(u)| \leq 2$ and $|\phi_\Delta^{-1}(v)| = 0$ for $v = v_0(\Delta), v_2(\Delta)$.

Case 2: P_Δ is a Hamiltonian path in G_Δ.
Let $v_0(\Delta)$ and $v_2(\Delta)$ be the endpoints of P_Δ so that P_Δ is oriented from $v_0(\Delta)$ to $v_2(\Delta)$ and let $v_1(\Delta)$ be the third outer vertex of G_Δ. Then by induction (item (i)) there exists a 1-assignment ϕ_Δ of G_Δ with $|\phi_\Delta^{-1}(v)| \leq 1$ for $v = v_1(\Delta), v_2(\Delta)$ and $|\phi_\Delta^{-1}(v_0(\Delta))| = 0$.

Case 3: P_Δ is a Hamiltonian path in $G_\Delta - v(\Delta)$.
Let $v_0(\Delta)$ and $v_2(\Delta)$ be the endpoints of P_Δ so that P_Δ is oriented from $v_0(\Delta)$ to $v_2(\Delta)$. Note that $\Delta = v_0(\Delta)v_1(\Delta)v_2(\Delta)$. Then by induction (item (iii)) there exists a 1-assignment ϕ_Δ of G_Δ with $|\phi_\Delta^{-1}(v_2(\Delta))| = 1$ and $|\phi_\Delta^{-1}(v)| = 0$ for $v = v_0(\Delta), v_1(\Delta)$.

Finally, define a map $\phi : X(G) \to V(G)$ by $\phi(\Delta_{\text{out}}) = v_1$ for items (i),(ii) and $\phi(\Delta_{\text{out}}) = v_2$ for item (iii) and $\phi(\Delta) = \phi_{\Delta'}(\Delta)$ for the unique $\Delta^\star \in Z$ with $\Delta \subset G_{\Delta^\star}$. It is straightforward to check that ϕ satisfies the desired requirements. $\square$

Having Lemma 4.3, we can deduce from Proposition 3.2 a $[2,3]^\star$-decomposition of any Hamiltonian triangulation. That is, we prove Theorem 1.2 stating that every Hamiltonian planar triangulation decomposes into two trees and one spanning tree of maximum degree 3.

Proof of Theorem 1.2. Let G be a Hamiltonian plane triangulation with Hamiltonian cycle C. Consider counterclockwise consecutive vertices v_1, v_2, v_0 on C, such that v_2 has no neighbors in the interior of C. The latter is possible because C with the edges in its interior is a maximal outerplanar graph and thus has a degree 2 vertex. Thus, $v_0v_1v_2$ is a facial triangle of G. We show that one edge of $v_0v_1v_2$ is in no separating triangle. Suppose, that v_1v_0 is in a separating triangle Δ. Then an edge of Δ incident with v_1 or v_0 has to lie entirely in the exterior of C - say v_1 is that vertex. Now, the edge v_1v_2 cannot lie in a separating triangle because all edges that could be used to form such a triangle lie in the exterior of C. In the other case v_2v_0 lies in no separating triangle.

Now, we embed G such that $v_0v_1v_2 = \Delta_{\text{out}}$ is the outer triangle of the embedding and after possibly renaming v_1, v_2, v_0 we have that G admits a Hamiltonian v_0-v_2-path and the edge v_0v_1 is in no separating triangle. Now, Lemma 4.3 gives that G admits a 1-assignment with $\phi(\Delta_{\text{out}}) = v_1$. Since v_0v_1 is in no separating triangle Proposition 3.2 yields the desired decomposition. □

5 Tightness

In the present section we show that all our results are best-possible. More precisely, any of the classes for that we show $[2, d]^\star$-decomposability contains members that are not even $(2, d-1)$-decomposable. Moreover, there are triangulations "close to the class" that are not $(2, d)$-decomposable.

