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Some topics on digraph colouring

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Quelques sujets sur la coloration des graphes orientés

· Résumé ·

On étudie le problème de partitionnement de digraphes en sous-graphes acycliques, et son paramètre associé : le nombre dichromatique. Ce paradigme, conçu par Neumann-Lara vers la fin des années 1970, nous a prodigué un bon nombre de généralisations de théorèmes classiques sur la coloration de graphes.
Nous dédions une attention spéciale aux bornes que l'on peut donner pour plusieurs classes de digraphes, comprenant les tournois locaux, les orientations des graphes de Kneser, les orientations des triangulations 2-planaires extérieures imbriquées, les digraphes aléatoires, et les digraphes aléatoires r-réguliers.
Nous nous intéressons aussi aux relations entre le nombre dichromatique et autres paramètres, comme le degré maximal et le nombre de biclique. En particulier, nous montrons qu'une version orientée de la conjecture de Borodin–Kostochka est valide pour des grands degrés maximaux, ce qui généralise un résultat de Reed.
Finalement, nous ajoutons quelques considérations sur les variantes circulaire, fractionnaire, et de liste du problème.

Mots clés : digraphe, coloration, ensemble acyclique, nombre dichromatique, digraphe aléatoire

Some topics on digraph colouring

· Abstract ·

We study the problem of partitioning digraphs into acyclic subgraphs, and its associated parameter: the dichromatic number. To this line of research, initiated by Neumann-Lara in the late 1970s, we owe many generalisations of classical results about graph colouring.
Our emphasis is placed on the bounds that can be given for several digraph classes, including local tournaments, orientations of Kneser graphs, orientations of nested 2-outerplanar triangulations, random digraphs, and random r-regular digraphs.
We also concern ourselves with the relationship between the dichromatic number and other digraph parameters, such as the maximum degree and the biclique number. In particular, we show that a directed version of the Borodin–Kostochka conjecture is true for large maximum degrees, thus generalising a result of Reed.
Finally, we include some considerations about the circular, the fractional, and the list variants of the problem.

Keywords: digraph, colouring, acyclic set, dichromatic number, random digraph

Contents

What is this thesis about?

The main object of this thesis is to study the bounds that can be given, under various hypotheses, for the dichromatic number: the minimum number of acyclic subgraphs into which a digraph can be partitioned. This parameter, conceived by Neumann-Lara in the 1970s as a generalisation of the usual chromatic number, has become increasingly popular in the last two decades. Researchers have realised that it yields natural extensions of many classical graph colouring results, sometimes with unexpected twists, that seem to confirm that this invariant adequately reflects the extra layer of complexity arising from the introduction of edge directions.

Under this new light, the graph colouring heritage becomes a vast territory to be re-explored. It seems a priority to discover where do new phenomena emerge, and, perhaps more importantly, to locate the new difficulties. Of special interest is thus finding those problems that we know how to solve for graphs, but that for digraphs seem to require fresh ideas. This probing task, still largely undone, is being actively carried out on several fronts. The present thesis can be regarded as a contribution to this collective effort.

The thesis is mainly based on [87, 88, 92], and other unpublished work from the last years. In what follows we briefly describe the contents of each chapter.

Chapter 1 contains all the basic definitions that are necessary for the other chapters, as well as a quick overview of the background on graph and digraph colouring that is relevant to the thesis.

In Chapter 2 we present some notable unresolved problems in the field. Some of them were raised in the early days of the theory, and already frame important challenges. The next three chapters address some special cases of these problems.

In Chapter 3 (based on joint work [92] with Ararat Harutyunyan) we complement a bound of Mohar and Wu for the dichromatic number of Kneser graphs. We also look at Borsuk graphs, and investigate the list dichromatic number of complete multipartite graphs and dense Kneser graphs. Chapter 4 deals with the dichromatic and acyclicity numbers of local tournaments, and Chapter 5 with a class of planar graphs obtained by means of a certain recursion.

Chapter 6 (based on joint work [87] with Ararat Harutyunyan, Ken-ichi Kawarabayashi and Lucas Picasarri-Arrieta) is devoted to a generalisation to digraphs of a theorem of Reed, which in turn is an extension of Brooks' theorem. For digraphs of large maximum degree Δ, we determine the obstructions to the existence of $(\Delta - 1)$-dicolourings.

Chapter 7 is a short chapter of an expository tone, covering circular and fractional dicolourings. It includes a couple of (to our knowledge) new basic results that might help better understanding these dicolouring variants, and in particular answer a question of Hochstättler, Schröder and Steiner.

The last two chapters concern random digraphs, and complement previous results of Dutta, Spencer and Subramanian, as well as classical results on colouring of random graphs. In Chapter 8, it is shown that the dichromatic number of sparse random digraphs is concentrated in two values. And finally, in Chapter 9 (which is based on joint work [88] with Ararat Harutyunyan and Colin McDiarmid), we determine the asymptotics of the dichromatic and acyclicity numbers of random regular digraphs. We also comment on a contiguity result linking random regular graphs and digraphs, and include a couple of independent results that bound the dichromatic and acyclicity numbers of arbitrary graphs in terms of their Laplacian eigenvalues.

Chapter 1

Preliminaries and background

In this chapter we introduce the terminology and notation, along with the background, needed for the understanding of the motivation and contributions of this thesis. The reader versed in graph theory will be familiar with most of it, except maybe with some parts of Sections 1.2 and 1.3.

1.1 Graph colouring

Graphs

A *simple undirected graph* G, or simply a *graph* G, is an ordered pair (X, Y), where X is a non-empty set and Y is a set of unordered pairs $\{u, v\} \subseteq X$ with $u \neq v$. X, denoted by $V(G)$, is the set of *vertices* of G, and G is a graph *on* X; Y, denoted by $E(G)$, is the set of *edges* of G. The cardinalities of $V(G)$ and $E(G)$ are the *order* and the *size* of G, respectively. An *n-vertex graph* is a graph of order n. Graphs considered in this thesis are of finite order, unless otherwise specified.

Two vertices u and v of G are *adjacent* if $\{u, v\}$ is an edge of G; this is denoted by $u \sim v$. In this case, u and v are the *endpoints* of the edge $\{u, v\}$, and are said to be *incident* to it. A vertex is *universal* if it is adjacent to all other vertices. The set of *neighbours* of u, or its *neighbourhood*, is the set $N(u)$ of vertices that are adjacent with u. The number of neighbours of u is the *degree* of u; this quantity is denoted by $\deg(v)$, or by $\deg_G(u)$ when G needs to be specified. If all vertices of G have degree k, G is *k-regular*, and if this happens for some integer k, then G is *regular*.

Example 1.1. A graph is *complete* if every two distinct vertices are adjacent. We denote by K_n the complete graph on $[n] = \{1, \ldots, n\}$. All vertices of K_n are universal, and K_n is $(n-1)$-regular.

If H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, and G is a *supergraph* of H. If, moreover, $E(H) = \{e \in E(G) \mid e \subseteq V(H)\}$, then H is an *induced subgraph* of G. The subgraph of G *induced* by a set of vertices $S \subseteq V(G)$, denoted by $G[S]$, is the graph on S with edge set $E(G[S]) = \{e \in E(G) \mid e \subseteq S\}$. The edges of $G[S]$ are said to be *spanned* by S. A subgraph or supergraph of G is *proper* if it is different from G.

A *walk* W of G is an alternating sequence $v_0, e_1, v_1, \dots, v_{\ell-1}, e_\ell, v_\ell$ of vertices and edges of G such that $e_i = \{v_{i-1}, v_i\}$ for each $1 \leq i \leq \ell$. The non-negative integer ℓ is the *length* of W. The *endpoints* of W are the vertices v_0 and v_ℓ; if $v_0 = v_\ell$, then W is *closed*. Let W' be a walk $v'_0, e'_1, v'_1, \dots, v'_{\ell'-1}, e'_{\ell'}, v'_{\ell'}$ of G such that $v'_0 = v_\ell$. The *concatenation* WW' of W and W' is the walk $v_0, e_1, \dots, e_\ell, v_\ell, e'_1, \dots, e'_\ell, v'_{\ell'}$. Edges may be concatenated as well, as if they were walks of length 1. When all the edges $e_1, \dots, e_\ell$ of the walk W are different, W is a *trail*. If, moreover, all the vertices $v_0, \dots, v_\ell$ are different, then W is a *path*. If, instead, $\ell \geq 3$, $v_0 = v_\ell$, and all other pairs of vertices are different, then W is a *cycle*, or an *ℓ-cycle*. A *forest* is a graph without cycles.

The graph G is *connected* if every two distinct vertices are the endpoints of a path. Otherwise, G is *disconnected*. The *connected components* of G are the maximal connected subgraphs of G. The *distance* between two vertices u and v of a connected graph is the smallest length of a path with endpoints u and v. The *distance* between u and a set of vertices S is the smallest distance between u and a vertex in S.

G is a *cycle graph* if $v_0, e_1, v_1, \dots, v_{\ell-1}, e_\ell, v_\ell$ is a cycle of G and all vertices and edges of G appear in this sequence; it is also called an *odd* or an *even cycle*, according to the parity of ℓ, or a *triangle* if $\ell = 3$, or an *ℓ-cycle*. In the special case that $v_1 = 1, \dots, v_\ell = \ell$, we denote G by C_ℓ. Similarly, if $v_0, e_1, v_1, \dots, v_{\ell-1}, e_\ell, v_\ell$ is a path of G and all vertices and edges of G appear in this sequence, G is a *path graph*. We often do not make a distinction between paths or cycles of G and their corresponding subgraphs of G.

Cliques, independent sets, and colourings

A *clique* of a graph G is a set of vertices $S \subseteq V(G)$ inducing a complete subgraph of G. The *clique number* of G, denoted by $\omega(G)$, is the largest size of a clique of G. Turán's theorem gives a bound on the number of edges that a graph of given order and clique number can have. A simplified version can be stated as follows.

Theorem 1.2. (TURÁN) [9, 154] *Let G be a graph of order n, size m, and* *clique number* $\omega(G) = k$. *Then,*

$$m \leq \left(1 - \frac{1}{k}\right)\frac{n^2}{2}.$$

An *independent set* of G is a set of vertices of G no two of which are adjacent. G is *k-partite* if its vertex set can be expressed as a disjoint union $V(G) = S_1 \uplus \ldots \uplus S_k$ of k independent sets. If, moreover, for any $1 \leq i < j \leq k$, all the vertices of S_i are adjacent to all the vertices of S_j, G is a *complete k-partite graph*, or simply a *complete multipartite graph*, and we call the sets $S_1, \ldots, S_k$ its *parts*. We denote by K_{s*k} the complete k-partite graph on $[sk]$, in which each part is formed by the vertices having the same residue modulo k. The graph K_{s*k} shows that the inequality of Theorem 1.2 is sharp.

2-partite graphs are also called *bipartite*, and K_{s*2} is more often denoted by $K_{s,s}$. There is a simple characterization of bipartite graphs in terms of their cycles.

Theorem 1.3. [83] *A graph is bipartite if and only if it does not have odd cycles.*

The *independence number* of G, denoted by $\alpha(G)$, is the largest size of an independent set of G. The Caro–Wei theorem provides a complementary formulation of Theorem 1.2, somewhat stronger. The *average degree* of G is the average of the degrees of the vertices of G.

Theorem 1.4. (Caro–Wei) [18, p. 100] *Let G be a graph of order n and average degree d. Then,*

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \geq \frac{n}{d+1}.$$

When G is *triangle-free*, i.e. the neighbourhood of every vertex of G is an independent set, the behaviour of $\alpha(G)$ undergoes a notable change. This was remarked by Ajtai, Komlós and Szemerédi [10], and later improved by Shearer.

Theorem 1.5. (Shearer) [142] *Let G be a triangle-free graph of order n and average degree d, and let $f : [0, \infty) \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{x \ln x - x + 1}{(x-1)^2} & \text{otherwise.} \end{cases}$$

Then, $\alpha(G) \geq nf(d)$.

This is sharp up to a constant factor, see Theorem 9.5.

Let G be a graph and k a positive integer. A *proper k-colouring* of G (*k-colouring* for short, or *proper colouring* when k is unspecified) is a mapping $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ for any two adjacent vertices u and v of G. If such an f exists, G is *k-colourable*. Any mapping of the form $f|_S$ with $S \subseteq V(G)$ is a *partial k-colouring* of G. The integers in $[k]$ are usually referred to as *colours*, and the sets $f^{-1}(i)$, $1 \leq i \leq k$, as *colour classes*. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G is k-colourable.

Example 1.6. The complete graph K_n has chromatic number $\chi(K_n) = n$. Thus, $\chi(G) \geq \omega(G)$ for any graph G.

Since the colour classes of any k-colouring are independent sets, the existence of a k-colouring of G is equivalent to the existence of a partition of $V(G)$ into k (possibly empty) independent sets. From this fact it follows the simple but important observation that $\alpha(G)\chi(G) \geq n$, where n is the order of G.

The *maximum degree* of G, denoted by $\Delta(G)$, is the maximum value among the degrees of the vertices of G. Its *minimum degree* is defined analogously, and is denoted by $\delta(G)$. The *greedy colouring algorithm*, illustrated in the proof of the next theorem, provides a basic method for finding $(\Delta(G) + 1)$-colourings of G.

Theorem 1.7. [126] *For any graph G, $\chi(G) \leq \Delta(G) + 1$.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of G. Since each vertex has degree at most $\Delta(G)$, the following recursion makes sense and defines a mapping $f : V(G) \to [\Delta(G)+1]$: given $1 \leq i \leq n$, we assume that $f(v_j)$ is defined for every $1 \leq j < i$, and set $f(v_i)$ to be the minimum colour not yet assigned to any neighbour of v_i, that is, $f(v_i) = \min\,[\Delta(G) + 1] \setminus \{f(v_j) \mid 1 \leq j < i,\ v_j \sim v_i\}$. Since f assigns different colours to adjacent vertices, it is a $(\Delta(G) + 1)$-colouring. □

A graph G is *k-degenerate* if each subgraph of G has a vertex with degree at most k. The *degeneracy* of G is the smallest integer k such that G is k-degenerate. With a little refinement, the argument above also shows that $\Delta(G)$ can be replaced by the degeneracy of G in the statement of Theorem 1.7. However, in terms of $\Delta(G)$, the bound of Theorem 1.7 is best-possible: when G is a complete graph, the equality is attained. Brooks' theorem says that, essentially, complete graphs and odd cycles are the only graphs for which that happens.

Theorem 1.8. (Brooks) [39, 126] *Let G be a graph with $\chi(G) = \Delta(G)+1$. Then, there is a connected component H of G for which one of the following holds:*

(*a*) *H is a complete graph of order* $\Delta(G)+1$*, or*

(*b*) $\Delta(G)=2$ *and H is an odd cycle.*

A celebrated result of Erdős implies that, unlike the clique number, the chromatic number cannot be deduced from local considerations. The *girth* of a graph G is the shortest length of a cycle of G; if G has no cycles, then its girth is ∞.

Theorem 1.9. [58] *For every two positive integers k and g, there exists a graph with chromatic number k and girth at least g.*

However, for triangle-free graphs, Theorem 1.7 can be significantly improved, similarly to what Shearer's theorem (Theorem 1.5) tells us about their independence number.

Theorem 1.10. (Johansson–Molloy) [123] *For every $\varepsilon > 0$ there exists some $\Delta_\varepsilon \in \mathbb{N}$ such that, for every triangle-free graph G with maximum degree $\Delta \geq \Delta_\varepsilon$, $\chi(G) \leq (1+\varepsilon)\Delta/\ln\Delta$.*

Graph homomorphisms

Let G and H be graphs. A *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ such that, for every edge $\{u, v\}$ of G, $\{\varphi(u), \varphi(v)\}$ is an edge of H. We write $G \to H$ to indicate that such a φ exists; otherwise, we write $G \not\to H$. When $G \to H$ and $H \to G$, G and H are *homomorphically equivalent.* If φ is bijective and φ^{-1} is also an homomorphism, φ is an *isomorphism.* In that case, we say that G and H are *isomorphic*, or that G is a *copy* of H, and we denote that by $G \cong H$.

A homomorphism φ from G to itself is an *endomorphism* of G. If φ is an isomorphism, then it is an *automorphism* of G.

A graph G is a *core* if $G \not\to H$ for any proper subgraph H of G. Consequently, all endomorphisms of G are automorphisms. Each graph is homomorphically equivalent to a core, which is unique up to isomorphism [96].

Homomorphisms give rise to several generalisations of proper colourings; Chapter 7 contains some details about that.

Planar graphs

An *embedded planar graph* is an ordered pair (G, ι) such that

(i) G is a graph;

(ii) $V(G) \subseteq \mathbb{R}^2$;

(iii) ι is a mapping associating to every edge $e \in E(G)$ an arc $\iota(e)$ of $\mathbb{R}^2$, that is, a topological subspace of $\mathbb{R}^2$ homeomorphic to the interval $[0, 1]$,

(iv) for every $e = \{u, v\} \in E(G)$, the endpoints of $\iota(e)$ are precisely u and v, and none of the other points of $\iota(e)$ is in $V(G)$;

(v) for every two distinct edges $e_1, e_2 \in E(G)$, $\iota(e_1) \cap \iota(e_2) \subseteq V(G)$.

Abusing of the terminology, we may identify (G, ι) with G when ι is understood, and the images by ι of the edges with the edges themselves. A graph G is *planar* if (G, ι) is an embedded planar graph for some ι. Any such ι is a *planar embedding* of G.

A *face* of an embedded planar graph (G, ι) is any of the connected components of $\mathbb{R}^2 \setminus V(G) \setminus \cup_{e \in E(G)} \iota(e)$. Each embedded planar graph has a unique unbounded face. A vertex v and a face F of (G, ι) are *incident* if v is in the boundary of F. (G, ι) is *outerplanar* if every vertex is incident to the unbounded face, and G is *outerplanar* if (G, ι) is outerplanar for some planar embedding ι. A face is *triangular* if either

(a) it is bounded, is homeomorphic to an open disc, and the edges on its boundary form a 3-cycle, or

(b) it is the unbounded face, is homeomorphic to $\mathbb{R}^2$ minus a point, and the edges on its boundary form a 3-cycle.

A *triangulation* is an embedded planar graph the faces of which are all triangular. A *triangulation* of a planar graph G is a triangulation of the form (G, ι).

Planar graphs have had a distinguished role in the historical development of the theory of graph colouring. The emblematic four colour theorem remains as a perennial symbol of that relationship.

Theorem 1.11. (FOUR COLOUR THEOREM) [19] *Every planar graph is 4-colourable.*

List colourings

Given a positive integer k, a *k-list assignment* to a graph G is a mapping L that assigns to every vertex of G a set of k positive integers. An *L-colouring* of G is a proper colouring f of G such that $f(v) \in L(v)$ for every vertex v of G; in this situation, we also say that f is *accepted* by L (or just *acceptable*, if L is clear from the context). G is *k-list colourable*, or *k-choosable*, if for every k-list assignment L to G there is an L-colouring of G. The *list chromatic number* $\chi_\ell(G)$ of G, also known as its *choice number*, is the smallest k such that G is k-list colourable.

This colouring variant was introduced by Vizing [156] and, independently, by Erdős, Rubin and Taylor [63]. Clearly, when L is the constant list assignment $L : V(G) \to \{[k]\}$, the L-colourings of G are precisely its proper k-colourings. In particular, $\chi_\ell(G) \geq \chi(G)$. However, $\chi_\ell(G)$ cannot be upper-bounded in terms of $\chi(G)$.

Theorem 1.12. [63] *Let k be a positive integer and $s = \binom{2k-1}{k}$. The complete bipartite graph $K_{s,s}$ is not k-choosable.*

The choice number of planar graphs has received considerable attention. Most notably, Thomassen gives the following sharp bound.

Theorem 1.13. [153] *Every planar graph is* 5*-choosable*

1.2 Digraph colouring

Digraphs

A *simple directed graph D*, or a *digraph* for short, is an ordered pair (X, R), where X is a non-empty set and R is an anti-reflexive relation over X, that is, a set of ordered pairs $(u, v) \in X \times X$ with $u \neq v$. X, denoted by $V(D)$, is the set of *vertices* of D, and D is a digraph *on X*. R is the set of *arcs* of D, and is denoted by $A(D)$. The cardinalities of $V(G)$ and $A(G)$ are the *order* and the *size* of D, respectively. An *n-vertex digraph* is a digraph of order n. The digraphs we consider are of finite order, unless otherwise specified.

An arc (u, v) of D may be denoted by uv. The *endpoints* of uv are u and v; we say that uv *leaves* u and *enters* v; if S and T are sets of vertices with $u \in S$ and $v \in T$, we say that uv *leaves* S and *enters* T. The vertices u and v are *adjacent*, and each of them is *incident* to uv. u is an *in-neighbour* of v, v is an *out-neighbour* of u, and each of them is a *neighbour* of the other. The *in-neighbourhood* and the *out-neighbourhood* of a vertex v, denoted by $N^-(v)$ and $N^+(v)$, are the set of in-neighbours and the set of out-neighbours of v. Their union $N^-(v) \cup N^+(v)$ is denoted by $N(v)$, and their intersection by $N^{\pm}(v)$. The *closed in-neighbourhood* of v and its *closed out-neighbourhood* are the sets $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$, and their union is denoted by $N[v]$. If S is a set of vertices, by $N^-(S)$ and $N^+(S)$ we denote the sets $\cup_{w \in S} N^-(w)$ and $\cup_{w \in S} N^+(w)$ of in-neighbours and out-neighbours of the elements of S. The *in-degree* and the *out-degree* of v, denoted by $\deg^-(v)$ and $\deg^+(v)$, are the number of in-neighbours and the number of out-neighbours of v. If all the vertices of D have in-degree and out-degree k, then D is *k-regular*.

The arc uv is *simple* if vu is not an arc of D; in this case, u is a *simple in-neighbour* of v and v is a *simple out-neighbour* of u. Otherwise, the set $\{uv, vu\}$ is a *digon* of D, and is denoted by $\langle u, v \rangle$; in this case, both u and v are *incident* to $\langle u, v \rangle$. A digraph is *symmetric* if it has no simple arcs.

To each graph G we can associate a symmetric digraph $\overleftrightarrow{G}$ by letting $V(G)$ be its set of vertices and $\{uv \mid \{u, v\} \in E(G)\}$ its set of arcs. Conversely, every digraph D has an *underlying graph $\underline{D}$*: the graph with vertex set $V(D)$ and edge set $\{\{u, v\} \mid uv \in A(D)\}$ (in order to stress the loss of the notion of directions, we might also call it the *underlying undirected graph* of D).

An *oriented graph* is a digraph D with no digons; in this case, if $\underline{D} = G$, D is an *orientation* of G.

Example 1.14. A digraph is *complete* if, for any two distinct vertices u and v, $\langle u, v\rangle$ is a digon. A *tournament* is an orientation of a complete graph. While a complete digraph of order n is always $(n-1)$-regular, a tournament of order n is not necessarily regular, although each of its vertices v satisfies $\deg^-(v) + \deg^+(v) = n - 1$.

A digraph D_1 is a *subgraph* of a digraph D_2 if $V(D_1) \subseteq V(D_2)$ and $A(D_1) \subseteq A(D_2)$. *Supergraphs*, *induced* and *proper subgraphs* of digraphs, and the arcs *spanned* by a set of vertices, are defined following the analogy with the homonymous graph notions. A *matching* of a digraph D is a set of arcs pairwise not sharing an endpoint. The maximum size of a matching of D is denoted by $\nu(D)$.

Similarly, a *directed walk* W of *length* ℓ of a digraph D is an alternating sequence $v_0, a_1, v_1, \dots, v_{\ell-1}, e_\ell, v_\ell$ of vertices and arcs of D satisfying $a_i = v_{i-1}v_i$ for every $1 \leq i \leq \ell$. For each such a_i, v_{i-1} is the *predecessor* of v_i in W, and v_i is the *successor* of v_{i-1}. Concatenation, directed paths and directed cycles are defined following the same analogy. $\vec{C}_\ell$ denotes the digraph on $[\ell]$ consisting in the vertices and arcs of the directed cycle $1, (1, 2), 2, \dots, \ell, (\ell, 1), 1$. An *acyclic digraph* is a digraph without directed cycles. Acyclic tournaments are often called *transitive tournaments*. We denote by TT_n the transitive tournament on $[n]$ with arc set $\{(i, j) \mid 1 \leq i < j \leq n\}$.

D is *connected* if its underlying graph $\underline{D}$ is connected, and it is *strongly connected* if every two distinct vertices of D are the endpoints of a directed path. The *(strongly) connected components* of D are its maximal (strongly) connected subgraphs. A classical theorem of Bondy links the length of the longest cycles of strongly connected digraphs to the chromatic number of their underlying graphs.

Theorem 1.15. (Bondy) [34] *Let G be a graph and D a strongly connected orientation of G without directed cycles of length greater than $s \geq 1$. Then, $\chi(G) \leq s$.*

Bicliques, acyclic sets, and dicolourings

A *biclique* of a digraph D is a set of vertices $S \subseteq V(D)$ inducing a complete subgraph of D. The *biclique number* of D, denoted by $\overleftrightarrow{\omega}(D)$, is the largest size of a clique of D. An *acyclic set* of D is a set $S \subseteq V(D)$ inducing an acyclic subgraph. The *acyclicity number* of D, denoted by $\vec{\alpha}(D)$, is the largest size of an acyclic set of D.

When D is a symmetric digraph, its bicliques and acyclic sets coincide with the cliques and independent set of $\underline{D}$. In particular, $\overleftrightarrow{\omega}(D) = \omega(\underline{D})$ and

$\vec{\alpha}(D) = \alpha(\underline{D})$. We have the following analogue of Theorem 1.4. The *average out-degree* of D is the average of the out-degrees of the vertices of D.

Theorem 1.16. [8, 100] *Let D be a digraph of order n and average out-degree d^+. Then,*

$$\vec{\alpha}(D) \geq \sum_{v \in V(D)} \frac{1}{\deg^+(v)+1} \geq \frac{n}{d^+ + 1}.$$

In the case of tournaments, much more can be said.

Theorem 1.17. [62]

(i) *For every tournament T of order n, $\vec{\alpha}(T) \geq \lfloor \log_2 n \rfloor + 1$.*

(ii) *For every positive integer n, there is a tournament T with $\vec{\alpha}(T) \leq 2\lfloor \log_2 n \rfloor + 1$*

Given a positive integer k, a *k-dicolouring* of a digraph D is a mapping $f : V(D) \to [k]$ such that $f^{-1}(i)$ is an acyclic set for each $1 \leq i \leq n$. If such an f exists, D is *k-dicolourable.* Mappings of the form $f|_S$ with $S \subseteq V(D)$, are *partial k-dicolourings* of D. The *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is the smallest k such that D is k-dicolourable.

k-dicolourings were introduced by Neumann-Lara in the early 1980s as a generalisation of proper colourings to digraphs [127]. Later rediscovered by Mohar [120], they have since then attracted a lot of attention, being the subject of numerous papers and various doctoral dissertations (see for instance [20, 84, 130, 140]). Aboulker and Harutyunyan are preparing a survey on the topic [85].

When D is a symmetric digraph, its k-dicolourings of coincide with the k-colourings of $\underline{D}$. In particular, $\vec{\chi}(D) = \chi(\underline{D})$. The behaviour of the dichromatic number resembles in many ways that of the chromatic number. The inequalities $\vec{\chi}(D) \geq \overleftrightarrow{\omega}(D)$ and $\vec{\alpha}(D)\vec{\chi}(D) \geq n$, valid for any digraph D of order n, are among the superficial manifestations of this fact. Another one comes from the adaptation of the greedy colouring algorithm. By $\Delta_{\min}(D)$ we denote the maximum, over all vertices v of D, of the quantity $\min\{\deg^-(v), \deg^+(v)\}$.

Theorem 1.18. *For any digraph D, $\vec{\chi}(D) \leq \Delta_{\min}(D) + 1$.*

Also in this case, we can replace $\Delta_{\min}(D)$ by k in the above statement whenever D is *k-degenerate*, that is, whenever each subgraph of D has a vertex with in-degree or out-degree at most k.

Digraphs achieving the equality in Theorem 1.18 have also been studied. For instance, in the case of oriented graphs, they must satisfy $\Delta_{\min}(D) \leq 1$ [131]. However, in general they seem to be difficult to describe [1]. The whole picture looks much nicer if the parameter $\Delta_{\min}(D)$ is substituted by $\Delta_{\max}(D)$, which denotes the maximum, over all vertices v of D, of the quantity $\max\{\deg^-(v), \deg^+(v)\}$.

Theorem 1.19. (Directed Brooks' theorem) [100, 121] *Let D be a digraph with $\vec{\chi}(D) = \Delta_{\max}(D) + 1$. Then, there is a connected component H of D for which one of the following holds:*

(*a*) *H is a complete digraph of order $\Delta_{\max}(D) + 1$,*

(*b*) *$\Delta_{\max}(D) = 2$ and H is a symmetric odd cycle, or*

(*c*) *$\Delta_{\max}(D) = 1$ and H is a directed cycle.*

In fact, in the above statement $\Delta_{\max}(D)$ can be replaced by the more precise $\lceil \tilde{\Delta}(D) \rceil$, where

$$\tilde{\Delta}(D) := \max_{v \in V(D)} \sqrt{\deg^-(v) \deg^+(v)}$$

is the *maximum geometric mean degree* of D. See [3, 75, 89] for other strengthenings of Theorem 1.19.

In order to further illustrate the parallelism with the chromatic number, we include two more results, resemblant of Theorems 1.9 and 1.3. The *digirth* of a digraph D is the length of a shortest directed cycle of D; if D has no directed cycles, its digirth is ∞.

Theorem 1.20. [28] *For every two positive integers k and g, there exists a digraph with dichromatic number k and digirth at least g.*

Theorem 1.21. [127] *If a digraph D has no directed cycles of odd length, or has no directed cycles of even length, then D is 2-dicolourable.*

Finally, concerning tournaments, we have the following complement of Theorem 1.17.

Theorem 1.22. [84]

(*i*) *For every $\varepsilon > 0$ there exists some $n_\varepsilon \in \mathbb{N}$ such that, for every tournament T of order $n \geq n_\varepsilon$, $\vec{\chi}(T) \leq (1 + \varepsilon)n / \log_2 n$.*

(*ii*) *For every positive integer n, there is a tournament T with $\vec{\chi}(T) \geq n/(2\log_2 n + 1)$.*

Digraph homomorphisms and list dicolourings

Let D and E be digraphs. A *homomorphism* from D to E is a mapping $\varphi : D \to E$ such that, for every arc (u, v) of D, $(\varphi(u), \varphi(v))$ is an arc of E. *Isomorphisms*, *endomorphisms*, *automorphisms* of digraphs, and all the related terminology and notation, are defined following the same analogy with the undirected setting. This generalises the concept of graph homomorphism, in the sense that, when D and E are symmetric, φ is an homomorphism from D to E if and only if it is a homomorphism from $\underline{D}$ to $\underline{E}$.

The notion of list colourings is adapted with the same philosophy. A *k-list assignment* L to a digraph D is a k-list assignment to $\underline{D}$. An *L-dicolouring* of D is a dicolouring f of D such that $f(v) \in L(v)$ for every vertex of D. The *list dichromatic number* of D, denoted by $\vec{\chi}_\ell(D)$, and all the related concepts, are defined following the same analogy. List dicolourings of digraphs have been studied for example in [24, 89].

1.3 Operations with graphs and digraphs

Some standard operations

It will be useful to fix the notation for some of the standard graph and digraph operations. We define them for digraphs; the definitions for graphs correspond to the case of symmetric digraphs, in the same vein as above. Let D and D' be digraphs.

Vertex deletion. Given a set S of vertices of D, $D - S$ denotes the subgraph of D induced by $V(D) \setminus S$. $D - \{v\}$ is also denoted by $D - v$.

Arc deletion. Given a set S of arcs of D, $D \setminus S$ denotes the digraph with vertex set $V(D)$ and arc set $A(D) \setminus S$.

Arc addition. Given a set S of ordered pairs $(u, v) \in V(D) \times V(D)$ with $u \neq v$, $D \cup S$ denotes the digraph with vertex set $V(D)$ and arc set $A(D) \cup S$.

Complement digraph. $\overline{D}$ denotes the digraph with vertex set $V(D)$ and arc set $\{(u, v) \in V(D) \times V(D) \mid u \neq v\} \setminus A(D)$.

Disjoint union. $D \sqcup D'$ denotes the digraph on $(\{1\} \times V(D)) \cup (\{2\} \times V(D'))$ with arc set $\{((1, u), (1, v)) \mid (u, v) \in A(D)\} \cup \{((2, u), (2, v)) \mid (u, v) \in A(D')\}$. We might abuse of the notation and denote by $V(D)$ and $V(D')$ the sets $\{1\} \times V(D)$ and $\{2\} \times V(D')$ when there is no danger of confusion.

Join. $D \boxplus D'$ denotes the digraph with vertex set $V(D \sqcup D')$ and arc set $A(D \sqcup D') \cup \{((1, v), (2, v')), ((2, v'), (1, v)) \mid v \in V(D),\ v' \in V(D')\}$.

Cartesian product. $D \square D'$ denotes the digraph with vertex set $V(D) \times V(D')$ and arc set $\{((u, u'), (v, v')) \mid (u = v \text{ and } (u', v') \in A(D')) \text{ or } ((u, v) \in A(D) \text{ and } u' = v')\}$.

Tensor product. $D \times D'$ denotes the digraph with vertex set $V(D) \times V(D')$ and arc set $\{((u, u'), (v, v')) \mid (u, v) \in A(D),\ (u', v') \in A(D')\}$.

There are also ways of constructing digraphs from other structures, for instance, from groups. Let Γ be a group and C a subset of Γ without the neutral element. The *Cayley digraph* of Γ with respect to C, denoted by $\mathrm{Cay}(\Gamma, C)$, is the digraph with vertex set Γ and arc set $\{(g,h) \in \Gamma \times \Gamma \mid \exists c \in C\ gc = h\}$. (If C is closed under taking inverses, the resulting digraph is symmetric. Its underlying graph, denoted by $\underline{\mathrm{Cay}}(\Gamma, C)$, is then the *Cayley graph* of Γ with respect to C.)

The dichromatic number of the digraphs resulting from these operations can sometimes be obtained directly from the dichromatic numbers of D and D'. For instance, $\vec{\chi}(D \sqcup D') = \max\{\vec{\chi}(D), \vec{\chi}(D')\}$ and $\vec{\chi}(D \boxplus D') = \vec{\chi}(D) + \vec{\chi}(D')$. We also have the following.

Proposition 1.23. [53, 140] *Let D and D' be digraphs. Then,*

(*i*) $\vec{\chi}(D \,\square\, D') = \max\{\vec{\chi}(D), \vec{\chi}(D')\}$ *and*

(*ii*) $\vec{\chi}(D \times D') \leq \min\{\vec{\chi}(D), \vec{\chi}(D')\}$.

(*i*) generalises to digraphs a well-known theorem of Sabidussi [137], while the graph version of (*ii*) is also common knowledge. It was a long-standing conjecture of Hedetniemi that, for graphs, the inequality in (*ii*) is in fact an inequality. This was refuted in 2019 by Shitov. Answering a question of Costa and Silva [53] (also posed independently in [92]), Picasarri-Arrieta showed that equality cannot be expected for oriented graphs either.

Theorem 1.24. [130, 143]

(*i*) *There exist two symmetric digraphs D and D' such that $\vec{\chi}(D \times D') < \min\{\vec{\chi}(D), \vec{\chi}(D')\}$.*

(*ii*) *There exist two oriented graphs D and D' such that $\vec{\chi}(D \times D') < \min\{\vec{\chi}(D), \vec{\chi}(D')\}$.*

In [92] it was also asked whether the fractional version of Hedetniemi's conjecture holds for digraphs, in view of a result of Zhu [160] stating that it holds for graphs (see Chapter 7 for the definition of fractional colourings). It turns out that this had already been answered in the negative by Severino [140].

Theorem 1.25. [140, 160]

(*i*) *For any two symmetric digraphs D and D', we have that $\vec{\chi}_f(D \times D') = \min\{\vec{\chi}_f(D), \vec{\chi}_f(D')\}$.*

(*ii*) *There exist two oriented graphs D and D' such that $\vec{\chi}_f(D \times D') < \min\{\vec{\chi}_f(D), \vec{\chi}_f(D')\}$.*

Back-arc graphs and feedback arc set graphs

We have seen that taking the symmetric digraph $\overleftrightarrow{G}$ of a graph G, and the underlying graph $\underline{D}$ of a digraph D, is a simple way of switching between the worlds of graphs and digraphs, that behaves particularly well with symmetric digraphs. However, for general digraphs, taking the underlying graph is not very meaningful. For instance, the transitive tournament TT_n is acyclic, but $\underline{TT_n}$ has no independent set of size larger than 1. Here we introduce a more refined technique that has been used implicitly or explicitly by various authors in order to take into account these idiosyncrasies.

Let D be a digraph and $\mathrm{O}(V(D))$ the set of total orders over $V(D)$. Given $\preceq\, \in \mathrm{O}(V(D))$, an arc uv of D is said to be *increasing* (or *forward*) if $u \preceq v$, and *decreasing* (or *backward*) otherwise. The *back-arc graph*[1] of D with respect to $\preceq$, denoted by $D^{\preceq}$, is the graph with vertex set $V(D)$ and edge set $\{\{u, v\} \mid uv \in A(D),\ v \preceq u\}$, that is, the underlying graph of the maximal subgraph of D having only backward arcs.

Clearly, every independent set of $D^{\preceq}$ is an acyclic set of D. And, for every acyclic set S of D, there is and order $\preceq\, \in \mathrm{O}(V(D))$ such that S is an independent set of $D^{\preceq}$. Similarly, every k-colouring of $D^{\preceq}$ is a k-dicolouring of D, and, for every k-dicolouring f of D, there is an order $\preceq\, \in \mathrm{O}(V(D))$ such that f is a k-colouring of $D^{\preceq}$. In particular, we have the following.

Proposition 1.26. *For any digraph D,*

(i) $\vec{\alpha}(D) = \max_{\preceq \in \mathrm{O}(V(D))} \alpha(D^{\preceq})$ *and*

(ii) $\vec{\chi}(D) = \min_{\preceq \in \mathrm{O}(V(D))} \chi(D^{\preceq})$.

At the end, the crude reason why this works is that each directed cycle of D yields an edge in $D^{\preceq}$, while $\preceq$ can chosen so that any given acyclic subgraph of D yields no edge. It makes sense, therefore, to try to push for a more refined notion. A *feedback arc set* of D is a set of arcs of D that contains an arc of every directed cycle of D. The *feedback arc set graph* of D with respect to a feedback arc set F is the graph D^F with vertex set $V(D)$ and edge set $\{\{u, v\} \mid (u, v) \in F\}$, that is, the result of turning the arcs of F into edges and throwing the rest of arcs away. A similar argument as above shows the following.

Proposition 1.27. *For any digraph D,*

(i) $\vec{\alpha}(D) = \max\{\alpha(D^F) \mid F \text{ is a feedback arc set of } D\}$ *and*

(ii) $\vec{\chi}(D) = \min\{\chi(D^F) \mid F \text{ is a feedback arc set of } D\}$.

[1] Frequently called 'backedge graph' in the literature.

Hence, for us it does not matter much which of the two notions we want to use. However, these concepts can have other applications; for instance, Aboulker, Aubian, Charbit and Lopes [4] define a new parameter generalising the clique number to digraphs by setting $\vec{\omega}(D) := \min_{\preceq \in \mathrm{O}(V(D))} \omega(D^{\preceq})$. Therefore, it would be desirable to have some understanding of how much can back-arc graphs and feedback arc set graphs differ. The following proposition tells us that, luckily, there are as few differences as one would hope for[2].

Proposition 1.28. *Let D be a digraph and F a minimal feedback arc set of D. Then, there is a total order $\preceq$ of $V(D)$ such that $D^{\preceq} = D^F$.*

Proof. We create a new digraph D' by reversing the direction of the arcs in F. That is, $V(D') = V(D)$ and $A(D') = A(D) \setminus F \cup F^{-1}$, where $F^{-1} := \{(v,u) \mid (u,v) \in F\}$. The key of the proof is the following.

Claim 1.28.1. *D' is acyclic.*

Proof. Suppose that D' has a directed cycle C. Assume that none of the arcs of C is the reverse a^{-1} of an arc $a \in F$. Then, C is a directed cycle of D, so it has an arc in F, say, a_C. But, by the definition of D', a_C must be in F^{-1}, a contradiction. Therefore, there is an integer $r \geq 1$ such that C can be expressed as a concatenation $P_1 a_1^{-1} \dots P_r a_r^{-1}$, where $a_1^{-1}, \dots, a_r^{-1}$ are the reverses of some arcs $a_1, \dots, a_r \in F$ and $P_1, \dots, P_r$ are directed paths of D' (possibly of length 0) without arcs in F^{-1}.

By the minimality of F, we can find directed cycles $C_1, \dots, C_r$ of D such that a_i is an arc of C_i for every $1 \leq i \leq r$. We claim that each C_i can be chosen so that a_i is its unique arc in F. Indeed, if, for some i, all directed cycles X of D using a_i have an arc $a_X \in F \setminus \{a_i\}$, then, $F \setminus \{a_i\} \subsetneq F$ would be a smaller feedback arc set, a contradiction. Hence, each C_i can be written as a concatenation $Q_i a_i$, where Q_i is a directed path of D with no arcs in F.

Now, $P_1 Q_1 \dots P_r Q_r$ is a closed directed walk of D with no arc in F. Therefore, there is a directed cycle of D with no arc in F, a contradiction. We conclude that D' must be acyclic. ■

By Claim 1.28.1, we can find an order $\preceq \in \mathrm{O}(V(D))$ such that $D'^{\preceq}$ has no edges. This implies that $D^{\preceq} = D^F$. □

1.4 Multigraphs and random graphs

Multigraphs and multidigraphs

Sometimes, the following more general notion of graph might be needed. A *multigraph* G is an ordered pair (X, μ), where X is a non-empty set and μ is

[2] We thank Pierre Aboulker for pointing us that this has already been remarked in [6], and actually has a much simpler proof.

a mapping $\{\{u, v\} \subseteq X\} \to \mathbb{N}$. X, denoted by $V(G)$, is the set of *vertices* of G; the set $\mu^{-1}(\mathbb{Z}^+)$, denoted by $E(G)$, is the set of *edges* of G; and, for each $e \in E(G)$, $\mu(e)$ is the *multiplicity* of e, denoted by $\mathrm{mult}(e)$. Edges of the form $\{v, v\}$ are *loops*, and edges with multiplicity greater than 1 are *multiple edges*. Multigraphs without loops and multiple edges are *simple*; we might identify them with graphs (which might be called *simple graphs* in order to stress this lack of loops and multiple edges).

The *degree* of a vertex v of G is the quantity $\deg(v) \coloneqq \sum_{v \in e \in E(G)} \mathrm{mult}(e) + \sum_{e=\{v\} \in E(G)} \mathrm{mult}(e)$. Other than that, we might use the basic graph concepts from Section 1.1 in this more general context of multigraphs in a way that needs no further specifications.

A *multidigraph* D is an ordered pair (X, μ), where X is a non-empty set and $\mu : X \times X \to \mathbb{N}$. The sets $V(D)$ and $A(D)$ of vertices and arcs of D, the multiplicity of the arcs of D, etc. are defined analogously. We denote by $\underline{D}$ the *underlying simple digraph* of D, that is, the digraph identified with the simple multidigraph $(V(D), \mu')$, where

$$\mu'((u, v)) = \begin{cases} 1 & \text{if } \mu((u, v)) \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(which we have to be careful not to mix up with the underlying undirected graph of a digraph). The *in-degree* and the *out-degree* of a vertex v of D are the quantities $\deg^-(v) \coloneqq \sum_{(u,v) \in A(D)} \mathrm{mult}(a)$ and $\deg^+(v) \coloneqq \sum_{(v,w) \in A(D)} \mathrm{mult}(a)$. We say that D is an *orientation* of a multigraph $G = (X, \mu_G)$ if $V(G) = V(D)$ and, for any $\{u, v\} \subseteq V(G)$,

$$\mu_G(\{u, v\}) = \begin{cases} \mu((u, v)) + \mu((v, u)) & \text{if } u \neq v \\ \mu((u, u)) & \text{if } u = v. \end{cases}$$

We can sometimes require an even more general setting. A *multigraph with labelled edges* G is an ordered triple (X, Y, ι), where X is a non-empty set and $\iota : Y \to \{\{u, v\} \subseteq X\}$. X is the set of *vertices* of G and Y is its set of *edges*. Thus, forgetting the edge labels of G yields a multigraph (X, μ), where $\mu(e) = |\iota^{-1}(e)|$. *Multidigraphs with labelled arcs* are defined analogously. A multidigraph with labelled arcs $D = (X', Y', \iota')$ is an *orientation* of G if $X' = X$, $Y' = Y$, and $\iota'(y) \in \{(u, v), (v, u)\}$ for every $y \in Y$ with $\iota(y) = \{u, v\}$.

Random graphs and digraphs

A *random (di)graph* is a discrete probability space $\mathbb{G}$ over a set of (di)graphs. In the usual case of interest, all graphs of $\mathbb{G}$ have the same order n, and one considers sequences of random graphs $(\mathbb{G}_n)_{n \in \mathbb{Z}^+}$ indexed by n. These may depend on another parameter ρ. Sometimes, ρ itself may be a function of n;

we denote that by writing $\rho = \rho(n)$. Let $(E_n)_{n\in\mathbb{Z}^+}$ be a sequence, where each E_n is an event of $\mathbb{G}_n$. We say that these events hold *with high probability* (*whp* for short) if $\mathbb{P}[E_n] \to 1$ as $n \to \infty$.

Let us fix our notation. We consider random graphs and digraphs over:

$\mathscr{G}(n)$, $\mathscr{D}(n)$ and $\mathscr{O}(n)$, the sets of all graphs, digraphs, and oriented graphs on $[n]$;

$\mathscr{G}^*_{\text{reg}}(n,r)$, $\mathscr{G}_{\text{reg}}(n,r)$, $\mathscr{D}_{\text{reg}}(n,r)$ and $\mathscr{O}_{\text{reg}}(n,r)$, the sets of all r-regular multigraphs, r-regular graphs, r-regular digraphs, and r-regular oriented graphs on $[n]$.

The random graphs and digraphs considered are the following. Most of them are introduced again whenever needed.

$\mathbb{G}(n,p)$, the random graph over $\mathscr{G}(n)$ obtained by incorporating each possible edge independently with probability p, known as *binomial random graph*;

$\mathbb{D}(n,p)$, the random digraph over $\mathscr{D}(n)$ obtained by incorporating each possible arc independently with probability p, known as *binomial random digraph*;

$\mathbb{O}(n,p)$, the random oriented graph over $\mathscr{O}(n)$ obtained from $\mathbb{G}(n,2p)$ by orienting each of its edges independently and uniformly at random (in this case, it is assumed that $0 \leq p \leq 1/2$);

$\mathbb{G}(n;M)$, the random graph over $\mathscr{G}(n)$ with exactly M edges, taken with the uniform distribution;

$\mathbb{D}(n;M)$, the random digraph over $\mathscr{D}(n)$ with exactly M arcs, taken with the uniform distribution;

$\mathbb{O}(n;M)$, the random oriented graph over $\mathscr{O}(n)$ obtained from $\mathbb{G}(n;M)$ by orienting each of its edges independently and uniformly at random;

$\mathbb{G}^*_{\text{reg}}(n,r)$, the random multigraph over $\mathscr{G}^*_{\text{reg}}(n,r)$ defined in Section 9.2;

$\mathbb{G}_{\text{reg}}(n,r)$, the random graph over $\mathscr{G}_{\text{reg}}(n,r)$ with the uniform distribution;

$\mathbb{D}^*_{\text{reg}}(n,r)$, the random r-regular multidigraph on $[n]$ defined in Section 9.2;

$\mathbb{D}_{\text{reg}}(n,r)$, the random graph over $\mathscr{D}_{\text{reg}}(n,r)$ with the uniform distribution;

$\mathbb{O}_{\text{reg}}(n,r)$, the random oriented graph over $\mathscr{O}_{\text{reg}}(n,r)$ with the uniform distribution;

$\operatorname{forg}\mathbb{D}^*_{\text{reg}}(n,r)$, the random multidigraph over $\mathscr{G}^*_{\text{reg}}(n,2r)$ obtained by forgetting the orientations of the arcs of $\mathbb{D}^*_{\text{reg}}(n,r)$;

$\operatorname{forg}\mathbb{O}_{\text{reg}}(n,r)$, the random graph over $\mathscr{G}_{\text{reg}}(n,2r)$ obtained by forgetting the orientations of the arcs of $\mathbb{O}_{\text{reg}}(n,r)$.

Chapter 2

Four open problems

In this chapter we review four important open problems in the theory of the dichromatic number, as well as some of their variants. They have been the drive of most of our investigations.

2.1 The relationship between the chromatic and the dichromatic numbers

Let G be a graph. The *dichromatic number* of G, denoted by $\vec{\chi}(G)$, is the maximum of $\vec{\chi}(D)$ over all orientations D of G. Erdős and Neumann-Lara wondered whether graphs with large chromatic number have large dichromatic number.

Question 2.1. [59] *Is there a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that, for every positive integer k and every graph G, $\chi(G) \geq f(k)$ implies that $\vec{\chi}(G) \geq k$?*

Clearly, one can take $f(1) = 1$ and $f(2) = 3$. However, it is already unknown whether $f(3)$ exists.

Mohar and Wu give a positive answer for the fractional version of the problem. The *fractional dichromatic number* of a graph G, denoted by $\vec{\chi}_f(G)$, is the maximum of $\vec{\chi}_f(D)$ over all orientations D of G. We refer the reader to Section 7.2 for the concrete details on fractional colourings.

Theorem 2.2. [122] *Let G be a graph with fractional chromatic number $\chi_f(G) = t$. Then,*

$$\vec{\chi}_f(G) \geq \frac{t}{8 \log_2 t + 4 + 4 \log_2 e}.$$

A short argument shows that a similar statement holds for the independence ratio. It relies on the following result of Manber and Tompa.

Theorem 2.3. [114] *The number of acyclic orientations of a graph G of order n and average degree d is at most*

$$\prod_{v \in V(G)} (\deg v + 1) \leq (d+1)^n.$$

By $\vec{\alpha}(G)$ we denote the minimum of $\vec{\alpha}(D)$ over all orientations D of the graph G.

Theorem 2.4. *Let G be a graph of order n with $n/\alpha(G) = t$. Then,*

$$\frac{n}{\vec{\alpha}(G)} \geq \min\left\{\frac{\ln 2}{2}t\,,\ \frac{t}{2\log_2 t + 2\log_2 e + 1}\right\}.$$

Proof. Let $s = \vec{\alpha}(G)$ and $u = n/\vec{\alpha}(G)$. We can assume that $t/u \geq 2/\ln 2$, for otherwise the result is immediate. We denote by $\mathscr{O}(G)$ the set of orientations of G, by d the average degree of G, and by d_S the average degree of $G[S]$, where S is any non-empty set of vertices of G. Since every orientation of G has an acyclic set of size s, and using Theorem 2.3,

$$\begin{aligned} 2^{\frac{1}{2}dn} = |\mathscr{O}(G)| &\leq \sum_{S \in \binom{V(G)}{s}} \big|\{D \in \mathscr{O}(G) \mid D[S] \text{ is acyclic}\}\big| \\ &\leq \sum_{S \in \binom{V(G)}{s}} (d_S+1)^s \cdot 2^{\frac{1}{2}dn - \frac{1}{2}d_S s}. \end{aligned} \tag{2.1}$$

By the Caro–Wei theorem (Theorem 1.4), for any $S \in \binom{V(G)}{s}$,

$$\frac{s}{d_S+1} \leq \alpha(G[S]) \leq \alpha(G) = \frac{n}{t},$$

which implies that $d_S \geq t/u - 1 \geq 2/\ln 2 - 1$. It is easy to check that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x+1) \cdot 2^{-\frac{x}{2}}$ is decreasing in the interval $[2/\ln 2 - 1, \infty)$. By (2.1),

$$1 \leq \sum_{S \in \binom{V(G)}{s}} f\left(\frac{t}{u} - 1\right)^s = \binom{n}{s}\left(\frac{t}{u} \cdot 2^{-\frac{1}{2}\left(\frac{t}{u}-1\right)}\right)^s \leq \left(et \cdot 2^{-\frac{1}{2}\left(\frac{t}{u}-1\right)}\right)^s.$$

Therefore, we get that

$$\begin{aligned} 1 &\leq e^2 t^2 \cdot 2^{-\frac{t}{u}+1} \\ u &\geq \frac{t}{2\log_2 t + 2\log_2 e + 1}. \end{aligned}$$

□

We believe that the list version of Question 2.1 has a positive answer as well. We define the *list dichromatic number* of a graph G as the maximum of $\vec{\chi}_\ell(D)$ over all orientations D of G, and we denote it by $\vec{\chi}_\ell(G)$.

Conjecture 2.5. *There is a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that, for every positive integer k and every graph G, $\chi_\ell(G) \geq f(k)$ implies that $\vec{\chi}_\ell(G) \geq k$.*

2.2 The dichromatic number of oriented graphs

Recall that, for a digraph D, we have always that $\vec{\chi}(D) \leq \Delta_{\min}(D) + 1$, and this is sharp. Erdős and Neumann-Lara conjectured that, for oriented graphs, much more can be said.

Conjecture 2.6. (Erdős–Neumann-Lara) [59] *There is a constant c such that, for any graph G with maximum degree Δ and any orientation D of G, $\vec{\chi}(G) \leq c\Delta / \ln \Delta$.*

Notice the similitude with the Johansson–Molloy theorem (Theorem 1.10). If true, the order of magnitude of the bound is best-possible, as shown for instance by random r-regular oriented graphs (see Chapter 9). It is already unknown whether $\vec{\chi}(D) = o(\Delta)$.

There are a few positive results in this direction. For the maximum geometric mean degree, Harutyunyan and Mohar show the following.

Theorem 2.7. [91] *There are positive constants Δ_0 and ε such that, for every oriented graph D with $\tilde{\Delta}(D) \geq \Delta_0$, $\vec{\chi}(D) \leq (1-\varepsilon)\tilde{\Delta}(D)$.*

This has been later improved by Golowich [74] and Steiner [147]. The best current result is due to Kawarabayashi and Picasarri-Arrieta.

Theorem 2.8. [104] *Let G be a graph and D an orientation of G. Then, $\vec{\chi}(D) \leq \min\left\{\frac{1}{3}\Delta(G) + 2, \frac{\sqrt{2}}{2}\tilde{\Delta}(D) + 2\right\}$.*

Working with $\tilde{\Delta}(D)$ is usually easier than with the *maximum out-degree* $\Delta^+(D)$ of D, defined as $\Delta^+(D) = \max_{v \in V(D)} \deg^+(v)$. In fact, the following is open.

Problem 2.9. [103] *Show the existence of positive constants Δ_0 and ε such that, for every oriented graph D with $\Delta^+(D) \geq \Delta_0$, $\vec{\chi}(D) \leq (1-\varepsilon)\Delta^+(D)$.*

These kind of results are closely related with a directed version of Reed's conjecture (Conjecture 6.2). Section 6.1 contains a few more details on that topic.

Aharoni, Berger and Kfir propose a complementary version of Conjecture 2.6 for the maximum size of an acyclic set. Compare it with Theorem 1.16; this time there is an analogy with Shearer's theorem (Theorem 1.5).

Conjecture 2.10. (Aharoni–Berger–Kfir) [8] *For every $\varepsilon > 0$ there exists some $d^+_\varepsilon \in \mathbb{R}$ such that, for every oriented graph D of order n and average out-degree $d^+ \geq d^+_\varepsilon$, $\vec{\alpha}(D) \geq (1-\varepsilon) n \log_2 d^+ / d^+$.*

We note that the condition $d^+ \geq d^+_\varepsilon$ is necessary. Indeed, there are oriented graphs D satisfying $\vec{\alpha}(D) < (1-\varepsilon) n \log_2 d^+ / d^+$ (for instance, the

Paley tournament on seven vertices [62]; see also [138]), and by taking the disjoint union of many copies of D one can obtain arbitrarily large examples.

Proving Conjecture 2.10 is likely to be very hard; the special case of tournaments is an old open problem. However, it would be interesting to know if it holds up to a constant a constant factor. This is the case, for example, of tournaments [62], as well as local tournaments (see Chapter 4) and random regular digraphs (Chapter 9), all witnessing that the order of magnitude of this bound cannot be improved.

The next conjecture is inspired by Bondy's theorem (Theorem 1.15) and a paper of Cordero-Michel and Galeana-Sánchez [52] on the dichromatic number of digraphs with forbidden cycle lengths.

Conjecture 2.11. [88] *There is a constant c such that, for any oriented graph D without directed cycles of length greater than $s \geq 2$, $\vec{\chi}(D) \leq cs/\ln s$.*

This has been verified for tournaments, and is best-possible, up to the value of c.

Theorem 2.12. [88]

(*i*) *For every $\varepsilon \in \mathbb{R}^+$ there is some $s_0 \in \mathbb{N}$ such that, for every $s \geq s_0$ and every tournament T without directed cycles of length greater than s, $\vec{\chi}_\ell(T) \leq (1+\varepsilon)s/\log_2 s$.*

(*ii*) *Let $s \leq n$ be two positive integers. There exists a tournament T of order n without directed cycles of length greater than s such that $\vec{\chi}(T) \geq s/(4\log_2 s + 4)$.*

2.3 The Neumann-Lara conjecture

Neumann-Lara [128] and, independently, Škrekovski [28], have conjectured the following analogue of the four colour theorem (Theorem 1.11) for oriented graphs.

Conjecture 2.13. (Neumann-Lara) *For every oriented planar graph D, $\vec{\chi}(D) \leq 2$.*

The conjecture has been verified for digraphs of order at most 26 [105] and for digraphs of digirth at least four [109], and it has been shown to be equivalent to the more general statement that all oriented K_5-minor-free graphs are 2-dicolourable [148]. Steiner has studied it in the dual context of flows [149], while, on other surfaces, analogous problems have been explored [5]. Albertson has proposed the following relaxation.

Problem 2.14. [84, 106, 119] *Is it true that, for every oriented planar graph D of order n, $\vec{\alpha}(D) \geq n/2$?*

Knauer, Valicov and Wenger [106] have constructed planar digraphs of arbitrary order and digirth with small acyclic sets, showing that, if true, the bound from Problem 2.14 is tight. On the other side, Esperet, Lemoine and Maffray [65] provide similar lower bounds for $\vec{\alpha}(D)$ in terms of the order and the digirth of D.

In Chapter 5 we prove that a certain class of oriented graphs satisfies Conjecture 2.13, by showing that their underlying undirected graphs have vertex-arboricity at most 2.

Also relating to this, Bensmail, Harutyunyan and Le have asked how large can the list dichromatic number of oriented planar graphs be.

Question 2.15. [24] *Does there exist an oriented planar graph D such that $\vec{\chi}_\ell(D) = 3$?*

2.4 The Erdős–Hajnal conjecture

Let s, t be two positive integers. The *Ramsey number* $R(s,t)$ is the least positive integer such that every graph of order $R(s,t)$ has an independent set of size s or a clique of size t. The existence of these numbers is guaranteed by Ramsey's theorem. It is known [64] that $\max\{\alpha(G), \omega(G)\} \geq \frac{1}{2}\log_2 n$ for every graph G of order n; however, already $\max\{\alpha(G), \omega(G)\} \leq 2\log_2 n$ for some G, given any $n \geq 2$ [60]. Erdős and Hajnal [61] wondered if forbidding induced subgraphs would change this behaviour significantly.

We say that a graph (or digraph) G is *H-induced-free* if no induced subgraph of G is isomorphic to H.

Conjecture 2.16. (Erdős–Hajnal) *For every graph H, there is a constant $\varepsilon(H) > 0$ such that every H-induced-free graph G of order n satisfies $\max\{\alpha(G), \omega(G)\} \geq n^{\varepsilon(H)}$.*

This has been verified for graphs up to five vertices [46, 47, 48, 129], as well as for graphs obtained from them with a certain 'substitution operation' (see below). Graphs H for which the above statement is satisfied are said to have the *Erdős–Hajnal property*. The previous conjecture can be reformulated in the context of tournaments. Indeed, it has been shown [17] to be equivalent to the following one.

Conjecture 2.17. [17] *For every tournament T, there is a constant $\varepsilon(T) > 0$ such that every T-induced-free tournament U of order n satisfies $\vec{\alpha}(U) \geq n^{\varepsilon(T)}$.*

Tournaments T for which the above is satisfied are said to have the *Erdős–Hajnal property*. One could ask if, more generally, every tournament has the *oriented Erdős–Hajnal property*, that is, whether for every tournament T there is a constant $\varepsilon(T) > 0$ such that $\vec{\alpha}(D) \geq n^{\varepsilon(T)}$ for every

T-induced-free oriented graph D of order n. In [88], an even stronger statement is conjectured.

A graph (or digraph) G is *H-subgraph-free* if no subgraph of G is isomorphic to H.

Conjecture 2.18. [88] *For every tournament T, there is a constant $\varepsilon(T) > 0$ such that every T-subgraph-free digraph D of order n satisfies $\vec{\alpha}(D) \geq n^{\varepsilon(T)}$.*

If a tournament T satisfies the above statement, we say that T has the *strong Erdős–Hajnal property*[1].

In [17] the Erdős–Hajnal property is shown to be stable under a certain operation called the *substitution* (or *replacement*) *operation.* The same argument works in the tournament setting [46], and, more generally, with T-subgraph-free digraphs. To make it explicit, we reproduce it here.

Let D_1 and D_2 be digraphs on disjoint vertex sets, and let u be a vertex of D_1. Let us denote by $D_1(u, D_2)$ the digraph on vertex set $V(D_1) \cup V(D_2) \setminus \{u\}$, where (v, w) is an arc if and only if either

(a) (v, w) is an arc of D_1 or D_2, or

(b) (u, w) is an arc of D_1 and $v \in V(D_2)$, or

(c) (v, u) is an arc of D_1 and $w \in V(D_2)$.

Theorem 2.19. *Let T_1 and T_2 be tournaments having the strong Erdős–Hajnal property, and let u be a vertex of T_1. Then, the tournament $T = T_1(u, T_2)$ also has the strong Erdős–Hajnal property. The same statement holds if 'strong' is replaced by 'oriented'.*

Proof. The proof is the same for the two cases; we assume that we are in the first one. Let k be the order of T_1, and let $\varepsilon(T_1)$ and $\varepsilon(T_2)$ be exponents for which T_1 and T_2 verify the strong Erdős–Hajnal property. We choose any $0 < \delta < 1$, and let $\varepsilon(T)$ denote the quantity $(1-\delta)\frac{\varepsilon(T_1)\varepsilon(T_2)}{\varepsilon(T_1)+k\varepsilon(T_2)}$. Let n be any integer large enough (with respect to the previous quantities), and D any T-subgraph-free digraph of order n. We are going to show that $\vec{\alpha}(D) \geq n^{\varepsilon(T)}$.

Consider any subset $S \subseteq V(D)$ of size $m = \lfloor n^{\varepsilon(T)\varepsilon(T_1)^{-1}(1-\delta)^{-1}} \rfloor > k$. We can assume that $D[S]$ has a subgraph isomorphic to T_1: otherwise, $\vec{\alpha}(D) \geq \vec{\alpha}(D[S]) \geq m^{\varepsilon(T_1)} \geq n^{\varepsilon(T)}$, and we are done. This implies that D has at least $\binom{n}{m}/\binom{n-k}{m-k}$ subgraphs isomorphic to T_1. Let $\mathrm{Mon}(T_1, D)$ and $\mathrm{Mon}(T_1 - u, D)$ be the sets of injective homomorphisms from T_1 to D and from $T_1 - u$ to D, respectively, and let $\Phi : \mathrm{Mon}(T_1, D) \to \mathrm{Mon}(T_1 - u, D)$ be the mapping defined by $\Phi(\varphi) = \varphi|_{V(T_1-u)}$. Since $|\mathrm{Mon}(T_1, D)| \geq \binom{n}{m}/\binom{n-k}{m-k}$

[1]The term 'strong Erdős–Hajnal property' has been used in the literature with other meanings.

and $|\mathrm{Mon}(T_1 - u, D)| \leq n(n-1)\dots(n-k+2)$, we can find some $\psi \in \mathrm{Mon}(T_1 - u, D)$ such that $|\Phi^{-1}(\psi)|$ is at least

$$\frac{\binom{n}{m} \cdot (n-k+1)!}{\binom{n-k}{m-k} \cdot n!} = \frac{n-k+1}{m(m-1)\dots(m-k+1)} \geq n^{1-\frac{k\varepsilon(T)}{(1-\delta)\varepsilon(T_1)}} = n^{\frac{\varepsilon(T)}{(1-\delta)\varepsilon(T_2)}}.$$

Therefore, there is a set $W \subseteq V(D)$ of at least that many vertices such that, for each $w \in W$, the mapping $\psi_w : V(T_1) \to V(D)$ defined by

$$\psi_w(v) = \begin{cases} w & \text{if } v = u \\ \psi(v) & \text{if } v \neq u \end{cases}$$

is an injective homomorphism from T_1 to D. Since D is T-subgraph-free, $D[W]$ is T_2-subgraph-free. Thus, using the strong Erdős–Hajnal property of T_2, we see that

$$\vec{\alpha}(D) \geq \vec{\alpha}(D[W]) \geq |W|^{\varepsilon(T_2)} \geq n^{\varepsilon(T)}.$$

□

Corollary 2.20. *For every positive integer k, the transitive tournament TT_k of order k has the strong Erdős–Hajnal property.*

Proof. Since TT_2 has the strong Erdős–Hajnal property, it is enough to apply Theorem 2.19 inductively. □

Corollary 2.20 makes us turn our attention to the following problem.

Problem 2.21. *Does the directed triangle $\vec{C}_3$ have the strong Erdős–Hajnal property?*

We can prove a weaker lower bound.

Proposition 2.22. *Let $\delta < \sqrt{\ln 4}$. There exists a constant c_δ such that, for every $\vec{C}_3$-subgraph-free digraph D of order n, $\vec{\alpha}(D) \geq c_\delta \exp(\delta\sqrt{\ln n})$.*

Proof. Let $f_\delta(n) := \exp(\delta\sqrt{\ln n})$. Let n_δ be a positive integer, that we assume to be conveniently large. By taking c_δ small enough, the statement holds automatically for every digraph D of order $n \leq n_\delta$.

Let us assume that D has order $n \geq n_\delta$, and that the statement is true for every digraph of smaller order. We can assume that $\Delta_{\min}(D) \geq 2n/f_\delta(n)$; otherwise

$$\vec{\alpha}(D) \geq \frac{n}{\vec{\chi}(D)} \geq \frac{n}{\Delta_{\min}(D)+1} \geq \frac{n}{2\frac{n}{f_\delta(n)}+1} \geq \frac{1}{3} f_\delta(n)$$

by Theorem 1.18. Let v be a vertex with $\min\{\deg^-(v), \deg^+(v)\} \geq 2n/f_\delta(n)$. Since D is $\vec{C}_3$-subgraph-free, there is no arc from $N^+(v)$ to $N^-(v)$. In particular, the set $N^+(v) \cap N^-(v)$ spans no arc, so $\vec{\alpha}(D) \geq |N^+(v) \cap N^-(v)|$. Hence, we can also assume that $|N^+(v) \cap N^-(v)| \leq f_\delta(n)$. Therefore, if n^+ and n^- are the orders of the subgraphs D^+ and D^- induced by $N^+(v) \setminus N^-(v)$ and $N^-(v) \setminus N^+(v)$, then $n^+, n^- \geq 2n/f_\delta(n) - f_\delta(n) \geq n/f_\delta(n)$. By applying the induction hypothesis on the disjoint subgraphs D^+ and D^-, one sees that

$$\vec{\alpha}(D) \geq \vec{\alpha}(D^+) + \vec{\alpha}(D^-) \geq c_\delta f_\delta(n^+) + c_\delta f_\delta(n^-) \geq 2c_\delta f_\delta\left(\frac{n}{f_\delta(n)}\right) \geq c_\delta f_\delta(n),$$

where the last inequality follows from the computation

$$\lim_{x\to\infty} \frac{2f_\delta\left(\frac{x}{f_\delta(x)}\right)}{f_\delta(x)} = 2\lim_{x\to\infty} \exp\left(\delta\sqrt{\ln x - \delta\sqrt{\ln x}} - \delta\sqrt{\ln x}\right)$$

$$= 2\exp\left(\delta \lim_{x\to\infty} x\left(\sqrt{1-\frac{\delta}{x}} - 1\right)\right) = 2\exp\left(\frac{-\delta^2}{2}\right) > 1.$$

□

A constant $\varepsilon \geq 0$ is an *EH-coefficient* (resp. an *oriented/strong EH-coefficient*) for a tournament T if there exists a $c > 0$ such that every T-induced-free tournament (resp. every T-induced-free oriented graph/every T-subgraph-free digraph) of order n has an acyclic set of size at least cn^ε.

The asymptotics of Ramsey numbers (the fact that, for $s \geq 3$ fixed, $R(s,t) = \Omega((t/\ln t)^{(s+1)/2})$ [144]) readily imply that the strong EH-coefficients for a tournament T of order k are at most $2/(k+1)$: it amounts to consider the symmetric digraph D corresponding to a graph G of order $n = R(k,t) - 1$ with $\omega(G) < k$ and $\alpha(G) < t$, for t large. EH-coefficients exhibit a different behaviour. Notably, the class of tournaments having 1 as an EH-coefficient is non-trivial; this class has actually been completely described [25]. The situation with oriented EH-coefficients is similar to the former case.

Theorem 2.23. [88] *Let T be a tournament of order $k \geq 2$ and $\varepsilon \geq 0$ an oriented EH-coefficient for T. Then, $\varepsilon \leq 2/k$.*

Remark 2.24. There is a simple argument that shows that $1/(k-1)$ is a strong EH-coefficient for the transitive tournament TT_k, improving on what one can get using Theorem 2.19. Let D be a TT_k-subgraph-free digraph of order n, and let G be the back-arc graph of D with respect to an arbitrary ordering of the vertex set. Note that G has no cliques of size k. Using the well-known upper bound $R(s,k) \leq \binom{s+k-2}{k-1} \leq s^{k-1}$ for Ramsey numbers [64],

$$\vec{\alpha}(D) \geq \alpha(G) > n^{1/(k-1)} - 1.$$

Chapter 3

Colouring orientations of Kneser graphs and complete multipartite graphs

3.1 Introduction

This chapter is based on joint work with Ararat Harutyunyan, which has appeared in [92]. Our background motivation is the question of Erdős and Neumann-Lara commented in Section 2.1.

Question 2.1. [59] *Is there a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that, for every positive integer k and every graph G, $\chi(G) \geq f(k)$ implies that $\vec{\chi}(G) \geq k$?*

Apart from answering the fractional version of Question 2.1 in the affirmative, Mohar and Wu [122] validate the statement for Kneser graphs (see Theorem 3.1). Good estimates of the dichromatic number of relevant graph families may provide insights on how to tackle the problem. In Section 3.2, we prove that the chromatic and the dichromatic numbers of Kneser graphs are within a constant factor of each other (Theorem 3.4), thus extending Mohar and Wu's result. We also give a general, optimal lower bound for the dichromatic number of Borsuk graphs (Theorem 3.5).

Next, in Section 3.3, we turn to list colourings. We recall that the list dichromatic number of a graph G, denoted by $\vec{\chi}_\ell(G)$, is the minimum k such that all orientations of G are k-list dicolourable. Extending a classical result of Alon [12], we give asymptotic bounds for the list dichromatic number of complete multipartite graphs (Theorem 3.7). We then adapt a recent result of Bulankina and Kupavskii on the list chromatic number of dense Kneser graphs [42], giving bounds for their list dichromatic numbers (Theorem 3.11). In both cases, the bounds are tight up to a constant factor.

3.2 The dichromatic number of Kneser graphs and Borsuk graphs

Kneser graphs

Given two positive integers $n \geq k$, the *Kneser graph* $KG(n,k)$ is the graph with vertex set $\binom{[n]}{k}$, where two vertices u and v are adjacent if and only if $u \cap v = \emptyset$. Confirming a famous conjecture of Kneser from the 1950s [107], Lovász [111] proved that, for any $1 \leq k \leq n/2$, $\chi(KG(n,k)) = n - 2k + 2$ (see also [76, 115, 161]). Mohar and Wu gave the following bound for the dichromatic number of Kneser graphs.

Theorem 3.1. [122] *Let n and k be integers with $1 \leq k \leq n/2$. Then,*

$$\vec{\chi}(KG(n,k)) \geq \left\lfloor \frac{n-2k+2}{8\log_2 \frac{n}{k}} \right\rfloor.$$

We note that, since $\vec{\chi}(G) \leq \chi(G)$ for any graph G, the estimate in Theorem 3.1 is tight up to a constant factor when k is a constant fraction of n. Theorem 3.4 below improves this bound for slower growth rates of k. This cannot be extended to $k = 1$ due to Theorem 1.22.

To prove Theorem 3.4, we shall adapt Greene's proof of Kneser's conjecture [76]. The argument relies on the following version of the Lusternik–Schnirelmann–Borsuk theorem. Here, by $\mathbb{S}^n$ we denote the unit sphere $S(0,1) = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\} \subseteq \mathbb{R}^{n+1}$ centred at the origin.

Lemma 3.2. [76] *If $\mathbb{S}^n$ is covered with $n+1$ sets, each of which is either open or closed, then one of the sets contains a pair of antipodal points.*

We will also need the following probabilistic lemma.

Lemma 3.3. *Let G be a graph of order $n \geq 2$ and D the random orientation of G obtained by orienting every edge independently with probability $1/2$. Let E_ℓ be the event that there exists a subgraph of G isomorphic to $K_{\ell,\ell}$ which is acyclic in D. If $5\log_2 n \leq \ell$, then $\mathbb{P}[E_\ell] < 1/2$.*

Proof. Each acyclic orientation of $K_{\ell,\ell}$ can be extended to a transitive tournament on the same vertex set, and different orientations always extend to different tournaments. Therefore, among the 2^{ℓ^2} possible orientations of $K_{\ell,\ell}$, at most $(2\ell)! \leq (2\ell)^{2\ell} \leq n^{2\ell}$ are acyclic. Since G has at most $\binom{n}{2\ell}(2\ell)! \leq n^{2\ell}$ copies of $K_{\ell,\ell}$, we have that $\mathbb{P}[E_\ell] \leq n^{4\ell} \cdot 2^{-\ell^2} \leq 2^{-\ell^2/5} \leq 2^{-5}$. □

Theorem 3.4. *There exist a positive integer n_0 such that, for all $n \geq n_0$ and $2 \leq k \leq n/2$, we have $\vec{\chi}(KG(n,k)) \geq \lfloor \chi(KG(n,k))/16 \rfloor$.*

Proof. We let $0 < c < 1/2$ be a constant and set $t = \frac{-1}{8\log_2 c}$; we will show that $\vec{\chi}(KG(n,k)) \geq \lfloor t\chi(KG(n,k)) \rfloor$ if c is smaller than a certain quantity.

Picking $c = 1/4$ will suffice, although there is some margin to choose larger values. If $cn \leq k \leq n/2$, then the result is implied by Theorem 3.1.

Now suppose that $2 \leq k \leq cn$. We assume for a contradiction that, for any given orientation of $KG(n,k)$, we can find a partition of its vertex set into $d = \lfloor t(n-2k+2) \rfloor - 1$ acyclic subsets $\mathcal{A}_1, ..., \mathcal{A}_d$. Let $X \subseteq \mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ be a set of n points on the unit sphere centered at the origin. We assume that these points together with the origin are in general position. In particular, there are no $d+1$ points of X in a common hyperplane through the origin. The set of vertices of $KG(n,k)$ is assumed to be $\binom{X}{k}$.

We let $s = tk + (1-t)(n/2+1)$ and $\ell = \left\lceil \frac{1}{d}\binom{\lfloor s \rfloor}{k} \right\rceil$ (we note that $d \geq 1$; otherwise, the result is immediate). We define U_i as the set of points $x \in \mathbb{S}^d$ for which there exist ℓ different vertices $A_1, \ldots, A_\ell \in \mathcal{A}_i$ such that $x \cdot y > 0$ for every $y \in A_1 \cup \ldots \cup A_\ell$. That is, U_i is the set of poles of the open hemispheres containing all the points of ℓ vertices of $\mathcal{A}_i$. It is clear that U_i is an open set of $\mathbb{S}^d$. Additionally, we define $F = \mathbb{S}^d \setminus U_1 \setminus \ldots \setminus U_d$. By Lemma 3.2, one of the sets $U_1, \ldots, U_d, F$ contains two antipodal points.

Suppose that U_i contains two antipodal points $x, -x$. Then, the hemispheres with pole $x, -x$ each contain the points of ℓ vertices of $\mathcal{A}_i$. Therefore, $KG(n,k)[\mathcal{A}_i]$ has a subgraph isomorphic to $K_{\ell,\ell}$. By Lemma 3.3, $\ell \leq 5 \log_2 \binom{n}{k} \leq 5k \log_2 n$. On the other hand,

$$\ell \geq \frac{\binom{\lfloor s \rfloor}{k}}{d} \geq \frac{1}{n} \frac{\lfloor s \rfloor (\lfloor s \rfloor - 1) \ldots (\lfloor s \rfloor - k + 1)}{k!} \geq \frac{1}{n} \left(\frac{\lfloor s \rfloor}{k} \right)^k \geq \frac{1}{n} \left(\frac{s-1}{k} \right)^k$$
$$\geq \frac{1}{n} \left(\frac{(1-t)n}{2k} \right)^k .$$

We distinguish two cases.

Case 1: $2 \leq k \leq n^{1/5}$.

In this case

$$\ell \geq \frac{1}{n} \left(\frac{(1-t)n}{2k} \right)^k \geq \frac{1}{n} \left(\frac{(1-t)n^{4/5}}{2} \right)^k \geq n^{1/5} \left(\frac{(1-t)n^{1/5}}{2} \right)^k ,$$

contradicting, when n is large, that $\ell \leq 5k \log_2 n \leq 5n^{1/5} \log_2 n$.

Case 2: $n^{1/5} \leq k \leq cn$.

In this case

$$\ell \geq \frac{1}{n} \left(\frac{(1-t)n}{2k} \right)^k \geq \frac{1}{n} \left(\frac{1-t}{2c} \right)^{n^{1/5}} .$$

Provided that $1 - t - 2c > 0$, this contradicts that $\ell \leq 5k \log_2 n \leq 5cn \log_2 n$ when n is large. By picking $c = 1/4$, we have $1 - t - 2c = 1 - 1/16 - 1/2 > 0$.

In conclusion, F must contain two antipodal points $x, -x$. But then the hemispheres with pole $x, -x$ each contain at most $\lfloor s \rfloor - 1$ points of X. Indeed, if one of them contained $\lfloor s \rfloor$ points, it would contain the points of $\binom{\lfloor s \rfloor}{k} > \left(\left\lceil \frac{1}{d}\binom{\lfloor s \rfloor}{k} \right\rceil - 1\right) d = (\ell - 1)d$ vertices, so at least ℓ vertices of the same colour would be involved. Hence, there are at least $n - 2(\lfloor s \rfloor - 1) \geq n - 2s + 2 = t(n - 2k + 2) \geq d + 1$ points of X on the hyperplane separating the two hemispheres, contradicting the general position of $X \cup \{0\}$. □

Borsuk graphs

Lovász's resolution of Kneser's conjecture was inspired by the analogy between Kneser graphs and Borsuk graphs. Let n be a natural number and $a \in (0, 2)$ a real number. The *Borsuk graph* with parameters $n+1$ and a, denoted by $BG(n+1, a)$, is the graph with vertex set $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, where two vertices x and y are adjacent if and only if $\|y - x\| \geq a$. The study of the chromatic number of Borsuk graphs can be linked with geometric packing/covering problems. If a is large enough, an $(n+2)$-colouring of $BG(n+1, a)$ can be obtained by projecting the faces of an inscribed $(n+1)$-dimensional simplex. It turns out that this cannot be improved, no matter how close a is to 2. Indeed, $\chi(BG(n+1, a)) \geq n+2$ for every $a \in (0, 2)$; interestingly, this fact is just a reformulation of the Borsuk–Ulam theorem [115]. The rest of the section is devoted to prove that the dichromatic number of Borsuk graphs admits the same general lower bound.

Theorem 3.5. *$\vec{\chi}(BG(n+1, a)) \geq n + 2$ for any $n \geq 1$ and any $a \in (0, 2)$.*

Proof. Let us denote by $B(x, r)$ the open ball $\{y \in \mathbb{R}^{n+1} \mid \|y - x\| < r\}$. Let $\delta \in (0, 2)$ such that every point in $B(x, \delta) \cap \mathbb{S}^n$ is adjacent to every point in $B(-x, \delta) \cap \mathbb{S}^n$ for any $x \in \mathbb{S}^n$. Let ℓ be an integer that for now remains unspecified, but that is assumed to be as large as desired. We define $m = \left\lceil \sqrt[n+1]{(\ell - 1)(n + 1)} \right\rceil + 1 \leq 2 \sqrt[n+1]{(\ell - 1)(n + 1)}$ and $c = \frac{\delta}{m\sqrt{n+1}}$.

An *open hypercube* of $\mathbb{R}^{n+1}$ is the image by a rigid transformation of a product of intervals $(0, \lambda)^{n+1} \subseteq \mathbb{R}^{n+1}$, where $\lambda \in \mathbb{R}^+$. The length of its *side* is λ and the length of its *longest diagonal* is its diameter (i.e. $\lambda\sqrt{n + 1}$). Let $\mathcal{Q}_c$ be the set of open hypercubes of side c of the form $(ck_1, ck_1 + c) \times \ldots \times (ck_{n+1}, ck_{n+1} + c)$ with $(k_1, \ldots, k_{n+1}) \in \mathbb{Z}^{n+1}$, i.e. the ones obtained by rescaling the integer lattice by a factor of c. We will make use of the following easy observations about $\mathcal{Q}_c$.

Claim 3.5.1. *For every $x \in \mathbb{R}^{n+1}$, $B(x, \delta)$ contains at least m^{n+1} hypercubes of $\mathcal{Q}_c$.*

Proof. Consider an open hypercube Q of longest diagonal δ with $x \in Q$. Clearly $Q \subseteq B(x, \delta)$ and the side of Q is $\frac{\delta}{\sqrt{n+1}}$. This implies the claim. ■

Claim 3.5.2. *$B(0, 1+2\delta)$ is contained in any open hypercube Q of side $2c\lceil(1+2\delta)/c\rceil$ centred at the origin. Moreover, one (in fact exactly one) such Q can be obtained as the interior of the closure of the union of $(2\lceil(1+2\delta)/c\rceil)^{n+1}$ hypercubes of $\mathcal{Q}_c$.* ■

Let $\mathcal{Q}'_c \subseteq \mathcal{Q}_c$ be the set of $(2\lceil(1+2\delta)/c\rceil)^{n+1}$ hypercubes from Claim 3.5.2. For each $Q \in \mathcal{Q}'_c$, we choose a point $x_Q \in Q$. Let y_Q be the point where the open ray starting at the origin and passing through x_Q intersects $\mathbb{S}^n$. Since $n \geq 1$, we can assume that the points x_Q have been chosen so that $y_Q \neq y_{Q'}$ if $Q \neq Q'$. Let $Y = \{y_Q \mid Q \in \mathcal{Q}'_c\}$. We note that Y has size

$$|\mathcal{Q}'_c| = \left(2\left\lceil\frac{(1+2\delta)m\sqrt{n+1}}{\delta}\right\rceil\right)^{n+1} \leq \left(\frac{8(1+2\delta)\sqrt{n+1}}{\delta}\right)^{n+1}(\ell-1)(n+1).$$

Claim 3.5.3. *For every $x \in \mathbb{S}^n$, $B(x,\delta)$ contains at least m^{n+1} points of Y.*

Proof. Since $B((1+\delta)x,\delta) \subseteq B(0, 1+2\delta)$, all hypercubes of $\mathcal{Q}_c$ intersecting $B((1+\delta)x,\delta)$ are in $\mathcal{Q}'_c$. Hence, by Claim 3.5.1, $B((1+\delta)x,\delta)$ contains m^{n+1} hypercubes of $\mathcal{Q}'_c$. The points in Y corresponding to these hypercubes all lie in $B(x,\delta)$. ■

We now consider the finite induced subgraph $H = BG(n+1,a)[Y]$ of $BG(n+1,a)$. It will be enough to show that $\vec{\chi}(H) \geq n+2$. Let us assume for a contradiction that each orientation of H admits a partition of Y into $n+1$ acyclic subsets $Y_1, \ldots, Y_{n+1}$. For $i \in [n+1]$ let $U_i = \{x \in \mathbb{S}^n \mid |B(x,\delta) \cap Y_i| \geq \ell\}$. Clearly, U_i is an open set of $\mathbb{S}^n$. Moreover, $\mathbb{S}^n = U_1 \cup \ldots \cup U_{n+1}$. Indeed, otherwise $B(x,\delta)$ would contain at most $(\ell-1)(n+1) < m^{n+1}$ points of Y for some $x \in \mathbb{S}^n$, contradicting Claim 3.5.3. Therefore, by Lemma 3.2, U_i contains two antipodal points x and $-x$ for some $i \in [n+1]$.

By the choice of δ, we know that in H there is a copy of $K_{\ell,\ell}$ of colour i. Now, $5\log_2|Y| \leq \ell$ if ℓ is large enough. By Lemma 3.3, there is an orientation of H such that every copy of $K_{\ell,\ell}$ in H has a directed cycle, a contradiction. □

3.3 The list dichromatic number of Kneser graphs and complete multipartite graphs

Complete multipartite graphs

We denote by K_{m*r} the graph with vertex set $[mr]$ where two vertices are adjacent if and only if they are not congruent modulo r; that is, K_{m*r} is a complete r-partite graph with m vertices in each part. Answering a question of Erdős, Rubin and Taylor [63], Alon determined, up to a constant factor, the list chromatic number of K_{m*r}.

Theorem 3.6. [12] *There exist two positive constants c_1 and c_2 such that, for every $r \geq 2$ and every $m \geq 2$,*

$$c_1 r \ln m \leq \chi_\ell(K_{m*r}) \leq c_2 r \ln m.$$

More precise results were obtained in [72]. Adapting Alon's proof, we find an analogous bound for the list dichromatic number of K_{m*r} when $r \geq 2$ and $m \geq \max\{\ln^\rho r, 2\}$, for any $\rho > 3$ (Theorem 3.7). Regarding the dichromatic number of K_{m*r}, some information can be found in [86].

Theorem 3.7. *For every $\rho > 3$ there exist two positive constants c_1 and c_2 such that, if $r \geq 2$ and $m \geq \max\{\ln^\rho r, 2\}$, then*

$$c_1 r \ln m \leq \vec{\chi}_\ell(K_{m*r}) \leq c_2 r \ln m.$$

Proof. Let $V_1, \ldots, V_r$ be the parts of K_{m*r}. The upper bound is implied by Theorem 3.6. For the lower bound, we can assume that m is large enough; otherwise, we get the job done by picking a suitable c_1.

Claim 3.7.1. *There is a constant c and an orientation D of K_{m*r} such that, if $\ell \geq c \ln(rm)$, each subgraph of K_{m*r} isomorphic to K_ℓ or to $K_{\ell,\ell}$ has a directed cycle in D.*

Proof. We orient the edges of K_{m*r} at random, independently and with probability 1/2. Let E (resp. E') be the event that each subgraph of K_{m*r} isomorphic to K_ℓ (resp. $K_{\ell,\ell}$) has a directed cycle. By Lemma 3.3, $\mathbb{P}[E], \mathbb{P}[E'] > 1/2$ if c is sufficiently large. Hence, $\mathbb{P}[E \cap E'] > 0$. ■

Let $k = \lfloor Cr \ln m \rfloor$, where $0 < C \leq 1$ is a constant for now unspecified. We start by showing that there exists an assignment of k-lists from a palette $\mathscr{C}$ of $\lfloor r \ln m \rfloor$ colours such that, for any given set $A \subseteq \mathscr{C}$ of at most $\frac{4}{3} \ln m$ colours, each part has at least $\frac{1}{2} m^{1-\delta}$ vertices that avoid the colours from A on their lists, where $\delta = 2C \ln 5$.

We assign to each vertex v of D a random k-list $L(v)$ chosen independently and uniformly among the $\binom{|\mathscr{C}|}{k}$ possible k-lists. Given $i \in [r]$ and $A \subseteq \mathscr{C}$, let us consider the random variable $X_{i,A} = |\{v \in V_i \mid L(v) \cap A = \emptyset\}|$. We note that there are exactly $\binom{|\mathscr{C}|-|A|}{k}$ k-lists avoiding the colours in A. Devoting ourselves to the case $|A| = \lfloor \frac{4}{3} \ln m \rfloor$, we have that

$$\mathbb{E}(X_{i,A}) = m \frac{\binom{|\mathscr{C}|-|A|}{k}}{\binom{|\mathscr{C}|}{k}} \geq m \left(\frac{|\mathscr{C}| - |A| - k}{|\mathscr{C}| - k} \right)^k = m \left(1 - \frac{|A|}{|\mathscr{C}| - k} \right)^k$$

$$\geq m \left(1 - \frac{\frac{4}{3} \ln m}{(1-C) r \ln m - 1} \right)^{Cr \ln m} \geq m \left(1 - \frac{4}{5} \right)^{2C \ln m} = m^{1-\delta}$$

if m is large enough and C is not too large, where we used the following.

Claim 3.7.2. *Let $a \in \mathbb{R}^+$. The function $f : (a, \infty) \to \mathbb{R}$ defined by $f(x) = (1 - \frac{a}{x})^x$ is increasing.*

Proof. $f'(x) = f(x)\left(\ln(1 - \frac{a}{x}) + \frac{a}{x-a}\right) \geq f(x)\left(\ln \frac{x-a}{x} + \ln \frac{x}{x-a}\right) = 0.$ ■

By the simple concentration bound (Lemma A.3),

$$\mathbb{P}\left[X_{i,A} < \frac{1}{2}m^{1-\delta}\right] \leq \mathbb{P}\left[|X_{i,A} - \mathbb{E}(X_{i,A})| > \frac{1}{2}m^{1-\delta}\right] \leq 2\exp\left(-\frac{1}{8}m^{1-2\delta}\right).$$

Let E be the event that $X_{i,A} < \frac{1}{2}m^{1-\delta}$ for some $i \in [r]$ and $A \subseteq \mathscr{C}$ with $|A| \leq \frac{4}{3}\ln m$. We have that $\mathbb{P}[E]$ is at most

$$r\binom{|\mathscr{C}|}{\left\lfloor \frac{4}{3}\ln m\right\rfloor} \cdot 2\exp\left(-\frac{1}{8}m^{1-2\delta}\right) \leq (r\ln m)^{\frac{4}{3}\ln m+1} \cdot 2\exp\left(-\frac{1}{8}m^{1-2\delta}\right)$$

$$\leq 2\exp\left\{\left(m^{\frac{1}{\rho}} + \ln\ln m\right)\left(\frac{4}{3}\ln m + 1\right) - \frac{1}{8}m^{1-2\delta}\right\}$$

$$\leq 2\exp\left(2m^{\frac{1}{\rho}}\ln m - \frac{1}{8}m^{1-2\delta}\right)$$

if m is large enough. Consequently, if $\delta < \frac{1}{2}(1 - \frac{1}{\rho})$ and m is large enough, there exists a list assignment L' satisfying the desired property. This is the assignment that we are going to use.

Now we let f be a proper colouring of D. We claim that there exists a set of indices $I \subseteq [r]$ of size at least $3r/4$ such that $|f(V_i)| \leq 4c\ln^2(rm)$ for each $i \in I$. Indeed, if more than $r/4$ parts are coloured with more than $4c\ln^2(rm)$ colours each, then one of the colours appears on more than $cr\ln^2(rm)/|\mathscr{C}| \geq c\ln^2(rm)/\ln m \geq c\ln(rm)$ parts. By the choice of D, f is not proper, a contradiction.

For each $i \in [r]$, let $A_i \coloneqq \{\gamma \in \mathscr{C} \mid |V_i \cap f^{-1}(\gamma)| \geq c\ln(rm)\}$. We claim that if f is acceptable then $|A_i| > \frac{4}{3}\ln m$ for every $i \in I$. Indeed, otherwise, by the choice of the lists, at least $\frac{1}{2}m^{1-\delta}$ vertices of V_i have been coloured with colours not from A_i. Thus one of these colours is used at least

$$\frac{\frac{1}{2}m^{1-\delta}}{4c\ln^2(rm)} \leq c\ln(rm)$$

times on V_i. If m is large enough, this implies that

$$m^{1-\delta} \leq 8c^2\ln^3(rm) \leq 8c^2\left(m^{\frac{1}{\rho}} + \ln m\right)^3 \leq 9c^2m^{\frac{3}{\rho}}.$$

If we further assume that $\delta < 1 - \frac{3}{\rho}$, we get a contradiction when m is large. Therefore $|A_i| > \frac{4}{3}\ln m$ for every $i \in I$.

Now, by the choice of D, the sets $A_1, \dots, A_r$ are mutually disjoint. But then

$$|\mathscr{C}| \geq \sum_{i=1}^{r} |A_i| \geq \sum_{i \in I} |A_i| > \frac{4}{3}|I| \ln m \geq r \ln m \geq |\mathscr{C}|.$$

This contradiction shows that there is no acceptable proper colouring for the k-list assignment L'. □

We do not know what happens for the values of m and r not covered by Theorem 3.7. In any case, the lower bound is not valid in general. Indeed, if $m \leq \ln r$, the following theorem implies that $\vec{\chi}_\ell(K_{m*r}) \leq \vec{\chi}_\ell(K_{mr}) \leq cr$ for some constant c.

Theorem 3.8. [24] *Let T be a tournament of order n. Then, $\vec{\chi}_\ell(T) \leq (1 + o(1))n/\log_2 n$.*

Kneser graphs

The list chromatic number of Kneser graphs has been studied by Bulankina and Kupavskii. They prove the following two results.

Theorem 3.9. [42] *For any positive integers n and k with $1 \leq k \leq n/2$, we have $\chi_\ell(KG(n, k)) \leq n \ln(n/k) + n$.*

Theorem 3.10. [42] *For every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that $\chi_\ell(KG(n, k)) \geq c_\varepsilon n \ln n$ for all n and k with $2 \leq k \leq n^{1/2-\varepsilon}$.*

However, good lower bounds for larger values of k are still unknown. Clearly, the upper bound of Theorem 3.9 trivially generalises to the dichromatic number. The rest of the section is devoted to the proof of Theorem 3.11, a directed analogue of Theorem 3.10. The proof is achieved by a sequence of lemmas, which involve the argument of Bulankina and Kupavskii, as well as ideas from Mohar and Wu [122].

Theorem 3.11. *For every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that $\vec{\chi}_\ell(KG(n, k)) \geq c_\varepsilon n \ln n$ for all n and k with $2 \leq k \leq n^{1/2-\varepsilon}$.*

Let $G = (V, E)$ be a graph, $\mathcal{C}$ a collection of subsets of V and s and t positive integers. We say that $\mathcal{C}$ is an (s, t)-*collection* of V if

(i) $|\mathcal{C}| \leq s$;

(ii) $|C| \leq t$ for every $C \in \mathcal{C}$.

Given a list assignment L to G, we denote by $U = \cup_{v \in V(G)} L(v)$ the total set of colours, referred to as the *palette*, and we set $u = |U|$. The partitions of V considered in the sequel will always have u (not necessarily non-empty) parts. (We will be mainly interested in partitions that are *acyclic* with respect to some specific orientation D of G, i.e., partitions in which each part induces an acyclic subgraph of D.) It will be convenient to regard as distinct any two partitions arising from different colourings. Thus, formally, here partitions should be thought of as indexed by U.

We say that a partition P of V is *covered* by an (s,t)-collection $\mathcal{C}$ (or that $\mathcal{C}$ is an (s,t)-*cover* of P) if each part determined by P is contained in some $C \in \mathcal{C}$. Let $P = (P_1, \dots, P_u)$ be a partition. We say that the list assignment L *accepts* P if, for every $i \in [u]$ and every $v \in P_i$, $i \in L(v)$. Otherwise, we say that L *rejects* P.

In what follows, ℓ_1 and ℓ_2 are integers. For convenience, we define the function

$$g(\ell_1, \ell_2, n, s, t, u) \coloneqq s^u \exp\left(-\frac{n}{2} \cdot 2^{-\frac{4\ell_2 tu}{(\ell_1-\ell_2)n}}\right).$$

Lemma 3.12. *Let $G = (V,E)$ be an undirected graph of order n, $\mathcal{C}$ an (s,t)-collection of V and $\mathcal{P}$ the family of partitions of V covered by $\mathcal{C}$. Let L_1 be an ℓ_1-list assignment for G from a palette of u colours and L_2 a random ℓ_2-list assignment for G where, for every $v \in V$, $L_2(v)$ is chosen independently and equiprobably among $\binom{L_1(v)}{\ell_2}$. If $4tu \le (\ell_1 - \ell_2)n$, then*

$$\mathbb{P}\left[L_2 \text{ accepts some } P \in \mathcal{P}\right] < g(\ell_1, \ell_2, n, s, t, u).$$

Proof. Let $\mathrm{C} = (C_1, \dots, C_u) \in \mathcal{C}^u$ be any u-tuple of elements of $\mathcal{C}$. For every $v \in V$, let $r_{\mathrm{C}}(v)$ be the number of indices $i \in [u]$ such that $v \in C_i$. We consider the set of vertices $W_{\mathrm{C}} = \{v \in V \mid r_{\mathrm{C}}(v) \le 2tu/n\}$. We claim that $|W_{\mathrm{C}}| > n/2$. Indeed, otherwise

$$tu \ge \sum_{i=1}^{u} |C_i| = \sum_{v \in V} r_{\mathrm{C}}(v) \ge \sum_{v \in V \setminus W_{\mathrm{C}}} r_{\mathrm{C}}(v) > tu.$$

Moreover, for any $v \in W_{\mathrm{C}}$, the probability $p_{\mathrm{C}}(v)$ that $v \notin \cup_{i \in L_2(v)} C_i$ is at least

$$\frac{\binom{\ell_1 - r_{\mathrm{C}}(v)}{\ell_2}}{\binom{\ell_1}{\ell_2}} = \prod_{k=1}^{\ell_2} \frac{\ell_1 - \ell_2 - r_{\mathrm{C}}(v) + k}{\ell_1 - \ell_2 + k} \ge \left(1 - \frac{r_{\mathrm{C}}(v)}{\ell_1 - \ell_2}\right)^{\ell_2}$$

$$\ge \left(1 - \frac{2tu}{(\ell_1 - \ell_2)n}\right)^{\ell_2} \ge \left(\frac{1}{2}\right)^{\frac{4\ell_2 tu}{(\ell_1-\ell_2)n}} =: p,$$

using Claim 3.7.2 and the inequality $4tu \le (\ell_1 - \ell_2)n$. Therefore, the probability that there is some u-tuple $\mathrm{C} = (C_1, \dots, C_u)$ of elements of $\mathcal{C}$ such that $v \in \cup_{i \in L_2(v)} C_i$ for every $v \in V$ is at most

$$\sum_{\mathrm{C} \in \mathcal{C}^u} \prod_{v \in W_{\mathrm{C}}} (1 - p_{\mathrm{C}}(v)) < s^u (1-p)^{\frac{1}{2}n} \le s^u e^{-\frac{1}{2}np}.$$

Since every $P \in \mathcal{P}$ is covered by $\mathcal{C}$, each of the u parts of any such P is contained in some $C \in \mathcal{C}$, so the result follows. □

Let G and H be graphs. We recall that the tensor product $G \times H$ of G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (v, x) and (w, y) are adjacent if and only if v and w are adjacent in G and x and y are adjacent in H. The tensor product of complete graphs $K_n \times K_n$ is going to play an auxiliary role; we denote it by G_n. Given $S \subseteq V(G_n)$, we call $\pi_1(S)$ and $\pi_2(S)$ the projection of S to the first and second coordinate, respectively. The *rows* (resp. *columns*) of S are the subsets of S of the form $S \cap (\{i\} \times [n])$ (resp. $S \cap ([n] \times \{i\})$), where $i \in [n]$. Now we give some properties of G_n.