Recall that for a graph G and integer $d \geq 0$ we say that G is $(2, d)$-decomposable (respectively $(2, d)^\star$-decomposable, $[2, d]^\star$-decomposable) if the edges of G can be partitioned into two forests and a graph (respectively a forest, tree) of maximum degree d. For this section we will furthermore say that G is $[2, d]$-decomposable if the edges of G can be partitioned into two forests and a connected graph of maximum degree d.

Let G be a planar triangulation. We prove in Theorem 1.3 that G is $[2, 4]^\star$-decomposable, in Theorem 1.2 that if G is Hamiltonian, then G is even $[2, 3]^\star$-decomposable, and in Theorem 1.1 that if G is 4-connected, then G is even $[2, 2]^\star$-decomposable. In this section we want to argue that Theorems 1.1 and 1.2 are best-possible in some sense, namely that these cannot be extended to triangulations that are "a decent amount away from" 4-connectivity, respectively Hamiltonicity. To this end let $G' \subseteq G$ be a subgraph on at least four vertices of G that is itself a triangulation. Let $n \geq 4$ be the number of vertices in G', and k be the number of faces of G' that are not faces of G, i.e., $k = |Y(G') - Y(G)|$.

Note that if G is 4-connected (and thus $[2, 2]^\star$-decomposable by Theorem 1.1), then necessarily $G' = G$ and thus $k = 0$. On the other hand, if G has a Hamiltonian cycle C (and is thus $[2, 3]^\star$-decomposable by Theorem 1.2), then along C between the interior vertices of any two faces in $Y(G') - Y(G)$ there is at least one vertex of G', showing that $k \leq n$. So we have shown $[2, d]^\star$-decomposability for $d = 2, 3$ in cases where k is relatively small. Next we show that indeed $[2, d]^\star$- and $(2, d)^\star$-decomposability for $d = 2, 3$ can be achieved *only if* k is relatively small.

As we also want to make a slightly stronger statement involving $[2,d]$- and $(2,d)$-decomposability for $d=2,3$, we introduce the following notation. For fixed triangulations $G' \subseteq G$, we say that a $[2,d]$- or $(2,d)$-decomposition (F_1,F_2,H_3) is *special*, if for every face $\Delta \in Y(G') - Y(G)$ we have that the subgraph of H_3 consisting of all edges in the interior of Δ but not on the boundary of Δ is a forest. Note in particular that every $(2,d)^\star$-decomposition and every $[2,d]^\star$-decomposition is special.

Proposition 5.1. *Let $G' \subseteq G$ be two triangulations, n be the number of vertices in G', and k be the number of faces of G' that are not faces of G, i.e., $k = |Y(G') - Y(G)|$. Then each of the following holds:*

(i) If $k \geq 6$, then G has no special $[2,2]$-decomposition.

(ii) If $k \geq 9$, then G has no special $(2,2)$-decomposition.

(iii) If $k \geq n+6$, then G has no special $[2,3]$-decomposition.

(iv) If $k \geq n+9$, then G has no special $(2,3)$-decomposition.

Proof. We shall prove all four items by contraposition, i.e., we assume that G has a special $(2,d)$- or $[2,d]$-decomposition for $d \in \{2,3\}$ and derive from this an upper bound on k. So let (F_1,F_2,H_3) be a special decomposition of $E(G)$ into two forests F_1,F_2 and a third graph H_3 of maximum degree d. As we shall count components, we consider each of F_1,F_2,H_3 to be a spanning subgraph of G, i.e., to contain all the vertices of G.

Claim 1. *Let J be a graph with n vertices and $kn-x$ edges. If the edges of J are partitioned into k spanning subgraphs whose components are only trees and cycles, then the total number of tree components in these k graphs is equal to x.*

Indeed, it is enough to observe that a spanning subgraph J' of J whose components are trees and cycles contains exactly $n-c$ edges of G, where c is the number of tree components of J'.

For a face Δ in $Y(G') - Y(G)$, let $G[\Delta]$ denote the maximal subgraph of G whose vertices are on or inside Δ and whose edges have at least one endpoint inside Δ. In other words $G[\Delta] = G_\Delta - E(\Delta)$.

Claim 2. *For every $\Delta \in Y(G') - Y(G)$ at least one of the following holds:*

- *Two vertices of Δ are in the same component of $F_1|_{G[\Delta]}$.*
- *Two vertices of Δ are in the same component of $F_2|_{G[\Delta]}$.*
- *A vertex of Δ is incident to an edge in $H_3|_{G[\Delta]}$.*

Indeed, if $G[\Delta]$ has n vertices, then it has $3n-9$ edges. As the edges of $G[\Delta]$ are partitioned into three forests $F_1|_{G[\Delta]}, F_2|_{G[\Delta]}, H_3|_{G[\Delta]}$ (the latter graph is a forest as the decomposition is special), by Claim 1 there are in total exactly 9 components. Now if the first item does not hold, then $F_1|_{G[\Delta]}$ has at least 3 components. Similarly, if the second items does not hold, then $F_2|_{G[\Delta]}$ has at least 3 components. Finally, if the third item does not hold, then $H_3|_{G[\Delta]}$ has at least 4 components. Thus if none of the items hold, then there are at least 10 components in total, which is a contradiction and proves the claim.

Taking the restrictions of F_1, F_2, H_3 to the subgraph G' of G, we obtain corresponding forests F_1', F_2' and graph H_3' partitioning $E(G')$. For a subgraph $J \in \{F_1, F_2, H_3, F_1', F_2', H_3'\}$ of G let $c(J)$ denote the number of its tree components that contain a vertex of G'. The crucial insight is that the number of faces $\Delta \in Y(G') - Y(G)$ for which the first, respectively second, item of Claim 2 holds, is at most $c(F_1') - c(F_1)$, respectively $c(F_2') - c(F_2)$. Every further face $\Delta \in Y(G') - Y(G)$ contributes at least one edge in $E(H_3) - E(H_3')$ incident to a vertex of G'.

Case 1: $d = 2$**.**

Claim 1 applied to the partition (F_1', F_2', H_3') of G' gives

$$c(F_1') + c(F_2') + c(H_3') = 6.$$

Since $G' \subseteq G$, we have $c(F_1') \geq c(F_1) \geq 1$, $c(F_2') \geq c(F_2) \geq 1$, and $c(H_3') \geq c(H_3)$, which implies that

$$c(H_3) \leq c(H_3') = 6 - c(F_1') - c(F_2') \leq 4.$$

For at most $c(F_1') - c(F_1)$, respectively $c(F_2') - c(F_2)$ faces $\Delta \in Y(G') - Y(G)$ the first, respectively second, item of Claim 2 holds. For at most $c(H_3') - c(H_3)$ faces $\Delta \in Y(G') - Y(G)$ two vertices of Δ are in the same component of $H_3|_{G[\Delta]}$. As H_3 has maximum degree at most $d = 2$, for at most $2c(H_3)$ further faces $\Delta \in Y(G') - Y(G)$ the last item of Claim 2 holds. Hence, the total number of faces in $Y(G') - Y(G)$ is at most

$$\begin{aligned} &c(F_1') - c(F_1) + c(F_2') - c(F_2) + c(H_3') - c(H_3) + 2c(H_3) \\ &\qquad\qquad\qquad\qquad \leq 6 - 2 + c(H_3) = 4 + c(H_3). \end{aligned}$$

Thus $k \leq 4 + c(H_3) = 5$ for $c(H_3) = 1$ which proves (i) and $k \leq 4 + c(H_3) \leq 8$ otherwise, which proves (ii).