Lemma 3.13. *For any $n \geq 2$, there is an orientation D_n of G_n (resp. of $K_2 \times G_n$) such that, for every $S, T \subseteq V(G_n)$ satisfying*

(i) $|S|, |T| \geq 30 \ln n$ and

(ii) $\pi_i(S) \cap \pi_i(T) = \emptyset$ for $i \in \{1, 2\}$,

the subgraph of D_n induced by $S \cup T$ (resp. by $(\{1\} \times S) \cup (\{2\} \times T)$) has a directed cycle.

Proof. Since in $G_n[S \cup T]$ (resp. in $(K_2 \times G_n)[(\{1\} \times S) \cup (\{2\} \times T)]$) all edges between S and T (resp. between $\{1\} \times S$ and $\{2\} \times T$) are present, the conclusion follows from Lemma 3.3. □

We define an (s_n, t_n)-collection $\mathcal{C}_{G_n}$ of $V(G_n)$ as follows. Let $\mathcal{L}_{G_n} = \{\{i\} \times [n] \mid i \in [n]\} \cup \{[n] \times \{i\} \mid i \in [n]\}$ be the set of rows and columns of $V(G_n)$, and let

$$\mathcal{Q}_{G_n} = \begin{cases} \left\{ A \times B \,\middle|\, A, B \in \binom{[n]}{\lfloor 124 \ln n \rfloor} \right\} & \text{if } 1 \leq \lfloor 124 \ln n \rfloor \leq n \\ \{V(G_n)\} & \text{otherwise.} \end{cases}$$

We set $\mathcal{C}_{G_n} = \{L \cup Q \mid L \in \mathcal{L}_{G_n},\ Q \in \mathcal{Q}_{G_n}\}$. We note that

(i) $|\mathcal{C}_{G_n}| \leq s_n := \max\left\{1, 2n\binom{n}{\lfloor 124 \ln n \rfloor}^2\right\}$ and

(ii) $|C| \leq t_n := n + \lfloor 124 \ln n \rfloor^2$ for every $C \in \mathcal{C}_{G_n}$.

Lemma 3.14. *There is an orientation D_n of G_n such that $\mathcal{C}_{G_n}$ covers all acyclic partitions of D_n.*

Proof. It can be assumed that $1 \leq \lfloor 124 \ln n \rfloor \leq n$. Let D_n be the orientation from Lemma 3.13, and let S be an acyclic set of D_n. We assume for a contradiction that S is not contained in any $C \in \mathcal{C}_{G_n}$. Let L be the largest row of S, or its largest column if it is larger than its largest row, and let $S' = S \setminus L$. Then S' is not contained in any $Q \in \mathcal{Q}_{G_n}$, so $|\pi_i(S')| > 124 \ln n > 90 \ln n + 2$ for some $i \in \{1, 2\}$. Let us assume that $i = 1$ (if $i = 2$, the argument below is repeated with rows instead of columns). Let

L' be the largest column of S'. We distinguish three cases. We will show that, in each of them, we can find two sets in S satisfying the hypotheses of Lemma 3.13. This will yield a contradiction since S is acyclic.

Case 1: $|L'| > 60 \ln n + 2$.

We recall that $|L| \geq |L'|$. Therefore, we can find a subset of L and a subset of L' satisfying the hypotheses of Lemma 3.13.

Case 2: $60 \ln n + 2 \geq |L'| \geq 30 \ln n$.

Since $|\pi_i(S')| > 90 \ln n + 2$, we can find a subset T of S' such that T and L' satisfy the hypotheses of Lemma 3.13.

Case 3: $|L'| \leq 30 \ln n$.

Let $\{T_1, \ldots, T_k\}$ be a minimal set of columns of S' with the property $|\pi_i(\cup_{j=1}^{k} T_j)| \geq 30 \ln n$. By minimality, $|\pi_i(\cup_{j=1}^{k} T_j)| \leq 60 \ln n$. Hence, as in Case 2, we can find a subset T of S' such that T and $\cup_{j=1}^{k} T_j$ satisfy the hypotheses of Lemma 3.13.

In any case, Lemma 3.13 yields a cycle in S, the desired contradiction. □

We can now determine the order of magnitude of $\vec{\chi}_\ell(KG(n,k))$ when k is bounded by a constant.

Lemma 3.15. *There is a constant $c \in \mathbb{R}^+$ such that $\vec{\chi}_\ell(KG(n,k)) \geq c(n/k)\ln(n/k)$ for every $2 \leq k \leq n/2$.*

Proof. First, we note that $G_{\lfloor n/k \rfloor}$ is isomorphic to a subgraph of $KG(n,k)$. Indeed, if we take $\lfloor n/k \rfloor + 1$ pairwise disjoint subsets $I, J_1, \ldots, J_{\lfloor n/k \rfloor} \subseteq [n]$ with $|I| = \lfloor n/k \rfloor$ and $|J_1| = \ldots = |J_{\lfloor n/k \rfloor}| = k-1$, then the set of vertices $S = \{\{i\} \cup J_j \mid i \in I,\ 1 \leq j \leq \lfloor n/k \rfloor\} \subseteq \binom{[n]}{k}$ induces a copy of $G_{\lfloor n/k \rfloor}$.

Thus, it suffices to show that $\vec{\chi}_\ell(G_{\tilde{n}}) \geq c\tilde{n}\ln\tilde{n}$ for some $c > 0$. We assume that $\tilde{n}$ is large enough. Given $G_{\tilde{n}}$, we consider the orientation $D_{\tilde{n}}$ from Lemma 3.14; we know that $\mathcal{C}_{G_{\tilde{n}}}$ covers all acyclic partitions of $D_{\tilde{n}}$. Let $u_{\tilde{n}} = \ell_{1,\tilde{n}} = \lfloor \tilde{n} \ln \tilde{n} \rfloor$ and $\ell_{2,\tilde{n}} = \lfloor c u_{\tilde{n}} \rfloor$, where $c < 1$ is a positive constant to be defined later. Let $L_{1,\tilde{n}}$ be the canonical $\ell_{1,\tilde{n}}$-list assignment to $D_{\tilde{n}}$ (i.e. $L_{1,\tilde{n}}(v) = [u_{\tilde{n}}]$ for every $v \in V(D_{\tilde{n}})$). It is clear that $4t_{\tilde{n}}u_{\tilde{n}} \leq (\ell_{1,\tilde{n}} - \ell_{2,\tilde{n}})\tilde{n}^2$ and

$$\ln g(\ell_{1,\tilde{n}}, \ell_{2,\tilde{n}}, \tilde{n}^2, s_{\tilde{n}}, t_{\tilde{n}}, u_{\tilde{n}}) \leq 330\tilde{n}\ln^3\tilde{n} - \frac{1}{2}\tilde{n}^{2-\frac{8c}{(1-c)\log_2 e}} < 0,$$

given that $\tilde{n}$ is large enough and c has been chosen so that $\frac{8c}{(1-c)\log_2 e} < 1$. Now, by Lemma 3.12, $\vec{\chi}_\ell(D_{\tilde{n}}) > \ell_{2,\tilde{n}}$. □

The previous lemma handles the case of small k. For larger values of k, we need to modify the definition of cover. Let $H = (V, E)$ be a graph, $\mathcal{C}$ an (s,t)-collection of V and $\lambda \in \mathbb{R}^+$. Let us consider the graph $K_2 \times H$ and one of its orientations D. We say that an acyclic partition P of $V(D)$ is *semicovered* by the pair $(\mathcal{C}, \lambda)$ (or that $(\mathcal{C}, \lambda)$ is an (s,t)-*semicover* of P) if, for every acyclic set $S = (\{1\} \times S_1) \cup (\{2\} \times S_2) \in P$, either $S_1 \subseteq C_1$ and $S_2 \subseteq C_2$ for some $C_1, C_2 \in \mathcal{C}$, or $S_i \subseteq C$ and $|S_i| < \lambda$ for some $i \in \{1,2\}$ and some $C \in \mathcal{C}$.

Lemma 3.16. *Let G and H be graphs, D an orientation of $K_2 \times H$, and $(\mathcal{C}, \lambda)$ an (s,t)-semicover of all the acyclic partitions of D. Let ℓ_1 and ℓ_2 be positive integers satisfying*

(i) $8t\ell_1 \leq (\ell_1 - \ell_2)n_H$,

(ii) $m_G\, g^2(\ell_1, \ell_2, n_H, s, t, 2\ell_1) < 1$ *and*

(iii) $\lambda \ell_1 \leq n_H$,

where m_G is the size of G and n_H is the order of H. If $\chi_\ell(G) > \ell_1$, then $\vec{\chi}_\ell(G \times H) > \ell_2$.

Proof. Suppose, for a contradiction, that $\vec{\chi}_\ell(G \times H) \leq \ell_2$. Let L_1 be any ℓ_1-list assignment to G. We consider a random ℓ_2-list assignment L_2 for $G \times H$, where, for each $v \in V(G)$ and each $x \in \{v\} \times V(H)$, $L_2(x)$ is chosen independently and equiprobably among the ℓ_2-element subsets of $L_1(v)$. For each edge $\{v, w\}$ of G, let $\mathcal{C}_{\{v,w\}} = \{(\{v\} \times C_1) \cup (\{w\} \times C_2) \mid C_1, C_2 \in \mathcal{C}\}$. We orient the subgraph induced by $\{v, w\} \times V(H)$ according to D (in any of the two possible ways). This results in an orientation of $G \times H$ that we will call $\overrightarrow{G \times H}$. By applying Lemma 3.12 to $(G \times H)[\{v, w\} \times V(H)]$ with a palette of size $u = 2\ell_1$, we see that the probability that $L_2|_{\{v,w\} \times V(H)}$ accepts some partition covered by $\mathcal{C}_{\{v,w\}}$ is smaller than $g(\ell_1, \ell_2, 2n_H, s^2, 2t, 2\ell_1) = g^2(\ell_1, \ell_2, n_H, s, t, 2\ell_1)$. Therefore, the probability that this happens for some edge $\{v, w\}$ of G is less than $m_G\, g^2(\ell_1, \ell_2, n_H, s, t, 2\ell_1) < 1$. Thus we can find a ℓ_2-list assignment L'_2 for $G \times H$ such that, for every $\{v, w\} \in E(G)$, $L'_2|_{\{v,w\} \times V(H)}$ rejects all partitions of $\{v, w\} \times V(H)$ covered by $\mathcal{C}_{\{v,w\}}$.

Since $\vec{\chi}_\ell(G \times H) \leq \ell_2$, $\overrightarrow{G \times H}$ has a colouring f'_2 which is accepted by L'_2 and produces no monochromatic cycles. Let us define a colouring f_1 for G as

$$f_1(v) = \begin{cases} \gamma_v & \text{if } \exists \gamma \ \forall C \in \mathcal{C} \ \ (f'_2)^{-1}(\gamma) \cap (\{v\} \times V(H)) \not\subseteq \{v\} \times C, \\ & \text{where } \gamma_v \text{ is any such } \gamma \\ \gamma_v^+ & \text{otherwise, where } \gamma_v^+ \text{ is any } \gamma \text{ maximizing} \\ & |(f'_2)^{-1}(\gamma) \cap (\{v\} \times V(H))|. \end{cases}$$

We note that $f_1(v) \in L_1(v)$ for each $v \in V(G)$. We will show that $f_1(v)$ is a proper colouring of G, and this contradiction will end the proof.

Let $\{v, w\}$ be an edge of G, and suppose for a contradiction that $f_1(v) = f_1(w)$. Since $L'_2|_{\{v,w\}\times V(H)}$ rejects all partitions of $\{v, w\} \times V(H)$ covered by $\mathcal{C}_{\{v,w\}}$, either $f_1(v) = \gamma_v$ or $f_1(w) = \gamma_w$. Without loss of generality, we can assume that $f_1(w) = \gamma_w$. Since f'_2 produces no monochromatic cycles, if $f_1(v) = \gamma_v$ then $(\mathcal{C}, \lambda)$ does not semicover all acyclic partitions of D, a contradiction. On the other hand, if $f_1(v) = \gamma_v^+$, then $|(f'_2)^{-1}(\gamma_v^+) \cap (\{v\} \times V(H))| \geq n_H/\ell_1 \geq \lambda$, also contradicting that $(\mathcal{C}, \lambda)$ semicovers all acyclic partitions of D.

Therefore, we have found a proper colouring f_1 accepted by the ℓ_1-list assignment L_1. Since L_1 was arbitrary, we conclude that $\chi_\ell(G) \leq \ell_1$, ending the proof. □

Lemma 3.17. *For every n there is an orientation D_n of $K_2 \times G_n$ such that all acyclic partitions of D_n are semicovered by $(\mathcal{C}_{G_n}, 2^{13}\ln^2 n)$.*

Proof. This proof is similar to that of Lemma 3.14. Let D_n be the orientation of $K_2 \times G_n$ from Lemma 3.13. We assume that $1 \leq \lfloor 124 \ln n \rfloor \leq n$; otherwise, the lemma is trivial. Let $S = (\{1\} \times S_1) \cup (\{2\} \times S_2)$ be an acyclic set in D_n. Suppose that S_1 is not contained in any $C \in \mathcal{C}_{G_n}$. As in the proof of Lemma 3.14 we can find in S_1 two sets $L_1, L'_1 \subseteq S_1$ satisfying the hypotheses of Lemma 3.13. We can assume that $|L_1| = |L'_1| = \lceil 30 \ln n \rceil$.

We argue by contradiction. Suppose that $|S_2| \geq 2^{13}\ln^2 n$ or that S_2 is not contained in any $C \in \mathcal{C}_{G_n}$. Let $T = \{(j_1, j_2) \in S_2 \mid j_1 \notin \pi_1(L_1 \cup L'_1)$ or $j_2 \notin \pi_2(L_1 \cup L'_1)\}$. We claim that $|T| \geq 60 \ln n$. We distinguish two cases.

Case 1: $|S_2| \geq 2^{13}\ln^2 n$.

We have that $|\pi_1(L_1 \cup L'_1)|, |\pi_2(L_1 \cup L'_1)| \leq 2\lceil 30 \ln n \rceil \leq 64 \ln n$. Therefore, $|T| \geq |S_2| - 64^2 \ln^2 n \geq 60 \ln n$.

Case 2: $S_2 \nsubseteq C$ for any $C \in \mathcal{C}_{G_n}$.

In this case, $S_2 \nsubseteq Q$ for any $Q \in \mathcal{Q}_{G_n}$. Therefore, $|\pi_i(S_2)| > 124 \ln n$ for some $i \in \{1, 2\}$. Since $|\pi_i(L_1 \cup L'_1)| \leq 2\lceil 30 \ln n \rceil$, we have that $|T| \geq 124 \ln n - 2\lceil 30 \ln n \rceil \geq 60 \ln n$.

Hence, $|T| \geq 60 \ln n$ in both cases. Let $T_1 = \{(j_1, j_2) \in T \mid j_1 \notin \pi_1(L_1),\ j_2 \notin \pi_2(L_1)\}$ and $T'_1 = T \setminus T_1$. We note that $T'_1 = \{(j_1, j_2) \in T \mid j_1 \notin \pi_1(L'_1),\ j_2 \notin \pi_2(L'_1)\}$ by the definition of T. Applying Lemma 3.13 yields the desired contradiction. Indeed, if $|T_1| \geq 30 \ln n$, then $(\{1\} \times L_1) \cup (\{2\} \times T_1)$ has a directed cycle, and if otherwise $|T'_1| \geq 30 \ln n$, then $(\{1\} \times L'_1) \cup (\{2\} \times T'_1)$ has a directed cycle. □

Proof of Theorem 3.11. We fix ε and assume that n is large enough (to deal with the case of n small, we just make sure that c_ε is small enough). If k is bounded by a constant, then Lemma 3.15 does the job. Therefore, we can assume that $k \geq 4$. We note that $KG(\lfloor n/2 \rfloor, k-2) \times G_{\lfloor n/4 \rfloor}$ is a subgraph of

$KG(n,k)$. Indeed, consider, for any positive integers n_1, n_2, k_1, k_2 satisfying $2k_1 \leq n_1$, $2k_2 \leq n_2$, $n_1 + n_2 \leq n$ and $k_1 + k_2 = k$, the set S of vertices of $KG(n,k)$ of the form $\{i_1, \ldots, i_{k_1}, j_1, \ldots, j_{k_2}\}$, where

$$\{i_1, \ldots, i_{k_1}\} \in \binom{[n_1]}{k_1} \quad \text{and} \quad \{j_1, \ldots, j_{k_2}\} \in \binom{n_1 + [n_2]}{k_2};$$

the subgraph induced by S is isomorphic to $KG(n_1, k_1) \times KG(n_2, k_2)$. By the proof of Lemma 3.15, if $k_2 \geq 2$ then $G_{\lfloor n_2/k_2 \rfloor}$ is isomorphic to a subgraph of $KG(n_2, k_2)$, so we can just take $n_1 = \lfloor n/2 \rfloor$, $n_2 = \lceil n/2 \rceil$, $k_1 = k - 2$ and $k_2 = 2$.

Let $m_n = \frac{1}{2}\binom{\lfloor n/2 \rfloor}{k-2}\binom{\lfloor n/2 \rfloor - k + 2}{k-2}$ be the number of edges of $KG(\lfloor n/2 \rfloor, k-2)$. Provided that n is sufficiently large, we can find an $\varepsilon' \in (0,1)$ such that $n^{1/2-\varepsilon} \leq \lfloor n/2 \rfloor^{1/2-\varepsilon'}$. Let $c_{\varepsilon'}$ be the corresponding constant from Theorem 3.10; let $c'_\varepsilon \in (0,1)$ be a constant for now unspecified, $\ell_{1,n} = \lfloor c_{\varepsilon'} \lfloor n/2 \rfloor \ln \lfloor n/2 \rfloor \rfloor - 1$ and $\ell_{2,n} = \lfloor c'_\varepsilon \ell_{1,n} \rfloor$. We apply Lemma 3.16 with the $(s_{\lfloor n/4 \rfloor}, t_{\lfloor n/4 \rfloor})$-semicover $(\mathcal{C}_{G_{\lfloor n/4 \rfloor}}, 2^{13} \ln^2 \lfloor n/4 \rfloor)$ of the family of acyclic partitions of $D_{\lfloor n/4 \rfloor}$, the orientation of $K_2 \times G_{\lfloor n/4 \rfloor}$ from Lemma 3.17. Clearly,

$$8t_{\lfloor n/4 \rfloor} \ell_{1,n} \leq (\ell_{1,n} - \ell_{2,n}) \left\lfloor \frac{n}{4} \right\rfloor^2,$$

$$2^{13} \ln^2 \left\lfloor \frac{n}{4} \right\rfloor \ell_{1,n} \leq \left\lfloor \frac{n}{4} \right\rfloor^2 \text{ and}$$

$$\ln m_n + 2 \ln g \left(\ell_{1,n}, \ell_{2,n}, \left\lfloor \frac{n}{4} \right\rfloor^2, s_{\lfloor n/4 \rfloor}, t_{\lfloor n/4 \rfloor}, 2\ell_{1,n} \right)$$

$$\leq 2k \ln n + 660 c_{\varepsilon'} n \ln^3 n - \frac{1}{25} n^{2 - \frac{50 c_{\varepsilon'} c'_\varepsilon}{(1 - c'_\varepsilon) \log_2 e}} < 0$$

if n is large enough and c'_ε has been chosen so that $\frac{50 c_{\varepsilon'} c'_\varepsilon}{(1-c'_\varepsilon)\log_2 e} < 1$. Thus, Lemma 3.16 and Theorem 3.10 imply that $\vec{\chi}_\ell(KG(\lfloor n/2 \rfloor, k-2) \times G_{\lfloor n/4 \rfloor}) > \ell_{2,n}$ for all n large enough. $\square$

Chapter 4

Acyclic sets and colourings of local tournaments

4.1 Introduction

An obvious class of oriented graphs satisfying, up to a constant factor, the Erdős–Neumann-Lara and the Aharoni–Berger–Kfir conjectures (Conjectures 2.6 and 2.10) is the class of orientations of triangle-free graphs, see Theorems 1.5 and 1.10. In fact, as remarked in [88], this can be easily extended to oriented graphs in which the in- and out-neighbourhoods of every vertex do not span any arc.

The *maximum total degree* of a digraph D is the quantity $\max\{\deg^-(v)+\deg^+(v) \mid v \in V(D)\}$.

Theorem 4.1. *For every $\varepsilon > 0$ there exist $d^+_\varepsilon, \Delta_\varepsilon \in \mathbb{R}$ such that, for any TT_3-subgraph-free oriented graph D of order n, average out-degree d^+ and maximum total degree Δ,*

(i) *if $d^+ \geq d^+_\varepsilon$, then $\vec{\alpha}(D) \geq (1-\varepsilon)n \ln d^+/d^+$, and*

(ii) *if $\Delta \geq \Delta_\varepsilon$, then $\vec{\chi}(D) \leq (1+\varepsilon)\Delta/\ln\Delta$.*

Proof. We first choose a total order $\preceq$ of $V(D)$ such that the average degree d of the back-arc graph $D^{\preceq}$ is at most d^+. Since D is TT_3-subgraph-free, all its oriented triangles are directed triangles, so $D^{\preceq}$ is triangle-free. Let f be the function from Theorem 1.5. We note that f is decreasing. Therefore,

$$\vec{\alpha}(D) \geq \alpha(D^{\preceq}) \geq nf(d) \geq nf(d^+) \geq (1-\varepsilon)\frac{n \ln d^+}{d^+},$$

assuming that d^+ is large enough. Similarly, let $\Delta^{\preceq}$ be the maximum degree of $D^{\preceq}$. By Theorem 1.10, and assuming that Δ is large enough,

$$\vec{\chi}(D) \leq \chi(D^{\preceq}) \leq \begin{cases} (1+\varepsilon)\Delta^{\preceq}/\ln\Delta^{\preceq} & \text{if } \Delta^{\preceq} \geq \Delta/\ln\Delta \\ \Delta/\ln\Delta + 1 & \text{if } \Delta^{\preceq} \leq \Delta/\ln\Delta, \end{cases}$$

which is at most $(1+\varepsilon)\Delta/\ln\Delta$ in both cases. □

The purpose of this short chapter is to show that Conjectures 2.6 and 2.10 also hold (up to a constant factor) for the opposite class of oriented graphs.

A *local in-tournament* (resp. a *local out-tournament*) is a digraph D in which $N^-[v]$ (resp. $N^+[v]$) induces a tournament for every vertex v of D. If D is both a local in- and a local out-tournament, then it is a *local tournament*. Local tournaments are a well-studied class of digraphs, see for instance [21].

Theorem 4.2. *There exist $c_1, c_2, c_3 \in \mathbb{R}^+$ such that the following hold.*

(*i*) *For every local tournament D of order n, average out-degree $d^+ > 0$ and $\Delta_{\max}(D) = \Delta > 0$,*

$$\vec{\alpha}(D) \geq c_1 \frac{n\ln(d^+ + 1)}{d^+} \quad \text{and} \quad \vec{\chi}(D) \leq c_2 \frac{\Delta}{\ln(\Delta+1)}.$$

(*ii*) *For every integer $d \geq 2$ there exist infinitely many strongly connected local tournaments D such that $\underline{D}$ is d-regular and*

$$\vec{\alpha}(D) \leq c_3 \frac{n\ln(d+1)}{d}.$$

Steiner and, independently, Aboulker, Aubian and Charbit, have obtained a nice result of a similar flavour, that demonstrates how structural constraints can radically affect the dichromatic number.

Theorem 4.3. [2, 150] *Let D be an oriented graph in which each out-neighbourhood induces a transitive tournament. Then, $\vec{\chi}(D) \leq 2$.*

A potential next step would be to introduce a small perturbation to our well-structured graphs from Theorems 4.1 and 4.2, and see if similar conclusions hold, in the spirit of [15]. More concretely, what can be said about the acyclicity and the dichromatic numbers of an oriented graph D in which both the in- and out-neighbourhoods of every vertex span few arcs? And of an oriented graph D' in which both the in- and out-neighbourhoods of every vertex span a lot of arcs?

Another natural thing to try would be to extend Theorem 4.2 to local out-tournaments. For that, it would be enough to generalise Proposition 4.6 below.

4.2 Proof of Theorem 4.2

Theorem 4.2 follows from Propositions 4.5, 4.6 and 4.7. For the first part of Theorem 4.2(*i*), we shall adapt Shearer's argument for the independence number of triangle-free graphs. We state the inductive step on a separate lemma that might be applicable to other digraph classes.

The *average total degree* of a digraph D is the average of $\deg^-(v) + \deg^+(v)$ over all the vertices v of D.

Lemma 4.4. *Let D be a digraph of order n and average total degree d, and let $f : [0, \infty) \to [0, 1]$ be a convex function, differentiable on $\mathbb{R}^+$. Assume that*

(i) for every proper induced subgraph D' of D of order n' and average total degree d', $\vec{\alpha}(D') \geq n' f(d')$, and

(ii) there is a non-empty acyclic set S of D and a set of vertices $S \subseteq N \subseteq V(D)$ satisfying:

(1) if $N \subsetneq V(D)$, then $S \cup S'$ is an acyclic set of D for every acyclic set S' of $D - N$;

(2) if A_N is the set of arcs of D with an endpoint in N, then

$$\frac{|S|}{|N|} - f(d) + \left(d - \frac{2|A_N|}{N}\right) f'(d) \geq 0.$$

Then, $\vec{\alpha}(D) \geq nf(d)$.

Proof. We argue by induction on the induced subgraphs of D. When D has no proper induced subgraphs, that is, when $n = 1$, clearly $\vec{\alpha}(D) = n \geq nf(d)$. For $n \geq 2$, we distinguish two cases. If $N = V(D)$, (2) yields $\vec{\alpha}(D) \geq |S| \geq nf(d)$ directly. So we assume that $N \subsetneq V(D)$, and let n' and d' be the order and the average total degree of $D' := D - N$. By the convexity of f, $f(d') \geq f(d) + (d' - d)f'(d)$. Thus, by (ii),

$$\vec{\alpha}(D) \geq |S| + \vec{\alpha}(D') \geq |S| + n'f(d') \geq |S| + n'f(d) + n'(d' - d)f'(d)$$
$$\geq |S| - |N|f(d) + (d|N| - 2|A_N|)f'(d) + nf(d) \geq nf(d).$$

□

Proposition 4.5. *Let D be a local out-tournament of order n and average total degree d, and let $f : [0, \infty) \to [0, 1]$ be the function defined by*

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\ln(x+1)}{x} & \text{otherwise.} \end{cases}$$

Then, $\vec{\alpha}(D) \geq cnf(d)$, where c is a positive constant not depending on D.

Proof. We note that f is decreasing and convex. We choose a $x_0 \geq 1$, large enough so we can find a $c \in \mathbb{R}^+$ satisfying

(i) $\frac{2}{x+2} \geq cf(x)$ for every $x \in [0, x_0]$;

(ii) $\frac{\log_2 x+2}{2x+2} \geq cf(x)$ for every $x \in [x_0, \infty)$;

(iii) $1 - (x^{1/2}+1) \cdot cf(x) + (x^{3/2}+x) \cdot cf'(x) \geq 0$ for every $x \in [x_0, \infty)$.

The proof is by induction on n. If $n = 1$, then $\vec{\alpha}(D) = n \geq cnf(d)$. So let us assume that $n \geq 2$ and that the statement holds for all local out-tournaments of smaller order. If $0 \leq d \leq x_0$, then

$$\vec{\alpha}(D) \geq \frac{n}{\frac{d}{2}+1} \geq cnf(d)$$

by Theorem 1.16 and (i), so from now on we will assume that $d \geq x_0$. We distinguish two cases.

Case 1: *every vertex of D has in-degree at least $d^{1/2}$.*

Given a vertex v, we denote the total degree $\deg^-(v)+\deg^+(v)$ of v by d_v. By Theorem 1.17(i), we can find $r_v := \lfloor \log_2 \deg^-(v) \rfloor + 1 \geq \log_2 d/2$ vertices $v_1, \ldots, v_{r_v} \in N^-(v)$ such that $S_v := \{v, v_1, \ldots, v_{r_v}\}$ is an acyclic set. Since D is a local out-tournament, $N^+(S_v) \subseteq N[v]$, so S_v and $N[v]$ satisfy (1) from Lemma 4.4. Let $A_{N[v]}$ be the set of arcs with an endpoint in $N[v]$, and $d_{N(v)}$ the average total degree $\frac{1}{d_v}\sum_{w \in N(v)} d_w$ of the neighbours of v. We have that

$$|A_{N[v]}| = \sum_{w \in N[v]} d_w - |A(D[N[v]])| \geq d_v d_{N(v)} + d_v - \binom{d_v+1}{2}$$

Therefore,

$$|S_v| - |N[v]| \cdot cf(d) + \left(d\,|N[v]| - 2\,|A_{N[v]}|\right) \cdot cf'(d)$$

$$\geq \frac{\log_2 d}{2} + 1 - (d_v+1) \cdot cf(d) + (dd_v + d - 2d_v d_{N(v)} + d_v^2 - d_v) \cdot cf'(d).$$

Note that $\sum_{v \in V(D)} d_v d_{N(v)} = \sum_{v \in V(D)} d_v^2 \geq nd^2$. Hence, by averaging over all vertices and then applying (ii), one sees that, for some vertex v, the previous expression is at least

$$\frac{\log_2 d}{2} + 1 - (d+1) \cdot cf(d) \geq 0;$$

that is, S_v and $N[v]$ also verify (2) from Lemma 4.4. Since every induced subgraph of D is a local out-tournament, by induction Lemma 4.4(i) holds as well, so $\vec{\alpha}(D) \geq cnf(d)$.

Case 2: *a vertex v of D has in-degree at most $d^{1/2}$.*

In this case, we apply Lemma 4.4 with the sets $S = \{v\}$ and $N = N^-[v]$. Let A_N be the set of arcs with an endpoint in N. By (iii), we see that

$$\begin{aligned} &|S| - |N| \cdot cf(d) + (d\,|N| - 2\,|A_N|) \cdot cf'(d) \\ &\geq 1 - (d^{1/2} + 1) \cdot cf(d) + (d^{3/2} + d) \cdot cf'(d) \geq 0. \end{aligned}$$

So again, we can conclude using induction and Lemma 4.4.

□

Proposition 4.6. *Let $f(n)$ be the maximum dichromatic number over all tournaments of order n. Then, any local tournament D satisfies $\vec{\chi}(D) \leq 2f(\Delta + 1)$, where $\Delta = \Delta_{\max}(D)$. In particular, for every $\varepsilon > 0$ there exists some $\Delta_\varepsilon \in \mathbb{N}$ such that, if $\Delta \geq \Delta_\varepsilon$, then $\vec{\chi}(D) \leq (2 + \varepsilon)\Delta/\log_2 \Delta$.*

Proof. We do induction on the order of D. If D has only one vertex, $\Delta = 0$ and certainly $\vec{\chi}(D) = 1 \leq 2f(1)$. So let us assume that D has more vertices and that the statement is true for all local tournaments of smaller order. Let v be an arbitrary vertex of D; we claim that $N^+(N^-[v]) \subseteq N[v]$. Let $x \in N^-[v]$ and $y \in N^+(x)$. Since $v, y \in N^+[x]$ and D is a local tournament, $y \in N[v]$. So indeed $N^+(N^-[v]) \subseteq N[v]$, and symmetrically $N^-(N^+[v]) \subseteq N[v]$.

By the induction hypothesis, there is a dicolouring g of $D - N[v]$ using at most $2f(\Delta_{\max}(D - N[v]) + 1) \leq 2f(\Delta + 1)$ colours. Since the subgraphs of D induced by $N^-[v]$ and by $N^+(v)$ are tournaments of order at most $\Delta + 1$, we can dicolour them using the colours from $[f(\Delta + 1)]$ and from $[2f(\Delta + 1)] \setminus [f(\Delta + 1)]$, respectively. By the previous paragraph, this yields and extension of g to a dicolouring of D.

Finally, the second part of the statement follows from Theorem 1.22. □

We note that the bound in Theorem 4.2(*ii*) can be obtained simply by considering disjoint unions of multiple copies of an appropriate tournament. A slight modification of this example yields strongly connected local tournaments. In turn, this admits many variations, so we just choose one with reasonably low acyclicity number.

Proposition 4.7. *Let $f(n)$ be the minimum acyclicity over all tournaments of order n. For each positive integer $d \geq 2$, there exists an infinite family $\mathscr{D}_d$ of strongly connected local tournaments such that, for each $D \in \mathscr{D}_d$, $\underline{D}$ is d-regular, and*

$$\vec{\alpha}(D) \leq \frac{n}{d+1}(f(d-1) + 2) \leq \frac{2n}{d+1}\left(\lfloor \log_2(d-1) \rfloor + \frac{3}{2}\right),$$

where n is the order of D.

Proof. Let T be a tournament of order $d-1$ with $\vec{\alpha}(T) = f(d-1)$. Given any positive integer k, we consider $3k$ disjoint sets $S_1, \dots, S_{3k}$, where, for each $1 \leq \ell \leq 3k$, S_ℓ is of size $d-1$ if ℓ is a multiple of 3 and of size 1 otherwise. We let D_k be a digraph with vertex set $S_1 \uplus \dots \uplus S_{3k}$ such that, for every $1 \leq \ell \leq 3k-1$, D_k has all the possible arcs from S_ℓ to $S_{\ell+1}$, as well as those from S_{3k} to S_1, and, moreover $D_k[S_\ell] \cong T$ whenever ℓ is a multiple of 3 (and no more arcs are present apart from these ones).

It is easy to check that D_k is a strongly connected local tournament of order $n := k(d+1)$ with a d-regular underlying graph. Moreover, by Theorem 1.17(*ii*),

$$\vec{\alpha}(D_k) \leq \sum_{\ell=1}^{3k} \vec{\alpha}(D_k[S_\ell]) = k(f(d-1)+2) \leq k(2\lfloor \log_2(d-1) \rfloor + 3).$$

□

Chapter 5

Nested 2-outerplanar triangulations are 2-arborisable

5.1 Introduction

This chapter is motivated by the Neumann-Lara conjecture, presented in Section 2.3. Let us recall it for the reader's convenience.

Conjecture 2.13. (Neumann-Lara) *For every oriented planar graph D, $\vec{\chi}(D) \leq 2$.*

Here our approach is purely in terms of undirected graphs. Let k be a positive integer and G a graph. We say that a function $f : V(G) \to [k]$ is a *k-arborisation*[1] of G if, for every $1 \leq i \leq k$, $f^{-1}(i)$ is an *acyclic set* of G, i.e., the subgraph of G induced by $f^{-1}(i)$ is a forest. If such a k-arborisation exists, then we say that G is *k-arborisable*. The *vertex-arboricity* $\mathrm{va}(G)$ of G is the minimum k such that G is k-arborisable.

One can use $\mathrm{va}(G)$ as a source of upper bounds for $\vec{\chi}(G)$. For instance, the following combined results of Raspaud, Wang, Huang and Shiu provide complementary evidence supporting Conjecture 2.13.

Theorem 5.1. [99, 134] *For each $k \in \{3, 4, 5, 6, 7\}$, every planar graph G without k-cycles has* $\mathrm{va}(G) \leq 2$.

However, a result of Chartrand and Kronk shows that this method alone is insufficient to confirm Conjecture 2.13.

Theorem 5.2. [44] *The maximum vertex-arboricity over all planar graphs is* 3.

[1] There is no general agreement on the terminology. Here we follow Bonamy, Kardoš, Kelly and Postle [33].

Raspaud and Wang proved that the smallest planar graphs attaining this maximum have 21 vertices [134]. Yet, Albertson and Berman suggest that the following might be true.

Conjecture 5.3. [11] *Every planar graph of order n has a set of at least $n/2$ vertices that induces a forest.*

The best known lower bound for Conjecture 5.3 is 2/5, due to Borodin [36]. This is also the best known positive result for Problem 2.14.

In view of Theorem 5.2, some efforts have been made in order to better understand which planar graphs have vertex-arboricity 2. The characterizations given by Stein [146] and Hakimi and Schmeichel [82] indicate that we should expect such class of graphs to be rather rich. Intuitively, this seems to be supported by the fact that planar graphs have exponentially many 3-arborisations [90]. One result in this direction is Theorem 5.4, due to Borradaile, Le and Sherman-Bennett, which concerns a class of graphs generalising outerplanar graphs.

Let k be a positive integer. A graph G is *k-outerplanar* if it has a planar embedding such that, for every vertex v, there is an alternating sequence $F_1, v_1, \dots, F_k, v_k$, where $F_1, \dots F_k$ are faces of the embedding and $v_1, \dots, v_k$ are vertices of G, starting with the unbounded face F_1 and ending with the vertex $v_k = v$, such that any two consecutive elements in the sequence are incident to each other. In particular, a graph is 1-outerplanar if and only if it is outerplanar.

From now onwards, whenever we consider planar embeddings of k-outerplanar graphs, we will be assuming that they satisfy the above property.

Theorem 5.4. [38] *Every 2-outerplanar graph has* $\mathrm{va}(G) \leq 2$.

The examples of Chartrand and Kronk [44] and Raspaud and Wang [134] show that Theorem 5.4 is optimal, in the sense that it cannot be extended to 3-outerplanar graphs.

Another approach would be to investigate how can one build 2-arborisable planar graphs from simpler instances. Given an embedded graph G, we can, for example, choose a triangular face F, and let G_F be the graph obtained from G by adding a new vertex and connecting it to all the vertices of F; clearly, this operation preserves planarity, and if $\mathrm{va}(G) \leq 2$, then $\mathrm{va}(G_F) \leq 2$. Informally speaking, we can replace triangular faces by copies of K_4. Can we replace triangular faces by more complicated graphs? Our main result is that, for 2-outerplanar triangulations, the answer is yes.

Theorem 5.5. *Let G be a 2-outerplanar triangulation and T the subgraph of G induced by the vertices of the unbounded face. Then, every 2-arborisation of T can be extended to a 2-arborisation of G.*

Before proceeding with the proof, we have some remarks. In first place, we stress that not all k-outerplanar graphs admit a k-outerplanar triangulation, although they admit a $(k+1)$-outerplanar one [26]. In particular, Theorem 5.4 is not a corollary of Theorem 5.5.

In the oriented setting, there exist results with a similar flavour regarding extendability from the border, see Li and Mohar [109, Lemma 2.3] and Steiner [149]. More precisely, Steiner posed the following conjecture, which he proved to be equivalent Conjecture 2.13 [149, Theorem 7.7].

Conjecture 5.6. [149, Conjecture 7.6(i)] *Let D be an oriented triangulation and T the subgraph of D induced by the vertices of the unbounded face. Then, every non-monochromatic 2-dicolouring of T can be extended to a 2-dicolouring of D.*

He then investigated the characteristics of minimal counterexamples. Below we include one of his results. A *separating triangle* of an (oriented) triangulation is an (oriented) 3-cycle which does not correspond to a face.

Theorem 5.7. [149, Theorem 7.8(i)] *Let D be a minimal counterexample to Conjecture 5.6. If T is a separating triangle of D, then the subgraph induced by the vertices of T and all the vertices in the region bounded by T is an orientation of the (embedded) octahedron graph H satisfying the following properties: the cycle T is not directed, while the cycles using exactly one arc from T, as well as the cycle formed by the three inner vertices, are directed (see Figure 5.1).*

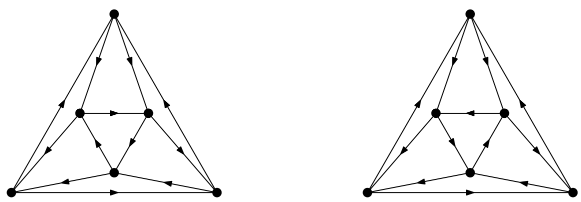

Figure 5.1: The two orientations of H satisfying the properties from Theorem 5.7, where the exterior cycle is T.

The research on 2-arborisable planar graphs could be continued in the following two directions.

Question 5.8. *Let G be a 2-arborisable triangulation and T the subgraph of G induced by the vertices of the unbounded face. Assume that not all 2-arborisations of T can be extended to a 2-arborisation of G. What can be said about G?*

An instance of such a graph can be found among the subgraphs of Raspaud and Wang's example [134].

Question 5.9. *Let G be an embedded planar graph such that the vertices of the unbounded face induce a 4-cycle C. Under which conditions does G satisfy that every 2-arborisation of C can be extended to a 2-arborisation of G?*

Due to the existence of graphs as simple as the 5-vertex wheel (i.e. the graph obtained by adding a universal vertex to a 4-cycle) failing to have this property, we expect partial answers to this version of the problem to be more intricate than in the version for 3-cycles.

5.2 Proof of Theorem 5.5

We will need the following simple "gluing lemma".

Lemma 5.10. *Let G be an embedded planar graph, C a cycle of G, G_{ext} the subgraph of G induced by the vertices of C together with the vertices outside the region delimited by C, and G_{int} the subgraph of G induced by the vertices of C together with the vertices inside the region delimited by C. Let f_{ext} be a k-arborisation of G_{ext} and f_{int} a k-arborisation of G_{int} such that $f_{\text{ext}}(x) = f_{\text{int}}(x)$ for every $x \in V(C)$, and consider the function $f : V(G) \to [k]$ defined by*

$$f(x) = \begin{cases} f_{\text{ext}}(x) & \text{if } x \in V(G_{\text{ext}}) \\ f_{\text{int}}(x) & \text{if } x \in V(G_{\text{int}}). \end{cases}$$

If, for every $1 \leq i \leq k$, $f^{-1}(i) \cap V(C)$ consists of consecutive vertices in C, then f is a k-arborisation of G.

Proof. We assume for a contradiction that f is not a k-arborisation. Let C_1 be a monochromatic cycle minimising the number of vertices in $V(C_1) \cap V(G_{\text{int}}) \setminus V(C)$. Since f_{ext} and f_{int} are k-arborisations, this set is non-empty, and C_1 and C have at least two vertices in common, say, u and v. We write C_1 as a concatenation PQ of two paths P and Q joining u and v and such that $V(Q) \cap V(G_{\text{int}}) \setminus V(C)$ is non-empty. By hypothesis, there is a monochromatic path R in C between u and v. Then, the concatenation PR defines a monochromatic closed walk W with $|V(W) \cap V(G_{\text{int}}) \setminus V(C)| < |V(C_1) \cap V(G_{\text{int}}) \setminus V(C)|$. Hence, we can find a monochromatic cycle C_2 with less vertices in $V(G_{\text{int}}) \setminus V(C)$ than C_1, a contradiction. □

We now prove Theorem 5.5. Let G be a 2-outerplanar triangulation and a, b and c the vertices of the unbounded face. Recall that we are given a 2-arborisation of $G[\{a, b, c\}]$, and we have to extend it to a 2-arborisation of G. We can assume that every vertex of G distinct from a, b and c has degree at least 4. We note that, since G is a 2-outerplanar triangulation, every vertex is adjacent to a, b or c.

We start by fixing some notation. Given an edge $\{x, y\}$, we denote by xy the corresponding plane curve of the embedding. And, given consecutive vertices $x_1, \dots, x_\ell$ of a cycle, we denote by $x_1 \dots x_\ell$ the bounded open region delimited by that cycle.

Let k^{ab} be the number of common neighbours of a and b distinct from c. For every common neighbour x of a and b, the bounded region abx is included in abc. It follows that there is a unique way of labelling the common neighbours of a and b as $c, v_1^{ab}, \dots, v_{k^{ab}}^{ab}$ so that $abc \supseteq abv_1^{ab} \supseteq \dots \supseteq abv_{k^{ab}}^{ab}$. We note that, since ab delimits two triangular faces, which are abc and $abv_{k^{ab}}^{ab}$, there is no vertex inside $abv_{k^{ab}}^{ab}$.

Consider now any $1 \le i \le k^{ab} - 1$. Each vertex inside $av_i^{ab}bv_{i+1}^{ab}$ is a neighbour of either a or b, but not of the other, and neither of c. Let $k_i^{a,b}$ be the number of neighbours of a inside $av_i^{ab}bv_{i+1}^{ab}$, and let us denote them by $v_{i,1}^{a,b}, \dots, v_{i,k_i^{a,b}}^{a,b}$ in a way that $av_i^{ab}v_{i,1}^{a,b}, av_{i,1}^{a,b}v_{i,2}^{a,b}, \dots, av_{i,k_i^{a,b}-1}^{a,b}v_{i,k_i^{a,b}}^{a,b}$, $av_{i,k_i^{a,b}}^{a,b}v_{i+1}^{ab}$ are faces of G (each of which shares an edge with the next one). Since every vertex has degree at least 4, there are no adjacencies between the vertices $v_i^{ab}, v_{i,1}^{a,b}, \dots, v_{i,k_i^{a,b}}^{a,b}, v_{i+1}^{ab}$ other than the ones determined by these faces, unless $k_i^{a,b} = 0$, in which case v_i^{ab} and v_{i+1}^{ab} are adjacent.

Similarly, we let $k_i^{b,a}$ be the number of neighbours of b inside $av_i^{ab}bv_{i+1}^{ab}$, that we denote by $v_{i,1}^{b,a}, \dots, v_{i,k_i^{b,a}}^{b,a}$ in an analogous manner. We note that $k_i^{a,b} = 0$ if and only if $k_i^{b,a} = 0$, because either of them implies that v_i^{ab} and v_{i+1}^{ab} are adjacent. Moreover, in the case $k_i^{a,b}, k_i^{b,a} \ge 1$, since there is no vertex inside $v_i^{ab}v_{i,1}^{a,b} \dots v_{i,k_i^{a,b}}^{a,b}v_{i+1}^{ab}v_{i,k_i^{b,a}}^{b,a} \dots v_{i,1}^{b,a}$, we have that $\{v_{i,1}^{a,b}, v_{i,1}^{b,a}\}$ and $\{v_{i,k_i^{a,b}}^{a,b}, v_{i,k_i^{b,a}}^{b,a}\}$ are edges of G.

We denote by G_i^{ab} the subgraph induced by a, v_i^{ab}, b and v_{i+1}^{ab} together with all the vertices inside $av_i^{ab}bv_{i+1}^{ab}$, and by H_i^{ab} the bipartite subgraph consisting in the vertices inside $av_i^{ab}bv_{i+1}^{ab}$ and the edges joining a neighbour of a with a neighbour of b. See the left of Figure 5.2 for an illustration.

We extend all this notation to the other pairs $\{b, c\}$ and $\{c, a\}$ of vertices of the unbounded face in the natural way.

Lemma 5.11. *Let $1 \le i \le k^{ab} - 1$, and let f be a 2-arborisation of $G[\{a, v_i^{ab}, b, v_{i+1}^{ab}\}]$ with $|f^{-1}(1)| = 2$. Then, f can be extended to a 2-arborisation g of G_i^{ab}.*

Proof. We distinguish two cases.

Case 1: $f^{-1}(1) = \{a, v_i^{ab}\}$, *or* $f^{-1}(1) = \{a, v_{i+1}^{ab}\}$.

By symmetry, we can assume that $f^{-1}(1) = \{a, v_i^{ab}\}$. And then it suffices to take g defined by $g^{-1}(1) = \{a, v_i^{ab}, v_{i,1}^{b,a}, \dots, v_{i,k_i^{b,a}}^{b,a}\}$ (see the right of Figure 5.2).

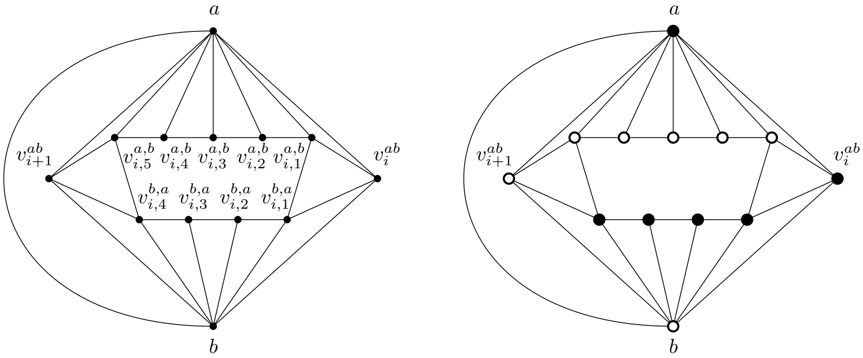

Figure 5.2: Left: illustration of G_i^{ab} with $k_i^{a,b} = 5$ and $k_i^{b,a} = 4$. The edges of H_i^{ab} do not appear, except $\{v_{i,1}^{a,b}, v_{i,1}^{b,a}\}$ and $\{v_{i,5}^{a,b}, v_{i,4}^{b,a}\}$. Right: the 2-arborisation g of G_i^{ab} from Case 1 of Lemma 5.11.

Case 2: $f^{-1}(1) = \{a, b\}$.

For this case we use the following claim.

Claim 5.11.1. *Let $\{x, y\}$ be an edge of H_i^{ab} and let h be a 2-arborisation of $G[\{a, b, x, y\}]$ such that $h(a) = h(b) = 1$. Then, h can be extended to a unique 2-arborisation of $G_i^{ab} - v_i^{ab} - v_{i+1}^{ab}$.*

Proof. Let us write $x = v_{i,j}^{a,b}$ and $y = v_{i,k}^{b,a}$. We first show how h can be *extend upwards*. Assume that $j < k_i^{a,b}$ or $k < k_i^{b,a}$; then, exactly one of $\{x, v_{i,k+1}^{b,a}\}$, $\{y, v_{i,j+1}^{a,b}\}$ is an edge of G. Let $\{w, z\}$ be this edge, with $w \in \{x, y\}$ and $z \in \{v_{i,k+1}^{b,a}, v_{i,j+1}^{a,b}\}$. We extend h to $h' : \{a, b, x, y, z\} \to \{1, 2\}$ by setting

$$h'(z) = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 2 & \text{if } h(x) \neq h(y) \end{cases}$$

(see Figure 5.3). Since only one neighbour of z has been assigned colour $h'(z)$, h' is a 2-arborisation of $G[\{a, b, x, y, z\}]$. Moreover, any 2-arborisation h'' of $G[\{a, b, x, y, z\}]$ must coincide with h'. Indeed, $h(x) \neq h(y)$ implies $h''(z) = 2$. And $h(x) = h(y) = 2$ forces $h''(z) = 1$. The case $h(x) = h(y) = 1$ is not possible because $h(a) = h(b) = 1$.

By a symmetric argument, h can be *extended downwards*. That is, assuming that $\min\{j, k\} > 1$, exactly one of $\{x, v_{i,k-1}^{b,a}\}$, $\{y, v_{i,j-1}^{a,b}\}$ is an edge of G. Let $\{w, z\}$ be this edge, with $w \in \{x, y\}$ and $z \in \{v_{i,k-1}^{b,a}, v_{i,j-1}^{a,b}\}$. Then, h can be extended to a unique 2-arborisation of $G[\{a, b, x, y, z\}]$, as shown above.

Now, by Lemma 5.10, h can be recursively extended upwards and downwards until a 2-arborisation of $G_i^{ab} - v_i^{ab} - v_{i+1}^{ab}$ is obtained. The

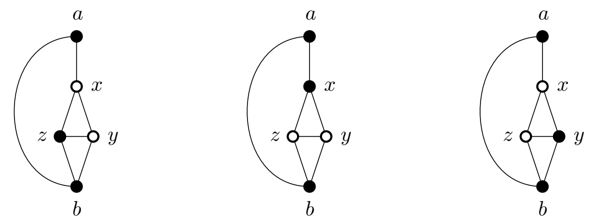

Figure 5.3: The upward extension h' of h described in Claim 5.11.1, in a situation where $w = x$.

uniqueness of the extensions at each step ensures the uniqueness of the final extension. ■

By Claim 5.11.1, there are exactly three 2-arborisations f_1, f_2 and f_3 of $G_i^{ab} - v_i^{ab} - v_{i+1}^{ab}$ extending $f|_{\{a,b\}}$; moreover, for each edge $\{x, y\}$ of H_i^{ab}, $f_1|_{\{x,y\}}$, $f_2|_{\{x,y\}}$ and $f_3|_{\{x,y\}}$ are pairwise distinct. This implies that, for some $1 \leq j \leq 3$, $1 \in f_j(\{v_{i,1}^{a,b}, v_{i,1}^{b,a}\}) \cap f_j(\{v_{i,k_i^{a,b}}^{a,b}, v_{i,k_i^{b,a}}^{b,a}\})$. Let us now consider $g : V(G) \to \{1, 2\}$ defined by

$$g(v) = \begin{cases} 2 & \text{if } v \in \{v_i^{ab}, v_{i+1}^{ab}\} \\ f_j(v) & \text{otherwise.} \end{cases}$$

Since at most one neighbour of v_i^{ab} and at most one of v_{i+1}^{ab} are assigned colour 2 by g, g is a 2-arborisation of G_i^{ab}.

We have thus seen that in both cases f can be extended to a 2-arborisation of G_i^{ab}. □

With Lemma 5.11 we can prove the theorem in the case that $v_1^{ab} = v_1^{bc} = v_1^{ca}$, i.e. when a, b and c have a common neighbour, see Lemma 5.15 and the paragraph thereafter. For the case in which such a vertex does not exist, we need to introduce some additional notation.

First of all, we note that, as long as there is a vertex in abc, $k^{ab}, k^{bc}, k^{ca} \geq 1$. Let k^a be the number of neighbours of a inside $av_1^{ab}bv_1^{bc}cv_1^{ca}$. We denote them by $v_1^a, \ldots, v_{k^a}^a$ in a way that $av_1^{ca}v_1^a, av_1^av_2^a, \ldots, av_{k^a-1}^av_{k^a}^a, av_{k^a}^av_1^{ab}$ are faces of G (each of which shares an edge with the next one). Since every vertex has degree at least 4, there are no adjacencies between the vertices $v_1^{ca}, v_1^a, \ldots, v_{k^a}^a, v_1^{ab}$ other than the ones determined by these faces, unless $k^a = 0$, in which case v_1^{ca} and v_1^{ab} are adjacent.

We define k^b and $v_1^b, \ldots, v_{k^b}^b$, and k^c and $v_1^c, \ldots, v_{k^c}^c$, in an analogous manner. Since every vertex of G is adjacent to a, b or c, each vertex inside $av_1^{ab}bv_1^{bc}cv_1^{ca}$ appears in one of these lists, or in $v_1^a, \ldots, v_{k^a}^a$. Also, by the above argument, all the neighbours of v_1^{bc} inside $av_1^{ab}bv_1^{bc}cv_1^{ca}$ are of the form v_i^a, with consecutive indices i. We denote by G_0 the subgraph of G induced by a, v_1^{ab}, b, v_1^{bc}, c and v_1^{ca} together with all the vertices inside $av_1^{ab}bv_1^{bc}cv_1^{ca}$ (see the left of Figure 5.4).

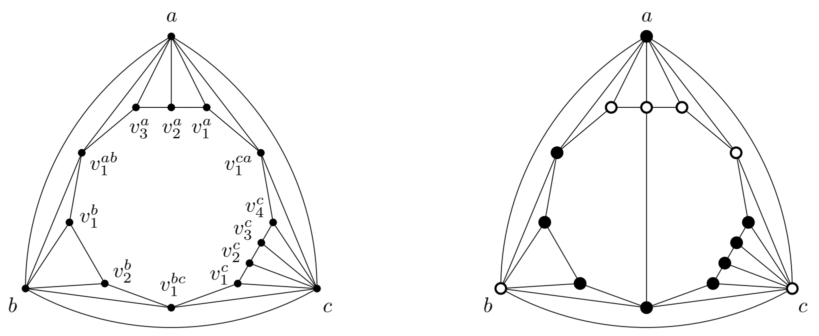

Figure 5.4: Left: illustration of G_0 with $k^a = 3$, $k^b = 2$ and $k^c = 4$. The edges inside $v_1^a v_2^a v_3^a v_1^{ab} v_1^b v_2^b v_1^{bc} v_1^c \dots v_4^c v_1^{ca}$ are not displayed. Right: the 2-arborisation g of G_0 from Case 1 of Lemma 5.12, in an instance with $i = 2$.

Lemma 5.12. *Assume that a, b and c do not have a common neighbour, and let f be a 2-arborisation of $G[\{a, b, c\}]$. If $k^a \geq 1$ and v_1^{bc} is adjacent to v_i^a for some $1 \leq i \leq k^a$, then f can be extended to a 2-arborisation g of G_0.*

Proof. We distinguish two cases.

Case 1: $f^{-1}(1) = \{a\}$.

In this case, we can take g defined by $g^{-1}(1) = \{a, v_1^{ab}, v_1^b, \dots, v_{k^b}^b, v_1^{bc}, v_1^c, \dots, v_{k^c}^c\}$ (see the right of Figure 5.4). It is not difficult to check that $\{a, v_1^{ab}, v_1^b, \dots, v_{k^b}^b, v_1^{bc}\}$ and $\{v_1^{bc}, v_1^c, \dots, v_{k^c}^c\}$ induce paths of G_0 with one endpoint in v_1^{bc}. Since v_1^{bc} is adjacent to v_i^a, $G_0[g^{-1}(1)] - v_1^{bc}$ is disconnected, so $g^{-1}(1)$ is an acyclic set. On the other hand, it is clear that $g^{-1}(2) = \{b, c, v_1^{ca}, v_1^a, \dots, v_{k^a}^a\}$ is also acyclic, so g is indeed a 2-arborisation of G_0.

Case 2: $f^{-1}(1) = \{a, b\}$. (The case $f^{-1}(1) = \{a, c\}$ is symmetric.)

Without loss of generality, we assume that i is the smallest index such that v_1^{bc} is adjacent to v_i^a. Let G' be the subgraph of G_0 induced by a, b, v_1^{bc}, v_i^a, all the vertices inside $abv_1^{bc}v_i^a$, and c. We consider the (embedded) graph G^* obtained from G' in the following way: first, we subdivide the edge $\{c, v_1^{bc}\}$ with a vertex z, and then we add edges from z to a, v_i^a and b. See the left of Figure 5.5 for an illustration.

We note that G^* is a 2-outerplanar triangulation, in which a, b and c are the vertices of the unbounded face. Moreover, any vertex of G^* different from a, b and c has degree at least 4; indeed, this is certainly the case for the vertices inside $abv_1^{bc}v_i^a$, and it is also true for v_1^{bc}, v_i^a and z. Thus, we can apply Lemma 5.11 with G^*. Let $(G^*)_1^{ab}$ be the

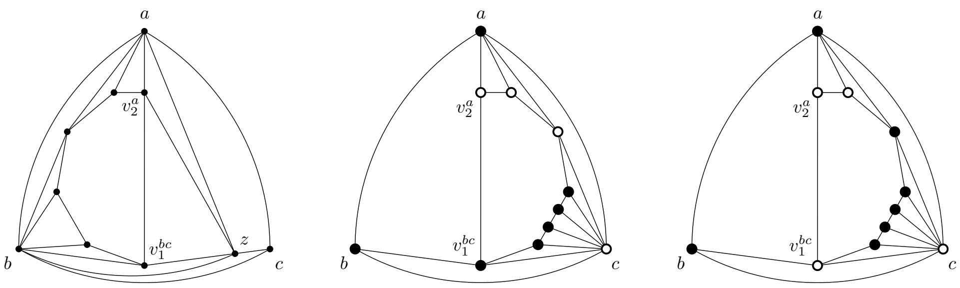

Figure 5.5: Continuation of the example from the right of Figure 5.4 (without displaying the edges that are not displayed there), as an illustration for Case 2 of Lemma 5.12. Left: G^*. Centre and right: the two possibilities for g_{ext}.

subgraph of G^* induced by a, z, b and v_1^{ab} together with all the vertices inside $azbv_1^{ab}$, and let h^* be a 2-arborisation of $G[\{a, b, v_i^a, v_1^{bc}\}]$ such that $h^*(a) = h^*(b) = 1$. By Claim 5.11.1, h^* can be extended to a unique 2-arborisation f^* of $(G^*)_1^{ab} - z - v_1^{ab}$; moreover, the restriction of f^* to the neighbourhood of v_1^{ab} in $(G^*)_1^{ab}$ determines h^*. Therefore, we can find a 2-arborisation g^* of $(G^*)_1^{ab} - z$ such that $g^*(a) = g^*(b) = 1$ and $g^*(v_1^{ab}) = g^*(v_i^a) = 2$.

We note that $(G^*)_1^{ab} - z$ is precisely the subgraph $(G_0)_{\text{int}}$ of G_0 induced by a, b, v_1^{bc}, v_i^a and all the vertices in $abv_1^{bc}v_i^a$. Let $(G_0)_{\text{ext}}$ be the subgraph of G_0 induced by the vertices of G_0 not in $abv_1^{bc}v_i^a$. Since $g^*(a) = g^*(b) = 1$, by Lemma 5.10, in order to find a 2-arborisation g of G_0, it will be enough to show that $g^*|_{\{a,b,v_1^{bc},v_i^a\}}$ can be extended to a 2-arborisation g_{ext} of $(G_0)_{\text{ext}}$.

And we take g_{ext} defined by

$$g_{\text{ext}}^{-1}(1) = \begin{cases} \{a, b, v_1^{bc}, v_1^c, \ldots, v_{k^c}^c\} & \text{if } g^*(v_1^{bc}) = 1 \\ \{v_1^c, \ldots, v_{k^c}^c, v_1^{ca}, a, b\} & \text{if } g^*(v_1^{bc}) = 2 \end{cases}$$

(see the centre and right of Figure 5.5); one can check that each colour class is a path.

Thus, in either of the cases f can be extended to a 2-arborisation of G_0. □

Lemma 5.13. *Assume that a, b and c do not have a common neighbour, and let f be a 2-arborisation of $G[\{a, b, c\}]$. If $k^a, k^b, k^c \geq 1$ and v_1^{bc} is not adjacent to v_i^a for any $1 \leq i \leq k^a$, v_1^{ca} is not adjacent to v_i^b for any $1 \leq i \leq k^b$, and v_1^{ab} is not adjacent to v_i^c for any $1 \leq i \leq k^c$, then f can be extended to a 2-arborisation g of G_0.*

Proof. Without loss of generality, we assume that $f^{-1}(1) = \{a, b\}$. Since v_1^{ab} is not adjacent to v_i^c for any $1 \leq i \leq k^c$, $v_{k^a}^a$ and v_1^b are adjacent. Similarly,

$v^b_{k^b}$ and v^c_1 are adjacent, and so are $v^c_{k^c}$ and v^a_1. Let i and i' be the maximum and minimum indices such that v^a_i is adjacent to v^c_ℓ for some $1 \le \ell \le k^c$ and $v^a_{i'}$ is adjacent to $v^b_{\ell'}$ for some $1 \le \ell' \le k^b$. Following the rotational symmetry, we define j, j', k and k' accordingly. Hence, $\{v^a_{i'}, v^b_j\}$, $\{v^b_{j'}, v^c_k\}$ and $\{v^c_{k'}, v^a_i\}$ are edges of G_0, and, since G is a triangulation, $i = i'$, $j = j'$ and $k = k'$. See the left of Figure 5.6 for an illustration.

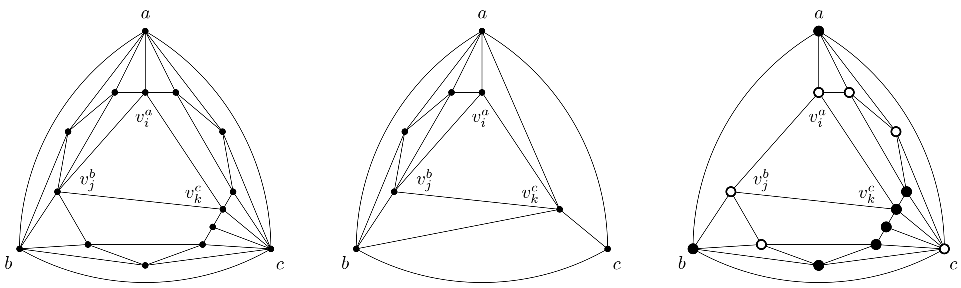

Figure 5.6: Left: an instance of G_0 satisfying the hypotheses of Lemma 5.13, with $k^a = 3$, $k^b = 2$, $k^c = 4$, $i = 2$, $j = 1$ and $k = 3$. The edges inside $v^a_1 v^a_2 v^a_3 v^b_1 v^b_2 v^c_1 \dots v^c_4$ and different from $\{v^a_i, v^b_j\}$, $\{v^b_j, v^c_k\}$ and $\{v^c_k, v^a_i\}$ are not displayed. Centre: G^*. Right: the 2-arborisation g_{ext} of $(G_0)_{\text{ext}}$ from Case 1 of Lemma 5.13.

Let G' be the subgraph of G_0 induced by a, b, v^b_j, v^a_i, all the vertices inside $abv^b_j v^a_i$, c and v^c_k. We consider the (embedded) graph G^* obtained from G' by linking v^c_k to a and b (see the centre of Figure 5.6).

We note that G^* is a 2-outerplanar triangulation, with a, b and c the vertices of its unbounded face. Moreover, the vertices of G^* different from a, b and c have degree at least 4; indeed, this is clearly true for the vertices inside $abv^b_j v^a_i$, and also holds for v^a_i, v^b_j and v^c_k. We apply Lemma 5.11 with G^*. Let $(G^*)^{ab}_1$ be the subgraph of G^* induced by a, v^c_k, b, v^{ab}_1 and all the vertices inside $av^c_k bv^{ab}_1$. It follows from Claim 5.11.1 that there is a 2-arborisation g^* of $(G^*)^{ab}_1 - v^c_k$ such that $g^*(a) = g^*(b) = 1$ and $g^*(v^{ab}_1) = g^*(v^a_i) = 2$. The argument is the same as the one used in Case 2 of Lemma 5.12.

We note that $(G^*)^{ab}_1 - v^c_k$ is precisely the subgraph $(G_0)_{\text{int}}$ of G_0 induced by a, b, v^b_j, v^a_i and all the vertices in $abv^b_j v^a_i$. Let $(G_0)_{\text{ext}}$ be the subgraph of G_0 induced by the vertices of G_0 not in $abv^b_j v^a_i$. Since $g^*(a) = g^*(b) = 1$, by Lemma 5.10, in order to find a 2-arborisation g of G_0, it is enough to show that $g^*|_{\{a,b,v^b_j,v^a_i\}}$ can be extended to a 2-arborisation g_{ext} of $(G_0)_{\text{ext}}$. We distinguish two cases.

Case 1: $g^*(v^b_j) = 2$.

In this case we can take g_{ext} defined by $g^{-1}_{\text{ext}}(1) = \{a, b, v^{bc}_1, v^c_1, \dots, v^c_{k^c}\}$ (see the right of Figure 5.6); one can check that each colour class is a path.

Case 2: $g^*(v_j^b) = 1$.

Given $d, e \in \{a, b, c\}$ with $(d, e) \in \{(a, b), (b, c), (c, a)\}$, and $1 \le \ell \le k^d$ and $1 \le m \le k^e$ such that v_ℓ^d and v_m^e are adjacent, we denote by $G_{\ell,m}^{de}$ the subgraph of G_0 induced by d, e, v_m^e, v_ℓ^d, and all the vertices inside $dev_m^e v_\ell^d$.

Claim 5.13.1. *Let d, e, ℓ, m be as above, and let h be a 2-arborisation of $G[\{d, e, v_m^e, v_\ell^d\}]$ with $h(d) = 1$ and $h(e) = 2$. Assume that one of the following holds.*

(a) *$k^d - \ell + 1$ and m are odd, and h does not satisfy that $h(d) = h(v_\ell^d)$ and $h(e) = h(v_m^e)$;*
(b) *$k^d - \ell + 1$ is odd and m is even, and h does not satisfy that $h(d) = h(v_\ell^d) = h(v_m^e)$;*
(c) *$k^d - \ell + 1$ is even and m is odd, and h does not satisfy that $h(e) = h(v_m^e) = h(v_\ell^d)$;*
(d) *$k^d - \ell + 1$ and m are even, and h does not satisfy that $h(d) = h(v_m^e)$ and $h(e) = h(v_\ell^d)$.*

Then, h can be extended to a 2-arborisation $h_{\ell,m}^{de}$ of $G_{\ell,m}^{de}$.

Proof. The proof is by induction on $(k^d - \ell + 1) + m$. The base case is when $(k^d - \ell + 1) + m = 2$; then, $\ell = k^d$ and $m = 1$, and we only need to colour the vertex v_1^{de}. By (a), it is enough to define

$$h_{\ell,m}^{de}(v_1^{de}) = \begin{cases} 1 & \text{if } h(v_\ell^d) = h(v_m^e) = 2 \\ 2 & \text{if } h(v_\ell^d) = h(v_m^e) = 1 \\ 2 & \text{if } h(v_\ell^d) = 2 \text{ and } h(v_m^e) = 1 \end{cases}$$

(see the left of Figure 5.7).

For $(k^d - \ell + 1) + m \ge 3$, we observe that either $m \ge 2$ and v_ℓ^d is adjacent to v_{m-1}^e, or $k^d - \ell + 1 \ge 2$ and v_m^e is adjacent to $v_{\ell+1}^d$. These two situations are symmetric, so we will assume that we are in the former. We distinguish two cases.

Case 1: $h(e) = h(v_m^e) = h(v_\ell^d)$.

In this case we can take $h_{\ell,m}^{de}$ defined by $(h_{\ell,m}^{de})^{-1}(1) = \{d, v_1^{de}, v_1^e, \ldots, v_{m-1}^e\}$ (see the right of Figure 5.7).

Case 2: *either $h(e) \ne h(v_m^e)$ or $h(e) \ne h(v_\ell^d)$.*

We consider the 2-arborisation h' of $G[\{d, e, v_{m-1}^e, v_\ell^d\}]$ defined as

$$h'(x) = \begin{cases} h(x) & \text{if } x \in \{d, e, v_\ell^d\} \\ 3 - h(v_m^e) & \text{if } x = v_{m-1}^e. \end{cases}$$

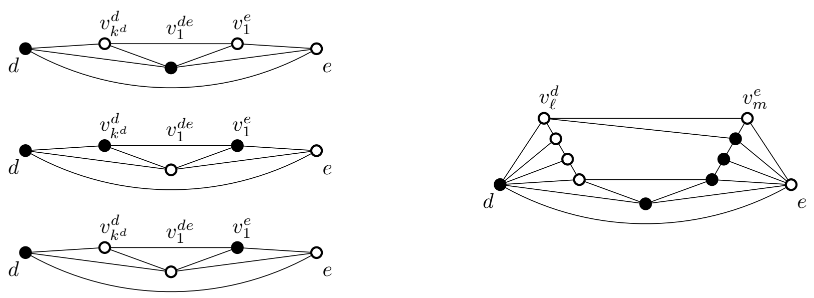

Figure 5.7: Illustration for Claim 5.13.1. Left: the three possibilities for $h^{de}_{\ell,m}$ in the base case. Right: $h^{de}_{\ell,m}$ as defined in Case 1 of the induction step, in an instance with $k^d-\ell+1 = m = 4$. The edges inside $v^d_\ell \dots v^d_{k^d} v^e_1 \dots v^e_{m-1}$ are not displayed.

By the induction hypothesis, there is a 2-arborisation $(h')^{de}_{\ell,m-1}$ of $G^{de}_{\ell,m-1}$ extending h', unless one of the following holds.

(a′) $k^d - \ell + 1$ and $m - 1$ are odd, $h'(d) = h'(v^d_\ell)$ and $h'(e) = h'(v^e_{m-1})$;

(b′) $k^d-\ell+1$ is odd, $m-1$ is even, and $h'(d) = h'(v^d_\ell) = h'(v^e_{m-1})$;

(c′) $k^d-\ell+1$ is even, $m-1$ is odd, and $h'(e) = h'(v^e_{m-1}) = h'(v^d_\ell)$;

(d′) $k^d - \ell + 1$ and $m - 1$ are even, $h'(d) = h'(v^e_{m-1})$ and $h'(e) = h'(v^d_\ell)$.

But, since one of (a)–(d) holds, none of the above does, so $(h')^{de}_{\ell,m-1}$ indeed exists. Since $h(e) = h(v^e_m) = h(v^d_\ell)$ does not hold, at most one among e, v^e_{m-1} and v^d_ℓ, the neighbours of v^e_m in $G^{de}_{\ell,m}$, is assigned colour $h(v^e_m)$ by $(h')^{de}_{\ell,m-1}$. Hence, we can take

$$h^{de}_{\ell,m}(x) = \begin{cases} (h')^{de}_{\ell,m-1}(x) & \text{if } x \in V(G^{de}_{\ell,m-1}) \\ h(v^e_m) & \text{if } x = v^e_m \end{cases}$$

as our 2-arborisation of $G^{de}_{\ell,m}$.

Hence the induction step holds in both cases. ■

We continue with Case 2 of the proof of Lemma 5.13. The strategy consists in applying Lemma 5.10 yet again, separating $(G_0)_{\mathrm{ext}}$ into $G^{bc}_{j,k}$ and the subgraph $\bar{G}^{bc}_{j,k}$ of $(G_0)_{\mathrm{ext}}$ induced by the vertices outside $bcv^c_k v^b_j$ (see the left of Figure 5.8). Since $f^{-1}(1) = \{a, b\}$, in what follows we will be allowed to assume that either i is odd or both i and $k^b - j + 1$ are even. We distinguish two subcases.

Subcase 2.1: *either $k^b - j + 1$ is even, or k is odd.*

In this case, we let h be the 2-arborisation of $G[\{b, c, v_k^c, v_j^b\}]$ defined by $h^{-1}(2) = \{c\}$. By Claim 5.13.1, h can be extended to a 2-arborisation $h_{j,k}^{bc}$ of $G_{j,k}^{bc}$.
Let us now consider the function $\bar{g} : V(\bar{G}_{j,k}^{bc}) \to \{1, 2\}$ given by

$$\bar{g}^{-1}(1) = \{a, b, v_j^b, v_k^c, \dots, v_{k^c}^c\}$$
$$\bar{g}^{-1}(2) = \{c, v_1^{ca}, v_1^a, \dots, v_i^a\}$$

(Figure 5.8, left). It is not difficult to check that $\bar{g}$ is a 2-arborisation of $\bar{G}_{j,k}^{bc}$. Since $\bar{g}$ and $h_{j,k}^{bc}$ coincide on $\{b, c, v_k^c, v_j^b\}$, by Lemma 5.10 they can be extended to a 2-arborisation g_{ext} of $(G_0)_{\text{ext}}$. Moreover, g_{ext} coincides with g^* on $\{a, b, v_j^b, v_i^a\}$.

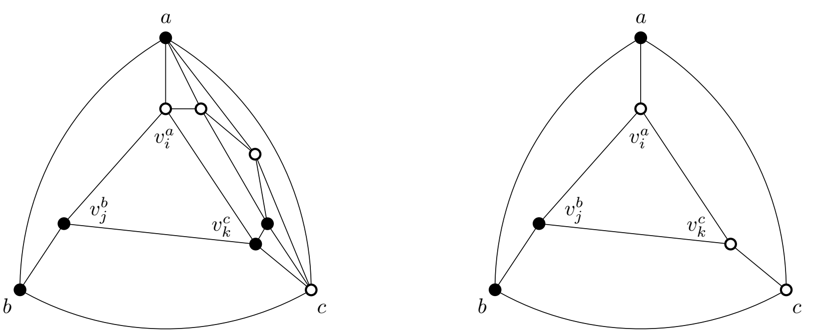

Figure 5.8: Continuation of the example in the left of Figure 5.6 as an illustration for Case 2 of Lemma 5.13. Left: the 2-arborisation $\bar{g}$ of $\bar{G}_{j,k}^{bc}$ as defined in Subcase 2.1. Right: combined representation of h and $\bar{h}$ from Subcase 2.2.

Subcase 2.2: $k^b - j + 1$ *is odd and* k *is even.*

In this case, we let h be the 2-arborisation of $G[\{b, c, v_k^c, v_j^b\}]$ defined by $h^{-1}(1) = \{b, v_j^b\}$. By Claim 5.13.1, h can be extended to a 2-arborisation $h_{j,k}^{bc}$ of $G_{j,k}^{bc}$.
Additionally, we consider the 2-arborisation $\bar{h}$ of $G[\{c, a, v_i^a, v_k^c\}]$ defined by $\bar{h}^{-1}(1) = \{a\}$. Since $k^b - j + 1$ is odd, i is odd by our assumption. By Claim 5.13.1, $\bar{h}$ can be extended to a 2-arborisation $\bar{h}_{k,i}^{ca}$ of $G_{k,i}^{ca}$. We extend $\bar{h}_{k,i}^{ca}$ to a function $\bar{g} : V(\bar{G}_{j,k}^{bc}) \to \{1, 2\}$ by setting

$$\bar{g}(x) = \begin{cases} 1 & \text{if } x \in \{b, v_j^b\} \\ \bar{h}_{k,i}^{ca}(x) & \text{it } x \in V(G_{k,i}^{ca}). \end{cases}$$

In $\bar{G}_{j,k}^{bc}$, the neighbours of v_j^b are v_i^a, v_k^c and b, and the neighbours of b are v_j^b, a and c, so $\bar{g}$ is a 2-arborisation of $\bar{G}_{j,k}^{bc}$. Since $\bar{g}$ and $h_{j,k}^{bc}$ coincide on $\{b, c, v_k^c, v_j^b\}$, by Lemma 5.10 they can be extended to a 2-arborisation g_{ext} of $(G_0)_{\text{ext}}$. As in Subcase 2.1, g_{ext} coincides with g^* on $\{a, b, v_j^b, v_i^a\}$.

Hence, not only in Case 1, but also in Subcases 2.1 and 2.2 we are able to find a 2-arborisation g_{ext} of $(G_0)_{\text{ext}}$ that extends $g^*|_{\{a,b,v_j^b,v_i^a\}}$. As discussed above, this ends the proof of Lemma 5.13. □

As a corollary of Lemmas 5.12 and 5.13, we obtain the following.

Remark 5.14. *Assume that a, b and c do not have a common neighbour, and let f be a 2-arborisation of $G[\{a, b, c\}]$. Then, f can be extended to a 2-arborisation g of G_0.*

Proof. If $k^a = k^b = k^c = 0$, then $G_0 = G[\{a, v_1^{ab}, b, v_1^{bc}, c, v_1^{ca}\}]$, and it is clear that the statement holds. If $k^a = 0 \neq \max\{k^b, k^c\}$, then, since G_0 is a triangulation, either v_1^{ab} is adjacent to v_i^c for some $1 \leq i \leq k^c$, or v_1^{ca} is adjacent to v_i^b for some $1 \leq i \leq k^b$. Hence, we are done by Lemma 5.12. Finally, if $k^a, k^b, k^c \geq 1$, then we can simply apply Lemmas 5.12 and 5.13. □

In order to prove Theorem 5.5, we need one last lemma. Given two different vertices $d, e \in \{a, b, c\}$ and $1 \leq i \leq k^{de}$, we denote by I_i^{de} the subgraph of G induced by d, e, v_i^{de}, and all the vertices in dev_i^{de}.

Lemma 5.15. *Let $1 \leq i \leq k^{ab}$. Any 2-arborisation f of $G[\{a, b, v_i^{ab}\}]$ can be extended to a 2-arborisation g of I_i^{ab}.*

Proof. The proof is by induction on $k^{ab} - i$. If $i = k^{ab}$, $I_i^{ab} = G[\{a, b, v_i^{ab}\}]$, so we assume that $i < k^{ab}$ and that the statement holds for $i + 1$. Let f' be the unique 2-arborisation of $G[\{a, v_i^{ab}, b, v_{i+1}^{ab}\}]$ extending f and satisfying $|f'^{-1}(1)| = 2$. By Lemma 5.11, f' can be extended to a 2-arborisation g_{ext} of G_i^{ab}. Now, by induction, we can extend $f'|_{\{a,b,v_{i+1}^{ab}\}}$ to a 2-arborisation g_{int} of I_{i+1}^{ab}. By Lemma 5.10, the function

$$g(x) = \begin{cases} g_{\text{ext}}(x) & \text{if } x \in V(G_i^{ab}) \\ g_{\text{int}}(x) & \text{if } x \in V(I_{i+1}^{ab}) \end{cases}$$

is a 2-arborisation of I_i^{ab}. □

Now, let f be any 2-arborisation of $G[\{a, b, c\}]$. If a, b and c have a common neighbour (i.e. $v_1^{ab} = v_1^{bc} = v_1^{ca}$), we let g be the unique 2-arborisation of $G[\{a, b, c, v_1^{ab}\}]$ extending f. Otherwise, we let g be the 2-arborisation of G_0 given by Remark 5.14. By Lemma 5.15 applied to I_1^{ab}, I_1^{bc} and I_1^{ca} and Lemma 5.10, g can be extended to a 2-arborisation of G. This ends the proof of Theorem 5.5.

Chapter 6

$(\Delta - 1)$-dicolouring of digraphs

This chapter is based on joint work with Ararat Harutyunyan, Ken-ichi Kawarabayashi and Lucas Picasarri-Arrieta [87].

6.1 Introduction

The relationships between the chromatic number $\chi(G)$, the clique number $\omega(G)$, and the maximum degree $\Delta(G)$ of a graph G have attracted a lot of attention during the last decades. Perhaps the first non-trivial result of this kind is Brooks' theorem (Theorem 1.8). In 1977, Borodin and Kostochka made the following conjecture, which, if true, would give a nice extension of Brooks' theorem for graphs of maximum degree at least 9.

Conjecture 6.1. (Borodin–Kostochka) [37] *Every graph G with maximum degree $\Delta(G) \geq 9$ and clique number $\omega(G) \leq \Delta(G) - 1$ satisfies $\chi(G) \leq \Delta(G) - 1$.*

The graph obtained from a 5-cycle by blowing up each vertex into a triangle witnesses that, if this is true, the lower bound on the maximum degree is tight. Twenty years later, Reed published two fundamental papers [135, 136] related to the Borodin–Kostochka conjecture. In the first one, Reed posed the following celebrated conjecture.

Conjecture 6.2. (Reed) [135] *Every graph G satisfies*

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil .$$

If true, it implies in particular that every graph G with chromatic number at least $\Delta(G)$ contains a clique of size $\Delta(G) - 2$, thereby approaching Conjecture 6.1. As a partial result, Reed shows that $\chi(G)$ is indeed bounded by a convex combination of $\Delta(G) + 1$ and $\omega(G)$ (up to ceiling).

Theorem 6.3. [135] *There exists an $\varepsilon > 0$ such that every graph G satisfies*

$$\chi(G) \le \lceil (1-\varepsilon)(\Delta(G)+1) + \varepsilon\omega(G) \rceil .$$

Theorem 6.3 implies the weaker version of the Borodin–Kostochka conjecture stating that every graph G with chromatic number at least $\Delta(G)$ contains a clique of size $\Delta(G) - C$, for some absolute constant C. One year later, Reed strengthened this by proving that Conjecture 6.1 holds for large values of $\Delta(G)$.

Theorem 6.4. [136] *There exists some $\Delta_0 \in \mathbb{N}$ such that the following holds. Every graph G with maximum degree $\Delta(G) \ge \Delta_0$ and clique number $\omega(G) \le \Delta(G) - 1$ satisfies $\chi(G) \le \Delta(G) - 1$.*

$\overleftrightarrow{\omega}$, $\tilde{\Delta}$, and $\vec{\chi}$

We provide extensions of Conjecture 6.1 and Theorem 6.4 to the digraph setting, substituting ω, Δ and χ by the digraph parameters $\overleftrightarrow{\omega}$, $\tilde{\Delta}$ and $\vec{\chi}$ (defined in Section 1.2). Similar to the undirected case, the relationships linking these digraph parameters have been gradually gaining interest. This line of research began with a result of Mohar (see Theorem 1.19), which states that $\vec{\chi}(D) \le \lceil \tilde{\Delta}(D) \rceil$ for every connected digraph D which is not a directed cycle, a symmetric odd cycle, nor a complete digraph, thereby generalising Brooks' theorem. Recently, Kawarabayashi and Picasarri-Arrieta posed that Reed's conjecture generalises to digraphs with these parameters.

Conjecture 6.5. [103] *Every digraph D satisfies*

$$\vec{\chi}(D) \le \left\lceil \frac{\tilde{\Delta}(D) + 1 + \overleftrightarrow{\omega}(D)}{2} \right\rceil .$$

If true, this would imply both Reed's conjecture in the case of symmetric digraphs and an independent conjecture posed by Harutyunyan and Mohar [89] when applied to digraphs with biclique number 1. As an intermediate result, Kawarabayashi and Picasarri-Arrieta generalised Theorem 6.3 to digraphs (up to the value of ε).

Theorem 6.6. [103] *There exists an $\varepsilon > 0$ such that every digraph D satisfies*

$$\vec{\chi}(D) \le \lceil (1-\varepsilon)(\tilde{\Delta}(D)+1) + \varepsilon\overleftrightarrow{\omega}(D) \rceil .$$

One might expect that, like Reed's conjecture, the Borodin–Kostochka conjecture directly extends to digraphs with these parameters; that is, for every integer $\Delta \ge 9$, digraphs D with maximum geometric mean degree $\tilde{\Delta}(D) \le \Delta$ and biclique number $\overleftrightarrow{\omega}(D) \le \Delta - 1$ have dichromatic number at most $\Delta - 1$. It turns out that this is not the case: for every integer $k \ge 3$, the digraph join $H_k = \vec{C}_3 \boxplus \overleftrightarrow{K}_{k-2}$ (consisting in a copy of $\vec{C}_3$, a copy of $\overleftrightarrow{K}_{k-2}$, and all the possible arcs between them) satisfies $\tilde{\Delta}(H_k) = k$, $\overleftrightarrow{\omega}(H_k) = k-1$, and $\vec{\chi}(H_k) = k$. In [87] it is conjectured that this is the only obstruction arising from the directed setting.

Conjecture 6.7. [87] *Let $\Delta \geq 9$ be an integer and D a digraph satisfying $\tilde{\Delta}(D) \leq \Delta$ and $\overleftrightarrow{\omega}(D) \leq \Delta - 1$. Then $\vec{\chi}(D) \leq \Delta - 1$ unless a subgraph of D is isomorphic to $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.*

If true, Conjecture 6.7 would imply Conjecture 6.1, since graphs in Conjecture 6.1 can be replaced by corresponding symmetric digraphs in Conjecture 6.7. In particular, the condition $\Delta \geq 9$ cannot be relaxed, as it is already best possible in the undirected case. Similarly to the undirected case, Theorem 6.6 implies the weaker version stating that digraphs D with $\vec{\chi}(D) \geq \tilde{\Delta}(D)$ have biclique number at least $\tilde{\Delta}(D) - C$ for some absolute constant C. Our goal in this chapter is to prove that Conjecture 6.7 holds for digraphs with large maximum geometric mean degree, thereby generalising Theorem 6.4 to digraphs (up to the value of Δ_0).

Theorem 6.8. *There exists some $\Delta_0 \in \mathbb{N}$ such that the following holds. Let $\Delta \geq \Delta_0$ be an integer and D a digraph with $\tilde{\Delta}(D) \leq \Delta$ and $\overleftrightarrow{\omega}(D) \leq \Delta - 1$. Then $\vec{\chi}(D) \leq \Delta - 1$ unless a subgraph of D is isomorphic to $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.*

The proof of Theorem 6.8 combines structural and probabilistic arguments. In particular, we use the following dense decomposition lemma, which generalises to the directed setting the so-called 'dense decomposition of graphs' appearing in Molloy and Reed's series of papers [66, 124, 125, 135, 136] (see also [126, Chapter 15]).

Before stating it, we need a few definitions. Let D be a digraph. We recall that $\Delta_{\max}(D)$ denotes the maximum of $\max\{\deg^+(v), \deg^-(v)\}$ over all $v \in V(D)$. Suppose that $\Delta_{\max}(D) = \Delta$, and let $0 \leq d \leq \Delta - 1$ be any real number. We say that a vertex v of D is *d-sparse* if the digraph induced by its out-neighbourhood contains at most $\Delta(\Delta-1) - d\Delta$ arcs. A vertex that is not d-sparse is *d-dense*. Given a set X of vertices of D, we denote by $\partial^+(X)$ (resp. $\partial^-(X)$) the set of arcs of D leaving X (resp. entering X). In what follows, by $\omega(1) \leq d \leq o(\Delta)$ we mean that d is any fixed sublinear function of Δ going to infinity.

Lemma 6.9. (Dense Decomposition Lemma) [87] *For every $0 < \varepsilon < 1/2$ and $\omega(1) \leq d \leq o(\Delta_{\max})$, there exists some $\Delta_0 \in \mathbb{N}$ such that the following holds. Every digraph D with $\Delta_{\max}(D) = \Delta_{\max} \geq \Delta_0$ admits a partition $X_1 \cup \ldots \cup X_t \cup S$ of its vertex set such that:*

(i) for every $i \in [t]$, $\Delta_{\max} - \frac{3}{\varepsilon}d < |X_i| < \Delta_{\max} + 1 + 4d$;

(ii) for every $i \in [t]$, $|\partial^+(X_i)| \leq \frac{8}{\varepsilon}d\Delta_{\max}$ and $|\partial^-(X_i)| \leq \frac{8}{\varepsilon}d\Delta_{\max}$;

(*iii*) *for every* $i \in [t]$ *and every* $v \in V(D)$, $v \in X_i$ *if and only if* $|N^+(v) \cap X_i| \geq (1-\varepsilon)\Delta_{\max}$*; and*

(*iv*) *vertices in* S *are* d*-sparse.*

We now outline the proof of Theorem 6.8. Consider a minimal counterexample D along with one of its dense decompositions $X_1 \uplus \ldots \uplus X_t \uplus S$ generated by Lemma 6.9, where ε is sufficiently small and d is approximately $\ln^3 \Delta$. It is easy to show that, because of the minimality of D, $\tilde{\Delta}(D)$ and $\Delta_{\max}(D)$ are close, so, for clarity, let us assume here that $\Delta = \Delta_{\max}(D) = \tilde{\Delta}(D)$.

Consider the random colouring process which consists of (i) assigning each vertex of D a colour uniformly at random from $\{1, \ldots, \Delta - 1\}$, and (ii) simultaneously uncolouring all vertices belonging to a monochromatic directed cycle. The core idea is to show that the resulting partial dicolouring can be extended to D with positive probability. This method is now standard in the field, and has been used for instance in [91, 103]. For a d-sparse vertex v, applying this process, it is highly likely that many colours are repeated in its out-neighbourhood, thereby allowing us to extend the obtained dicolouring to v. This argument is central to the proof of Theorem 6.6 in [103], along which it is established that a sparse digraph H can be dicoloured using only a fraction of $\Delta_{\max}(H)$ colours.

Thus, the main challenge in proving Theorem 6.8 lies in handling the d-dense vertices. Furthermore, we know that all such vertices belong to some X_i, and that each X_i can be coloured almost independently from the rest of the digraph. This is mainly due to property (*iii*) of Lemma 6.9 and the fact that ε is sufficiently small. Building on this observation, we progressively constrain the possible structure of each X_i until reaching a point where it becomes likely that the random colouring process described above yields a partial dicolouring that can be extended to X_i. At that stage, using the Lovász local lemma, we conclude that the obtained partial dicolouring can be extended simultaneously to all sparse vertices and all X_i's with positive probability, leading to the desired contradiction.

Analogues for other extensions of Δ

A legitimate question is whether $\tilde{\Delta}$ can be replaced by other, potentially more natural, definitions of the maximum degree of a digraph in Conjecture 6.7 and Theorem 6.8, so let us briefly comment on that. Observe that, by definition, all digraphs D satisfy $\Delta_{\max}(D) \geq \tilde{\Delta}(D)$. Hence, replacing $\tilde{\Delta}$ with $\Delta_{\max}$ yields weaker statements. A natural candidate is then the maximum out-degree Δ^+.

Indeed, Δ^+ and $\tilde{\Delta}$ are independent parameters, in the sense that one cannot be bounded by any function of the other. Furthermore, when only the out-degree of a digraph is bounded, probabilistic methods are unlikely to work. Intuitively, if the maximum in-degree of a digraph is arbitrarily large compared to its maximum out-degree, one cannot expect to apply the Lovász local lemma, as it seems unlikely to bound the degree of "bad events". That is why finding upper bounds on the dichromatic number of a digraph D involving $\Delta^+(D)$ is usually significantly harder than ones involving $\tilde{\Delta}(D)$. The most emblematic open problem of this kind is probably the following one, proposed by Kawarabayashi and Picasarri-Arrieta, whose version with $\tilde{\Delta}$ has been proved first by Harutyunyan and Mohar (see Theorems 2.7 to 2.8), and appears to be a particular case of Theorem 6.6.

Problem 2.9. [103] *Show the existence of positive constants Δ_0 and ε such that, for every oriented graph D with $\Delta^+(D) \geq \Delta_0$, $\vec{\chi}(D) \leq (1-\varepsilon)\Delta^+(D)$.*

Nevertheless, Conjecture 6.7 might remain valid when $\tilde{\Delta}$ is replaced by Δ^+. Building on Theorem 6.8, in [87] it is proved that this is indeed the case for Δ^+ large.

Theorem 6.10. [87] *There exists some $\Delta_0 \in \mathbb{N}$ such that the following holds. Let $\Delta \geq \Delta_0$ be an integer and D a digraph with $\Delta^+(D) \leq \Delta$ and $\overleftrightarrow{\omega}(D) \leq \Delta - 1$. Then $\vec{\chi}(D) \leq \Delta - 1$ unless a subgraph of D is isomorphic to $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.*

Theorem 6.10 yields a second (and independent) generalisation of Theorem 6.4. The proof combines structural and discharging arguments on a well-chosen counterexample, which in particular does not satisfy the requirements of Theorem 6.8.

Finally, Theorem 6.10 has been pushed further to obtain various conditions under which a digraph D satisfies $\vec{\chi}(D) \leq \Delta_{\min}(D) - 1$. For instance:

Theorem 6.11. [87] *Let Δ_0 be as in Theorem 6.10. The following holds for every integer $\Delta \geq \Delta_0$. Let D be a digraph with underlying graph G. If $\Delta_{\min}(D) \leq \Delta$ and $\omega(G) \leq \Delta - 1$, then $\vec{\chi}(D) \leq \Delta - 1$.*

This constitutes a third independent generalisation of Theorem 6.4.

6.2 Proof of Theorem 6.8

For the reader's convenience, let us recall the statement of the main theorem.

Theorem 6.8. *There exists some $\Delta_0 \in \mathbb{N}$ such that the following holds. Let $\Delta \geq \Delta_0$ be an integer and D a digraph with $\tilde{\Delta}(D) \leq \Delta$ and $\overleftrightarrow{\omega}(D) \leq \Delta - 1$. Then $\vec{\chi}(D) \leq \Delta - 1$ unless a subgraph of D is isomorphic to $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.*

We do not explicitly give the value of Δ_0; we simply assume that it is large enough so that all inequalities claimed along the proof actually hold.

We assume the statement does not hold, which means that there exists an integer $\Delta \geq \Delta_0$ and a digraph D satisfying $\vec{\chi}(D) \geq \Delta \geq \tilde{\Delta}(D)$ that does not contain any copy of $\overleftrightarrow{K}_\Delta$ nor $\vec{C_3} \boxplus \overleftrightarrow{K}_{\Delta-2}$. Among all such digraphs, we choose D for which $|V(D)| + |A(D)|$ is minimum. We further set $d = \ln^3 \Delta$.

In Subsection 6.2.1, with Lemma 6.9 in hand, we show a collection of structural properties on D. In Subsection 6.2.2, we use the probabilistic method to show that a digraph satisfying these specific properties can be partially coloured with at most $\Delta-1$ colours, in such a way that the obtained dicolouring can be greedily extended to the whole digraph, thus yielding a contradiction.

6.2.1 Structure of a minimum counterexample

A digraph is *k-critical* if it has dichromatic number k and all its proper subgraphs are $(k-1)$-dicolourable. We first remark that D is Δ-critical, for otherwise we can remove an arc or a vertex without decreasing the dichromatic number, thus yielding a smaller counterexample. We start with two well-known basic properties of critical digraphs.

Lemma 6.12. *Every vertex $v \in V(D)$ satisfies $\Delta - 1 \leq \deg^-(v), \deg^+(v) \leq \Delta + 1$. Moreover, we have $\Delta \leq \Delta_{\max}(D) \leq \Delta + 1$.*

Proof. If some vertex $v \in V(D)$ satisfies $\min\{\deg^-(v), \deg^+(v)\} \leq \Delta - 2$, we can extend a $(\Delta-1)$-dicolouring of $D - v$ (which exists as D is Δ-critical) to D by choosing for v a colour that is not appearing in the in- or out-neighbourhood of v, thus contradicting $\vec{\chi}(D) = \Delta$. This shows the lower bound. As for the upper bound, assume for a contradiction that v satisfies $\max\{\deg^-(v), \deg^+(v)\} \geq \Delta + 2$. Using the lower bound we just proved, we thus have

$$\deg^+(v) \cdot \deg^-(v) \geq (\Delta-1)(\Delta+2) > \Delta^2 \geq \tilde{\Delta}(D)^2.$$

This is a contradiction to the definition of $\tilde{\Delta}(D)$. This shows the first part of the statement.