Case 2: $d = 3$**.**

As H_3' has maximum degree at most $d = 3$, we have $|E(H_3')| = 3n/2 - y$ for some y. Claim 1 applied to the partition (F_1', F_2') of $G' - E(H_3')$ gives $c(F_1') + c(F_2') = n/2 + 6 - y$ and thus

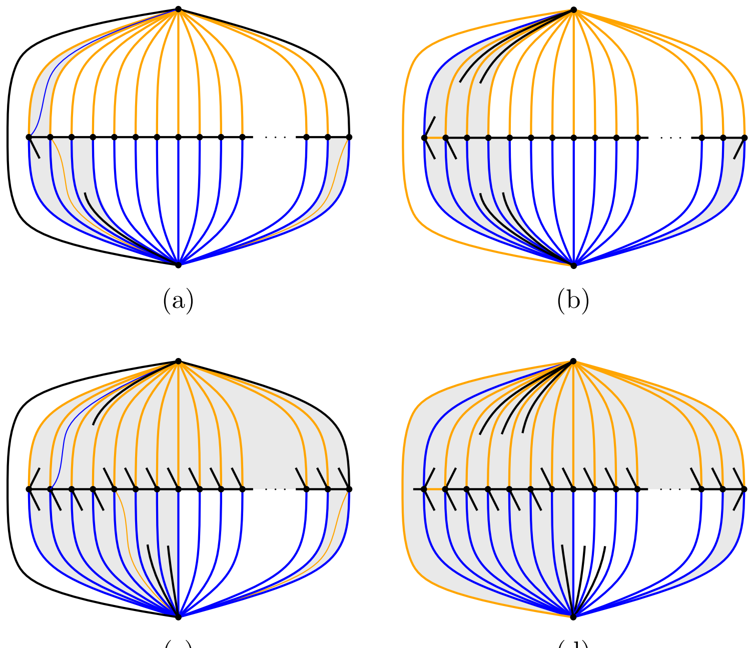
$$y = n/2 + 6 - c(F_1') - c(F_2') \leq n/2 + 4.$$

Moreover, $c(H_3') \geq \max\{1, n - |E(H_3')|\}$ and thus

$$y = \frac{3}{2}n - |E(H_3')| \leq n/2 + c(H_3').$$

Clearly, $c(F_1), c(F_2) \geq 1$, and thus for at most $c(F_1') - c(F_1) + c(F_2') - c(F_2) \leq n/2 + 4 - y$ faces $\Delta \in Y(G') - Y(G)$ the first or second item of Claim 2 holds.

Now consider the set $S = \{(v, e) \mid v \in V(G'), e \in E(H_3), v \in e\}$. As H_3 has maximum degree at most $d = 3$, we have $|S| \leq 3n$. There are exactly $2|E(H_3')| = 3n - 2y$ pairs $(v, e) \in S$ with $e \in E(H_3')$. For at most $c(H_3') - c(H_3)$ faces $\Delta \in Y(G') - Y(G)$ two vertices of Δ are in the same component of $H_3|_{G[\Delta]}$, which corresponds to at least $2(c(H_3') - c(H_3))$



(a) (b) (c) (d)

Figure 10: Examples showing that the bounds on $k = |Y(G') - Y(G)|$ in Proposition 5.1 are best-possible. The triangulation $G' \subset G$ is drawn with thick edges, gray areas indicate faces in $Y(G') - Y(G)$, and edge colors indicate a claimed decomposition. (a) $k = 5$ and G is $[2,2]^\star$-decomposable. (b) $k = 8$ and G is $(2,2)^\star$-decomposable. (c) $k = n+5$ and G is $[2,3]^\star$-decomposable. (d) $k = n+8$ and G is $(2,3)^\star$-decomposable.

further pairs $(v, e) \in S$ with $e \in E(G[\Delta])$ for such faces Δ. Every further face $\Delta \in Y(G') - Y(G)$ for which the last item of Claim 2 holds corresponds to at least one further pair $(v, e) \in S$ with $e \in E(G[\Delta])$, giving that there are at most $|S| - (3n - 2y) - 2(c(H_3') - c(H_3)) = 2(y - c(H_3') + c(H_3))$ such faces $\Delta \in Y(G') - Y(G)$. Hence, for the total number k of faces in $Y(G') - Y(G)$ we have

$$\begin{aligned} k &\leq (n/2 + 4 - y) + (c(H_3') - c(H_3)) + 2(y - c(H_3') + c(H_3)) \\ &= n/2 + 4 + y + (c(H_3) - c(H_3')) \\ &\leq n + 4 + \min\{c(H_3), 4\}. \end{aligned}$$