For the second part, it remains to justify that $\Delta_{\max}(D) \neq \Delta - 1$. Suppose that $\Delta_{\max}(D) = \Delta - 1$, then by the directed Brooks' theorem (Theorem 1.19), and because $\vec{\chi}(D) \geq \Delta$, some connected component of D must be isomorphic to $\overleftrightarrow{K}_\Delta$, a contradiction. □

Lemma 6.13. *If uv is a simple arc of D, then v has a simple out-neighbour, and u has a simple in-neighbour.*

Proof. By criticality of D, let φ be a $(\Delta-1)$-dicolouring of $D \setminus \{uv\}$. Since φ is not a dicolouring of D, D coloured with φ contains a monochromatic directed cycle, and among all of them we let $\mathscr{C}$ be an induced one. Clearly, $\mathscr{C}$ contains uv, for otherwise $\mathscr{C}$ is a monochromatic directed cycle of $D \setminus \{uv\}$ coloured with φ. Since $\mathscr{C}$ is induced and uv is simple, the successor of v in $\mathscr{C}$ is indeed one of its simple out-neighbours. Similarly, the predecessor of u in $\mathscr{C}$ is one of its simple in-neighbours. □

By definition, D does not contain any copy of $\overleftrightarrow{K}_\Delta$ nor of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$. The following lemma shows that this remains true even when we add one or two arcs between two vertices of D.

Lemma 6.14. *For every pair of vertices $u, v \in V(D)$, $D \cup \{uv, vu\}$ does not contain any copy of $\overleftrightarrow{K}_\Delta$ nor of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.*

Proof. Throughout the proof, we will often rely on the fact that, for every vertex w of D, $\min\{\deg^-(w), \deg^+(w)\} \leq \Delta$. Let us see first that $D \cup \{uv, vu\}$ does not contain any induced copy of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$. (If $D \cup \{uv, vu\}$ contains a non-induced copy of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$, then it contains a copy of $\overleftrightarrow{K}_\Delta$ and we will deal with this later.) Assume for a contradiction that this is not true. Let T be a set of vertices inducing a directed 3-cycle in $D \cup \{uv, vu\}$ and W a biclique of $D \cup \{uv, vu\}$ of size $\Delta - 2$, such that all arcs between T and W are in $A(D) \cup \{uv, vu\}$. We note that $u, v \in T \cup W$; otherwise, D itself would contain a copy of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$, a contradiction. Let φ be a $(\Delta-1)$-dicolouring of $D - T - W$, which exists due to the Δ-criticality. We are going to extend φ to a $(\Delta-1)$-dicolouring of D, yielding a contradiction.

Since T induces a directed 3-cycle in $D \cup \{uv, vu\}$, at least one of $\{u, v\}$, say v, belongs to W. Let x and y be two distinct vertices in $T \setminus \{u\}$. Since none of $\{\langle x, y\rangle, \langle u, v\rangle\}$ is a digon of D, and each of these four vertices has at most three out-neighbours outside $T \cup W$, we can extend φ to $V(D) \setminus W \cup \{u, v\}$ in a way that $\varphi(x) = \varphi(y) \neq \varphi(u) = \varphi(v)$. Then, each $w \in W \setminus \{u, v\}$ has at most $\Delta - 2$ forbidden colours, so we can extend φ to $V(D)$ greedily.

Let us now see that $D \cup \{uv, vu\}$ does not contain any copy of $\overleftrightarrow{K}_\Delta$. Suppose for a contradiction that we can find a set W of $\Delta - 2$ vertices such that $X = \{u, v\} \cup W$ is a biclique of $D \cup \{uv, vu\}$ of size Δ.

Claim 6.14.1. *For every $w \in W$, $\deg^-(w) = \deg^+(w) = \Delta$.*

Proof. Since $\tilde{\Delta}(D) \leq \Delta$, it is enough to show that $\deg^-(w), \deg^+(w) \geq \Delta$. Since D is Δ-critical, there is a $(\Delta - 1)$-dicolouring φ of $D - X$. Note that $\langle u, v\rangle$ is not a digon, otherwise X would be a biclique of D, a contradiction. Moreover, by Lemma 6.12, both u and v have at most three out-neighbours outside W. Therefore, φ can be extended to $V(D) \setminus W$ by assigning the same colour to u and v. Now, if there was a vertex $w \in W$ with $\min\{\deg^-(w), \deg^+(w)\} \leq \Delta - 1$, then, due to u and v having the same colour, we could extend φ to a $(\Delta - 1)$-dicolouring of D greedily, by taking care that w is coloured in the last place, a contradiction. ■

Let us denote by $w_1, \dots, w_{\Delta-2}$ the vertices in W, and, for $1 \leq i \leq \Delta - 2$, let w_i^- and w_i^+ be the unique in-neighbour and out-neighbour of w_i in $V(D) \setminus X$, given by Claim 6.14.1. We note that w_i^- and w_i^+ are not necessarily distinct, and that some (non-trivial) identities of the kind

$w_i^- = w_j^-$, $w_i^+ = w_j^+$, or $w_i^- = w_j^+$ may hold. Generalising the argument used in the proof of Claim 6.14.1, we actually obtain the following.

Claim 6.14.2. *Let φ be a $(\Delta-1)$-dicolouring of $D - X$. Then,*

(i) for every $1 \le i \le \Delta - 2$ there is a monochromatic directed path from w_i^+ to w_i^- in $D - X$ (possibly of length 0), and

(ii) $\varphi(w_1^-) = \varphi(w_1^+) = \ldots = \varphi(w_{\Delta-2}^-) = \varphi(w_{\Delta-2}^+)$.

Proof. Assume that for some $1 \le i \le \Delta-2$ property (i) does not hold. Then we can extend φ to a $(\Delta-1)$-dicolouring of D as follows. First, we assign the same colour to u and v (we have already justified in the proof of Claim 6.14.1 that this is possible), and then, we colour the vertices in $W \setminus \{w_i\}$ greedily. Finally, we assign to w_i a colour not assigned to any vertex in $X \setminus \{w_i\}$. This last step does not create any monochromatic directed cycle in $D[X]$, nor, by our assumption, any involving w_i and the vertices outside X. So φ can be extended to a $(\Delta-1)$-dicolouring of D, a contradiction.

Now let us see (ii). By (i) we already know that $\varphi(w_i^-) = \varphi(w_i^+)$ for every $1 \le i \le \Delta - 2$. Assume that $\varphi(w_i^+) \neq \varphi(w_j^+)$ for some $i \neq j$. Then we can extend φ to a $(\Delta-1)$-dicolouring of D as follows. First, we assign the same colour to u and v, with the additional requirement that this colour is different from $\varphi(w_j^+)$. The justification that this can be done without creating monochromatic directed cycles is the same as before. Then, we assign to w_i the colour $\varphi(w_j^+)$. By assumption, this does not create any monochromatic directed cycle. We can then colour the vertices in $W \setminus \{w_j\}$ greedily, because they have two out-neighbours sharing a colour and one uncoloured out-neighbour. Finally, w_j has two pairs of out-neighbours sharing a colour, namely, u, v and w_i, w_j^+, so w_j can also be coloured greedily using one of the original $\Delta - 1$ colours, a contradiction. ■

The point is that now some structure is forced.

Claim 6.14.3. *Let $1 \le i, j \le \Delta - 2$.*

(i) If $w_i^- \neq w_j^+$ and $w_j^- \neq w_i^+$, then the digraph D' defined as $D' = D - X \cup \{w_i^- w_j^+, w_j^- w_i^+\}$ contains a copy of $\overleftrightarrow{K}_\Delta$ or of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.

(ii) If $w_i^- \neq w_j^+$ and $w_j^- = w_i^+$, then the digraph $D - X \cup \{w_i^- w_j^+\}$ contains a copy of $\overleftrightarrow{K}_\Delta$ or of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$.

Proof. We prove (i); (ii) can be proved analogously. Let us begin by showing that $\vec{\chi}(D') \ge \Delta$. Assume for a contradiction that there is a $(\Delta-1)$-dicolouring φ of D'. Since φ is also a dicolouring of $D - X$, by Claim 6.14.2 we can find directed paths from w_i^+ to w_i^- and from w_j^+ to w_j^-, in which all the vertices have been assigned the colour $\varphi(w_i^+)$. Together with the arcs

$w_i^- w_j^+$ and $w_j^- w_i^+$, this yields a monochromatic directed cycle in D'. Hence, $\vec{\chi}(D') \geq \Delta$.

By construction, observe that both inequalities $\deg^-_{D'}(x) \leq \deg^-_D(x)$ and $\deg^+_{D'}(x) \leq \deg^+_D(x)$ hold for all vertices $x \in V(D')$. Therefore, D' satisfies that $\vec{\chi}(D') \geq \Delta \geq \tilde{\Delta}(D')$. The choice of D implies that D' contains a copy of $\overleftrightarrow{K}_\Delta$ or of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$. ■

As a consequence, we obtain the following.

Claim 6.14.4. *Let $1 \leq i,j \leq \Delta - 2$ such that $w_i^- \neq w_j^+$. Then, either $N^\pm(w_i^-) \cap N^\pm(w_j^+)$ contains a biclique of $D - X$ of size $\Delta - 4$, or $w_j^- \neq w_i^+$ and $N^\pm(w_j^-) \cap N^\pm(w_i^+)$ contains such a biclique.*

Proof. Let K be the copy of $\overleftrightarrow{K}_\Delta$ or of $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$ found by (*i*) or (*ii*) of Claim 6.14.3. Recall that K cannot be a subgraph of D, so it uses at least one arc of $\{w_i^- w_j^+, w_j^- w_i^+\}$. Assume that $w_i^- w_j^+ \in A(K)$, the other case being symmetric.

Note that, in K, for each pair of vertices $\{x, y\}$, there exists a biclique of size $\Delta - 3$ in $N^\pm_{D'}(x) \cap N^\pm_{D'}(y)$. In particular K contains a biclique Y_{ij} of size $\Delta - 3$ in $N^\pm_{D'}(w_i^-) \cap N^\pm_{D'}(w_j^+)$. Note that Y_{ij} is not necessarily a biclique of D as, in K, it may contain the arc $w_j^- w_i^+$. However, either $Y_{ij} \setminus \{w_j^-\}$ or $Y_{ij} \setminus \{w_i^+\}$ is a biclique of D in $N^\pm_D(w_i^-) \cap N^\pm_D(w_j^+)$. The result follows. ■

With the next claim, we make sure that Claim 6.14.4 can be used.

Claim 6.14.5. *There exist $1 \leq i,j \leq \Delta - 2$ such that $w_i^- \neq w_j^+$.*

Proof. Suppose that the statement is not true. Then, there is a vertex w such that $w_i^- = w_i^+ = w$ for every $1 \leq i \leq \Delta - 2$. In other words, w forms a digon with every vertex of W, just as u and v. Observe that the subgraph of D induced by $\{u, v, w\}$ is acyclic. Indeed, if $D[\{u, v, w\}]$ had a digon, the vertices of the digon together with W would form a biclique of D of size Δ. And if, otherwise, $D[\{u, v, w\}]$ was a copy of $\vec{C}_3$, then $D[X \cup \{w\}]$ would be isomorphic to $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$, a contradiction.

Now, by the Δ-criticality of D, there is a $(\Delta-1)$-dicolouring of $D - X - w$. Since the set $\{u, v, w\}$ is acyclic (and because, by Lemma 6.12, u, v and w have at most 3 out-neighbours outside W each), φ can be extended to a $(\Delta - 1)$-dicolouring of $D - W$ by assigning the same colour to u, v, and w. Then, by Claim 6.14.1, this can be extended to a $(\Delta - 1)$-dicolouring of D by colouring the vertices in W greedily, a contradiction. ■

The final claim shows the existence of vertices with in-degree and out-degree at least $\Delta + 2$. This contradicts Lemma 6.12, thus completing the proof of Lemma 6.14.

Claim 6.14.6. *There exists some $1 \leq i \leq \Delta - 2$ and two sets of vertices $Z^-, Z^+ \subseteq V(D)$ such that*

(*i*) $|Z^-| + |Z^+| \geq 11$;

(*ii*) $w_i^- \in Z^-$ *and, unless* Z^+ *is empty,* $w_i^+ \in Z^+$;

(*iii*) *every vertex in* $Z^- \cup Z^+$ *is of the form* w_j^- *or* w_j^+ *for some* $1 \leq j \leq \Delta - 2$.

(*iv*) *for every* $z \in Z^- \setminus \{w_i^-\}$, $N^\pm(w_i^-) \cap N^\pm(z)$ *contains a biclique* Y_z^- *of* $D - X$ *of size* $\Delta - 4$;

(*v*) *for every* $z \in Z^+ \setminus \{w_i^+\}$, $N^\pm(w_i^+) \cap N^\pm(z)$ *contains a biclique* Y_z^+ *of* $D - X$ *of size* $\Delta - 4$.

Proof. The proof is by induction on $|Z^-| + |Z^+|$. Let us prove first that the statement holds if (*i*) is replaced by $|Z^-| \geq 2$. By Claims 6.14.4 and 6.14.5, it is enough to take $Z^- = \{w_i^-, w_j^+\}$ and $Z^+ = \emptyset$ for some adequate $1 \leq i, j \leq \Delta - 2$.

Let us now assume that the statement holds if (*i*) is replaced by $|Z^-| + |Z^+| \geq t$, with $2 \leq t \leq 10$, and prove that it is also true with $|Z^-| + |Z^+| \geq t + 1$. By Lemma 6.12, each vertex $z \in Z^- \cup Z^+$ has at most five in-neighbours and five out-neighbours in W. Since Δ is large, there exists some $1 \leq k \leq \Delta - 2$ with $w_k^-, w_k^+ \notin Z^- \cup Z^+$. We apply Claim 6.14.4 to w_i^- and w_k^+. There are two possible outcomes. If $N^\pm(w_i^-) \cap N^\pm(w_k^+)$ contains a biclique of $D - X$ of size $\Delta - 4$, then we simply add w_k^+ to Z^-. Else, $N^\pm(w_k^-) \cap N^\pm(w_i^+)$ contains a biclique of $D - X$ of size $\Delta - 4$, and we add w_k^- to Z^+, together with w_i^+ if it was not already in Z^+. ■

Let $1 \leq i \leq \Delta - 2$ and $Z^-, Z^+ \subseteq V(D)$ given by Claim 6.14.6. Let us assume that $|Z^-| \geq 6$; if $|Z^+| \geq 6$, the proof goes the same way. We can further assume that Z^- has exactly 6 elements; otherwise, we simply truncate it. Observe that, for any $z \in Z^- \setminus \{w_i^-\}$, the biclique Y_z^- given by Claim 6.14.6 is disjoint from Z^-: indeed, by Claims 6.14.2 and 6.14.6, any $(\Delta - 1)$-dicolouring of $D - X$ is constant on Z^-, and z forms a digon with every vertex of Y_z^-. Now, by Lemma 6.12, for any $z \in Z^- \setminus \{w_i^-\}$, $|N^\pm(w_i^-) \setminus Y_z^-| \leq 4$. Therefore,

$$\left| \bigcap_{z \in Z^- \setminus \{w_i^-\}} Y_z^- \right| \geq (\Delta - 4) - 4(|Z^-| - 2) > 0.$$

Let thus $y \in \cap_{z \in Z^- \setminus \{w_i^-\}} Y_z^-$ and $z \in Z^- \setminus \{w_i^-\}$. Note that y forms a digon with every vertex in $Y_z^- \uplus Z^- \setminus \{y\}$. Thus $|N^\pm(y)| \geq \Delta + 1$, a contradiction to $\tilde{\Delta}(D) \leq \Delta$. □

With the same type of arguments, we now forbid copies of $\vec{C_3} \boxplus \overleftrightarrow{K}_{\Delta-3}$ in D.

Lemma 6.15. *The digraph D does not contain any copy of $\vec{C_3} \boxplus \overleftrightarrow{K}_{\Delta-3}$.*

Proof. Let us assume for a contradiction that there exist two disjoint sets $\{x, y, z\}, W \subseteq V(D)$ such that $D[\{x, y, z\}]$ contains a 3-cycle, W is a biclique of size $\Delta - 3$, and D contains all possible digons between $\{x, y, z\}$ and W. We let $X = \{x, y, z\} \cup W$. Note that, by Lemma 6.14, $D[\{x, y, z\}]$ contains at most one digon. We thus assume without loss of generality that both xy and yz are simple arcs.

Claim 6.15.1. *For every $w \in W$, $\deg^-(w) = \deg^+(w) = \Delta$.*

Proof. The proof is similar to that of Claim 6.14.1. By Lemma 6.12, it is enough to show that $\deg^-(w), \deg^+(w) \geq \Delta$. Let φ be a $(\Delta-1)$-dicolouring of $D - X$. We extend φ to $V(D) \setminus W$ by assigning the same colour to x and y. Now, if there was a vertex $w \in W$ with $\min\{\deg^-(w), \deg^+(w)\} \leq \Delta - 1$, then, due to x and y having the same colour, we could extend φ to a $(\Delta-1)$-dicolouring of D greedily, by taking care that w is coloured in the last place, a contradiction. ■

We label $w_1, \ldots, w_{\Delta-3}$ the vertices of W, and for every $i \in [\Delta - 3]$, we denote by w_i^- and w_i^+ the unique in- and out-neighbour of w_i outside X. Following exactly the proof of Claim 6.14.2, we obtain the following. We give the details for completeness.

Claim 6.15.2. *Let φ be a $(\Delta - 1)$-dicolouring of $D - X$. Then,*

(i) for every $1 \leq i \leq \Delta - 3$ there is a monochromatic directed path from w_i^+ to w_i^- in $D - X$ (possibly of length 0), and

(ii) $\varphi(w_1^-) = \varphi(w_1^+) = \ldots = \varphi(w_{\Delta-3}^-) = \varphi(w_{\Delta-3}^+)$.

Proof. If *(i)* does not hold for some $i \in [\Delta - 3]$, then we can extend φ to a $(\Delta - 1)$-dicolouring of D as follows. First, we assign a same colour to x and y, and then, we colour the vertices in $\{z\} \cup W \setminus \{w_i\}$ greedily, starting with z. Finally, we assign to w_i a colour not assigned to any vertex in $X \setminus \{w_i\}$.

For *(ii)*, assume that $\varphi(w_i^+) \neq \varphi(w_j^+)$ for some $i \neq j$. Then we can extend φ to a $(\Delta-1)$-dicolouring of D as follows. First, we assign the same colour c_1 to x and y, and a colour c_2 to z, with the additional requirement that both c_1 and c_2 are different from $\varphi(w_j^+)$. Then, we assign to w_i the colour $\varphi(w_j^+)$. We can then colour the vertices in $W \setminus \{w_j\}$ greedily. At the end, w_j has two pairs of out-neighbours sharing a colour, namely, x, y and w_i, w_j^+, so w_j can also be coloured greedily, a contradiction. ■

Actually, all vertices w_i^-, w_i^+ must be the same.

Claim 6.15.3. $w_1^- = w_1^+ = \ldots = w_{\Delta-3}^- = w_{\Delta-3}^+$

Proof. Assume first that, for some $i \in [\Delta - 3]$, $w_i^- \neq w_i^+$. Let D' be the digraph obtained from $D - X$ by further adding the arc $w_i^- w_i^+$. Note that $\tilde{\Delta}(D) \geq \tilde{\Delta}(D')$, and that D' does not contain any copy of $\overleftrightarrow{K}_\Delta$ nor of $\vec{C_3} \boxplus \overleftrightarrow{K}_{\Delta-2}$ by Lemma 6.14. Therefore, the minimality of $|V(D)| + |A(D)|$ implies the existence of a $(\Delta - 1)$-dicolouring of D', which by construction is a dicolouring of $D - X$ without any monochromatic directed path from w_i^+ to w_i^-, a contradiction to Claim 6.15.2.

Assume then that, for some i, j, $w_i^- = w_i^+ \neq w_j^- = w_j^+$. Similarly, let D' be the digraph obtained from $D - X$ by further adding the digon $\langle w_i^-, w_j^- \rangle$. Since $w_i^- = w_i^+$ and $w_j^- = w_j^+$, we have $\tilde{\Delta}(D) \geq \tilde{\Delta}(D')$, and D' does not contain any copy of $\overleftrightarrow{K}_\Delta$ nor of $\vec{C_3} \boxplus \overleftrightarrow{K}_{\Delta-2}$ by Lemma 6.14. Therefore, the minimality of $|V(D)| + |A(D)|$ implies the existence of a $(\Delta - 1)$-dicolouring φ of D', which by construction is a dicolouring of $D - X$ satisfying $\varphi(w_i^-) \neq \varphi(w_j^-)$, a contradiction to Claim 6.15.2. ■

Let $w \in V(D) \setminus X$ be such that $W \subseteq N^{\pm}(w)$. Note that D contains all possible digons between W and $\{x, y, z, w\}$. By Lemma 6.14, we get that, in $D[\{x, y, z, w\}]$, no vertex is incident to two distinct digons. It is straightforward to check that, in particular, $\overline{D}[\{x, y, z, w\}]$ admits a matching $\{x_1 y_1, x_2 y_2\}$.

Let φ be a $(\Delta - 1)$-dicolouring of $D - (X \cup \{w\})$. We can extend φ to D as follows. We first give a same colour c_1 to both x_1 and y_1 that is not appearing in $N^+(x_1) \cup N^+(y_1)$, and similarly a same colour c_2 to both x_2 and y_2. Now, two colours are repeated in the out- and in-neighbourhoods of every vertex of W, so φ can be greedily extended to D, yielding the contradiction. □

For the next lemma, we need a few specific definitions. We say that $X \subseteq V(D)$ is a *quasi-biclique* of D if one of the following holds:

(1) X is a biclique;

(2) $D[X] \cup \{xy\} \cong \overleftrightarrow{K}_{|X|}$ for some $x, y \in X$;

(3) $D[X] \cong \vec{C_3} \boxplus \overleftrightarrow{K}_{|X|-3}$;

(4) $D[X] - x \cong \overleftrightarrow{K}_{|X|-1}$ and $|N^+(x) \cap X| \geq \frac{99}{100}\Delta$ hold for some $x \in X$; or

(5) X admits a partition (R, W, K) such that $|R| \leq 4$, $|K| > \frac{1}{2}|X|$, $W \cup K$ is a biclique, vertices in K have in-degree exactly $\Delta + 1$, and R *dominates* K, that is, D has all possible arcs leaving R and entering K.

For i $\in$ [5], we say that X is a *quasi-biclique of type (i)* if X satisfies property (i) above and none of the properties (j) for j < i.

In what follows, by applying the dense decomposition lemma to D, we show that D is made of a collection of pairwise disjoint quasi-bicliques and a set of sparse vertices. Recall that d was set to $\ln^3 \Delta$ at the beginning of the section.

Lemma 6.16. *There is a partition $X_1 \uplus \dots \uplus X_t \uplus S$ of $V(D)$ such that:*

(*i*) *for every $i \in [t]$, X_i is a quasi-biclique of D of size $|X_i| > \Delta - 300d$;*

(*ii*) *for every $i \in [t]$ and $u \in V(D) \setminus X_i$, $|N^+(u) \cap X_i| < \frac{100}{101}\Delta$;*

(*iii*) *vertices in S are d-sparse.*

Proof. From now on, we denote $\Delta_{\max}(D)$ by $\Delta_{\max}$ for conciseness. Recall that, by Lemma 6.12, $\Delta \leq \Delta_{\max} \leq \Delta + 1$. We apply Lemma 6.9 with $\varepsilon = 1/100$ and $d = \ln^3 \Delta$. We thus obtain a partition $X_1 \uplus \dots \uplus X_t \uplus S$ of $V(D)$ such that:

- for every $i \in [t]$, $|X_i| > \Delta - 300d$;
- for every $i \in [t]$ and every $u \in V(D)$, $u \in X_i$ if and only if $|N^+(u) \cap X_i| \geq \frac{99}{100}\Delta_{\max}$; and
- vertices in S are d-sparse.

In particular, the second property implies that vertices $u \in V(D) \setminus X_i$ satisfy $|N^+(u) \cap X_i| < \frac{99}{100}(\Delta + 1) < \frac{100}{101}\Delta$. Clearly, it thus remains to prove that each X_i is a quasi-biclique. We thus fix $i \in [t]$, and denote $D[X_i]$ by D_i. To prove that X_i is a quasi-biclique, our strategy is first to show that the complement $\overline{D_i}$ of D_i has no matching of size 3. We first prove that $\nu(\overline{D_i})$ is indeed at most a small fraction of Δ.

Claim 6.16.1. $\nu(\overline{D_i}) < \dfrac{1}{50}\Delta_{\max}$.

Proof. We assume for a contradiction that $\nu(\overline{D_i}) \geq \Delta_{\max}/50$, and let $M = \{u_1v_1, \dots, u_\ell v_\ell\}$ be a matching of $\overline{D_i}$ of size exactly $\ell = \lceil \Delta_{\max}/50 \rceil$. Note that, for every $j \in [\ell]$, there is at most one arc between u_j and v_j in D.

Let φ be a $(\Delta - 1)$-dicolouring of $D - X_i$, which exists as D is Δ-critical. We will show that φ can be extended into a $(\Delta - 1)$-dicolouring of D, thus yielding a contradiction. We first colour the vertices covered by M by sequentially choosing, for every $j \in [\ell]$, a common colour $c_j \in [\Delta - 1]$ for both u_j and v_j that is not already appearing in $N^+(u_j) \cup N^+(v_j)$. Note that this does not create any monochromatic directed cycle as there exists at most one arc between u_j and v_j. Let us show that such a colour c_j exists. By Lemma 6.9(*iii*), both u_j and v_j have at most $\Delta_{\max}/100$ out-neighbours in $V(D) \setminus X_i$. Moreover, at most $j-1$ colours appear in X_i when colouring $\{u_j, v_j\}$ (namely $c_1, \dots, c_{j-1}$). Hence, the number of colours that appear in $N^+(u_j) \cup N^+(v_j)$ is at most

$$\frac{1}{50}\Delta_{\max} + \ell - 1 \leq \frac{1}{25}\Delta_{\max} < \Delta - 1.$$

Once the vertices covered by M are coloured, we colour the remaining vertices of X_i greedily, by choosing for each of them a colour of $[\Delta - 1]$ that is not appearing in its out-neighbourhood. We claim that this is always possible, so let us assume for a contradiction that, for some $x \in X_i \setminus V(M)$, all $\Delta - 1$ colours appear in $N^+(x)$ when x is considered. Let $\iota(x)$ be the number of indices j such that $\{u_j, v_j\} \subseteq N^+(x)$. Since any two vertices matched in M received the same colour, the number of colours appearing in $N^+(x)$ is at most $\deg^+(x) - \iota(x)$. We thus have $\deg^+(x) - \iota(x) \geq \Delta - 1$, which together with Lemma 6.12 implies $\iota(x) \leq 2$. Therefore, at least $\ell - 2$ vertices covered by M do not belong to $N^+(x)$. Together with Lemma 6.9(i), this implies

$$|N^+(x) \cap X_i| \leq |X_i| - \ell + 2 < \frac{98}{100}\Delta_{\max} + 4d + 4 < \frac{99}{100}\Delta_{\max}.$$

By Lemma 6.9(iii), x does not belong to X_i, a contradiction. The claim follows. ■

We can now prove that, indeed, $\nu(\overline{D_i}) \leq 2$.

Claim 6.16.2. $\nu(\overline{D_i}) \leq 2$.

Proof. Let $M = \{u_1v_1, \dots, u_{|M|}v_{|M|}\}$ be a maximum matching of $\overline{D_i}$, and assume for a contradiction that $|M| \geq 3$. By Claim 6.16.1, $|M| < \frac{1}{50}\Delta_{\max}$. We denote by Y the vertices of X_i that are not covered by M. Note that, by maximality of M, Y is a biclique of D. We define Z as the set of vertices in Y that are out-neighbours of all vertices $u_1, u_2, u_3, v_1, v_2, v_3$, so formally

$$Z = Y \cap \bigcap_{j=1}^{3} N^+(u_j) \cap \bigcap_{j=1}^{3} N^+(v_j).$$

Subclaim 6.16.2.1. $|Z| \geq 3$.

Proof. Let u be any vertex covered by M. Recall that, by Lemma 6.9(iii),

$$|N^+(u) \cap X_i| \geq \frac{99}{100}\Delta_{\max}.$$

Together with Claim 6.16.1, this implies that $|N^+(u) \cap Y| \geq \frac{95}{100}\Delta_{\max}$. Since, by Lemma 6.9($i$), $|Y| < |X_i| \leq \Delta_{\max} + 1 + 4d$, we obtain

$$|Y \setminus N^+(u)| < \frac{5}{100}\Delta_{\max} + 4d + 1 < \frac{6}{100}\Delta_{\max}.$$

Since this holds for every u saturated by M, and in particular for every $u \in \{u_1, u_2, u_3, v_1, v_2, v_3\}$, we have

$$|Y \setminus Z| < \frac{36}{100}\Delta_{\max}. \tag{6.1}$$

Recall that the size of Y is larger than $|X_i| - \frac{4}{100}\Delta_{\max}$ by Claim 6.16.1, which together with Lemma 6.9(i) implies

$$|Y| > \frac{96}{100}\Delta_{\max} - 300d. \tag{6.2}$$

Combining (6.1) and (6.2), straightforward calculations imply that $|Z| > \frac{1}{2}\Delta_{\max} \geq 3$, as desired. ♦

We let φ be a $(\Delta - 1)$-dicolouring of $D - Y$, with the extra property that $\varphi(u_j) = \varphi(v_j)$ holds for every $j \in [|M|]$. We skip the proof that φ exists, it is shown by taking a $(\Delta - 1)$-dicolouring of $D - X_i$ and extending it to $D - Y$ by choosing, for every $j \in [|M|]$, a colour that is not appearing in $N^+(u_j) \cup N^+(v_j)$, exactly as in the proof of Claim 6.16.1.

We now prove that φ can be extended into a $(\Delta - 1)$-dicolouring of D, thus yielding the contradiction. We first greedily colour the vertices in $Y \setminus Z$ by choosing for each of them a colour that does not appear in its out-neighbourhood. Let us show that this is possible. Note that, since Y is a biclique, and because $|Z| \geq 3$ by Subclaim 6.16.2.1, when we consider $v \in Y \setminus Z$, it has at least three uncoloured neighbours (namely the vertices of Z). Since, by Lemma 6.12, $\deg^+(v) \leq \Delta + 1$, this implies that the number of colours appearing in $N^+(v)$ (when considered) is at most $\Delta - 2$.

We then extend the obtained dicolouring to D by choosing for every vertex $z \in Z$ a colour that does not appear in its in-neighbourhood. Let us show that, again, this is possible. Since all vertices $u_1, u_2, u_3, v_1, v_2, v_3$ belong to $N^-(z)$ (by definition of Z), and because $\varphi(u_j) = \varphi(v_j)$ for every $j \in [3]$, the number of colours appearing in the in-neighbourhood of z (when considered) is at most $\deg^-(z) - 3$, which is at most $\Delta - 2$ by Lemma 6.12. This yields a contradiction, and concludes the proof of Claim 6.16.2. ■

We will now prove that X_i is a quasi-biclique. We distinguish two cases, depending on the value of $\nu(\overline{D_i})$.

Case 1: $\nu(\overline{D_i}) \leq 1$.

If $\nu(\overline{D_i}) = 0$, then X_i satisfies (1). We thus assume that $\nu(\overline{D_i}) = 1$ and let u and v be two vertices with at most one arc between them. We observe first that, if $X_i \setminus \{u, v\} \subseteq N^{\pm}(u)$, then X_i satisfies (4), and X_i is a quasi-biclique. It can be thus assumed that there exists $x_u \in X_i \setminus \{u, v\}$ such that there is at most one arc between x_u and u. Similarly, we can assume that there exists $x_v \in X_i \setminus \{u, v\}$ such that there is at most one arc between x_v and v. Since $\nu(\overline{D_i}) = 1$, we must have $x_u = x_v$, and D contains every possible digon between $\{u, v, x_u\}$ and $X_i \setminus \{u, v, x_u\}$. Note that $D[\{u, v, x_u\}]$ does not contain any digon by construction. Hence, if it contains a directed cycle, it has size exactly 3, and X_i satisfies (3).

Assume now that $D[\{u, v, x_u\}]$ is acyclic and let us reach a contradiction. Let φ be any $(\Delta-1)$-dicolouring of $D - X_i$. We can extend φ to $\{u, v, x_u\}$ in such a way that $\varphi(u) = \varphi(v) = \varphi(x_u)$. Again, this is possible because $D[\{u, v, x_u\}]$ is acyclic, and because D contains all digons between $\{u, v, x_u\}$ and $X_i \setminus \{u, v, x_u\}$. Now, φ can be extended to the remaining vertices of X_i because each such vertex has one colour used three times in both its in- and out-neighbourhood.

Case 2: $\nu(\overline{D_i}) = 2$.

Let $\{u_1v_1, u_2v_2\}$ be a maximum matching of $\overline{D_i}$, $R = \{u_1, v_1, u_2, v_2\}$, and K be the set of vertices in $X_i \setminus R$ that are in the out-neighbourhood of all vertices in R, so formally

$$K = (X_i \setminus R) \cap \bigcap_{u \in R} N^+(u).$$

Note that, by definition, K is disjoint from R. Let $W = X_i \setminus (R \cup K)$. We are now going to show that the partition (R, W, K) of X_i satisfies (5). By construction, we have:

- $|R| = 4$,
- $W \cup K$ is a biclique (as M is maximum), and
- R dominates K.

It remains to show that $K > \frac{1}{2}|X_i|$ and that every vertex in K has in-degree $\Delta+1$. We first show the former. The proof is similar to that of Subclaim 6.16.2.1. Let y be any vertex in R. By Lemma 6.9(iii), $|N^+(y) \cap X_i| \geq \frac{99}{100}\Delta_{\max}$. Since, by Lemma 6.9($i$), $|X_i| < \Delta_{\max}+1+4d$, we have $|(X_i \setminus N^+(y)) \cup R| < \frac{1}{100}\Delta_{\max} + 4 + 4d < \frac{2}{100}\Delta_{\max}$. Since this holds for every $y \in R$, we have

$$|X_i \setminus K| < \frac{8}{100}\Delta_{\max}. \tag{6.3}$$

Recall that $\Delta_{\max} - 300d < |X_i| < \Delta_{\max} + 4d + 1$ by Lemma 6.9(i). Hence, (6.3) implies $|K| > \frac{1}{2}|X_i|$, as desired.

To prove that every vertex $x \in K$ has in-degree at least $\Delta+1$, assume for a contradiction that $\deg^-(x) \leq \Delta$ for some $x \in K$. Let φ be a $(\Delta-1)$-dicolouring of $D - (K \cup W)$, with the extra property that $\varphi(u_1) = \varphi(v_1)$ and $\varphi(u_2) = \varphi(v_2)$.

Let y be any vertex in K distinct from x. Since two colours (namely $\varphi(u_1)$ and $\varphi(u_2)$) are repeated in the in-neighbourhood of both x and y, and because $d^-(x) \leq \Delta$, one can extend φ to D by first colouring the vertices in $K \cup W \setminus \{x, y\}$ (choosing for each of them a colour that is not appearing either in its in- or out-neighbourhood), and then colouring y and x in this order. This yields the desired contradiction, shows that X_i is a biclique of type (5), and concludes the proof of Lemma 6.16. □

From now on, we fix a decomposition $X_1 \uplus \ldots \uplus X_t \uplus S$ of $V(D)$ satisfying the properties of Lemma 6.16. For the next lemma, we need a few additional technical definitions. Let X be a quasi-biclique of D, and let r be any positive real number. We define $\mathscr{Z}_r(X)$ as the set of vertices in $V(D) \setminus X$ with less than r in- and out-neighbours in X, that is,

$$\mathscr{Z}_r(X) = \left\{ z \in V(D) \setminus X \;\middle|\; \max\left\{ |N^-(z) \cap X|, |N^+(z) \cap X| \right\} < r \right\}.$$

The *saving part* $\mathscr{S}(X)$ of X is defined as

$$\mathscr{S}(X) = \begin{cases} X & \text{if } X \text{ is of type (1)} \\ X \setminus \{x, y\} & \text{if } X \text{ is of type (2)} \\ X \setminus T & \text{if } X \text{ is of type (3)} \\ X \setminus \{x\} & \text{if } X \text{ is of type (4)} \\ K & \text{if } X \text{ is of type (5),} \end{cases}$$

where x, y and K are as in the definition of quasi-biclique, and T corresponds to $V(\vec{C}_3)$ in that definition. We say that a vertex $x \in \mathscr{S}(X)$ is an *r-saviour* of X if one of the following holds:

(a) $\deg^+(x) = \Delta - 1$ and $\mathscr{Z}_r(X) \cap N^+(x) \neq \emptyset$;

(b) $\deg^-(x) = \Delta - 1$ and $\mathscr{Z}_r(X) \cap N^-(x) \neq \emptyset$;

(c) $\deg^+(x) \leq \Delta$ and $|\mathscr{Z}_r(X) \cap N^+(x)| \geq 2$; or

(d) $\deg^-(x) \leq \Delta$ and $|\mathscr{Z}_r(X) \cap N^-(x)| \geq 2$.

Intuitively speaking, if r is small enough compared to Δ, when applying the pseudo-random colouring process of the next subsection, a saviour will have a quite high probability to be uncoloured and to have either two repeated colours in its out- or in-neighbourhood (if (c) or (d) holds) or only one repeated colour but one missing arc (if (a) or (b) holds). If this property holds for at least three saviours of X, and if these saviours are in the out-neighbourhood of every vertex of X, we will be able to extend a partial dicolouring of D to X.

The following two lemmas say that, in a minimum counterexample to Theorem 6.8, large quasi-bicliques indeed have many saviours. From now on, we fix $r = \ln^4 \Delta$.

Lemma 6.17. *For all $i \in [t]$, if X_i is of type (5) with partition (R_i, K_i, W_i), then the elements of K_i are r-saviours.*

Proof. We fix $i \in [t]$ such that X_i is of type (5). For the sake of conciseness, we respectively denote X_i, R_i, K_i, and W_i by X, R, K, and W. We further denote $\mathscr{S}(X)$ by $\mathscr{S}$ and $\mathscr{Z}_r(X)$ by $\mathscr{Z}_r$. Recall that, by definition, $|R| \leq 4$, R dominates K, $K \cup W$ is a biclique, $|K| > \frac{1}{2}|X|$, and vertices in K have in-degree $\Delta + 1$. We claim that all vertices in K satisfy (a), thus implying the statement.

Assume for a contradiction that $s \in K$ does not satisfy (a). By definition of Δ and because $\deg^-(s) = \Delta + 1$, we have $\deg^+(s) = \Delta - 1$. Since $\deg^-(s) \neq \deg^+(s)$, s is incident to a simple arc, and by Lemma 6.13 we know that s has a simple out-neighbour y. Since R dominates K, and $K \cup W$ is a biclique, we thus necessarily have $y \in V(D) \setminus X$. As s does not satisfy (a), $y \notin \mathscr{Z}_r$, that is, y has at least r out-neighbours or r in-neighbours in X. Let u_y be any vertex in $K \cup W \setminus N^+(y)$ distinct from s. Note that u_y exists because $y \notin X$, so $|N^+(y) \cap X| < \frac{100}{101}\Delta$, while $|K \cup W| = |X| - |R| > \Delta - 300d - 4$.

Let φ be a $(\Delta - 1)$-dicolouring of $D - (X \cup \{y\})$. We claim that there exists a colour c that is not appearing either in $N^-(y) \cup N^-(u_y)$ or in $N^+(y) \cup N^+(u_y)$. To see this, assume that y has r out-neighbours in X. Then y has r uncoloured out-neighbours, and u_y has at least $|K \cup W| - 1 \geq |X| - 5$ uncoloured out-neighbours. The total number of colours appearing in $N^+(y) \cup N^+(u_y)$ is thus at most

$$(\Delta + 1) - r + (\Delta + 1) - (\Delta - 301d) = \Delta - \ln^4 \Delta + 301 \ln^3 \Delta + 2,$$

which is less than $\Delta - 1$ when Δ is large enough. Similarly, if y has r in-neighbours in X, then at most $\Delta + 1 - r$ colours appear in its in-neighbourhood, while at most $301d$ colours appear in the in-neighbourhood of u_y.

We thus extend φ by first choosing colour c for both y and u_y. Then, we greedily colour the vertices of R and W in this order. This is possible because K is large, uncoloured, and in the out-neighbourhood of all vertices of X. We then colour the vertices of K, keeping s for the end. Since the vertices in $K \setminus \{s\}$ have out-degree $\Delta - 1$ and an uncoloured out-neighbour, this is always possible for them. For s, this is possible because $\deg^+(s) = \Delta - 1$ and one colour (namely c) is repeated in its out-neighbourhood. This yields a contradiction. □

Lemma 6.18. *For every $i \in [t]$, if X_i is of type (1)–(4), it has at least $\frac{1}{2}|X_i|$ r-saviours.*

Proof. We fix $i \in [t]$ and assume that X_i is of type (1)–(4). Using ideas analogous to the one used in the proof of Lemma 6.17, we first show a collection of structural results around non-r-saviour vertices. We then leverage these to show that many vertices must indeed be r-saviours.

For the sake of conciseness, again, we denote X_i by X and $\mathscr{S}(X)$ by $\mathscr{S}$. Similarly, for every r', we denote $\mathscr{Z}_{r'}(X)$ by $\mathscr{Z}_{r'}$. We further denote by $\mathscr{R}$ the set of r-saviours of X. Finally, we let Y be $V(D) \setminus X$, unless X is of type (4), in which case Y is $(V(D) \setminus X) \cup \{x\}$, where x is the special vertex witnessing that X is of type (4).

Along the proof, we will use many times that $|\mathscr{S}| \geq |X| - 3$ and that every vertex $s \in \mathscr{S}$ is linked with digons to all vertices in $X \setminus (Y \cup \{s\})$. Note that these two properties hold because X is of type (1)–(4). We further have the following.

Claim 6.18.1. *Every vertex $s \in \mathscr{S}$ satisfies $|N(s) \cap (X \setminus Y)| \leq \Delta - 2$.*

Proof. If X is of type (1) or (4), this is because D does not contain any biclique of size Δ. Else, if X is of type (2), this is a consequence of Lemma 6.14. Finally, if X is of type (3), this is a consequence of Lemma 6.15. ■

Intuitively, and similarly to the proof of Lemma 6.17, when some vertex $y \in Y$ has many neighbours in $X \setminus Y$, we mainly want to argue that we can extend a $(\Delta - 1)$-dicolouring of $D - (X \cup \{y\})$ by first colouring y and a vertex $u_y \in X \setminus Y$ with a same colour, and then greedily colour the remaining vertices in X in a specific order. The first step consists of showing that, actually, we can find such a vertex u_y that is not linked with a digon to y.

Claim 6.18.2. *Let $s \in \mathscr{S}$ and let $y \in Y$ be a neighbour of s. There exists a vertex $u_y \in X \setminus Y$, distinct from s, such that $\langle y, u_y \rangle$ is not a digon of D. Moreover, if $y_1, y_2 \in Y$ are two distinct neighbours of s, then there exist distinct vertices $u_1, u_2 \in X \setminus Y$, both distinct from s, such that none of $\{\langle y_1, u_1 \rangle, \langle y_2, u_2 \rangle\}$ is a digon of D.*

Proof. We first prove the first part of the claim. Assume that this is not the case, so y is linked with a digon to all vertices in $X \setminus (Y \cup \{s\})$. Since vertices outside X have less than $\frac{100}{101}\Delta$ out-neighbours in X, and because $|X| > \Delta - 300d$, we have that $y \in X$. Since y is chosen in Y, it means that $Y \cap X \neq \emptyset$; hence X is of type (4), and y is the special vertex such that $X \setminus \{y\}$ is a biclique of D. Since y is a neighbour of s, adding at most one arc between y and s to $D[X]$ yields a complete digraph. Hence X is either of type (1) or (2), a contradiction (recall that X being of type (i) implies that X is not of type (j) for any j < i).

To show the second part, note that at most one of $\{y_1, y_2\}$ belongs to X, so assume by symmetry that $y_2 \notin X$. We apply the first part of the statement to y_1 and thus find u_1. Since $y_2 \notin X$, we can let u_2 be any vertex of $X \setminus (N^+(y_2) \cup Y \cup \{s, u_1\})$, which is non-empty as y_2 has less than $\frac{100}{101}\Delta$ out-neighbours in X, and $|X| > \Delta - 300d$. ■

From now on, let us show some structure around vertices in $\mathscr{S} \setminus \mathscr{R}$, i.e. non-$r$-saviours of $\mathscr{S}$.

Claim 6.18.3. *Every vertex $s \in \mathscr{S} \setminus \mathscr{R}$ satisfies* $\deg^-(s) = \deg^+(s) = \Delta$.

Proof. Assume for a contradiction that $\min\{\deg^+(s), \deg^-(s)\} = \Delta - 1$. The proof is the same as that of Lemma 6.17, but we give it for completeness.

If $\deg^+(s) = \Delta - 1$, we let y be any out-neighbour of s in Y, otherwise $\deg^-(s) = \Delta - 1$ and we let y be any in-neighbour of s in Y. In both cases, y exists by Claim 6.18.1. Since $s \notin \mathscr{R}$, we know that $y \notin \mathscr{Z}_r$, for otherwise s would satisfy either (a) or (b). By definition of $\mathscr{Z}_r$, either $y \in X$, $|N^-(y) \cap X| \geq r$, or $|N^+(y) \cap X| \geq r$. If $y \in X$, then X being a quasi-biclique implies $|N^+(y) \cap X| \geq r$, so we can ignore this case. Let us thus assume that $|N^+(y) \cap X| \geq r$ or $|N^-(y) \cap X| \geq r$.

Let u_y be any vertex in $X \setminus (\{s\} \cup Y)$ such that $\langle y, u_y \rangle$ is not a digon of D, which exists by Claim 6.18.2. Let v be any vertex in $N(y) \cap \mathscr{S} \setminus \{u_y, s\}$, which exists as $|\mathscr{S}| \geq |X| - 3$.

Let φ be a $(\Delta-1)$-dicolouring of $D - (X \cup \{y\})$. We greedily extend φ to D, thus yielding the contradiction. We first give a same colour c to both y and u_y that is not appearing in $N^+(y) \cup N^+(u_y)$ or $N^-(y) \cup N^-(u_y)$. This is possible because y has at least r uncoloured in- or out-neighbours and u_y is linked with digons to at least $|X| - 2 > \Delta - 301d$ uncoloured vertices. We then greedily colour the vertices of X, starting with the possible vertex in $Y \cap X$, and keeping v and s for the end (in this order). Since both v and s are linked with digons to all vertices in $X \setminus Y \setminus \{v, s\}$, it is possible to choose for each of them. For v, if $\deg^+(v) = \Delta - 1$ or $\deg^-(v) = \Delta - 1$, then we can pick a colour that is not appearing in $N^+(v)$ or in $N^-(v)$ as $\langle v, s \rangle$ is a digon and s is uncoloured. Else, $\deg^-(v) = \deg^+(v) = \Delta$, s is an uncoloured in-neighbour of v, and v has a repeated colour (namely c) in its in- or out-neighbourhood. We can thus choose a colour that is not appearing in $N^-(v)$ or in $N^+(v)$. We finally consider s, for which either $\deg^+(s) = \Delta - 1$ and one colour (namely c) is repeated in $N^+(s)$, or $\deg^-(s) = \Delta - 1$ and c is repeated in $N^-(s)$. In both cases, we can extend φ to D, a contradiction. ∎

For vertices s with in-degree and out-degree Δ, the argument used above does not work anymore, as we can only save one colour for s. However, if two neighbours y_1, y_2 outside $X \setminus Y$ indeed have a lot of neighbours in X, then we can manage to save two colours for s, and the argument works.

Claim 6.18.4. *For every vertex $s \in \mathscr{S}$ with* $\max\{\deg^+(s), \deg^-(s)\} \leq \Delta$,

$$|N^+(s) \cap Y \setminus \mathscr{Z}_{r-3}| \leq 1 \quad \textit{and} \quad |N^-(s) \cap Y \setminus \mathscr{Z}_{r-3}| \leq 1.$$

Proof. Assume for a contradiction that some $s \in \mathscr{S}$ with $\deg^+(s) \leq \Delta$ has two distinct out-neighbours $y_1, y_2 \in Y$, both of them having at least $r - 3$ in- or out-neighbours in X. The case of s having $\deg^-(s) \leq \Delta$ and two in-neighbours $y_1, y_2 \in Y$, none of them belonging to $\mathscr{Z}_{r-3}$, is proved analogously.

Let φ be any $(\Delta-1)$-dicolouring of $D-(X\cup\{y_1,y_2\})$. We will show that φ extends to D, thus yielding the contradiction. By Claim 6.18.2, we let u_1 and u_2 be two distinct vertices of $X \setminus Y$, both distinct from s, such that none of $\{\langle y_1, u_1\rangle, \langle y_2, u_2\rangle\}$ is a digon of D. We finally let v be any vertex in $N(y_1)\cap\mathscr{S}$ distinct from u_1, u_2, and s. Note that we excluded the case of X being of type (5), so $|\mathscr{S}| \geq |X|-3$, and v exists as $|N(y_1)\cap\mathscr{S}\setminus\{u_1,u_2,s\}| \geq r-9 > 0$. Observe that, by definition, $\langle s, v\rangle$ is a digon of D, and that D contains all possible digons between $\{s, v\}$ and $\{u_1, u_2\}$.

We first extend φ by choosing for y_1 and u_1 a common colour that is not appearing in $N^+(y_1) \cup N^+(u_1)$ if y_1 has $r-3$ out-neighbours in X, or in $N^-(y_1) \cup N^-(u_1)$ otherwise. Similarly, we choose for y_2 and u_2 a common colour that is not appearing either in $N^+(y_2)\cup N^+(u_2)$ or $N^-(y_2)\cup N^-(u_2)$. To see that this is possible, note that both y_1 and y_2 have at least $r-4$ uncoloured in- or out-neighbours, and that both u_1 and u_2 have at least $\Delta - 301d$ uncoloured in- and out-neighbours (the ones in $X \setminus \{u_1\}$). The number of forbidden colours is thus at most

$$(\Delta+1)-(r-4)+(\Delta+1)-(\Delta-301d) = \Delta - \ln^4\Delta + 301\ln^3\Delta + 6,$$

which is smaller than $\Delta - 1$ when Δ is large enough. We finally greedily colour the vertices in X that are not already coloured, starting with the possible vertex in Y (if it is not y_1 or y_2), and finishing with v and s in this order. The possible vertex in Y can be coloured because it has at least $\frac{99}{100}\Delta - 2$ uncoloured out-neighbours. Vertices in $X \setminus (Y \cup \{v, s\})$ can be coloured because they are linked with digons to both v and s, which are uncoloured.

At the end, v has a colour repeated in its in- or out-neighbourhood (the colour of u_1 and y_1) and one uncoloured neighbour s to which v is linked with a digon. Therefore, a colour is missing either in $N^-(v)$ or in $N^+(v)$. Finally, two colours are repeated in the out-neighbourhood of s (namely the colour of y_1 and u_1, and the colour of y_2 and u_2). Since we assumed $\deg^+(s) \leq \Delta$, we can actually colour s, yielding the contradiction. ■

Recall that $|\mathscr{S}| \geq |X|-3$ and that every vertex $s \in \mathscr{S}$ is linked with digons to all vertices in $X \setminus Y$. Therefore, since $|X| > \Delta - 300d$, the statement follows from Claims 6.18.3 and 6.18.4 when every vertex s of $\mathscr{S}$ satisfies $|N^{\pm}(s) \cap X \setminus Y| < \Delta - 2$ (note that $\mathscr{Z}_{r-3} \subseteq \mathscr{Z}_r$). Since vertices in $\mathscr{S}$ have at most $\Delta - 2$ neighbours in $X \setminus Y$ by Claim 6.18.1, we are done unless one of the following holds:

1. $D[X \setminus Y]$ is a biclique of size $\Delta - 1$ (if X is of type (1) or (4));

2. $D[X] \cup \{xy\} \cong \overleftrightarrow{K}_{\Delta-1}$ for some $x, y \in X$ (if X is of type (2)); or

3. $D[X] \cong \vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-4}$ (if X is of type (3)).

From now on, we thus assume that we are in one of the three cases above. Furthermore, combining Claims 6.18.1, 6.18.3, and 6.18.4, we obtain that every vertex $s \in \mathscr{S} \setminus \mathscr{R}$ has exactly two in-neighbours y_s^-, z_s^- and exactly two out-neighbours y_s^+, z_s^+ in Y (we may have $y_s^- = y_s^+$ and/or $z_s^- = z_s^+$). Furthermore, as $s \notin \mathscr{R}$ and by Claim 6.18.4, exactly one of $\{y_s^-, z_s^-\}$ belongs to $\mathscr{Z}_{r-3}$, and the other one has at least r in- or out-neighbours in X. Similarly, exactly one of $\{y_s^+, z_s^+\}$ belongs to $\mathscr{Z}_{r-3}$, and the other one has at least r in- or out-neighbours in X.

Henceforth, for every vertex $s \in \mathscr{S} \setminus \mathscr{R}$, we thus denote by z_s^- the unique vertex in $N^-(s) \cap \mathscr{Z}_{r-3}$ and by y_s^- the other vertex in $N^-(s) \cap Y$. Similarly, we denote by z_s^+ the unique vertex in $N^+(s) \cap \mathscr{Z}_{r-3}$ and by y_s^+ the other vertex in $N^+(s) \cap Y$. It turns out that z_s^- and z_s^+ are necessarily the same vertex.

Claim 6.18.5. *For every vertex $s \in \mathscr{S} \setminus \mathscr{R}$, $z_s^- = z_s^+$.*

Proof. Assume for a contradiction that, for some $s \in \mathscr{S} \setminus \mathscr{R}$, $z_s^- s$ and $s z_s^+$ are two simple arcs. Let D' be the digraph obtained from D by removing $X \cup \{y_s^-, y_s^+\}$ and adding the arc $z_s^- z_s^+$ (note that we may have $y_s^- = y_s^+$). Note that $\tilde{\Delta}(D') \leq \tilde{\Delta}(D)$, and that, by Lemma 6.14, D' does not contain any copy of $\overleftrightarrow{K}_\Delta$ nor $\vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-2}$. The minimality of D implies the existence of a $(\Delta - 1)$-dicolouring φ of D'.

Hence, φ is a dicolouring of $D - (X \cup \{y_s^-, y_s^+\})$. Let us show that it extends to D, thus yielding the contradiction. Let u_y be any vertex in $\mathscr{S}$ such that $\langle y_s^-, u_y \rangle$ is not a digon of D, the existence of which is guaranteed by Claim 6.18.2. We first extend φ by colouring y_s^- and u_y with a common colour c that is not appearing either in $N^+(y_s^-) \cup N^+(u_y)$ or in $N^-(y_s^-) \cup N^-(u_y)$. If $y_s^- \neq y_s^+$, we further colour y_s^+ with a colour that is not appearing in $N^+(y_s^+) \cup \{z_s^-\}$ or in $N^-(y_s^-) \cup N^-(u_y)$.

Next, we greedily extend the obtained dicolouring to $D - \{s, v\}$ by choosing for every vertex a colour that is not appearing either in its in- or out-neighbourhood, where v is an arbitrary fixed vertex in $N(y_s^-) \cap \mathscr{S} \setminus \{s, u_y\}$. Since s is uncoloured, $\langle s, v \rangle$ is a digon, and one colour is repeated in either $N^-(v)$ or $N^+(v)$, we can further extend this dicolouring to $D - s$. Finally, if one colour is not appearing in $N^-(s)$, we choose this colour for s, and we are done. If not, every colour in $[\Delta - 1]$ is appearing exactly once in $N^-(s)$, except the colour of u_y and y_s^-, which is appearing exactly twice. We thus choose the colour $\varphi(z_s^-)$ for s. We claim that the obtained colouring ψ is a dicolouring of D, so assume for a contradiction that D, coloured with ψ, contains a monochromatic directed cycle $\mathscr{C}$.

Since X is of type (1) to (4), we have $X \setminus (Y \cup \{s\}) \subseteq N^\pm(s)$. Moreover, we explicitly fixed $\psi(y_s^+)$ to a colour distinct from $\psi(z_s^-)$, so $V(\mathscr{C}) \setminus V(D')$ is exactly $\{s\}$. In particular, the successor of s in $\mathscr{C}$ is z_s^+. Since ψ is equal to φ on $V(D')$, this implies that D', coloured with φ, contains a monochromatic path from z_s^+ to z_s^-. Since D' contains the arc $z_s^- z_s^+$, this is a contradiction. ■

Henceforth, for better readability, given a vertex $s \in \mathscr{S} \setminus \mathscr{R}$, we thus let $z_s = z_s^- = z_s^+$.

Claim 6.18.6. *For every vertex $s \in \mathscr{S} \setminus \mathscr{R}$, $|N^+(z_s) \cap X| \leq 3$ and $|N^-(z_s) \cap X| \leq 3$.*

Proof. Assume for a contradiction that, for some $s \in \mathscr{S} \setminus \mathscr{R}$, $|N^+(z_s) \cap X| \geq 4$, the other case being proved symmetrically.

Let u_y be any vertex in $X \setminus Y$, distinct from s, such that $\langle y_s^-, u_y \rangle$ is not a digon of D, the existence of which is guaranteed by Claim 6.18.2. If possible, we further choose u_y such that $u_y \notin N(y_s^-)$. We then let u_z be any vertex in $N(y_s^-) \cap \mathscr{S}$ distinct from vertices in $N^+(z_s) \cup \{u_y\}$ (note that it is also distinct from s as $s \in N^+(z_s)$). Let us justify that u_z exists. Recall that $z_s \in \mathscr{Z}_{r-3}$, and $|\mathscr{S} \setminus X| \leq 3$, hence

$$|N(y_s^-) \cap \mathscr{S} \setminus (N^+(z_s) \cup \{u_y\})| > |N(y_s^-) \cap X| - r - |N(y_s^-) \cap \{u_y\}|.$$

If u_y is not a neighbour of y_s^-, this is non-negative as $|N(y_s^-) \cap X| \geq r$. If u_y is a neighbour of y_s^- then, by choice of u_y, $X \setminus Y \subseteq N(y_s^-)$, which implies $|N(y_s^-) \cap X| \geq |X| - 1 \geq \Delta - 2$ and, again, $|N(y_s^-) \cap \mathscr{S} \setminus (N^+(z_s) \cup \{u_y\})| > 0$. This shows the existence of u_z.

We finally let v be any vertex in $N(y_s^-) \cap \mathscr{S}$ that is distinct from s, u_y, and u_z. Let φ be a $(\Delta - 1)$-dicolouring of $D - (X \cup \{y_s^-, z_s\})$. Let us show that it extends to D, thus yielding the contradiction.

We first give to z_s and u_z a common colour c that is not appearing in $N^+(z_s)$, and that is neither appearing in one of $\{N^-(u_z), N^+(u_z)\}$. Note that this does not create any monochromatic directed cycle because $zu_z \notin A(D)$. To see that such a colour c exists, note that at most $\Delta - 3$ colours appear in $N^+(z_s)$, and at most one colour appears either in $N^-(u_z)$ or in $N^+(u_z)$. For u_z, this is because $u_z \in \mathscr{S} \cap N(y_s^-)$, so u_z is linked with digons to at least $\Delta - 2$ vertices in X, and it is adjacent to y_s^-, which is uncoloured.

We then give to y_s^- and u_y a common colour that is not appearing either in $N^-(u_y) \cup N^-(y_s^-)$ or in $N^+(u_y) \cup N^+(y_s^-)$, which is possible because y_s^- has r in- or out-neighbours in X. We finally greedily colour the vertices in X that are not already coloured, starting with the possible vertex in Y, and finishing with v and s in this order. The possible vertex in Y can be coloured because it has at least $\frac{99}{100}\Delta - 2$ uncoloured out-neighbours. Vertices in $X \setminus (Y \cup \{v, s\})$ can be coloured because they are linked with digons to both v and s, which are uncoloured. At the end, v has a colour repeated in its in- or out-neighbourhood (the colour of y_s^- and u_y) and one uncoloured neighbour s to which v is linked with a digon. Hence, v can be coloured. Finally, two colours are repeated in the in-neighbourhood of s (namely the colour of y_s^- and u_y, and the one of z_s and u_z). Since $\deg^-(s) = \Delta$ by Claim 6.18.3, we can actually colour s, yielding the contradiction. ∎

Claim 6.18.7. *For every vertex $s \in \mathscr{S} \setminus \mathscr{R}$, we have $|N(y_s^-) \cap \mathscr{S} \setminus \mathscr{R}| \leq 10$.*

Proof. Assume for a contradiction that, for some $s \in \mathscr{S} \setminus \mathscr{R}$, $|N(y_s^-) \cap \mathscr{S} \setminus \mathscr{R}| \geq 11$. Therefore, either $N^- := N^-(y_s^-) \cap \mathscr{S} \setminus \mathscr{R}$ or $N^+ := N^+(y_s^-) \cap \mathscr{S} \setminus \mathscr{R}$ has order at least 6. We let $N = N^-$ if $|N^-| > |N^+|$, and $N = N^+$ otherwise.

Let u_y be any vertex in $X \setminus Y$, distinct from s, such that $\langle y_s^-, u_y \rangle$ is not a digon of D, the existence of which is guaranteed by Claim 6.18.2. We further let s' be any vertex in $N \setminus (N^+(z_s) \cup \{u_y, s\})$, which is non-empty by Claim 6.18.6. We finally let v be any vertex in $N(y_s^-) \cap X \setminus Y$ distinct from s, u_y, and s'.

Note that s' is chosen in $\mathscr{S} \setminus \mathscr{R}$, which allows us to consider vertex $z_{s'}$. Thus, let D' be the digraph obtained from D by first removing $X \cup \{y_s^-\}$, and further adding the digon $\langle z_s, z_{s'} \rangle$. Note that z_s and $z_{s'}$ are indeed distinct because $s' \notin N^+(z_s)$, and $z_{s'}$ is an in-neighbour of s' by definition. Moreover, $z_{s'}$ is distinct from y_s^- by Claim 6.18.6 applied to s'. By Claim 6.18.5, both z_s and $z_{s'}$ are linked with a digon to a vertex in X (namely s and s'), so in particular $\tilde{\Delta}(D') \leq \tilde{\Delta}(D)$. Furthermore, by Lemma 6.14, D' does not contain a copy of $\overleftrightarrow{K}_\Delta$ nor $\vec{C_3} \boxplus \overleftrightarrow{K}_{\Delta-2}$. The minimality of D thus implies the existence of a $(\Delta - 1)$-dicolouring φ of D'. Let us show that it extends to D, thus yielding the contradiction.

We first give to s' colour $\varphi(z_s)$. This is possible because $s' \in (\mathscr{S} \setminus \mathscr{R}) \cap N(y_s^-)$, so exactly one colour appears in both the in- and out-neighbourhood of s', namely the one of $z_{s'}$, and $\varphi(z_{s'}) \neq \varphi(z_s)$ because $\langle z_s, z_{s'} \rangle$ is a digon of D'.

We then give to y_s^- and u_y a common colour that is not appearing either in $N^+(u_y) \cup N^+(y_s^-)$ or in $N^-(u_y) \cup N^-(y_s^-)$. We finally greedily colour the vertices in X that are not already coloured, starting with the possible vertex in Y, and finishing with v and s in this order. Again, this is always possible using the fact that we gave the same colour to both z_s and s', and the same to both y_s^- and u_y. ∎

With Claim 6.18.7 in hand, we are now able to prove that almost all vertices in $\mathscr{S}$ are in $\mathscr{R}$. To do this, let us count the following sum in two different ways:

$$S = \sum_{s \in \mathscr{S} \setminus \mathscr{R}} |N(y_s^-) \cap \mathscr{R}|.$$

Let s be any vertex in $\mathscr{S} \setminus \mathscr{R}$. Recall that $|X \setminus \mathscr{S}| \leq 3$. Hence, together with Claim 6.18.7, we have

$$|N(y_s^-) \cap \mathscr{R}| \geq |N(y_s^-) \cap X| - 13 \geq r - 13.$$

On the other hand, recall that vertices $s \in \mathscr{S}$ (and, a fortiori, vertices in $\mathscr{R}$) satisfy $|N^{\pm}(s) \cap X \setminus Y| \geq \Delta - 2$, because we are in one of the following cases:

1. $D[X \setminus Y]$ is a biclique of size $\Delta - 1$ (if X is of type (1) or (4));
2. $D[X] \cup \{xy\} \cong \overleftrightarrow{K}_{\Delta-1}$ for some $x, y \in X$ (if X is of type (2)); or
3. $D[X] \cong \vec{C}_3 \boxplus \overleftrightarrow{K}_{\Delta-4}$ (if X is of type (3)).

This implies in particular that vertices $s \in \mathscr{S}$ have at most four neighbours in Y, i.e. $|N(s) \cap Y| \leq 4$. Since, for every vertex $s \in \mathscr{S} \setminus \mathscr{R}$, y_s^- belongs to Y, we have $S \leq 4|\mathscr{R}|$. Together with the inequality above, we thus have

$$4|\mathscr{R}| \geq S \geq |\mathscr{S} \setminus \mathscr{R}| \cdot (r - 13),$$

which implies

$$|\mathscr{R}| \cdot \left(\frac{4}{r-13} + 1\right) \geq |\mathscr{S}| \geq |X| - 3.$$

Since $r = \ln^4 \Delta$ and $|X| > \Delta - 300d$, and because Δ is arbitrarily large, we obtain $|\mathscr{R}| > \frac{1}{2}|X|$, concluding the proof of Lemma 6.18. □

Now that we know that each X_i has a large number of saviours, we can derive the following. For every $i \in [t]$, we denote by $X_i^\star$ the set of vertices of X_i that are in the closed out-neighbourhood of all vertices of X_i, so formally

$$X_i^\star = X_i \cap \bigcap_{x \in X_i} N^+[x].$$

Lemma 6.19. *For every $i \in [t]$, there exists a collection $\mathscr{T}_i$ of $\lceil \frac{1}{50r}\Delta \rceil$ k_i-tuples of vertices of D, such that every vertex appears in at most one tuple and only once within that tuple, and such that one of the following holds.*

(A) *$k_i = 2$, and every $(x, u) \in \mathscr{T}_i$ satisfies $x \in X_i^\star$, $\deg^+(x) = \Delta - 1$, and $u \in N^+(x) \cap \mathscr{Z}_r(X_i)$.*

(B) *$k_i = 2$, and every $(x, u) \in \mathscr{T}_i$ satisfies $x \in X_i^\star$, $\deg^-(x) = \Delta - 1$, and $u \in N^-(x) \cap \mathscr{Z}_r(X_i)$.*

(C) *$k_i = 3$, and every $(x, u, v) \in \mathscr{T}_i$ satisfies $x \in X_i^\star$, $\deg^+(x) \leq \Delta$, and $u, v \in N^+(x) \cap \mathscr{Z}_r(X_i)$.*

(D) *$k_i = 3$, and every $(x, u, v) \in \mathscr{T}_i$ satisfies $x \in X_i^\star$, $\deg^-(x) \leq \Delta$, and $u, v \in N^-(x) \cap \mathscr{Z}_r(X_i)$.*

Proof. Let us fix $i \in [t]$ and let $\mathscr{S}_i^\star = X_i^\star \cap \mathscr{S}(X_i)$. By Lemma 6.16, X_i is a quasi-biclique. If it is a quasi-biclique of type (1), (2) or (3), then $\mathscr{S}_i^\star = \mathscr{S}(X_i)$ and $|\mathscr{S}_i^\star| \geq |X_i| - 3$ by definition of $\mathscr{S}(X_i)$. If it is of type (4), then $|\mathscr{S}_i^\star| \geq \frac{99}{100}\Delta$ by definition.

Therefore, in cases (1)–(4), since $\Delta - 300d < |X_i| \leq \Delta + 1$ (the upper bound coming from the fact that D does not contain a biclique of size Δ), and because Δ is large enough, we have $|\mathscr{S}_i^\star| \geq \frac{98}{100}|X_i|$, and in particular $|\mathscr{S}_i \setminus \mathscr{S}_i^\star| \leq \frac{2}{100}|X_i|$. By Lemma 6.18, $\mathscr{S}_i$ contains at least $\frac{1}{2}|X_i|$ r-saviours of X_i. Using the remark above, $\mathscr{S}_i^\star$ contains at least $\frac{48}{100}|X_i|$ r-saviours.

If X_i is of type (5) and (R_i, K_i, W_i) is the given partition of X_i, then by definition $K_i = \mathscr{S}_i^\star$, and $|K_i| > \frac{1}{2}|X_i|$. Therefore, by Lemma 6.17, $\mathscr{S}_i^\star$ contains more than $\frac{48}{100}|X_i|$ r-saviours again.

Since each r-saviour satisfies (a), (b), (c), or (d), by the pigeonhole principle, let $\gamma \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ be such that $\mathscr{S}_i^\star$ contains at least $\frac{12}{100}|X_i|$ r-saviours, all satisfying property (γ), and let Y be the set of such r-saviours. We now greedily construct $\mathscr{T}_i$ by repeating the following process as long as Y is non-empty:

1. choose any vertex $x \in Y$;

2. choose any vertex u in $N^+(x) \cap \mathscr{Z}_r(X_i)$ if $\gamma \in \{\mathrm{a}, \mathrm{c}\}$ and in $N^-(x) \cap \mathscr{Z}_r(X_i)$ otherwise;

3. choose any vertex $v \neq u$ in $N^+(x) \cap \mathscr{Z}_r(X_i)$ if $\gamma = \mathrm{c}$ and in $N^-(x) \cap \mathscr{Z}_r(X_i)$ if $\gamma = \mathrm{d}$;

4. if $\gamma \in \{\mathrm{a}, \mathrm{b}\}$, add (x, u) to $\mathscr{T}_i$ and remove $N(u)$ from Y;

5. else if $\gamma \in \{\mathrm{c}, \mathrm{d}\}$, add (x, u, v) to $\mathscr{T}_i$ and remove $N(u) \cup N(v)$ from Y.

Note that steps 2 and 3 above are always possible by definition of x being an r-saviour. By construction, the obtained $\mathscr{T}_i$ thus satisfies one of (A), (B), (C) or (D). Furthermore, steps 4 and 5 guarantee that the elements of $\mathscr{T}_i$ are pairwise disjoint. Finally, at each step, the size of Y decreases by at most $4r$. Hence,

$$|\mathscr{T}_i| \geq \frac{3}{100r}|X_i| \geq \frac{1}{50r}\Delta,$$

where in the second inequality we used $|X_i| \geq \Delta - 300d$ and the fact that Δ is large enough. The result follows. □

From now on, for every $i \in [t]$, we fix a collection $\mathscr{T}_i$ as in the statement of Lemma 6.19. We note, for future use, that the proof of Lemma 6.19 implies the following.

Remark 6.20. *For every $i \in [t]$, $|X_i^\star| \geq \frac{49}{100}\Delta$.*

6.2.2 The probabilistic analysis

Let φ be a partial $(\Delta - 1)$-dicolouring of D. For an integer $i \in [t]$, we say that φ is *i-extendable* if there exist three distinct uncoloured vertices $x_1, x_2, x_3 \in X_i^\star$ such that, for each $1 \leq j \leq 3$, one of the following is satisfied:

(a) $\deg^+(x_j) = \Delta - 1$ and one colour is repeated in $N^+(x_j)$,

(b) $\deg^-(x_j) = \Delta - 1$ and one colour is repeated in $N^-(x_j)$,

(c) $\deg^+(x_j) \leq \Delta$ and two colours are repeated in $N^+(x_j)$, or

(d) $\deg^-(x_j) \leq \Delta$ and two colours are repeated in $N^-(x_j)$.

We say that φ is *extendable* if it is i-extendable for every $i \in [t]$ and, for every d-sparse vertex $s \in S$, three colours are repeated in $N^+(s)$.

Let us briefly justify that an extendable partial dicolouring of D can actually be extended to D. We colour the uncoloured vertices of $X_1, \dots, X_t, S$ in this order. When colouring the vertices of X_i, we keep the three special vertices described above for the end. Since they are uncoloured and belong to $X_i^\star$, for each other vertex of X_i, there is a colour not appearing in its out-neighbourhood. When we end up with these three vertices, clearly we can choose a colour that does not appear in either the in- or out-neighbourhood. At the end, we can colour each vertex $s \in S$ with a colour that does not appear in its out-neighbourhood, as three colours are repeated in it.

The following lemma thus implies $\vec{\chi}(D) \leq \Delta - 1$, yielding a contradiction and concluding the proof of Theorem 6.8.

Lemma 6.21. *The digraph D has an extendable partial $(\Delta - 1)$-dicolouring.*

Proof. Let ψ be a random colouring of the vertices of D, where each vertex is assigned a colour from $[\Delta - 1]$ independently and uniformly at random. Let V_ψ be the set of vertices x such that $\psi(x) = \psi(x^-) = \psi(x^+)$ for some $x^- \in N^-(x)$ and some $x^+ \in N^+(x)$. We consider the random partial dicolouring φ obtained from ψ by uncolouring simultaneously all the vertices in V_ψ. By construction, φ is indeed a partial $(\Delta - 1)$-dicolouring of D. We are going to show, using the Lovász local lemma, that φ is extendable with positive probability.

Let us define the events that we want to avoid. For each d-sparse vertex $s \in S$, let A_s be the event that fewer than three colours are repeated in $N^+(s)$. And, for each $i \in [t]$, let B_i be the event that φ is not i-extendable. We note that A_s depends only on the colour choices for the vertices within distance 2 of s in the underlying undirected graph $\underline{D}$ of D. Likewise, B_i depends only on the colour choices for the vertices within distance 2 of X_i in $\underline{D}$. Therefore, each of these events is mutually independent of all the others, except at most $(2\Delta_{\max})^5$ of them. By the Lovász local lemma (Lemma A.8) and Lemma 6.12, it will be enough to show that each of these events holds with probability at most Δ^{-6}. This is the purpose of Claims 6.21.1 and 6.21.2.

Claim 6.21.1. *For any vertex $s \in S$, A_s holds with probability at most $e^{-\ln^2 \Delta}$.*

Proof. We let M_s be the random variable counting the number of colours c such that:

- c is assigned to two distinct out-neighbours x and y of s not inducing a digon, and
- for every distinct $x, y \in N^+(s)$ not inducing a digon, if x and y are assigned colour c, then both of them retain the colour.

It suffices to show that $M_s < 3$ holds with probability at most $e^{-\ln^2 \Delta}$. To do so, we first estimate $\mathbb{E}(M_s)$ and then show that M_s is concentrated around its expectation. In what follows, we denote by U_s the set of unordered pairs of distinct out-neighbours of s not inducing a digon.

Subclaim 6.21.1.1. $\mathbb{E}(M_s) \geq \dfrac{|U_s|}{e^6(\Delta-1)}$.

Proof. Let $\tilde{M}_s$ be the number of colours c such that:

- c is assigned to exactly two out-neighbours x and y of s,
- $\langle x, y \rangle$ is not a digon, and
- both x and y retain colour c.

Observe that $M_s \geq \tilde{M}_s$. Given a colour $c \in [\Delta-1]$ and two distinct out-neighbours x and y of s such that $\langle x, y \rangle$ is not a digon, let us consider the event $E_{\{x,y\},c}$ that x and y are assigned colour c, no other vertex in $N^+(s)$ is assigned colour c, and both x and y retain the colour. Let $\mathscr{W}_{x,y} = N^+(s) \cup N(x) \cup N(y) \setminus \{x, y\}$. Since $\langle x, y \rangle$ is not a digon, we have

$$\mathbb{P}\left[E_{\{x,y\},c}\right] \geq \mathbb{P}\left[\text{for every } w \in \mathscr{W}_{x,y},\ \psi(w) \neq c = \psi(x) = \psi(y)\right]$$
$$\geq \left(\frac{1}{\Delta-1}\right)^2 \left(\frac{\Delta-2}{\Delta-1}\right)^{5\Delta_{\max}} \geq \frac{1}{e^6(\Delta-1)^2},$$

where in the last inequality we used that Δ is large enough, $\lim\limits_{\Delta\to\infty} \left(\frac{\Delta-2}{\Delta-1}\right)^{5\Delta} = \frac{1}{e^5} > \frac{1}{e^6}$, and $\Delta \leq \Delta_{\max} \leq \Delta + 1$. By linearity of the expectation, we conclude that

$$\mathbb{E}(M_s) \geq \mathbb{E}(\tilde{M}_s) \geq (\Delta-1) \cdot |U_s| \cdot \frac{1}{e^6(\Delta-1)^2} = \frac{|U_s|}{e^6(\Delta-1)}.$$

♦

Subclaim 6.21.1.2. *For any* $\lambda > 504\sqrt{\frac{|U_s|}{\Delta-1}} + 2752$, *we have*

$$\mathbb{P}\left[|M_s - \mathbb{E}(M_s)| > \lambda\right] \leq 8\exp\left(\frac{-\lambda^2}{512\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right).$$

Proof. We consider two auxiliary random variables: $M_s^\star$, which counts the number of colours c assigned to both x and y for some $\{x, y\} \in U_s$, and M_s^{del}, which counts the number of colours c such that, for some $\{x, y\} \in U_s$, c is assigned to x and y, but not retained by both of them. Observe that $M_s = M_s^\star - M_s^{\mathrm{del}}$. Let us show that these auxiliary random variables are concentrated, hence implying that M_s is concentrated as well.

Given $\{x, y\} \in U_s$ and a colour $c \in [\Delta - 1]$, let $E^\star_{\{x,y\},c}$ be the event that x and y are assigned colour c. We have that

$$\mathbb{P}\left[E^\star_{\{x,y\},c}\right] = \frac{1}{(\Delta-1)^2}.$$

Thus, by the union bound,

$$\mathbb{E}(M_s^\star) \leq (\Delta - 1) \cdot |U_s| \cdot \frac{1}{(\Delta-1)^2} = \frac{|U_s|}{\Delta - 1}. \tag{6.4}$$

We note that changing the outcome of any colour assignment affects $M_s^\star$ by at most 1. Moreover, whenever $M_s^\star \geq k$ for some integer k, this can be certified by revealing $2k$ colour assignments: for each colour c counted by $M_s^\star$, it suffices to exhibit two vertices x and y with $\{x, y\} \in U_s$ that have been assigned colour c. By Talagrand's inequality (Lemma A.6) and (6.4),

$$\begin{aligned}\mathbb{P}\left[|M_s^\star - \mathbb{E}(M_s^\star)| > \lambda\right] &\leq 4\exp\left(\frac{-\lambda^2}{32 \cdot 2 \cdot (\mathbb{E}(M_s^\star) + \lambda)}\right)\\ &\leq 4\exp\left(\frac{-\lambda^2}{64\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right)\end{aligned} \tag{6.5}$$

for any $\lambda > 126\sqrt{2\frac{|U_s|}{\Delta-1}} + 688 \geq 126\sqrt{2\mathbb{E}(M_s^\star)} + 344 \cdot 2$.

Similarly, M_s^{del} is affected by at most 1 when the outcome of any colour assignment is changed. Moreover, whenever $M_s^{\mathrm{del}} \geq k$, this can be certified by revealing at most $4k$ colour assignments: for each one of the k colours, it suffices to exhibit two vertices x and y with $\{x, y\} \in U_s$, and one in-neighbour x^- and one out-neighbour x^+ of x, such that x, y, x^- and x^+ have all been assigned that colour. By Talagrand's inequality (Lemma A.6), (6.4), and the fact that $M_s^{\text{del}} \leq M_s^\star$, we have

$$\begin{aligned}\mathbb{P}(|M_s^{\text{del}} - \mathbb{E}(M_s^{\text{del}})| > \lambda) &\leq 4\exp\left(\frac{-\lambda^2}{32 \cdot 4 \cdot (\mathbb{E}(M_s^{\text{del}}) + \lambda)}\right) \\ &\leq 4\exp\left(\frac{-\lambda^2}{128\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right)\end{aligned} \tag{6.6}$$

for any $\lambda > 252\sqrt{\frac{|U_s|}{\Delta-1}} + 1376 \geq 126\sqrt{4\mathbb{E}(M_s^{\text{del}})} + 344 \cdot 4$.

Recall that $M_s = M_s^\star - M_s^{\text{del}}$. Hence, by linearity of the expectation and the triangle inequality, we have that $|M_s - \mathbb{E}(M_s)| \leq |M_s^\star - \mathbb{E}(M_s^\star)| + |M_s^{\text{del}} - \mathbb{E}(M_s^{\text{del}})|$. Combining (6.5) and (6.6), we thus obtain

$$\begin{aligned}\mathbb{P}\left[|M_s - \mathbb{E}(M_s)| > \lambda\right] &\leq \mathbb{P}\left[|M_s^\star - \mathbb{E}(M_s^\star)| + |M_s^{\text{del}} - \mathbb{E}(M_s^{\text{del}})| > \lambda\right] \\ &\leq \mathbb{P}\left[|M_s^\star - \mathbb{E}(M_s^\star)| > \tfrac{\lambda}{2}\right] + \mathbb{P}\left[|M_s^{\text{del}} - \mathbb{E}(M_s^{\text{del}})| > \tfrac{\lambda}{2}\right] \\ &\leq 4\exp\left(\frac{-\lambda^2}{256\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right) + 4\exp\left(\frac{-\lambda^2}{512\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right) \\ &\leq 8\exp\left(\frac{-\lambda^2}{512\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right)\end{aligned}$$

for any $\lambda > 504\sqrt{\frac{|U_s|}{\Delta-1}} + 2752$. ♦

To end the proof of the claim, we note that, since s is d-sparse, out of the $\deg^+(s)(\deg^+(s) - 1)$ possible arcs between the vertices of $N^+(s)$, at least

$$\deg^+(s)(\deg^+(s) - 1) - (\Delta_{\max}(\Delta_{\max} - 1) - d\Delta_{\max})$$

$$\geq (\Delta - 1)(\Delta - 2) - (\Delta + 1)\Delta + d(\Delta - 1) \geq \frac{d\Delta}{2}$$

of them are not present in D. This implies that $|U_s| \geq \frac{d\Delta}{4}$. In particular, if we take $\lambda = \frac{|U_s|}{e^6(\Delta-1)} - 3$, the condition of Subclaim 6.21.1.2 holds. By Subclaims 6.21.1.1 and 6.21.1.2, we obtain that $\mathbb{P}\left[A_s\right]$ is at most

$$\mathbb{P}\left[M_s < 3\right] \leq \mathbb{P}\left[\mathbb{E}(M_s) - M_s > \lambda\right] \leq 8\exp\left(\frac{-\lambda^2}{512\left(\frac{|U_s|}{\Delta-1} + \lambda\right)}\right) \leq e^{-\ln^2 \Delta},$$

as desired. ■

Claim 6.21.2. *For any $i \in [t]$, B_i holds with probability at most $e^{\frac{-\Delta}{r^3}}$.*

Proof. Let $\mathscr{T}_i$ be a set given by Lemma 6.19 (of size exactly $\lceil \frac{\Delta}{50r} \rceil$). For the sake of conciseness, we write the proof assuming that $\mathscr{T}_i$ satisfies (C); the other cases can be argued analogously. Let T_i be the set of vertices appearing in some triple of $\mathscr{T}_i$. Let M_i be the random variable counting the number of triples $(x, u, v) \in \mathscr{T}_i$ such that:

(i) vertex x is uncoloured;

(ii) both u and v retain their colours, which both appear on some vertex of $X_i^\star$; and

(iii) for every vertex of the triple, its colour is not assigned to any other vertex of T_i.

We note that, by (i) and (ii), φ is i-extendable with probability at least $\mathbb{P}[M_i \geq 3]$. (In cases (B) and (D), we take into account that $X_i^\star$ is a biclique.) Subclaim 6.21.2.1 shows that M_i is large on average, and Subclaim 6.21.2.2 that it is concentrated around its expected value. The claim is then obtained by combining these two facts. The purpose of condition (iii) is to prepare the ground for using Azuma's inequality.

Subclaim 6.21.2.1. $\mathbb{E}(M_i) \geq \dfrac{\Delta}{e^{37} r}$.

Proof. Given a triple $(x, u, v) \in \mathscr{T}_i$, three distinct vertices $x' \in X_i^\star \setminus T_i$, $u' \in X_i^\star \setminus (T_i \cup N(u))$, and $v' \in X_i^\star \setminus (T_i \cup N(v))$, and three distinct colours $c_1, c_2, c_3 \in [\Delta - 1]$, let $E_{x,u,v,x',u',v',c_1,c_2,c_3}$ be the event that

- c_1 is assigned to x and x', but not to any vertex in $X_i^\star \cup T_i \setminus \{x, x'\}$;
- c_2 is assigned to u and u', but not to any vertex in $X_i^\star \cup T_i \cup N(u) \cup N(u') \setminus \{u, u'\}$;
- c_3 is assigned to v and v', but not to any vertex in $X_i^\star \cup T_i \cup N(v) \cup N(v') \setminus \{v, v'\}$.

Since $\langle x, x' \rangle$ is a digon, these conditions ensure that the triple (x, u, v) is counted by M_i. Since $|T_i| = 3\lceil \frac{\Delta}{50r} \rceil$, we have that

$$\mathbb{P}\left[E_{x,u,v,x',u',v',c_1,c_2,c_3}\right] \geq \left(\frac{1}{\Delta - 1}\right)^6 \left(\frac{\Delta - 4}{\Delta - 1}\right)^{|X_i^\star| + |T_i| + 8\Delta} \geq \left(\frac{1}{\Delta - 1}\right)^6 e^{-30}.$$

By Remark 6.20 and the fact that $u, v \in \mathscr{Z}_r(X_i)$, there are at least $\left(\frac{48}{100}\Delta\right)^3$ ways of choosing vertices x', u', and v'; moreover, there are $(\Delta - 1)(\Delta - 2)(\Delta - 3)$ possible ways of choosing colours c_1, c_2, and c_3. Since for every fixed $(x, u, v) \in \mathscr{T}_i$ the events of the form $E_{x,u,v,x',u',v',c_1,c_2,c_3}$ are pairwise disjoint, by linearity of the expectation we have

$$\mathbb{E}(M_i) \geq |\mathscr{T}_i| \left(\frac{48}{100}\Delta\right)^3 (\Delta - 1)(\Delta - 2)(\Delta - 3) \left(\frac{1}{\Delta - 1}\right)^6 e^{-30} \geq \frac{\Delta}{e^{37} r}.$$

♦

Subclaim 6.21.2.2. *For any* $\lambda \geq 0$, $\mathbb{P}\left[|M_i - \mathbb{E}(M_i)| > \lambda\right] \leq 2\exp\left(\frac{-\lambda^2}{44\Delta}\right)$.

Proof. We will be interested in how much M_i can vary when the colour assignment of an arbitrary vertex is altered. For that, we first fix some notation. Let $\tau, \tau' : V(D) \to [\Delta - 1]$ be two arbitrary colour assignments that differ everywhere. We consider the event E that $\psi = \tau$ and, for every vertex w of D, the event E_w that $\psi|_{V(D)\setminus\{w\}} = \tau|_{V(D)\setminus\{w\}}$ and $\psi(w) = \tau'(w)$. We let $\mathscr{M}$ and $\mathscr{M}_w$ be the sets of triples of $\mathscr{T}_i$ satisfying (i), (ii) and (iii) when E and E_w hold, respectively. In particular, $|\mathscr{M}|$ and $|\mathscr{M}_w|$ are the values $M_i(E)$ and $M_i(E_w)$.

We are going to consider the random outcomes defining ψ in a specific order, depending on the given $i \in [t]$. We fix a labelling $w_1, \dots, w_n$ of the vertices of D such that $w_1, \dots, w_q \in V(D) \setminus (X_i^\star \cup T_i)$, and $w_{q+1}, \dots w_n \in X_i^\star \cup T_i$ for some q. For $1 \leq j \leq n$, we denote by E_j and E'_j the events that $\psi(w_1) = \tau(w_1), \dots, \psi(w_j) = \tau(w_j)$ and that $\psi(w_1) = \tau(w_1), \dots, \psi(w_{j-1}) = \tau(w_{j-1})$ and $\psi(w_j) = \tau'(w_j)$.

Let us now discuss how much M_i can vary when changing the random assignment of an arbitrary vertex; in other words, let us fix $1 \leq j \leq n$ and examine the quantity $\big||\mathscr{M}| - |\mathscr{M}_{w_j}|\big|$. We note that a triple $(x,u,v) \in \mathscr{T}_i$ with $\{\tau(x), \tau(u), \tau(v)\} \cap \{\tau(w_j), \tau'(w_j)\} = \emptyset$ belongs to $\mathscr{M}$ if and only if it belongs to $\mathscr{M}_{w_j}$. On the other hand, by (iii), both $\mathscr{M}$ and $\mathscr{M}_{w_j}$ contain at most two triples $(x,u,v) \in \mathscr{T}_i$ with $\{\tau(x), \tau(u), \tau(v)\} \cap \{\tau(w_j), \tau'(w_j)\} \neq \emptyset$. Therefore,

$$\Big|M_i(E) - M_i(E_{w_j})\Big| \leq 2. \tag{6.7}$$

For $1 \leq j \leq q$, the previous bound (6.7) can be improved to 0 in some situations. Let us denote by $\mathcal{N}_j$ the set of neighbours of w_j in $X_i^\star \cup T_i$, and let F_j be the event that some vertex in $\mathcal{N}_j$ is assigned colour $\tau(w_j)$ or $\tau'(w_j)$. Let us discuss when do $M_i(E)$ and $M_i(E_{w_j})$ differ. If $\mathscr{M} \neq \mathscr{M}_{w_j}$, then, since w_j is not in $X_i^\star \cup T_i$, changing the outcome of $\psi(w_j)$ from $\tau(w_j)$ to $\tau'(w_j)$ has to affect the instruction of uncolouring a vertex y from $X_i^\star \cup T_i$. In particular, y must be adjacent to w_j, and its assigned colour $\tau(y)$ under E and E_{w_j} must be $\tau(w_j)$ or $\tau'(w_j)$. Hence, $E, E_{w_j} \subseteq F_j$. Therefore, for every $1 \leq j \leq q$, changing the outcome of $\psi(w_j)$ does not affect M_i on $\overline{F_j}$. We deduce that

$$\mathbb{E}\left(M_i \,\middle|\, E_j \cap \overline{F_j}\right) = \mathbb{E}\left(M_i \,\middle|\, E'_j \cap \overline{F_j}\right). \tag{6.8}$$

We also need some control over the probability of F_j, for every $1 \leq j \leq q$. We note that, by the definition of the ordering $w_1, \dots, w_n$, F_j is independent of both E_j and E'_j. Therefore, by the union bound,

$$\mathbb{P}\left[F_j \mid E_j\right] = \mathbb{P}\left[F_j \mid E'_j\right] = \mathbb{P}\left[F_j\right] \leq \frac{2|\mathcal{N}_j|}{\Delta - 1} \tag{6.9}$$

for every $1 \leq j \leq q$. We now bound

$$\sum_{j=1}^{n} \left|\mathbb{E}(M_i \mid E_j) - \mathbb{E}(M_i \mid E_j')\right|^2,$$

which will allow us to apply Azuma's inequality. We split the sum into two chunks. For the terms $q+1 \leq j \leq n$, (6.7) yields

$$\sum_{j=q+1}^{n} \left|\mathbb{E}(M_i \mid E_j) - \mathbb{E}(M_i \mid E_j')\right|^2 \leq 2^2 |X_i^\star \cup T_i| \leq 5\Delta.$$

For $1 \leq j \leq q$, (6.7), (6.8), and (6.9) yield

$$\begin{aligned}
&\sum_{j=1}^{q} \left|\mathbb{E}\left(M_i \mid E_j\right) - \mathbb{E}\left(M_i \mid E_j'\right)\right|^2 \\
&\quad \leq \sum_{j=1}^{q} 2\left|\mathbb{E}\left(M_i \mid E_j\right) - \mathbb{E}\left(M_i \mid E_j'\right)\right| \\
&\quad = 2\sum_{j=1}^{q} \Big|\mathbb{E}\left(M_i \mid E_j \cap F_j\right) \mathbb{P}\left[F_j \mid E_j\right] + \mathbb{E}\left(M_i \,\middle|\, E_j \cap \overline{F_j}\right) \mathbb{P}\left[\overline{F_j} \,\middle|\, E_j\right] \\
&\qquad\qquad - \mathbb{E}\left(M_i \mid E_j' \cap F_j\right) \mathbb{P}\left[F_j \mid E_j'\right] - \mathbb{E}\left(M_i \,\middle|\, E_j' \cap \overline{F_j}\right) \mathbb{P}\left[\overline{F_j} \,\middle|\, E_j'\right]\Big| \\
&\quad = 2\sum_{j=1}^{q} \left|\mathbb{E}\left(M_i \mid E_j \cap F_j\right) \mathbb{P}\left[F_j \mid E_j\right] - \mathbb{E}\left(M_i \mid E_j' \cap F_j\right) \mathbb{P}\left[F_j \mid E_j'\right]\right| \\
&\quad \leq 2\sum_{j=1}^{q} \left|\mathbb{E}\left(M_i \mid E_j \cap F_j\right) - \mathbb{E}\left(M_i \mid E_j' \cap F_j\right)\right| \frac{2|\mathcal{N}_j|}{\Delta - 1} \\
&\quad \leq \frac{8}{\Delta - 1} \sum_{j=1}^{q} |\mathcal{N}_j| \leq \frac{16}{\Delta - 1} \Delta_{\max} |X_i^\star \cup T_i| \leq 17\Delta.
\end{aligned}$$

Altogether,

$$\sum_{j=1}^{n} \left|\mathbb{E}\left(M_i \mid E_j\right) - \mathbb{E}\left(M_i \mid E_j'\right)\right|^2 \leq 22\Delta.$$

The statement now follows from Azuma's inequality (Lemma A.5). ♦

To end the proof of the Claim 6.21.2, we combine Subclaims 6.21.2.1 and 6.21.2.2, and obtain that $\mathbb{P}\left[B_i\right]$ is at most

$$\mathbb{P}\left[M_i < 3\right] \leq \mathbb{P}\left[\mathbb{E}(M_i) - M_i > \frac{\Delta}{e^{37} r} - 3\right] \leq 2\exp\left(\frac{-\left(\frac{\Delta}{e^{37} r} - 3\right)^2}{44\Delta}\right) \leq e^{\frac{-\Delta}{r^3}},$$

as desired. ■

It follows from Claims 6.21.1 and 6.21.2 that each of the bad events occurs with probability at most Δ^{-6}. As discussed above, this allows us to conclude the proof of Lemma 6.21 by applying the Lovász local lemma (Lemma A.8), and, with it, the proof of Theorem 6.8. □

Chapter 7

Circular colourings and fractional colourings

There exist a number of variants of graph colouring. Proper k-colourings of a graph G are precisely the homomorphisms from G to K_n, so homomorphisms themselves can be regarded as a generalisation of colourings. Indeed, some colouring variants are based on this feature. When translating this into the directed setting, one cannot take the shortest route: a k-dicolouring does not always correspond to a homomorphism to $\overleftrightarrow{K}_k$ (consider for example transitive tournaments). Instead, other notions are used: acyclic homomorphisms and circular homomorphisms, which still correspond to usual homomorphisms when restricting to symmetric digraphs.

In this chapter we review two graph colouring variants related to homomorphisms, and their generalisations to digraphs.

7.1 Circular colourings

Let $p \geq q$ be two positive integers. The *circular complete graph* $C(p,q)$ is the graph with vertex set $[p]$ where two vertices i and j are adjacent if and only if $q \leq |j-i| \leq p-q$. We note that, in order for $C(p,q)$ to have any edges, $p \geq 2q$; we can nicely express $C(p,q)$ as a copy of the Cayley graph $\underline{\mathrm{Cay}}(\mathbb{Z}_p, \{\overline{q}, \overline{q+1}, \dots, \overline{p-q}\})$. A *circular* (p,q)*-colouring* of a graph G is a homomorphism $\varphi : G \to C(p,q)$. In particular, the circular $(k,1)$-colourings of G are precisely its proper k-colourings. Finally, the *circular chromatic number* of G is defined as

$$\chi_c(G) = \inf \left\{ \frac{p}{q} \;\middle|\; \exists p,q \in \mathbb{Z}^+ \; p \geq q,\; G \to C(p,q) \right\}.$$

This parameter, originally called the 'star chromatic number', was introduced in 1988 by Vince [155] as a refinement of the usual chromatic number. The following theorem gathers some of its fundamental properties.

Theorem 7.1. [95, 155] *For any graph G and positive integers p, q, p', q' with $\frac{p}{q}, \frac{p'}{q'} \geq 2$, the following hold.*

(i) The infimum in the definition of $\chi_c(G)$ is attained; in particular, $\chi_c(G)$ is a rational number.

(ii) $\lceil \chi_c(G) \rceil = \chi(G)$.

(iii) $C(p,q) \to C(p',q')$ if and only if $\frac{p}{q} \leq \frac{p'}{q'}$; in particular, $\chi_c(C(p,q)) = \frac{p}{q}$.

In fact, the circular chromatic number admits various (non-trivially) equivalent definitions, each one illuminating some of its particular aspects (see [95, Section 6.1]). More details on the topic of circular colouring can be found in Zhu's survey [158] and book chapter [159]. Before turning to the directed setting, it will be convenient to briefly mention the following well-known property of circular complete graphs.

Proposition 7.2. *Let p and q be positive integers with $\frac{p}{q} \geq 2$. $C(p,q)$ is a core if and only if p and q are coprime.*

Proof. Suppose that $\mathrm{mcd}(p,q) = 1$. If $q = 1$, then $C(p,q)$ is a complete graph. If $q > 1$, according to [95, Lemma 6.6],

$$\chi_c(C(p,q) - x) = \frac{p'}{q'} < \frac{p}{q}$$

for any vertex x of $C(p,q)$, where p' and q' are the unique integers with $0 < p' < p$, $0 < q' < q$ and $pq' - qp' = 1$. By (*iii*) of Theorem 7.1, $C(p,q)$ cannot have any homomorphism to a proper subgraph.

The other implication follows from Theorem 7.1(*iii*). □

Now the question is: is there any reasonable extension of the circular chromatic number to digraphs? By an extension, we mean a digraph parameter such that, if D is a symmetric digraph and $\underline{D}$ is the undirected graph obtained from D by replacing its digons by edges, then this digraph parameter on D is equal to $\chi_c(\underline{D})$. As it turns out, three different alternatives have been investigated; we will look at them in the chronological order of apparition.

The first one was introduced by Mohar [120] and Bokal et al. [28] in the 2000s. We will define it in terms of the following auxiliary notion. An *acyclic homomorphism* from a digraph D to a digraph D' is a mapping $\varphi : V(D) \to V(D')$ such that

(i) for every arc (u,v) of D, either $\varphi(u) = \varphi(v)$ or $(\varphi(u), \varphi(v))$ is an arc of D'; and

(ii) for every vertex v of D', $\varphi^{-1}(v)$ is an acyclic set of D.