Thus $k \leq n + 4 + \min\{c(H_3), 4\} = n + 5$ for $c(H_3) = 1$ which proves (iii) and $k \leq n + 4 + \min\{c(H_3), 4\} \leq n + 8$ otherwise, which proves (iv). $\square$

Let us mention that Proposition 5.1 is tight in the following sense: Figure 10 shows triangulations $G' \subset G$ with $k = |Y(G') - Y(G)|$ being one less than the bound in Proposition 5.1 and yet G has a special $[2, d]$- or $(2, d)$-decomposition (even stronger: $[2, d]^\star$-, respectively $(2, d)^\star$-decomposition) for $d \in \{2, 3\}$ as in the respective case in the theorem.

Finally, Proposition 5.1 implies Theorem 1.4 stating that (i) some 4-connected triangulations are not $(2, 1)$-decomposable, (ii) some Hamiltonian triangulations are not $(2,2)$-decomposable, and (iii) some planar triangulations are not $(2,3)$-decomposable.

Proof of Theorem 1.4. The first item is a simple counting argument.

(i) Let G_1 be any 4-connected triangulation on $n \geq 9$ vertices. Note that every n-vertex $(2,1)$-decomposable graph has at most $2(n-1)+n/2$ edges, since it decomposes into two forests and a matching. On the other hand, G_1 has $3n-6$ edges, which is strictly more than $2(n-1)+n/2$ for $n \geq 9$, and thus G_1 is not $(2,1)$-decomposable.

The key observation for (ii) and (iii) is that if $G' \subseteq G$ are triangulations and every face $\Delta \in Y(G') - Y(G)$ contains exactly one vertex of $V(G) - V(G')$, then every $(2,d)$-decomposition of G is special. Hence we can use Proposition 5.1 to argue that such G admits no $(2,d)$-decomposition for $d \in \{2,3\}$, provided $k = |Y(G') - Y(G)|$ is big enough.

(ii) Let G_2' be any Hamiltonian triangulation on an even number $n \geq 18$ vertices and let C be a Hamiltonian cycle in G_2'. Now for every other edge e in C add a new vertex v_e in one of the faces incident to e, making v_e adjacent to all vertices of this face. As we picked every other edge, we have $v_e \neq v_{e'}$ for $e \neq e'$. The resulting graph G_2 is a triangulation satisfying $G_2' \subset G_2$ and $k = n/2 \geq 9$. Hence, by Proposition 5.1(ii) G_2 has no special $(2,2)$-decomposition, and as every $\Delta \in Y(G_2') - Y(G_2)$ contains exactly one vertex of $V(G_2) - V(G_2')$, G_2 has no $(2,2)$-decomposition at all. Finally, G_2 is Hamiltonian as the cycle C can be easily rerouted to also contain every vertex $v_e \in V(G_2) - V(G_2')$.

(iii) Let G_3' be any triangulation on $n \geq 12$ vertices. Let G_3 be the triangulation arising from G_3' by adding a new vertex in each face, making it adjacent to all vertices of this face. Note that $G_3' \subset G_3$ and $k = |Y(G_3') - Y(G_3)| = 2n - 3 \geq n + 12 - 3 = n + 9$. Hence, by Proposition 5.1(iv) G_3 has no special $(2,3)$-decomposition, and as every $\Delta \in Y(G_3') - Y(G_3)$ contains exactly one vertex of $V(G_3) - V(G_3')$, G_3 has no $(2,3)$-decomposition at all.

□

6 Conclusions

Gonçalves [5] showed that every planar graph admits a $(2,4)^{\star}$-decomposition. In this paper we showed that structural properties of planar triangulations allow for $[2,d]^{\star}$-decompositions with $d < 4$. Moreover, note that our results are slightly stronger than just showing decomposability into forests, since for planar triangulations we give decompositions into trees and the bounded degree tree is spanning. In light of these results and the Nine Dragon Tree Theorem one can ask under what conditions a graph is coverable by k trees and a bounded degree d tree or connected graph.

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