We write that as $\varphi : D \overset{\mathrm{a}}{\to} D'$, and, when such a φ exists, we write $D \overset{\mathrm{a}}{\to} D'$. Thus, any digraph homomorphism is an acyclic homomorphism, and the converse also holds when D is symmetric. Moreover, it is not hard to see that acyclic homomorphisms are closed under composition, and that the k-dicolourings of a digraph D are the acyclic homomorphisms from D to $\overleftrightarrow{K}_k$. Acyclic homomorphisms were first studied by Feder, Hell and Mohar [67].

Given two positive integers $p \geq q$, we let $\vec{C}(p,q)$ be the digraph with vertex set $[p]$, where there is an arc from i to j if and only if $j - i$ is congruent to one of $q, q+1, \dots, p-1$ modulo p. Hence, $\vec{C}(p,q)$ has no arc unless $p > q$, and we can express it as a copy of the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_p, \{\overline{q}, \overline{q+1}, \dots, \overline{p-1}\})$. A *circular (p,q)-dicolouring* of a digraph D is an acyclic homomorphism $\varphi : D \overset{\mathrm{a}}{\to} \vec{C}(p,q)$. By taking $(p,q) = (k,1)$ we recover the k-dicolourings of D, and when D is symmetric we recover the circular (p,q)-colourings of $\underline{D}$. The *circular dichromatic number* of D is defined as

$$\vec{\chi}_c(D) = \inf \left\{ \frac{p}{q} \;\middle|\; \exists p, q \in \mathbb{Z}^+ \; p \geq q,\; D \overset{\mathrm{a}}{\to} \vec{C}(p,q) \right\}.$$

As in the undirected setting, there are several equivalent definitions of this parameter [28, 120]. Moreover, we have the following analogue of Theorem 7.1.

Theorem 7.3. [28] *For any digraph D and positive integers p, q, p', q' with $\frac{p}{q}, \frac{p'}{q'} \geq 1$, the following hold.*

(i) The infimum in the definition of $\vec{\chi}_c(D)$ is attained.

(ii) $\lceil \vec{\chi}_c(D) \rceil = \vec{\chi}(D)$.

(iii) $\vec{C}(p,q) \overset{\mathrm{a}}{\to} \vec{C}(p',q')$ if and only if $\frac{p}{q} \leq \frac{p'}{q'}$; in particular, $\vec{\chi}_c(C(p,q)) = \frac{p}{q}$.

Recall that the dichromatic number of a digraph is the maximum dichromatic number over all its strongly connected components. Hochstättler and Steiner observed that the circular dichromatic number no longer has this nice property. Given a digraph D, we denote by D^s the digraph obtained from D by adding an extra vertex s and all possible arcs from s to the rest of the vertices.

Proposition 7.4. [98] *For any digraph D, $\vec{\chi}_c(D^s) = \vec{\chi}(D)$.*

The second extension of the circular chromatic number arose in the work of Steiner [149, Chapter 6]. Let $p \geq q$ be positive integers and D be a digraph. A *strict acyclic (p,q)-dicolouring* of D is a function $f : V(D) \to [p]$ such that the subgraph $D_{(p,q),f}$ obtained from D by removing the arcs in

$$\{(u,v) \in A(D) \mid q \leq |f(v) - f(u)| \leq p - q\}$$

is acyclic. The *strict star dichromatic number*[1] of D is defined as

$$\vec{\chi}^{\circledast}(D) = \inf \left\{ \frac{p}{q} \;\middle|\; \begin{array}{c}\text{there is a strict acyclic } (p,q)\text{-dicolouring} \\ \text{of } D \text{ for some positive integers } p \geq q\end{array} \right\}.$$

We note that in this case $\vec{\chi}^{\circledast}(D)$ can be computed from the strongly connected components of D. Moreover, it has a natural formulation in terms of back-arc graphs.

Observation 7.5. *For any digraph D, $\vec{\chi}^{\circledast}(D) = \min_{\preceq \in \mathrm{O}(V(D))} \chi_c(D^{\preceq})$, where $\mathrm{O}(V(D))$ is the set of total orders on $V(D)$.*

Proof. If f is a strict acyclic (p,q)-dicolouring of D, the subgraph $D_{(p,q),f}$ is acyclic, so we can find an order $\preceq \in \mathrm{O}(V(D))$ such that all arcs of $D_{(p,q),f}$ are increasing. Thus, $(D_{(p,q),f})^{\preceq}$ has no edges, so f is a circular (p,q)-colouring of $D^{\preceq}$. This implies that $\vec{\chi}^{\circledast}(D) \geq \min_{\preceq \in \mathrm{O}(V(D))} \chi_c(D^{\preceq})$.

Conversely, given any $\preceq \in \mathrm{O}(V(D))$ and any circular (p,q)-colouring φ of $D^{\preceq}$, $(D_{(p,q),\varphi})^{\preceq}$ has no edges, so $D_{(p,q),\varphi}$ is acyclic and φ is a strict acyclic (p,q)-dicolouring of D. Hence, $\vec{\chi}^{\circledast}(D) \leq \min_{\preceq \in \mathrm{O}(V(D))} \chi_c(D^{\preceq})$. □

Steiner points out that, unlike the circular dichromatic number, the strict star dichromatic number does not make a distinction between directed cycles of different lengths. Indeed, for $n \geq 2$, $\vec{\chi}_c(\vec{C}_n) = \vec{\chi}_c(\vec{C}(n, n-1)) = 1 + \frac{1}{n-1}$, while $\vec{\chi}^{\circledast}(\vec{C}_n) = 2$.

More generally, we observe the following discrepancy between the two parameters: for $p > q > 1$, $\vec{\chi}^{\circledast}(\vec{C}(p,q)) > \frac{p}{q} = \vec{\chi}_c(\vec{C}(p,q))$. This only needs to be shown for p and q coprime (for each positive integer k, $\vec{C}(p,q)$ is isomorphic to a subgraph of $\vec{C}(kp,kq)$). Since $p > q$, the sequence $p, p-1, \ldots, 2, 1, p$ defines a directed p-cycle C of $\vec{C}(p,q)$. For every order $\preceq \in \mathrm{O}(V(\vec{C}(p,q)))$, $C^{\preceq}$ has at least one edge e. Since $q > 1$, e is not an edge of $C(p,q)$, so $\vec{C}(p,q)^{\preceq}$ is a proper supergraph of $C(p,q)$ on the same vertex set. Proposition 7.2 implies that $\vec{C}(p,q)^{\preceq} \not\to C(p,q)$, so $\chi_c(\vec{C}(p,q)^{\preceq}) > \frac{p}{q}$.

On the other hand, using the trick of Hochstättler and Steiner (Proposition 7.4), one can easily find digraphs D with $\vec{\chi}_c(D) > \vec{\chi}^{\circledast}(D)$. So, in general, the circular dichromatic number and the strict star dichromatic number are incomparable.

The third extension of the circular chromatic number was introduced by Hochstättler and Steiner [98], and further studied by the same authors and Schröder in [97]. It is associated to another type of homomorphism. A *circular homomorphism* from a digraph D to a digraph D' is a mapping $\varphi : V(D) \to V(D')$ such that, for every acyclic set $A \subseteq V(D')$ of D', $\varphi^{-1}(A)$ is an acyclic set of D. (Equivalently, for each directed cycle C of D, $D'[\varphi(V(C))]$ has a directed cycle.) We write that as $\varphi : D \xrightarrow{c} D'$, and, when

[1] Originally named *star dichromatic number* by Steiner.

such a φ exists, we write $D \overset{c}{\to} D'$. We note that this notion generalises acyclic homomorphisms. Analogously as before, the composition of circular homomorphisms is again a circular homomorphism, they coincide with digraph homomorphisms when D is symmetric, and they are k-dicolourings of D when $D' = \overleftrightarrow{K}_k$.

Given positive integers $p \geq q$, an *acyclic* (p, q)*-dicolouring* of a digraph D is a circular homomorphism $\varphi : D \overset{c}{\to} \vec{C}(p, q)$. The *star dichromatic number* of D is defined as

$$\vec{\chi}^*(D) = \inf \left\{ \frac{p}{q} \,\middle|\, \exists p, q \in \mathbb{Z}^+ \ p \geq q,\ D \overset{c}{\to} \vec{C}(p, q) \right\}.$$

Again, this is accompanied with its own kit of basic properties.

Theorem 7.6. [98] *For any digraph D and positive integers p, q, p', q' with $\frac{p}{q}, \frac{p'}{q'} \geq 1$, the following hold.*

(*i*) *The infimum in the definition of $\vec{\chi}^*(D)$ is attained.*

(*ii*) $\lceil \vec{\chi}^*(D) \rceil = \vec{\chi}(D)$.

(*iii*) $\vec{C}(p, q) \overset{c}{\to} \vec{C}(p', q')$ *if and only if* $\frac{p}{q} \leq \frac{p'}{q'}$; *in particular,* $\vec{\chi}^*(C(p, q)) = \frac{p}{q}$.

Moreover, since both circular (p, q)-dicolourings and strict acyclic (p, q)-dicolourings are acyclic (p, q)-dicolourings, $\vec{\chi}^*(D) \leq \min\{\vec{\chi}_c(D), \vec{\chi}^{\circledast}(D)\}$ for any digraph D. The same arguments as above show that the inequality is sometimes strict.

The *acyclicity complex* of a digraph D, denoted by $\mathcal{A}(D)$, is the family of all the acyclic sets of D. The particular nature of circular homomorphisms makes $\vec{\chi}^*(D)$ depend only on $\mathcal{A}(D)$, rather than on the specific disposition of the arcs of D. It turns out that this extra freedom is the only reason why $\vec{\chi}^*(D)$ can get ahead of $\vec{\chi}_c(D)$.

Proposition 7.7. *For any digraph D,*

$$\vec{\chi}^*(D) = \min\{\vec{\chi}_c(D') \mid D' \text{ is a digraph on } V(D) \text{ with } \mathcal{A}(D') \subseteq \mathcal{A}(D)\}.$$

Proof. Let D' be any digraph with vertex set $V(D)$ satisfying $\mathcal{A}(D') \subseteq \mathcal{A}(D)$, and let $p \geq q$ be positive integers for which there is an acyclic homomorphism $\varphi : D' \overset{a}{\to} \vec{C}(p, q)$. Then φ is a circular homomorphism from D to $\vec{C}(p, q)$, proving the inequality $\leq$.

To prove the other inequality, let us assume that there is a circular homomorphism $\varphi : D \overset{c}{\to} \vec{C}(p, q)$. For each $1 \leq k \leq p$, we choose an arbitrary total order $\leq_k$ on $\varphi^{-1}(k)$. Let $\mathcal{C}(D)$ be the family directed cycles of D. For each $C \in \mathcal{C}(D)$, we label its vertices as $v_1^C, \ldots, v_{\ell(C)}^C$, where $\ell(C)$ is the length of C, in a way that $\varphi(v_1^C) \leq \ldots \leq \varphi(v_{\ell(C)}^C)$ and that, for all $1 \leq i \leq j \leq \ell(C)$, if $\varphi(v_i^C) = \varphi(v_j^C) = k$ then $v_i^C \leq_k v_j^C$.

Now we define a digraph D' with vertex set $V(D') = V(D)$ by setting

$$A(D') = \bigcup_{C \in \mathcal{C}(D)} \left\{ \left(v_1^C, v_2^C\right), \ldots, \left(v_{\ell(C)-1}^C, v_{\ell(C)}^C\right), \left(v_{\ell(C)}^C, v_1^C\right) \right\}.$$

D' may depend on the choices of the orders $\leq_k$, but it always satisfies that $\mathcal{A}(D') \subseteq \mathcal{A}(D)$. Indeed, suppose that there is an acyclic set X of D' which is not an acyclic set of D. This means that there is a directed cycle C of D with $V(C) \subseteq X$. By definition of D', $D'[V(C)]$ has a directed cycle, contradicting that $D'[X]$ is acyclic.

We will end the proof by showing that the function $\psi : V(D') \to V(\vec{C}(p,q))$ defined by $\psi(x) = p + 1 - \varphi(x)$ is an acyclic homomorphism from D' to $\vec{C}(p,q)$. Let us assume that, for some $1 \leq k \leq p$, $\psi^{-1}(k)$ is not an acyclic set of D'. We can then find an arc $(x,y) \in A(D')$ with $\psi(x) = \psi(y) = k$ and $x >_k y$. Thus, we have that $(x,y) = \left(v_{\ell(C)}^C, v_1^C\right)$ for some $C \in \mathcal{C}(D)$. This implies that $\varphi(V(C)) \subseteq \{p + 1 - k\}$, contradicting that $\varphi : D \overset{c}{\to} \vec{C}(p,q)$.

Now, let (x,y) be any arc of D' such that $\psi(x) \neq \psi(y)$. If $\psi(x) < \psi(y)$, then $\varphi(x) > \varphi(y)$, so $(x,y) = \left(v_{\ell(C)}^C, v_1^C\right)$ for some $C \in \mathcal{C}(D)$. Hence, $\varphi(V(C)) \subseteq \{\varphi(y), \varphi(y)+1, \ldots, \varphi(x)\}$. Since $\varphi : D \overset{c}{\to} \vec{C}(p,q)$,

$$q \leq \varphi(x) - \varphi(y) = \psi(y) - \psi(x) \leq p - 1.$$

This implies that $(\psi(x), \psi(y)) \in A(\vec{C}(p,q))$.

If, instead, $\psi(x) > \psi(y)$, then $\varphi(x) < \varphi(y)$, so there is some $C \in \mathcal{C}(D)$ and some $1 \leq i \leq \ell(C) - 1$ such that $(x,y) = \left(v_i^C, v_{i+1}^C\right)$. Hence, $\varphi(V(C)) \subseteq \{\varphi(y), \varphi(y)+1, \ldots, p, 1, 2, \ldots, \varphi(x)\}$. Since $\varphi : D \overset{c}{\to} \vec{C}(p,q)$,

$$q \leq p + \varphi(x) - \varphi(y) = p + \psi(y) - \psi(x) \leq p - 1,$$

so $(\psi(x), \psi(y)) \in A(\vec{C}(p,q))$. Thus, ψ is an acyclic homomorphism from D' to $\vec{C}(p,q)$, as we wanted to see. □

7.2 Fractional colourings

Let $n \geq k$ be two positive integers. The *Kneser graph* $KG(n,k)$ is the graph with vertex set $\binom{[n]}{k}$ where two vertices are adjacent if and only if they are disjoint. Of course, the interesting case is when $n \geq 2k$. A *k-tuple n-colouring* of a graph G is a homomorphism $\varphi : G \to KG(n,k)$. Note that the 1-tuple colourings of G correspond to its proper colourings. The *fractional chromatic number* of G is defined as

$$\chi_f(G) = \inf \left\{ \frac{n}{k} \,\middle|\, \exists n, k \in \mathbb{Z}^+ \; n \geq k, \; G \to KG(n,k) \right\}.$$

Here are some of the properties of this parameter.

Theorem 7.8. [95] *For any graph G and positive integers n, k, n', k' with $\frac{n}{k}, \frac{n'}{k'} \geq 2$, the following hold.*

(*i*) *The infimum in the definition of $\chi_f(G)$ is attained.*

(*ii*) $\frac{|V(G)|}{\alpha(G)} \leq \chi_f(G) \leq \chi_c(G)$.

(*iii*) *If* $\frac{n}{k} > \frac{n'}{k'}$, $KG(n,k) \not\to KG(n',k')$; *in particular,* $\chi_f(KG(n,k)) = \frac{n}{k}$.

Alternatively, the fractional chromatic number can be defined in terms of a linear program. Let $\mathcal{I}(G)$ be the family of independent sets of G. Then, $\chi_f(G)$ coincides with the optimum of

$$\begin{array}{lll} \underset{(x_I)_{I\in\mathcal{I}(G)}\in\mathbb{R}^{|\mathcal{I}(G)|}}{\text{minimise}} & \sum_{I\in\mathcal{I}(G)} x_I & \\ \text{subject to} & \forall v \in V(G) & \sum_{I \ni v} x_I \geq 1, \\ & \forall I \in \mathcal{I}(G) & x_I \geq 0 \end{array} \tag{7.1}$$

(see for instance [95]). If the minimisation is over $\mathbb{Z}^{|\mathcal{I}(G)|}$ instead, we again recover $\chi(G)$.

This notion has also been brought to the directed setting. For positive integers $n \geq k$, a *k-tuple n-dicolouring* of a digraph D is a function $f : V(D) \to \binom{[n]}{k}$ with the property that, for each non-empty family of k-sets $B \subseteq \binom{[n]}{k}$ with $\cap B \neq \emptyset$, $f^{-1}(B)$ induces an acyclic subgraph of D. The *fractional dichromatic number* of D is defined as

$$\vec{\chi}_f(D) = \inf \left\{ \frac{n}{k} \;\middle|\; \begin{array}{c} \text{there is a } k\text{-tuple } n\text{-dicolouring of } D \\ \text{for some positive integers } n \geq k \end{array} \right\}.$$

As before, $\vec{\chi}_f(D)$ can be defined in terms of a linear program [140, Chapter 5]: in (7.1), one just has to replace G by D, and $\mathcal{I}(G)$ by the acyclicity complex $\mathcal{A}(D)$ of D. We also have the following version of Theorem 7.8.

Theorem 7.9. [98, 140] *For any digraph D, the following hold.*

(*i*) *The infimum in the definition of $\vec{\chi}_f(D)$ is attained.*

(*ii*) $\frac{|V(D)|}{\vec{\alpha}(D)} \leq \vec{\chi}_f(D) \leq \vec{\chi}^*(D)$.

Unlike in the case of circular colourings and circular complete graphs, when fractional colourings are generalised to digraphs no clear directed analogue of Kneser graphs seems to show up. This prompted Severino [140, Chapter 7] and Hochstättler, Schröder and Steiner [97] to ask for the existence of such digraphs.

Let us introduce some new concepts that will allow us to formulate this question with more precision. Let $n \geq k$ be positive integers and K a digraph with vertex set $\binom{[n]}{k}$. We say that

- K has the *acyclic tuple-dicolouring property* (for short, $\mathrm{P_a}$) if, for every digraph D and every $B \subseteq \binom{[n]}{k}$, there exists a k-tuple n-dicolouring $f : V(D) \to B$ of D if and only if $D \xrightarrow{\mathrm{a}} K[B]$;
- K has the *circular tuple-dicolouring property* (for short, $\mathrm{P_c}$) if, for every digraph D and every $B \subseteq \binom{[n]}{k}$, there exists a k-tuple n-dicolouring $f : V(D) \to B$ of D if and only if $D \xrightarrow{\mathrm{c}} K[B]$;
- K has the *intersecting–acyclic property* (for short, P) if, for every non-empty $B \subseteq \binom{[n]}{k}$, $\cap B \neq \emptyset$ if and only if B is an acyclic set of K.

Proposition 7.10. *Let $n \geq k$ be positive integers and K a digraph with vertex set $\binom{[n]}{k}$.*

(i) If K has $\mathrm{P_a}$, then it has P.

(ii) K has $\mathrm{P_c}$ if and only if it has P*, if and only if, for every digraph D, the k-tuple n-dicolourings of D are precisely the circular homomorphisms from D to K.*

Proof. We first show (i) and one direction of (ii): we assume that K has $\mathrm{P_a}$ or $\mathrm{P_c}$, and want to see that K has P. Let $* \in \{\mathrm{a}, \mathrm{c}\}$ and $B \subseteq \binom{[n]}{k}$ non-empty. Let us assume for a contradiction that $\cap B \neq \emptyset$ but $K[B]$ is not acyclic. Then, $\vec{C}_\ell \to K[B]$ for some $\ell \geq 2$. By P_*, there is a k-tuple n-dicolouring $f : V(\vec{C}_\ell) \to B$ of $\vec{C}_\ell$. Thus, $f^{-1}(B)$ is acyclic, a contradiction. Now let us assume, again for a contradiction, that $\cap B = \emptyset$ but $K[B]$ is acyclic. We consider a bijection $f : V(\vec{C}_{|B|}) \to B$. We have that, for every $B' \subsetneq B$, $f^{-1}(B')$ is an acyclic set of $\vec{C}_{|B|}$. Hence, f is a k-tuple n-dicolouring of $\vec{C}_{|B|}$. By P_*, $\vec{C}_{|B|} \xrightarrow{*} K[B]$, a contradiction. Thus, K has P.

Let us now suppose that K has P. Then, for any digraph D and any $B \subseteq \binom{[n]}{k}$, by definition, a function $f : V(D) \to B$ is a k-tuple n-dicolouring of D if and only if it is a circular homomorphism from D to $K[B]$, so K has P. □

Hochstättler, Schröder and Steiner [97] give directed n-cycles with vertex set $\binom{[n]}{n-1}$ as examples of digraphs with the intersecting–acyclic property. It turns out that such digraphs exist only for a few values of k.

Theorem 7.11. *Let $n \geq k$ be positive integers. There exists a digraph K with vertex set $\binom{[n]}{k}$ satisfying the intersecting–acyclic property if and only if $k \in \{1, 2, n-1, n\}$. Moreover, in each of these cases, K is unique up to isomorphism.*

Proof. If $k = 1$, it is clear that we can take K to be the complete digraph on $[n]$, and this is the only possibility. If $k = n - 1$, it is also clear that we can take K to be any directed n-cycle on $\binom{[n]}{n-1}$, and these are the only possibilities. And if $k = n$, K is the one-vertex digraph.

For $k = 2$, we construct a digraph $KD(n, 2)$ as follows. Let $\overleftrightarrow{KG}(n, 2)$ be the digraph obtained from the Kneser graph $KG(n, 2)$ after replacing each edge by a digon. For every $i \in [n]$, let T_i be the transitive tournament with vertex set $\{\{i, j\} \in \binom{[n]}{k} \mid j \in [n] \setminus \{i\}\}$ that has a directed path along the vertex sequence

$$\{i, i+1\}, \{i, i+2\}, \ldots, \{i, n\}, \{i, 1\}, \{i, 2\}, \ldots \{i, i-1\}.$$

The vertex set of $KD(n, 2)$ is defined as $\binom{[n]}{k}$ and its arc set as

$$A(KD(n, 2)) = A\left(\overleftrightarrow{KG}(n, 2)\right) \cup \bigcup_{i=1}^{n} A(T_i).$$

We claim that $KD(n, 2)$ has the intersecting–acyclic property. Let us assume for a contradiction that there is a non-empty $B \subseteq \binom{[n]}{k}$ with $\cap B \neq \emptyset$ which is not acyclic. Let $i \in \cap B$. Since $B \subseteq V(T_i)$, this implies that there are two distinct vertices $u, v \in B$ inducing a digon. Since $i \in u \cap v$, neither (u, v) nor (v, u) is an arc of $\overleftrightarrow{KG}(n, 2)$. Thus, u and v are vertices of T_j for some $j \in [n] \setminus \{i\}$. But then, since $k = 2$, $u = v = \{i, j\}$, a contradiction. Let us now consider a non-empty $B \subseteq \binom{[n]}{k}$ with $\cap B = \emptyset$, and show that B is not acyclic. If B has two disjoint elements u and v, then they induce a digon, because $\{u, v\}$ is an edge of $KG(n, 2)$. We can thus assume that $\{i_0, i_1\}, \{i_0, i_2\} \in B$ for some pairwise distinct $i_0, i_1, i_2 \in [n]$. Moreover, B has another element $\{i_3, i_4\}$ with $i_0 \notin \{i_3, i_4\}$. We distinguish two cases.

Case 1: $i_1 \notin \{i_3, i_4\}$.

Then $\{i_0, i_1\} \cap \{i_3, i_4\} = \emptyset$, so $\{i_0, i_1\}$ and $\{i_3, i_4\}$ induce a digon.

Case 2: $i_1, i_2 \in \{i_3, i_4\}$.

Then, $\{i_0, i_1\}, \{i_0, i_2\}, \{i_1, i_2\} \in B$. By symmetry, we can assume that $i_0 < i_1 < i_2$. Since $(\{i_0, i_1\}, \{i_0, i_2\})$ is an arc of T_{i_0}, $(\{i_1, i_2\}, \{i_1, i_0\})$ is an arc of T_{i_1} and $(\{i_2, i_0\}, \{i_2, i_1\})$ is an arc of T_{i_2}, $KD(n, 2)[B]$ has a directed triangle.

We have thus shown that $KD(n, 2)$ has the intersecting–acyclic property.

Now we will show that, if $k = 2$ and K has the intersecting–acyclic property, $K \cong KD(n, 2)$. It is clear from the definition that, under these assumptions, all arcs of $\overleftrightarrow{KG}(n, 2)$ are arcs of K. We note that, for any $i \in [n]$, the subgraph $K^{(i)}$ of K induced by $\{\{i, j\} \in \binom{[n]}{k} \mid j \in [n] \setminus \{i\}\}$ is a transitive tournament. Indeed, $K^{(i)}$ is acyclic by definition, and any two distinct vertices $\{i, i'\}$ and $\{i, i''\}$ of $K^{(i)}$ form a directed triangle with $\{i', i''\}$. We claim that $K^{(1)}$ determines K; then, it will follow that $K \cong KD(n, 2)$. A quick analysis shows that this is true for $n \in \{2, 3, 4\}$. Let us see that, for $n \geq 5$, the direction of the arc between any two distinct intersecting vertices

$\{i_0, i_1\}$ and $\{i_0, i_2\}$ of K is determined by $K^{(1)}$. Indeed, it is enough to look its direction in the subgraph of K induced by $\binom{\{1,i_0,i_1,i_2\}}{2}$; by the above argument, this subgraph is determined by $K^{(1)}$. Thus, $K \cong KD(n, 2)$.

Finally, let us assume for a contradiction that $3 \le k \le n-2$ and there is a digraph K on $\binom{[n]}{k}$ with the intersecting–acyclic property. We consider the set of vertices $B = \{[k+1]\setminus\{1\}, [k+1]\setminus\{2\}, \dots, [k+1]\setminus\{k+1\}\}$. We note that $K[B]$ is a directed cycle; in particular, each vertex of B has exactly one in-neighbour and one out-neighbour in B. Since $|B| = k + 1 \ge 4$, we can choose two distinct vertices $v_i = [k+1]\setminus\{i\}$ and $v_j = [k+1]\setminus\{j\}$ of B that are not consecutive in $K[B]$. Hence, if we let v_i^- and v_j^- be their respective in-neighbours in $K[B]$, then v_i, v_i^-, v_j and v_j^- are pairwise distinct. Now, we consider the set of vertices $B' = B\setminus\{v_i, v_j\}\cup\{[k+2]\setminus\{i, j\}\}$; here we use that $k \le n-2$. Again, $K[B']$ is a directed cycle. Hence, both v_i^- and v_j^- are in B', and one of them must have an out-neighbour in $B'\setminus\{[k+2]\setminus\{i, j\}\}$, the desired contradiction. □

Observation 7.12. *Given an integer $n \ge 2$, let $KD(n, 2)$ be the digraph from Theorem 7.11.*

(*i*) $\vec{\chi}_f(KD(n, 2)) = \frac{n}{2}$.

(*ii*) *Any circular homomorphism from $KD(n, 2)$ to itself is a bijection.*

Proof. (*i*) Since the identity function on $\binom{[n]}{2}$ is a 2-tuple n-dicolouring of $KD(n, 2)$, $\vec{\chi}_f(KD(n, 2)) \le \frac{n}{2}$. Since $\overleftrightarrow{KG}(n, 2)$ is a subgraph of $KD(n, 2)$, $\vec{\chi}_f(KD(n, 2)) \ge \frac{n}{2}$.

(*ii*) Let $\varphi : KD(n, 2) \xrightarrow{c} KD(n, 2)$; we have that φ is an endomorphism of $KG(n, 2)$. It is well-known that, for $n > 2k$, the Kneser graph $KG(n, k)$ is a core [73, Theorem 7.9.1]. Therefore, we only need to prove our claim for $n = 4$.

Let us assume for a contradiction that $\varphi(u) = \varphi(v)$ for some distinct $u, v \in V(KD(4, 2))$. This implies that $u \cap v \ne \emptyset$, so their symmetric difference $w = u \,\Delta\, v$ is a vertex of $KD(4, 2)$. Since the set $\{u, v, w\}$ is not acyclic, $\{\varphi(u), \varphi(w)\}$ is not acyclic either, so $\varphi(w) = [n]\setminus\varphi(u)$. But then φ sends all the elements of the non-acyclic set $\{[n]\setminus u, [n]\setminus v, [n]\setminus w\}$ to $[n]\setminus\varphi(u)$, a contradiction. □

Corollary 7.13. *Let $n \ge k$ be positive integers.*

(*i*) *There exists a digraph K with vertex set $\binom{[n]}{k}$ satisfying the acyclic tuple-dicolouring property if and only if $k \in \{1, n\}$.*

(*ii*) *There exists a digraph K with vertex set $\binom{[n]}{k}$ satisfying the circular tuple-dicolouring property if and only if $k \in \{1, 2, n-1, n\}$.*

Proof. (ii) is a consequence of Theorem 7.11 and Proposition 7.10(ii). For (i), if $k \in \{1, n\}$ one can take as K the complete digraph on n/k vertices. So let us assume that $2 \leq k \leq n-1$ and show that such a K does not exist. Theorem 7.11 and Proposition 7.10(i) imply that K can only exist if $k \in \{2, k\}$, and in these cases it must be isomorphic to $\vec{C}_n$ or $KD(n, 2)$, the digraph from Theorem 7.11. Let K^s be the digraph obtained from K by adding a vertex s and all possible arcs from s to the rest of the vertices. Since $K^s \xrightarrow{c} K$, by Theorem 7.11 K^s has a k-tuple n-dicolouring. It will be enough to show that there is no acyclic homomorphism $\varphi : K^s \xrightarrow{a} K$.

If such φ exists, it is surjective. Indeed, this is clear if $K \cong \vec{C}_n$, and, if $K \cong KD(n, 2)$, it follows from Observation 7.12(ii). But then, there must be an arc from $\varphi(s)$ to every other vertex of K, a contradiction. □

Although this answers the question of Hochstättler, Schröder and Steiner, it is still unclear whether the existence of k-tuple n-colourings is equivalent to the existence of an acyclic or circular homomorphism to some specific digraph.

Problem 7.14. *Let $n \geq k$ be positive integers.*

- *Is there a digraph K such that, for every digraph D, D has a k-tuple n-dicolouring if and only if $D \xrightarrow{a} K$?*
- *Is there a digraph K such that, for every digraph D, D has a k-tuple n-dicolouring if and only if $D \xrightarrow{c} K$?*

We believe that the answer is negative in all cases, except maybe in the second question when $\frac{n}{k} \leq 2$.

Chapter 8

Concentration of the dichromatic number of random digraphs

8.1 Introduction

We point the reader to Section 1.4 for the basic definitions about random graphs and digraphs.

The chromatic number of the binomial random graph $\mathbb{G}(n,p)$ has received a lot of attention. In 1975, Grimmet and McDiarmid [78] determined the typical order of magnitude of $\chi(\mathbb{G}(n,p))$ for p constant. Later, Bollobás gave its precise asymptotics.

Theorem 8.1. [31] *For any fixed* $0 < p < 1$*, with probability tending to* 1 *as* $n \to \infty$*,*

$$\chi(\mathbb{G}(n,p)) \sim \frac{\ln \frac{1}{1-p}}{2} \frac{n}{\ln n}.$$

A similar behaviour was observed by Łuczak in the case $p = p(n) \to 0$.

Theorem 8.2. [113] *There exists a constant* d_0 *such that, if* $p = p(n) \to 0$ *and* $d = d(n) = np \geq d_0$*, then, with probability tending to* 1 *as* $n \to \infty$*,*

$$\frac{d}{2\ln d}\left(1 + \frac{\ln\ln d - 1}{\ln d}\right) < \chi(\mathbb{G}(n,p)) < \frac{d}{2\ln d}\left(1 + \frac{30\ln\ln d}{\ln d}\right).$$

Another question that has been studied is: regardless of its typical values, how much concentrated is $\chi(\mathbb{G}(n,p))$? This goes back to a paper of Shamir and Spencer from 1987 [141], in which they show that, for any fixed p and any function f going to infinity, $\chi(\mathbb{G}(n,p))$ lies on an unspecified interval I_n of length $f(n)\sqrt{n}$ with high probability. This was later improved to $f(n)\sqrt{n}/\ln n$, see [18, Section 7.9, Exercise 3] and [139].

In the sparse case, the length of such an interval can be further reduced. For $p = p(n) = n^{-\alpha}$ with $0 < \alpha < 1/2$ fixed, and f as above, Shamir and Spencer show in their paper that length $f(n)p\sqrt{n}\ln n$ is enough. Surya and Warnke [152] have recently improved this to $f(n)p\sqrt{n}/\ln n$. The case $p = n^{-\alpha}$ for $1/2 < \alpha < 1$ fixed is perhaps the most striking one. Here Shamir and Spencer show that, with high probability, $\chi(\mathbb{G}(n,p))$ falls on an interval of constant length $\frac{2\alpha+1}{2\alpha-1}$. Łuczak [112] reduces it to just two consecutive integers when $\alpha > 5/6$, and Alon and Krivelevich [14] extend this to all $\alpha > 1/2$.

In 2005, Achlioptas and Naor were able to determine these two mysterious integers in the case np constant.

Theorem 8.3. [7] *Given a positive real number d, let k_d be the smallest integer k such that $2k\ln k > d$. With probability tending to 1 as $n \to \infty$,*

$$\chi(\mathbb{G}(n, d/n)) \in \{k_d, k_d + 1\}.$$

Coja-Oghlan, Panagiotou and Steger [49] have later extended this to the case $p \le n^{-\alpha}$ for $\alpha > 3/4$ fixed, providing three explicit possible values.

Regarding the opposite type of results, that is, about non-concentration, not much was known until Heckel's breakthrough [93], who proved that, for $p = 1/2$ and any constant $c < 1/4$, there is no sequence of intervals $(I_n)_{n\in\mathbb{Z}^+}$ of length $\sup I_n - \inf I_n = n^c$ such that I_n contains $\chi(\mathbb{G}(n,p))$ with high probability. In a follow-up paper, Heckel and Riordan [94] extend this to all constant $0 < p < 1$ and also increase the lengths of the intervals, bringing them near the upper bound mentioned above.

Much less is known about the analogous questions for random digraphs. A first obstacle is that computing the probability that a given subset of vertices is acyclic is already a challenging problem. In spite of that, there do exist relevant asymptotic results concerning acyclic sets in random digraphs. The acyclicity number of random digraphs was first investigated by Subramanian [151]. Spencer and Subramanian [145] found its asymptotics, and Dutta and Subramanian [56] have later given more precise bounds.

Theorem 8.4. [56, 145] *For every constant $W > 2$, there exists a suitably large constant d_0 such that, for $d_0/n \le p = p(n) \le 1/2$ and $d = d(n) = np$,*

$$\frac{2}{\ln \frac{1}{1-p}} (\ln d - W) \le \vec{\alpha}(\mathbb{O}(n,p)) \le \frac{2}{\ln \frac{1}{1-p}} (\ln d + 3e)$$

with probability tending to 1 as $n \to \infty$. The same statement holds with $\mathbb{D}(n,p)$ in the place of $\mathbb{O}(n,p)$.

The upper bound from Theorem 8.4, together with the fact that the random back-arc graphs $\mathbb{O}(n,p)^{\le}$ and $\mathbb{D}(n,p)^{\le}$ have the same distribution as $\mathbb{G}(n,p)$, readily yields the following analogues of Theorems 8.1 and 8.2 (see for instance [24]).

Corollary 8.5. *For any fixed* $0 \le p \le 1/2$*, with probability tending to* 1 *as* $n \to \infty$*,*

$$\vec{\chi}(\mathbb{O}(n,p)) \sim \frac{\ln \frac{1}{1-p}}{2} \frac{n}{\ln n}.$$

The same statement holds with $\mathbb{D}(n,p)$ *in the place of* $\mathbb{O}(n,p)$*.*

Corollary 8.6. *There exists a constant* d_0 *such that, if* $p = p(n) \to 0$ *and* $d = d(n) = np > d_0$*, then, with probability tending to* 1 *as* $n \to \infty$*,*

$$\frac{d}{2\ln d}\left(1 - \frac{3e}{\ln d + 3e}\right) \le \vec{\chi}(\mathbb{O}(n,p)) < \frac{d}{2\ln d}\left(1 + \frac{30 \ln\ln d}{\ln d}\right).$$

The same statement holds with $\mathbb{D}(n,p)$ *in the place of* $\mathbb{O}(n,p)$*.*

In a detailed study [55], Dogval, de Panafieu, Ralaivaosaona, Rasendrahasina and Wagner examine random digraphs of constant average degree using the symbolic method. They obtain neat asymptotic expressions of the probability that a random digraph is acyclic (see their paper for the precise statement).

Theorem 8.7. [55, Theorem 7.1] *Let* $p = p(n) = d/n$ *with* $d \ge 0$ *fixed. Then, as* $n \to \infty$*, the probability that* $\mathbb{O}(n,p)$ *is acyclic is asymptotically equivalent to*

$$\begin{cases} \delta_1(d)(1-d) & \text{if } 0 \le d < 1 \\ b_1\delta_1(1)n^{-1/3} & \text{if } d = 1 \\ b_2(d)\delta_2(d)n^{-1/3}\exp\left(-\alpha(d)n + a_1\beta(d)n^{1/3}\right) & \text{if } d > 1, \end{cases}$$

where

$$\alpha(d) = \frac{d^2-1}{2d} - \ln d, \quad \beta(d) = 2^{-1/3}d^{-1/3}(d-1),$$

b_2*,* δ_1 *and* δ_2 *are other functions, and* $a_1 < 0$ *and* b_1 *are constants. The same statement holds with* $\mathbb{D}(n,p)$ *in the place of* $\mathbb{O}(n,p)$*, for some different* δ_1 *and* δ_2*.*

More references concerning the enumeration of acyclic digraphs can be found in [55, 80]. With Theorem 8.7 in hand, one can extend the bounds of Corollary 8.6 to all constant d using a first moment argument, see Corollary 8.9 below.

Although the bounds of Corollary 8.9 are, in absolute terms, far from each other, it does not seem too adventurous to believe that some directed analogue of Theorem 8.3 can be obtained. In Section 8.3 we adapt Alon and Krivelevich's result [14] to the directed setting, showing that the dichromatic number of sparse random digraphs is indeed two-point concentrated (Theorem 8.13).

In denser regimes, a vanilla vertex exposure martingale argument still yields that $\vec{\chi}(\mathbb{O}(n,p))$ is concentrated in an interval of any length $\omega(\sqrt{n})$, but this can likely be improved.

Proposition 8.8. *Let $p = p(n) \in [0,1]$ and $f(n)$ be functions of n with $f(n) \to \infty$ as $n \to \infty$, and let $X = \vec{\chi}(\mathbb{O}(n, p/2))$. Then,*

$$|X - \mathbb{E}(X)| \leq f(n)\sqrt{n}$$

with probability tending to 1 as $n \to \infty$. The same statement holds with $\mathbb{D}(n,p)$ in the place of $\mathbb{O}(n,p/2)$.

Proof. The claim is immediate from the fact that the function $L : \mathscr{D}(n) \to \mathbb{R}$ defined by $L(D) = \vec{\chi}(D)$ satisfies the condition from Lemma B.3. □

These results provide evidence of the similarity between the chromatic number of random graphs and the dichromatic number of random digraphs. It would be interesting to try to push them and see when do dissimilarities start appearing. One reason to believe that this might happen is that there is indeed a second order discrepancy between $\alpha(\mathbb{G}(n,p))$ and $\vec{\alpha}(\mathbb{O}(n,p))$: compare Theorem 8.4 with Frieze's estimate of the independence number of random graphs (Theorem 9.5). A question in this direction would be, for example, whether the upper bound from Corollary 8.9 is optimal.

Another aspect that would be worth investigating is the concentration of the acyclicity number of random digraphs. It is a classical result of Matula [116] and Bollobás and Erdős [32] (see also [30]) that the independence number of dense random graphs is two-point concentrated; this has recently been extended to sparser regimes by Bohman and Hofstad [27]. Dutta and Subramanian [57] have shown that the largest size of a transitive tournament in a random digraph is also two-point concentrated.

8.2 More explicit bounds

In this section we prove the following consequence of Theorem 8.7, which complements Corollaries 8.5 and 8.6.

Corollary 8.9. *Given a positive real number d, let k_d be the smallest integer k such that $2k \ln k > d$. With probability tending to 1 as $n \to \infty$,*

$$\frac{d}{\sqrt{\ln^2 d + 1} + \ln d} \leq \vec{\chi}(\mathbb{O}(n, d/n)) \leq k_d + 1.$$

The same statement holds with $\mathbb{D}(n,p)$ in the place of $\mathbb{O}(n,p)$.

Proof. The upper bound follows from Theorem 8.3 and the fact that the random back-arc graphs $\mathbb{O}(n,p)^{\leq}$ and $\mathbb{D}(n,p)^{\leq}$ have the same distribution as $\mathbb{G}(n,p)$. We prove the lower bound for $\mathbb{O}(n,p)$; the proof for $\mathbb{D}(n,p)$ is analogous.

We note that the lower bound can be rewritten as $d\sqrt{\ln^2 d + 1} - d \ln d$. This is strictly increasing for $d \neq 0$, so for $0 < d \leq 1$ the bound is trivial.

Hence, we can assume that $d > 1$. We consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x\alpha(x) & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 1, \end{cases}$$

where α is as in Theorem 8.7. It is a routine exercise to check that f is twice differentiable, and that $f' \leq 0$ and $f'' \leq 0$. Given a positive integer $k < d$, we choose an integer M large enough. We denote by $\mathscr{P}$ the set of partitions of $[n]$ into k classes. Given such a partition $P = \{P_1, \dots, P_k\} \in \mathscr{P}$, let us denote $|P_i|$ by n_{P_i}, dn_{P_i}/n by d_{P_i}, and $\max\{L/M \mid L \in \mathbb{N},\ L/M \leq d_{P_i}\}$ by d'_{P_i}. Let δ_{P_i} be 1 if $d'_{P_i} > 1$, and 0 otherwise. By Lemma B.1 and Theorem 8.7,

$$\begin{aligned}
\mathbb{P}\left[P_i \text{ is acyclic in } \mathbb{O}(n, d/n)\right] &= \mathbb{P}\left[\mathbb{O}(n_{P_i}, d_{P_i}/n_{P_i}) \text{ is acyclic}\right] \\
&\leq \mathbb{P}\left[\mathbb{O}(n_{P_i}, d'_{P_i}/n_{P_i}) \text{ is acyclic}\right] \\
&\leq \left(\frac{1}{n_{P_i}} \exp\left(-n_{P_i}\alpha(d'_{P_i})\right)\right)^{\delta_{P_i}} \\
&\leq \left(\frac{1}{n_{P_i}}\right)^{\delta_{P_i}} \exp\left(\frac{n}{d} f(d'_{P_i})\right),
\end{aligned}$$

given that n_{P_i} is larger than a constant depending only on d and M (and in fact, we only need n_{P_i} to be large if $d'_{P_i} > 1$). Let E_P be the event that P corresponds to a k-dicolouring of $\mathbb{O}(n, d/n)$. Using that P always has a class P_i of size at least n/k (and thus with $\delta_{P_i} = 1$), and Jensen's inequality, we have that, for n large enough,

$$\begin{aligned}
\mathbb{P}\left[\vec{\chi}(\mathbb{O}(n, d/n)) \leq k\right] &\leq \sum_{P \in \mathscr{P}} \mathbb{P}\left[E_P\right] \\
&\leq k^n \prod_{i=1}^{k} \left(\frac{1}{n_{P_i}}\right)^{\delta_{P_i}} \exp\left(\frac{n}{d} f(d'_{P_i})\right) \\
&\leq k^n \frac{k}{n} \exp\left(\frac{n}{d} \sum_{i=1}^{k} f\left(d_{P_i} - \frac{1}{M}\right)\right) \\
&\leq k^n \frac{k}{n} \exp\left(\frac{nk}{d} f\left(\frac{d}{k} - \frac{1}{M}\right)\right) \\
&\leq k^n \frac{k}{n} \exp\left(n\left(-\frac{d}{2k} + \frac{1}{M} + \frac{k}{2d} + \ln\frac{d}{k}\right)\right).
\end{aligned}$$

The following condition on k is therefore sufficient to ensure that $\mathbb{O}(n, d/n)$

is not k-dicolourable whp:

$$\begin{aligned} k\exp\left(-\frac{d}{2k}+\frac{1}{M}+\frac{k}{2d}+\ln\frac{d}{k}\right) &\leq 1 \\ -\frac{d}{2k}+\frac{1}{M}+\frac{k}{2d}+\ln d &\leq 0 \\ k^2+2d\left(\ln d+\frac{1}{M}\right)k-d^2 &\leq 0. \end{aligned}$$

Hence, whp $\vec{\chi}(\mathbb{O}(n,d/n))$ is at least

$$\left\lceil \frac{d}{\sqrt{\left(\ln d+\frac{1}{M}\right)^2+1}+\left(\ln d+\frac{1}{M}\right)} \right\rceil.$$

Since this holds for every M large enough, the claim follows. □

8.3 Concentration

This section is devoted to the proof of Theorem 8.13. Essentially, the same ideas of Łuczak [112] and Alon and Krivelevich [14] work in the directed setting. It may be worth remarking that, in our case, Łuczak's short argument applies to a wider range of p.

We need some preliminary lemmas. We include the proofs of the first two for illustrative purposes; they are analogous to their counterparts in the undirected setting.

Lemma 8.10. *Let n be a positive integer, $0 \leq p \leq 1$ and $0 < \varepsilon \leq 1$, and let t be a positive integer with*

$$\mathbb{P}\left[\vec{\chi}(\mathbb{O}(n,p/2)) \leq t\right] > \varepsilon.$$

For a $D \in \mathscr{D}(n)$, let $L(D)$ denote the minimum number of vertices that have to be removed from D in order to obtain a t-dicolourable digraph, and let X be the random variable $L(\mathbb{O}(n,p/2))$ Then,

$$\mathbb{P}\left[X \geq 2\lambda\sqrt{n}\right] \leq \varepsilon,$$

where $\lambda = \sqrt{-2\ln\varepsilon}$. The same statement holds with $\mathbb{D}(n,p)$ in the place of $\mathbb{O}(n,p/2)$.

Proof. Clearly, $L : \mathscr{D}(n) \to \mathbb{R}$ satisfies the condition from Lemma B.3, so we have that

(i) $\mathbb{P}\left[X \leq \mathbb{E}(X) - \lambda\sqrt{n}\right] \leq e^{-\lambda^2/2} = \varepsilon$ and

(ii) $\mathbb{P}\left[X \geq \mathbb{E}(X) + \lambda\sqrt{n}\right] \leq e^{-\lambda^2/2}$.

Since by hypothesis $\mathbb{P}[X=0] > \varepsilon$, by (i) we have that $\mathbb{E}(X) \leq \lambda\sqrt{n}$. Then, $\mathbb{P}[X \geq 2\lambda\sqrt{n}] \leq \mathbb{P}[X \geq \mathbb{E}(X) + \lambda\sqrt{n}] \leq \varepsilon$ by (ii). □

Lemma 8.11. *For every $\delta, \varepsilon \in \mathbb{R}^+$ there exists some $\zeta \in (0,1)$ such that, for every $C \in \mathbb{R}^+$, for every integer n large enough and every $0 \leq p \leq n^{-3/4-\delta}$,*

$$\mathbb{P}[E] \geq 1 - \varepsilon,$$

where E is the event that every subgraph H of $\mathbb{O}(n,p)$ of order $h \leq C\sqrt{n}$ has at least ζh vertices with at most one in-neighbour in H. The same statement holds with $\mathbb{D}(n,p)$ in the place of $\mathbb{O}(n,p)$.

Proof. We fix a ζ as small as we want, and then, given C, we assume that n is arbitrarily large. We note that, for every set of $h \geq 3$ vertices of $\mathbb{O}(n,p)$, the probability that the subgraph they induce has more than $(1-\zeta)h$ vertices with in-degree at least 2 is at most

$$x_h := \binom{h}{\lceil (1-\zeta)h \rceil} \left(\binom{h-1}{2} p^2 \right)^{\lceil (1-\zeta)h \rceil}.$$

Hence, the probability that this happens for some set of vertices of $\mathbb{O}(n,p)$ of size at most $C\sqrt{n}$ is at most

$$\sum_{3 \leq h \leq C\sqrt{n}} \binom{n}{h} x_h \leq \sum_{3 \leq h \leq C\sqrt{n}} \left(\frac{en}{h}\right)^h \left(\frac{e}{1-\zeta}\right)^h \left(h^2 n^{-3/2-2\delta}\right)^{(1-\zeta)h}$$

$$\leq \sum_{3 \leq h \leq C\sqrt{n}} \left(\frac{e^2}{1-\zeta} h n^{-1/2-\delta}\right)^h \leq C\sqrt{n} \max_{3 \leq h \leq C\sqrt{n}} \left(e^3 h n^{-1/2-\delta}\right)^h \leq \varepsilon,$$

since every term is at most $(e^3 n^{-1/2-\delta} \ln n)^3$ when $3 \leq h \leq \ln n$ and at most $(Ce^3 n^{-\delta})^{\ln n}$ when $\ln n \leq h \leq C\sqrt{n}$. □

We will also make use of the following deterministic result of Alon and Krivelevich [14, Proposition 3.1]. We note that their statement is slightly different, but the proof works the same way.

Proposition 8.12. *For every $\delta'_0, c \in \mathbb{R}^+$, there exists some $n_0 \in \mathbb{N}$ such that the following holds for every integer $n \geq n_0$. Let $\delta'_0 \leq \delta' \leq 3/8$ and let G be a graph of order n and average degree d, with $n^{1/2-\delta'}/(10 \ln n) \leq d \leq 10n^{1/2-\delta'}$. Assume that the following hold.*

- *(i) There is some $r = r(\delta'_0) \in \mathbb{R}^+$ such that, for every $C \in \mathbb{R}^+$, if n is large enough (depending on C), then every $i \leq C\sqrt{n}$ vertices of G span less than ri edges.*
- *(ii) If $\delta' \geq 2/9$, then, for any $C \in \mathbb{R}^+$, if n is large enough, every $i \leq Cn^{1-\delta'} \ln n$ vertices of G span less than $in^{1/10}$ edges.*

(*iii*) *Every vertex of* G *has degree at most* $100d\ln n$.

(*iv*) *If* $\delta' \geq 1/6$, *then the number of trails of length three between any two vertices of* G *is at most* $\ln n$; *if* $\delta'_0 \leq \delta' < 1/6$, *then the number of trails of length three between any two vertices of* G *is at most* $d^3 \ln^4 n/n$.

(*v*) $\chi(G) \geq d/(2\ln n)$, *and there exists an integer* $\chi(G) \geq t \geq d/(2\ln n)$ *and a set of vertices* $U \subseteq V(G)$ *of size at most* $c\sqrt{n}$ *such that the subgraph of* G *induced by* $V(G) \setminus U$ *is* t*-colourable.*

Then, G *is* $(t+1)$*-colourable.*

We need two more last results. For a non-negative integer k, the k-*core* of a graph G is the (if it exists, unique) maximal subgraph of G with minimum degree at least k. Pittel, Spencer and Wormald [132] study the sharp threshold for the appearance of a k-core in a random graph. More precisely, for $k \geq 3$, they determine

$$\begin{aligned} c_k \coloneqq \sup\{c \in \mathbb{R} \mid \forall M = M(n) \in (0, cn/2) \cap \mathbb{Z} \\ \mathbb{G}(n; M) \text{ does not have a } k\text{-core whp}\} \end{aligned}$$

(which indeed are real constants). For our purposes, it will be enough to know that $c_4 > 51/10$.

Given two non-negative integers k_1 and k_2, the (k_1, k_2)*-core* of a digraph D is the (if it exists, unique) maximal subgraph of D with minimum in-degree at least k_1 and minimum out-degree at least k_2. The emergence of (k_1, k_2)-cores in random digraphs has been investigated by Pittel and Poole [133]. For $\max\{k_1, k_2\} \geq 2$, they determine

$$\begin{aligned} c_{k_1,k_2} \coloneqq \sup\{c \in \mathbb{R} \mid \forall M = M(n) \in (0, cn/2) \cap \mathbb{Z} \\ \mathbb{D}(n; M) \text{ does not have a } (k_1, k_2)\text{-core whp}\} \end{aligned}$$

(which indeed are real constants again). For us, it will be enough to know that $c_{2,2} > 76/10$.

Recall that a digraph D is k*-degenerate* if every induced subgraph of D has a vertex with in-degree or out-degree at most k. We note that D is k-degenerate if and only if it does not have a $(k+1, k+1)$-core. Analogously, a graph G is k-degenerate if and only if it does not have a $(k+1)$-core.

Theorem 8.13. *For every* $\delta, \varepsilon \in \mathbb{R}^+$ *there exists some* $n_0 \in \mathbb{N}$ *such that, for every integer* $n \geq n_0$ *and every* $0 \leq p \leq n^{-1/2-\delta}$, *there is an integer* t *satisfying*

$$\mathbb{P}\left[t \leq \vec{\chi}(\mathbb{O}(n,p)) \leq t+1\right] \geq 1 - \varepsilon.$$

The same statement holds with $\mathbb{D}(n,p)$ *in the place of* $\mathbb{O}(n,p)$.

Proof. We assume that δ and ε are small enough. Once their values are given, we further assume that n is large enough so that all the claims involving it actually hold. The proof is divided into three cases.

Case 1: $np \leq 5/2$.

We start with $\mathbb{O}(n,p)$. As commented above,

$$\mathbb{P}\left[\mathbb{G}(n; \lfloor 51n/20 \rfloor) \text{ does not have a 4-core}\right] \geq 1 - \varepsilon'$$

for any $\varepsilon' \in \mathbb{R}^+$, if n is large enough. By Lemmas B.2 and B.1, and using that the underlying undirected graph of $\mathbb{O}(n,p)$ has the same distribution as $\mathbb{G}(n, 2p)$,

$$1 - \varepsilon \leq \mathbb{P}\left[\mathbb{G}(n, 5/n) \text{ does not have a 4-core}\right]$$

$$\leq \mathbb{P}\left[\mathbb{O}(n,p) \text{ is 1-degenerate}\right] \leq \mathbb{P}\left[\vec{\chi}(\mathbb{O}(n,p) \leq 2\right].$$

For $\mathbb{D}(n,p)$ the argument is similar. As commented above,

$$\mathbb{P}\left[\mathbb{D}(n; \lfloor 76n/20 \rfloor) \text{ does not have a } (2,2)\text{-core}\right] \geq 1 - \varepsilon'$$

for any $\varepsilon' \in \mathbb{R}$, if n is large enough. By Lemmas B.2 and B.1,

$$1 - \varepsilon \leq \mathbb{P}\left[\mathbb{D}(n, 3/n) \text{ does not have a } (2,2)\text{-core}\right] \leq \mathbb{P}\left[\vec{\chi}(\mathbb{D}(n,p) \leq 2\right].$$

For Cases 2 and 3, i.e. when $1 \leq np \leq n^{1/2-\delta}$, the argument is the same for $\mathbb{O}(n,p)$ and $\mathbb{D}(n,p)$. In both of them, we let t be the minimum positive integer such that

$$\mathbb{P}\left[\vec{\chi}(\mathbb{O}(n,p)) \leq t\right] \geq \varepsilon/3.$$

We note that, by Theorem 8.7 and Lemma B.1, $t \geq 2$. Let $\lambda = \sqrt{-2\ln(\varepsilon/3)}$. By Lemma 8.10, with probability at least $1-\varepsilon/3$, there exists a set of vertices $S \subseteq [n]$ of size at most $2\lambda\sqrt{n}$ such that, if S is removed from $\mathbb{O}(n,p)$, the resulting digraph is t-dicolourable.

Case 2: $1 \leq np \leq n^{1/4-\delta}$.

We define a sequence of sets $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_m$ as follows:

- if S exists, $U_0 = S$; otherwise, $U_0 = \emptyset$;
- if the subgraph of $\mathbb{O}(n,p)$ induced by $N^+(U_i) \setminus U_i$ has no directed cycle, then the sequence ends with $m = i$; otherwise, $U_{i+1} = U_i \cup \{w_i^1, \ldots, w_i^{r_i}\}$, where $w_i^1, \ldots, w_i^{r_i} \in N^+(U_i) \setminus U_i$ are the vertices of a directed cycle written consecutively.

We choose a constant $C \in \mathbb{R}^+$ large enough, and let ℓ be the maximum index such that $|U_\ell| \leq C\sqrt{n}/2$ (we note that $\ell \geq 0$ always, so ℓ is well-defined). By Lemma 8.11, for some $\zeta \in (0,1)$ not depending on C, the probability of the event E that every subgraph H of $\mathbb{O}(n,p)$ of order $h \leq C\sqrt{n}$ has at least ζh vertices with in-degree at most 1 is at least $1-\varepsilon/3$. We claim that E implies $\ell = m$. Let us assume that $\ell < m$; then, there exists a set of vertices U such that

(i) $|U| = \lfloor C\sqrt{n}/2 \rfloor + 1$ and
(ii) the subgraph of $\mathbb{O}(n,p)$ induced by U has at most $2\lambda\sqrt{n}+1$ vertices of in-degree at most 1.

Indeed, since $m \geq 1$, one can just take $U = U_\ell \cup \{w_\ell^1, \ldots, w_\ell^r\}$, where $1 \leq r \leq r_\ell$ is the index such that (i) is satisfied. Clearly, the existence of U contradicts E, so E implies $\ell = m$.

Moreover, E implies that $\mathbb{O}(n,p)[U_m]$ is 1-degenerated. Now, let E' be the event that S exists and E holds. Under E', we can obtain a $(t+1)$-dicolouring of $\mathbb{O}(n,p)$ by assigning colours 1 and 2 to the vertices in U_m, colour $t+1 \geq 3$ to all the vertices in $N^+(U_m) \setminus U_m$, and colours $1, \ldots, t$ to the rest of the vertices. Thus,

$$\mathbb{P}\left[t \leq \vec{\chi}(\mathbb{O}(n,p)) \leq t+1\right] \geq \mathbb{P}\left[E'\right] - \mathbb{P}\left[\vec{\chi}(\mathbb{O}(n,p)) \leq t-1\right] \geq 1-\varepsilon.$$

Case 3: $n^{1/8+\delta} \leq np \leq n^{1/2-\delta}$.

With probability at least $1-\varepsilon/3$, there exists a total order $\preceq$ of $[n]$ such that, if S is removed from the back-arc graph $\mathbb{O}(n,p)^{\preceq}$, the resulting graph is t-colourable. We extend the random order $\preceq$ to all the probability space in an arbitrary way. Our aim is to apply Proposition 8.12 to $\mathbb{O}(n,p)^{\preceq}$ with $\delta_0' = \delta$, $c = 2\lambda$, $\delta' = -\log_n p - 1/2$, d the average degree of $\mathbb{O}(n,p)^{\preceq}$ and t as defined above. If its hypotheses hold with probability at least $1-2\varepsilon/3$, we will have that

$$\mathbb{P}\left[t \leq \vec{\chi}(\mathbb{O}(n,p)) \leq t+1\right] \geq 1-\varepsilon,$$

ending the proof.

The upper bound of $np/(2\ln n) \leq d \leq 2np$ follows directly from Chernoff's inequality (Lemma A.2). (Here, and throughout this paragraph, we are talking about things that hold whp, where the speed of convergence is bounded by a function of δ). The lower bound is a consequence of the Caro–Wei theorem (Theorem 1.4) and Theorem 8.4 (see the proof in [145] for the claim about the speed of convergence):

$$\frac{n}{d+1} \leq \alpha(\mathbb{O}(n,p)^{\preceq}) \leq \vec{\alpha}(\mathbb{O}(n,p)) \leq \frac{2}{\ln \frac{1}{1-p}}\left(\ln(np) + 3e\right) \leq \frac{\ln n}{p}.$$

From this we also get the first part of (v),

$$\chi(\mathbb{O}(n,p)^{\preceq}) \geq \frac{n}{\alpha(\mathbb{O}(n,p)^{\preceq})} \geq \frac{np}{\ln n} \geq \frac{d}{2\ln n},$$

and the second part of (v) has already been established. For the rest of the hypotheses, we use that $\mathbb{O}(n,p)^{\preceq}$ is a subgraph of the underlying undirected (simple) graph of $\mathbb{O}(n,p)$, which has the same distribution as $\mathbb{G}(n,2p)$ (or $\mathbb{G}(n,2p-p^2)$ in the case of $\mathbb{D}(n,p)$, but this does not affect the argument). Alon and Krivelevich [14, Lemma 2.2] show that, for any $\delta_0', \varepsilon' \in \mathbb{R}^+$ there exists an n_0' such that, if $n \geq n_0'$, the random graph $\mathbb{G}(n,p)$ (with $p = n^{-1/2-\delta'}$ and $\delta_0' \leq \delta' \leq 3/8$) satisfies (i)–(iv) of Proposition 8.12 with probability at least $1-\varepsilon'$ (again, we note that their statement is slightly different, but the proof works the same way). Their proof also shows that the same is true for $\mathbb{G}(n,2p)$, maybe with another n_0', and that actually (iii) and the second half of (iv) can be strengthened to

- (iii') *every vertex of* $\mathbb{G}(n,2p)$ *has degree at most* $3n(2p) \leq 12d\ln n$*;*
- (iv') (2nd half) *if* $\delta_0' \leq \delta' < 1/6$*, then the number of trails of length three between any two vertices of* $\mathbb{G}(n,2p)$ *is at most* $n^2(2p)^3\ln n/$ $/100 \leq d^3\ln^4 n/n$

(always with probability at least $1-\varepsilon'$ if $n \geq n_0'$). Hence, we conclude that $\mathbb{O}(n,p)^{\preceq}$ also satisfies (i)–(iv) with probability as close to 1 as desired, ending the proof.

□

A concentration on two integers is best-possible in general. Indeed, since for any positive integer n and any set $\mathcal{Q} \subseteq \mathscr{D}(n)$ of digraphs on $[n]$ $\mathbb{P}[\mathbb{O}(n,p) \in \mathcal{Q}]$ is a continuous function of p, if n is large enough then, for any positive integer $t \leq n/(4\log_2 n)$, there exist some $p_t \in [0,1/2]$ such that $\mathbb{P}[\vec{\chi}(\mathbb{O}(n,p_t)) \leq t] = 1/2$ (similarly for $\mathbb{D}(n,p)$). However, for some p, $\vec{\chi}(\mathbb{O}(n,p))$ is concentrated on just one integer. All this has been pointed out by Alon and Krivelevich in the undirected setting [14].

Chapter 9

Acyclic sets and colourings of random r-regular graphs

This chapter is based on joint work with Ararat Harutyunyan and Colin McDiarmid, some of which can be found in [88].

9.1 Introduction

Let r and n be positive integers. A digraph D is *r-regular* if every vertex of D has in-degree and out-degree equal to r. We denote by $\mathscr{D}_{\text{reg}}(n,r)$ be the set of r-regular digraphs with vertex set $[n]$ and by $\mathbb{D}_{\text{reg}}(n,r)$ the random graph with the uniform distribution on $\mathscr{D}_{\text{reg}}(n,r)$. Similarly, we denote by $\mathscr{O}_{\text{reg}}(n,r)$ the set of r-regular oriented graphs on $[n]$ and by $\mathbb{O}_{\text{reg}}(n,r)$ the random digraph with the uniform distribution on $\mathscr{O}_{\text{reg}}(n,r)$.

In this chapter we show the following.

Theorem 9.1. *Let $\varepsilon > 0$. There exists a constant r_ε such that, for every integer $r \geq r_\varepsilon$, the following are satisfied with probability tending to 1 as $n \to \infty$:*

(i) $\displaystyle \frac{2n}{r}(\ln r - \ln\ln r + 1 - \ln 2 - \varepsilon) \leq \vec{\alpha}(\mathbb{O}_{\text{reg}}(n,r)) \leq \frac{2n}{r}(\ln r + 2)$,

(ii) $\displaystyle \frac{r}{2\ln r}\left(1 - \frac{2}{\ln r + 2}\right) \leq \vec{\chi}(\mathbb{O}_{\text{reg}}(n,r)) \leq \frac{r}{2\ln r}\left(1 + \frac{32\ln\ln r}{\ln r}\right)$.

The same statement holds with $\mathbb{D}_{\text{reg}}(n,r)$ in the place of $\mathbb{O}_{\text{reg}}(n,r)$.

Asymptotically (as $r \to \infty$), this matches to the bounds that Dutta, Subramanian and Spencer give for the random digraphs $\mathbb{O}(n, r/n)$ and $\mathbb{D}(n, r/n)$ (see Theorem 8.4 and Corollary 8.6). Theorem 9.1 thus provides an infinite family of examples witnessing that, if Conjectures 2.6 and 2.10 are true, they are tight, up to multiplication by a constant factor.

Our proof of Theorem 9.1 is based on Frieze and Łuczak's proof of an homologous result for random regular graphs.

Theorem 9.2. [71] *Let $\varepsilon > 0$. There exists a constant r_ε such that, for every $r = r(n)$ with $r_\varepsilon \leq r = o(n^\theta)$, where θ is a constant with $\theta < 1/3$, the following are satisfied with probability tending to 1 as $n \to \infty$:*

$$(i)\ \left|\alpha(\mathbb{G}_{\mathrm{reg}}(n,r)) - \frac{2n}{r}(\ln r - \ln\ln r + 1 - \ln 2)\right| \leq \frac{\varepsilon n}{r},$$

$$(ii)\ \frac{r}{2\ln r} \leq \chi(\mathbb{G}_{\mathrm{reg}}(n,r)) \leq \frac{r}{2\ln r}\left(1 + \frac{32\ln\ln r}{\ln r}\right).$$

In view of Theorem 9.2, it seems that an interesting direction in which our result could be improved would be to aim for more precise bounds. Also in the case of binomial random (di)graphs, precise estimates appear to be harder to obtain in the directed case, see Section 8.1. Another possible direction would be to extend it to a larger range of r (allowing r to grow with n). Indeed, Frieze and Łuczak's theorem itself has been extended to $r_\varepsilon \leq r \leq 9n/10$, see [51, 108].

In contrast with that of binomial random graphs, the definition of random regular graphs is not particularly amenable. This initial obstacle can be overcome with the so-called 'configuration model', introduced by Bollobás [29] (see also [23]). A directed version of this method has also been developed, see for instance [22, 45, 50, 118]. When it comes to the uniform generation of r-regular oriented graphs, some extra precision is needed. In this case, one can resort to Creed's estimates of the number of Eulerian orientations of random r-regular graphs [54]. We give all the necessary details about that in Section 9.2. Next, in Section 9.3, we prove Theorem 9.1.

As a second goal of the chapter, in Section 9.4 we highlight the contiguity of the models $\mathbb{G}_{\mathrm{reg}}(n, 2r)$ and $\operatorname{forg}\mathbb{O}_{\mathrm{reg}}(n,r)$, the random regular graph obtained by forgetting the orientations of $\mathbb{O}_{\mathrm{reg}}(n,r)$ (see Corollary 9.11). This might significantly reduce the effort required to prove certain properties about $\mathbb{O}_{\mathrm{reg}}(n,r)$. As an illustration, we give a weaker, but constructive, version of Theorem 9.1 (Corollary 9.16). The proof of Corollary 9.16 is based on spectral techniques, that we detail in a separate section. These yield bounds for the maximum size of an acyclic set, and the dichromatic number, of an arbitrary oriented graph, in terms of the Laplacian eigenvalues of its underlying undirected graph (Propositions 9.13 and 9.14).

9.2 The configuration model

Configurations

By a *configuration* on a set Z we mean an involution on Z without fixed elements, i.e., a mapping $F : Z \to Z$ such that $F \circ F$ is the identity and $F(z) \neq z$ for every $z \in Z$. The set of configurations on Z is denoted by Φ_Z. Given two positive integers r and n such that rn is even, we denote by $\mathscr{G}^*_{\text{reg}}(n,r)$ the set of all r-regular multigraphs with vertex set $[n]$. For a configuration $F \in \Phi_{[rn]}$, let $\mu(F)$ be the r-regular multigraph on $[n]$ which has an edge $\{\lceil s/r \rceil, \lceil t/r \rceil\}$ for each $\{s,t\} \subseteq [rn]$ such that $F(s) = t$. We define a probability space $\mathbb{G}^*_{\text{reg}}(n,r)$ over $\mathscr{G}^*_{\text{reg}}(n,r)$ by assigning to each $G \in \mathscr{G}^*_{\text{reg}}(n,r)$ the measure

$$P_{n,r}(G) = \frac{|\mu^{-1}(G)|}{|\Phi_{[rn]}|}.$$

In other words, to sample an element of $\mathbb{G}^*_{\text{reg}}(n,r)$, it amounts to consider $\mu(F)$, where F is the uniform random configuration from $\Phi_{[rn]}$.

One can check that every simple r-regular graph on $[n]$ appears with the same probability. Thus, if we condition $\mathbb{G}^*_{\text{reg}}(n,r)$ on the set $\mathscr{G}_{\text{reg}}(n,r) \subseteq \mathscr{G}^*_{\text{reg}}(n,r)$ of simple r-regular graphs on $[n]$, we obtain the uniform random r-regular graph $\mathbb{G}_{\text{reg}}(n,r)$. Crucially, as $n \to \infty$, $P_{n,r}(\mathscr{G}_{\text{reg}}(n,r))$ tends to a positive constant, implying that any property that holds with high probability for $\mathbb{G}^*_{\text{reg}}(n,r)$ also holds whp for $\mathbb{G}_{\text{reg}}(n,r)$. We refer the reader to [102, Chapter 9] for a proper introduction to the configuration model.

Directed configurations

In the directed setting, things are very similar, but we will need to prepare some extra notation for Section 9.3. A *directed configuration* on a pair (Z^+, Z^-) of disjoint sets of the same cardinality is a bijection $Z^+ \to Z^-$. The set of directed configurations on (Z^+, Z^-) is denoted by Φ_{Z^+,Z^-}. Given $F \in \Phi_{Z^+,Z^-}$, we may use the set-theoretical notation $(x^+, x^-) \in F$ for $F(x^+) = x^-$.

Given two positive integers r and n, we set $W^+ = [rn]$ and, for $1 \le i \le n$, $W_i^+ = \{r(i-1)+1, r(i-1)+2, \ldots, ri\}$ (when using this notation, r and n will be clear from the context). Symmetrically, we set $W^- = -[rn]$ and $W_i^- = -W_i^+$ for every $1 \le i \le n$. For a $w \in W^+ \cup W^-$, we denote by $\psi(w) \in [n]$ the index such that $w \in W^+_{\psi(w)} \cup W^-_{\psi(w)}$.

Let $F \in \Phi_{Z^+,Z^-}$, with $Z^+ \subseteq W^+$ and $Z^- \subseteq W^-$ of the same size. We now denote by $\mu(F)$ the multidigraph with vertex set $[n]$ which has an arc (i,j) for each $x \in Z^+ \cap W_i^+$ with $F(x) \in W_j^-$. If F is chosen uniformly at random from Φ_{W^+,W^-}, $\mu(F)$ is a random r-regular multidigraph, that will be denoted by $\mathbb{D}^*_{\text{reg}}(n,r)$. Lemma 9.4 assures us that, again, all simple digraphs from $\mathscr{D}_{\text{reg}}(n,r)$ appear with the same probability as $\mathbb{D}^*_{\text{reg}}(n,r)$, and the probability that $\mathbb{D}^*_{\text{reg}}(n,r)$ is an oriented graph tends to a positive constant as $n \to \infty$.

Eulerian orientations

For the proof of Lemma 9.4 we need to introduce yet another random graph: $\text{forg}\,\mathbb{D}^*_{\text{reg}}(n,r)$, the random $2r$-regular multigraph obtained from $\mathbb{D}^*_{\text{reg}}(n,r)$

by forgetting the orientations of its arcs. We denote by $Q_{n,r}$ the probability measure on $\mathscr{G}^*_{\mathrm{reg}}(n,2r)$ associated with $\operatorname{forg}\mathbb{D}^*_{\mathrm{reg}}(n,r)$.

When considering orientations of multigraphs, we have to clarify whether the edges are labelled or not. In what follows, unless specified, we will make no distinction between multiple copies of the same edge. An *Eulerian orientation* of a multidigraph G is an orientation D of G such that $\deg^+_D(v) = \deg^-_D(v)$ for every vertex v. Let $E^*_{n,2r} : \mathscr{G}^*_{\mathrm{reg}}(n,2r) \to \mathbb{N}$ be the function counting the number of arc-labelled Eulerian orientations of each $G \in \mathscr{G}^*_{\mathrm{reg}}(n,2r)$, with the convention that loops can be oriented in two ways. By 'arc-labelled' we mean that the edges of G are supposed to be labelled; in other words, here we do make a distinction between multiple edges linking two vertices.

Below we compute the expectation of various functions $\mathscr{G}^*_{\mathrm{reg}}(n,2r) \to \mathbb{N}$, as if they were random variables from $\mathbb{G}^*_{\mathrm{reg}}(n,2r)$ or $\operatorname{forg}\mathbb{D}^*_{\mathrm{reg}}(n,r)$. We write $\mathbb{E}_P$ to indicate that the expectation is being taken in $\mathbb{G}^*_{\mathrm{reg}}(n,2r)$, and $\mathbb{E}_Q$ when it is being taken in $\operatorname{forg}\mathbb{D}^*_{\mathrm{reg}}(n,r)$.

Lemma 9.3. *Let r and n be positive integers. Then,*

$$\frac{E^*_{n,2r}}{\mathbb{E}_P(E^*_{n,2r})} = \frac{Q_{n,r}}{P_{n,2r}}.$$

Proof. For any $G \in \mathscr{G}^*_{\mathrm{reg}}(n,2r)$,

$$P_{n,2r}(G) = \frac{(2r)!^n}{2^{\ell(G)}|\Phi_{[2rn]}|} \prod_{e\in E(G)} \frac{1}{\mathrm{mult}(e)!},$$

where $|\Phi_{[2rn]}| = \frac{(2rn)!}{2^{rn}(rn)!}$, $\ell(G)$ is the number of loops of G, and $E(G)$ is its set of edges. Similarly,

$$Q_{n,r}(G) = \frac{r!^{2n}}{|\Phi_{[rn],-[rn]}|} \sum_{D\in\mathscr{O}_{\mathrm{E}}(G)} \prod_{a\in A(D)} \frac{1}{\mathrm{mult}(a)!},$$

where $|\Phi_{[rn],-[rn]}| = (rn)!$, $\mathscr{O}_{\mathrm{E}}(G)$ is the set of Eulerian orientations of G, and $A(D)$ is the set of arcs of D. On the other hand,

$$E^*_{n,2r}(G) = 2^{\ell(G)} \sum_{D\in\mathscr{O}_{\mathrm{E}}(G)} \prod_{e\in E'(G)} \binom{\mathrm{mult}_G(e)}{\mathrm{mult}_D(e^+)},$$

where $E'(G)$ is the set of non-loop edges of G and, for each $e \in E'(G)$, e^+ is a fixed orientation of e (notice that the previous expression is independent of this choice). The claim follows from the fact that

$$\mathbb{E}_P(E^*_{n,2r}) = \frac{2^{rn}\binom{2r}{r}^n}{\binom{2rn}{rn}}$$

(see the proof of Theorem 3.47 in [54]; we note that, there, $E^*_{n,2r}$ is defined in an alternative way). □

Lemma 9.4. *Let r and n be positive integers.*

(i) *Every simple r-regular digraph on $[n]$ has the same probability of appearing as $\mathbb{D}^*_{\mathrm{reg}}(n,r)$.*

(ii) $\lim_{n\to\infty} \mathbb{P}\left[\mathbb{D}^*_{\mathrm{reg}}(n,r) \in \mathscr{O}_{\mathrm{reg}}(n,r)\right] = e^{-r^2-\frac{1}{2}}$.

Proof. (i) We use the notation from above. Let Γ^+ the group of permutations π^+ of W^+ with $\pi^+(W_i^+) \subseteq W_i^+$ for every $1 \le i \le n$, and Γ^- the group of permutations π^- of W^- with $\pi^-(W_i^-) \subseteq W_i^-$ for every $1 \le i \le n$. These groups act on the set Φ_{W^+,W^-} of directed configurations and, for each $F \in \Phi_{W^+,W^-}$, the orbit $\Gamma^+ \cdot F \cdot \Gamma^-$ corresponds via μ to an r-regular multidigraph D with vertex set $[n]$. Thus, any such D arises from precisely $r!^{2n} \prod_{a\in A(D)} \frac{1}{\mathrm{mult}(a)!}$ directed configurations.

(ii) Given two non-negative integers k and j, we denote by $(k)_j$ the truncated factorial $k(k-1)\ldots(k-j+1)$. For every positive integer i, let $X_{i,n} : \mathscr{G}^*_{\mathrm{reg}}(n,2r) \to \mathbb{N}$ be the function counting the number of cycles of length i. It is shown in [54, Lemma 3.51] that

$$\lim_{n\to\infty} \frac{\mathbb{E}_P\big(E^*_{n,2r}(X_{1,n})_{j_1}\ldots(X_{k,n})_{j_k}\big)}{\mathbb{E}_P\big(E^*_{n,2r}\big)} = \prod_{i=1}^{k} \mu_i^{j_i}$$

for any k and any $j_1,\ldots,j_k$, where $\mu_i = \frac{(2r-1)^i+1}{2i}$. By Lemma 9.3,

$$\frac{\mathbb{E}_P\big(E^*_{n,2r}(X_{1,n})_{j_1}\ldots(X_{k,n})_{j_k}\big)}{\mathbb{E}_P\big(E^*_{n,2r}\big)} = \mathbb{E}_P\left(\frac{Q_{n,r}}{P_{n,2r}}(X_{1,n})_{j_1}\ldots(X_{k,n})_{j_k}\right)$$

$$= \sum_{G\in\mathscr{G}^*_{\mathrm{reg}}(n,2r)} \left(\frac{Q_{n,r}}{P_{n,2r}}(X_{1,n})_{j_1}\ldots(X_{k,n})_{j_k}\right)(G)\, P_{n,2r}(G)$$

$$= \mathbb{E}_Q\big((X_{1,n})_{j_1}\ldots(X_{k,n})_{j_k}\big).$$

Therefore, by the method of moments (see [102, Theorem 6.10]), the random variables $X_{i,n}(\mathrm{forg}\,\mathbb{D}^*_{\mathrm{reg}}(n,r))$ converge in distribution to X_i as $n\to\infty$, jointly for all i, where $X_i \sim \mathrm{Pois}(\mu_i)$ are independent Poisson random variables (see also [102, Lemma 9.17]). Hence,

$$\lim_{n\to\infty} \mathbb{P}\left[\mathbb{D}^*_{\mathrm{reg}}(n,r) \in \mathscr{O}_{\mathrm{reg}}(n,r)\right] = \lim_{n\to\infty} \mathbb{P}\left[\mathrm{forg}\,\mathbb{D}^*_{\mathrm{reg}}(n,r) \in \mathscr{G}_{\mathrm{reg}}(n,2r)\right]$$

$$= \lim_{n\to\infty} \mathbb{P}\left[X_{1,n}(\mathrm{forg}\,\mathbb{D}^*_{\mathrm{reg}}(n,r)) = X_{2,n}(\mathrm{forg}\,\mathbb{D}^*_{\mathrm{reg}}(n,r)) = 0\right] = e^{-\mu_1-\mu_2}.$$

□

9.3 Proof of Theorem 9.1

We split the proof of Theorem 9.1 into Theorems 9.6 and 9.7. The proof of the first one relies on the following result of Frieze on the independence number of sparse random graphs, as well as on the corresponding result of Łuczak about their chromatic number (Theorem 8.2).

Theorem 9.5. [70] *Let $\varepsilon > 0$. There exists a constant d_ε such that, if $p = p(n) \to 0$ and $d = d(n) = np \geq d_\varepsilon$, then, with probability tending to* 1 *as $n \to \infty$,*

$$\left|\alpha(\mathbb{G}(n,p)) - \frac{2n}{d}\left(\ln d - \ln\ln d - \ln 2 + 1\right)\right| \leq \frac{\varepsilon n}{d}.$$

Theorem 9.6. *Let $\varepsilon > 0$. There exists a constant r_ε such that, for every integer $r \geq r_\varepsilon$, the following are satisfied with probability tending to* 1 *as $n \to \infty$:*

(*i*) $\vec{\alpha}(\mathbb{O}_{\text{reg}}(n,r)) \geq \dfrac{2n}{r}\left(\ln r - \ln\ln r + 1 - \ln 2 - \varepsilon\right)$ *and*

(*ii*) $\vec{\chi}(\mathbb{O}_{\text{reg}}(n,r)) \leq \dfrac{r}{2\ln r}\left(1 + \dfrac{32\ln\ln r}{\ln r}\right)$

The same statement holds with $\mathbb{D}_{\text{reg}}(n,r)$ in the place of $\mathbb{O}_{\text{reg}}(n,r)$.

Proof. (*i*) The argument for (*i*) follows [71] and it is organised into two parts. First (up until Claim 9.6.4), we generate a uniform random $F \in \Phi_{W^+,W^-}$ with some artifice. An analysis of the construction reduces then our problem to showing that sparse random graphs have large independent sets, and, at that point, the claim swiftly follows from the result of Frieze mentioned above (Theorem 9.5). For the first part, we only need to assume that r and n are positive integers.

We define the quantities $r_1 = r - \lceil r^{1/2}\ln r\rceil$ and $m_1 = r_1 n$, and, for each $1 \leq t \leq m_1$, we choose $x_t^+ \in [n]$ and $x_t^- \in -[n]$ uniformly at random, where all the choices are independent. We define $\mathbb{D}_1$ to be the random multidigraph with vertex set $[n]$ which has an arc (i,j) for each $1 \leq t \leq m_1$ such that $(i,j) = (x_t^+, -x_t^-)$.

Given $j \in [n]$, we denote by $d_{1,j}^+$ and $d_{1,j}^-$ the out-degree and in-degree of j in $\mathbb{D}_1$. Let $\{Y_1^+,\dots,Y_n^+\}$ and $\{Y_1^-,\dots,Y_n^-\}$ be the two partitions of $Y^+ := [m_1]$ and $Y^- := -[m_1]$ defined as follows:

$$Y_1^+ = [d_{1,1}^+] \text{ and } Y_i^+ = [\textstyle\sum_{k=1}^{i} d_{1,k}^+] \setminus [\sum_{k=1}^{i-1} d_{1,k}^+] \text{ for } 2 \leq i \leq n;$$

$$Y_1^- = -[d_{1,1}^-] \text{ and } Y_i^- = -[\textstyle\sum_{k=1}^{i} d_{1,k}^-] \setminus -[\sum_{k=1}^{i-1} d_{1,k}^-] \text{ for } 2 \leq i \leq n.$$

We now construct a random directed configuration F_1 on (Y^+, Y^-) with the following algorithm.

begin
  $F_1 \coloneqq \emptyset$; for each $1 \leq i \leq n$, $\tilde{Y}_i^+ \coloneqq Y_i^+$ and $\tilde{Y}_i^- \coloneqq Y_i^-$;
  for t from 1 to m_1 do
    choose $p_t^+ \in \tilde{Y}^+_{x_t^+}$ uniformly at random; $\tilde{Y}^+_{x_t^+} \coloneqq \tilde{Y}^+_{x_t^+} \backslash \{p_t^+\}$;
    choose $p_t^- \in \tilde{Y}^-_{-x_t^-}$ uniformly at random; $\tilde{Y}^-_{-x_t^-} \coloneqq \tilde{Y}^-_{-x_t^-} \backslash \{p_t^-\}$;
    $F_1 \coloneqq F_1 \cup \{(p_t^+, p_t^-)\}$;
  end
end

Claim 9.6.1. *Let* $\mathbf{x}^+ \in [n]^{m_1}$ *and* $\mathbf{x}^- \in -[n]^{m_1}$. *Conditioning on the event that* $(x_1^+, \ldots, x_{m_1}^+) = \mathbf{x}^+$ *and* $(x_1^-, \ldots, x_{m_1}^-) = \mathbf{x}^-$, F_1 *is uniformly distributed on* Φ_{Y^+,Y^-}.

Proof. We consider the random permutations

$$\pi^+ = \begin{pmatrix} 1 & \ldots & m_1 \\ p_1^+ & \ldots & p_{m_1}^+ \end{pmatrix} \quad \text{and} \quad \pi^- = \begin{pmatrix} 1 & \ldots & m_1 \\ p_1^- & \ldots & p_{m_1}^- \end{pmatrix}.$$

We note that F_1 is given by $\pi^-(\pi^+)^{-1}$. Therefore, it is enough to see that π^+ and π^- are uniformly distributed on the set of permutations of $[m_1]$. And indeed, for each $1 \leq t \leq m_1 - 1$, swapping p_t^+ and p_{t+1}^+ does not alter the distribution of π^+. For π^- it is analogous. ∎

For $p \in Y^+ \cup Y^-$, we let sign(p) be the sign of p and $1 \leq \ell(p) \leq n$ the index such that $Y_{\ell(p)}^{\mathrm{sign}(p)}$ is the class $\overline{p}$ of p in the partition $\{Y_1^+, Y_1^-, \ldots, Y_n^+, Y_n^-\}$. The *rank* of p is defined as $\rho(p) = |p| - \min_{t \in \overline{p}} |t| + 1$. In other words, p is the $\rho(p)$-th smallest element from $\overline{p}$ in absolute value. Additionally, we define $\sigma(p) = \mathrm{sign}(p)((\ell(p) - 1)r + \rho(p))$ and the random directed configurations

$$F_1' = \{(p^+, p^-) \in F_1 \mid \rho(p^+), \rho(p^-) \leq r\} \quad \text{and}$$

$$F_2 = \{(\sigma(p^+), \sigma(p^-)) \mid (p^+, p^-) \in F_1'\}.$$

Claim 9.6.2. *Let* $Z^+ \subseteq W^+$ *and* $Z^- \subseteq W^-$ *with* $|Z^+| = |Z^-|$. *All the elements of* Φ_{Z^+,Z^-} *have the same probability of appearing as* F_2.

Proof. Let $\mathbf{x}^+ \in [n]^{m_1}$ and $\mathbf{x}^- \in -[n]^{m_1}$. It is enough to prove the claim conditioning on the event E that $(x_1^+, \ldots, x_{m_1}^+) = \mathbf{x}^+$ and $(x_1^-, \ldots, x_{m_1}^-) = \mathbf{x}^-$. Under this condition, ρ and σ are completely determined. In particular, the sets $Y_{\leq r}^+ = \{p^+ \in Y^+ \mid \rho(p^+) \leq r\}$ and $Y_{\leq r}^- = \{p^- \in Y^- \mid \rho(p^-) \leq r\}$ are determined.

We first analyse F_1'. We fix $\tilde{Z}^+ \subseteq Y^+$ and $\tilde{Z}^- \subseteq Y^-$ with $|\tilde{Z}^+| = |\tilde{Z}^-|$, and, given any $G' \in \Phi_{\tilde{Z}^+,\tilde{Z}^-}$, we denote by $\mathrm{Ext}(G')$ the set

$$\{G \in \Phi_{Y^+,Y^-} \mid G \cap (Y^+_{\leq r} \times Y^-_{\leq r}) = G'\}.$$

We note that, for any $G \in \mathrm{Ext}(G')$ and $H' \in \Phi_{\tilde{Z}^+,\tilde{Z}^-}$, $H' \cup G|_{Y^+ \setminus \tilde{Z}^+}$ is a directed configuration on (Y^+, Y^-) that belongs to $\mathrm{Ext}(H')$. This defines a bijection between $\mathrm{Ext}(G')$ and $\mathrm{Ext}(H')$, so these sets have the same size. Now, by Claim 9.6.1,

$$\mathbb{P}\left[F_1' = G' \mid E\right] = \mathbb{P}\left[F_1 \in \mathrm{Ext}(G') \mid E\right] = \sum_{G \in \mathrm{Ext}(G')} \mathbb{P}\left[F_1 = G \mid E\right] = \frac{|\mathrm{Ext}(G')|}{|\Phi_{Y^+,Y^-}|}.$$

Hence, conditioning on E, all the elements of $\Phi_{\tilde{Z}^+,\tilde{Z}^-}$ have the same probability of appearing as F_1'.

Let $p, q \in Y^+ \cup Y^-$. If $\sigma(p) = \sigma(q)$, then $\mathrm{sign}(p) = \mathrm{sign}(q)$ and $\rho(p) \equiv \rho(q) \mod r$. If, moreover, $p, q \in Y^+_{\leq r} \cup Y^-_{\leq r}$, then $\rho(p) = \rho(q)$ and $\ell(p) = \ell(q)$. Hence, σ is injective on $Y^+_{\leq r} \cup Y^-_{\leq r}$. Depending on the conditional event E, we distinguish two possible cases.

Case 1: *there exist $\tilde{Z}^+ \subseteq Y^+_{\leq r}$ and $\tilde{Z}^- \subseteq Y^-_{\leq r}$ with $\sigma(\tilde{Z}^+) = Z^+$ and $\sigma(\tilde{Z}^-) = Z^-$.*

By the paragraph above, σ induces bijections $\tilde{Z}^+ \to Z^+$ and $\tilde{Z}^- \to Z^-$. Since all the elements of $\Phi_{\tilde{Z}^+,\tilde{Z}^-}$ have the same probability of appearing as F_1', all the elements of Φ_{Z^+,Z^-} have the same probability of appearing as $\sigma(F_1') = F_2$.

Case 2: *the opposite.*

In this case, for any $G \in \Phi_{Z^+,Z^-}$,

$$\mathbb{P}\left[F_2 = G\right] \leq \mathbb{P}\left[\sigma(\mathrm{dom}(F_1')) = Z^+ \text{ and } \sigma(\mathrm{im}(F_1')) = Z^-\right] = 0.$$

■

We denote by $\mathbb{D}_2$ the random multidigraph $\mu(F_2)$; thus, $\mathbb{D}_2$ is a subgraph of $\mathbb{D}_1$. In what follows we recursively enlarge F_2 into a random directed configuration on (W^+, W^-) that will finally generate r-regular multidigraphs. Given a directed configuration G on (Z^+, Z^-), where $Z^+ \subsetneq W^+$ and $Z^- \subsetneq W^-$, and two elements $w^+ \in W^+ \setminus Z^+$ and $w^- \in W^- \setminus Z^-$, the following algorithm returns a random directed configuration F_G on $(Z^+ \cup \{w^+\}, Z^- \cup \{w^-\})$.

ADD(G, w^+, w^-)
begin
  choose $(z^+, z^-) \in G$ uniformly at random;

$$F_G := \begin{cases} G \cup \{(w^+, w^-)\} \text{ with probability } \frac{1}{|G|+1} \\ G \cup \{(w^+, z^-), (z^+, w^-)\} \setminus \{(z^+, z^-)\} \text{ otherwise;} \end{cases}$$

output F_G;

end

Claim 9.6.3. *Let $Z^+ \subsetneq W^+$, $Z^- \subsetneq W^-$, $w^+ \in W^+ \setminus Z^+$ and $w^- \in W^- \setminus Z^-$ with $|Z^+| = |Z^-|$, and let F be a random directed configuration from Φ_{Z^+,Z^-} with the uniform distribution. Then,* $\mathrm{ADD}(F, w^+, w^-)$ *is uniformly distributed on $\Phi_{Z^+\cup\{w^+\},Z^-\cup\{w^-\}}$.*

Proof. We define a mapping $f : \Phi_{Z^+\cup\{w^+\},Z^-\cup\{w^-\}} \to \Phi_{Z^+,Z^-}$ by setting

$$f(\tilde{G}) = \tilde{G} \cup \{(\tilde{G}^{-1}(w^-), \tilde{G}(w^+))\} \setminus \{(w^+, \tilde{G}(w^+)), (\tilde{G}^{-1}(w^-), w^-)\}.$$

Let any $G \in \Phi_{Z^+,Z^-}$, and let $F_G = \mathrm{ADD}(G, w^+, w^-)$. We note that $F_G \in f^{-1}(G)$. Moreover, any $\tilde{G} \in f^{-1}(G)$ is a possible outcome of F_G. Indeed, if $(w^+, w^-) \in \tilde{G}$, then $\tilde{G} = G \cup \{(w^+, w^-)\}$, and if $(w^+, w^-) \notin \tilde{G}$, then $\tilde{G}$ is of the form $G \cup \{(w^+, z^-), (z^+, w^-)\} \setminus \{(z^+, z^-)\}$ for some $(z^+, z^-) \in G$. Therefore, $f^{-1}(G)$ is the set of possible outcomes of F_G, $|f^{-1}(G)| = |G| + 1$, and F_G is uniformly distributed on $f^{-1}(G)$.

Let $\tilde{G}$ be any element of $\Phi_{Z^+\cup\{w^+\},Z^-\cup\{w^-\}}$. We have that

$$\begin{aligned} \mathbb{P}\left[\mathrm{ADD}(F, w^+, w^-) = \tilde{G}\right] &= \sum_{G \in \Phi_{Z^+,Z^-}} \mathbb{P}\left[\mathrm{ADD}(G, w^+, w^-) = \tilde{G} \text{ and } F = G\right] \\ &= \mathbb{P}\left[\mathrm{ADD}(f(\tilde{G}), w^+, w^-) = \tilde{G} \text{ and } F = f(\tilde{G})\right] \\ &= \mathbb{P}\left[\mathrm{ADD}(f(\tilde{G}), w^+, w^-) = \tilde{G}\right] \cdot \mathbb{P}\left[F = f(\tilde{G})\right] \\ &= \frac{1}{|\tilde{G}|} \cdot \frac{1}{|\Phi_{Z^+,Z^-}|} = \frac{1}{|\Phi_{Z^+\cup\{w^+\},Z^-\cup\{w^-\}}|}. \end{aligned}$$

■

Let s be the random variable $|W^+| - |F_2|$. We denote by $w_1^+, \ldots, w_s^+$ the elements of $W^+ \setminus \mathrm{dom}(F_2)$, and by $w_1^-, \ldots, w_s^-$ the elements of $W^- \setminus \mathrm{im}(F_2)$, ordered increasingly. We define $\mathbb{D}$ to be the random r-regular multidigraph $\mu(F)$, where F is the random directed configuration on Φ_{W^+,W^-} obtained with the following algorithm.

FINISH

begin

$F := F_2$;

for t from 1 to s do $F := \mathrm{ADD}(F, w_t^+, w_t^-)$;

end

Claim 9.6.4. *F is uniformly distributed on Φ_{W^+,W^-}. In particular, $\mathbb{D}$ and $\mathbb{D}^*_{\mathrm{reg}}(n,r)$ have the same distribution.*

Proof. Let $\mathscr{Z}$ be the set of pairs (Z^+, Z^-) such that $Z^+ \subseteq W^+$, $Z^- \subseteq W^-$, $|Z^+| = |Z^-|$, and one of the possible outcomes of F_2 is in Φ_{Z^+,Z^-}. Let (Z^+, Z^-) be any element of $\mathscr{Z}$. It will be enough to see that, conditioning on the event E_{Z^+,Z^-} that $F_2 \in \Phi_{Z^+,Z^-}$, F is uniformly distributed on Φ_{W^+,W^-}.

Under E_{Z^+,Z^-}, s is determined, and so are $w_1^+, \dots, w_s^+$ and $w_1^-, \dots, w_s^-$. By Claim 9.6.2, F_2 is uniformly distributed on Φ_{Z^+,Z^-}. We get the desired conclusion by recursively applying Claim 9.6.3. ∎

Up until this point, we have been working with general positive integers r and n. From now on, we may implicitly assume that r is larger than a suitable function of ε, and that n is larger than a suitable function of r.

Given a multidigraph D, we denote by $\underline{D}$ the simple digraph obtained from D by deleting loops and merging parallel arcs. For a real number δ, we let α_δ denote the quantity $\frac{2n}{r}(\ln r - \ln\ln r + 1 - \ln 2 - \delta)$. By Lemma 9.4, it will be enough to show that $\vec{\alpha}(\underline{\mathbb{D}}) \geq \alpha_\varepsilon$ with probability tending to 1 as $n \to \infty$.

We now use back-arc graphs as a bridge towards the undirected world. Let $\mathbb{G}_1$ be a random graph obtained from $\underline{\mathbb{D}_1}$ by choosing, for every outcome D_1 of $\underline{\mathbb{D}_1}$, a graph among $\underline{D_1}^{\leq}$ and $\underline{D_1}^{\geq}$ minimising the number of edges.

Claim 9.6.5. *For any $\delta > 0$ there is a constant c_δ such that, if $r \geq c_\delta$, then $\alpha(\mathbb{G}_1) \geq \alpha_\delta$ with probability tending to 1 as $n \to \infty$.*

Proof. Let $\mathbb{G}(n;M)$ denote the uniform random graph with vertex set $[n]$ and M edges. Let $\underline{m_1}$, $\underline{m_1^{\leq}}$ and $\underline{m_1^{\geq}}$ be the random variables counting the number of arcs/edges of $\underline{\mathbb{D}_1}$, $\underline{\mathbb{D}_1}^{\leq}$ and $\underline{\mathbb{D}_1}^{\geq}$. Since $\underline{\mathbb{D}_1}$ does not have any loop, we immediately get that

$$\underline{m_1^{\leq}} + \underline{m_1^{\geq}} = \underline{m_1} \leq m_1.$$

We note that, for $\preceq \in \{\leq, \geq\}$, $\underline{\mathbb{D}_1}^{\preceq}$ and $\mathbb{G}(n; \underline{m_1^{\preceq}})$ have the same distribution. Hence, $\mathbb{G}(n; \lfloor m_1/2 \rfloor)$ can be obtained by adding to $\mathbb{G}_1$ a random number of extra edges (precisely, $\lfloor m_1/2 \rfloor - \min\{\underline{m_1^{\leq}}, \underline{m_1^{\geq}}\}$) uniformly at random. Therefore, it is enough to show that, for r large enough, $\alpha(\mathbb{G}(n; \lfloor m_1/2 \rfloor)) \geq \alpha_\delta$ with probability tending to 1 as $n \to \infty$.

But this is well-known. Let $p = r/n$, and let $\mathbb{G}(n,p)$ be the binomial random graph with vertex set $[n]$, i.e., every possible edge of $\mathbb{G}(n,p)$ appears independently with probability p. By Theorem 9.5, for every $\delta > 0$, there is a constant c_δ such that, if $r \geq c_\delta$, then $\alpha(\mathbb{G}(n,p)) \geq \alpha_\delta$ with probability tending to 1 as $n \to \infty$. By Lemmas B.1 and B.2, this implies that, if $r \geq c_\delta$, then $\alpha(\mathbb{G}(n; \lfloor m_1/2 \rfloor)) \geq \alpha_\delta$ with probability tending to 1 as $n \to \infty$. ∎

Given $j \in [n]$, we denote by $d^+_{2,j}$ and $d^-_{2,j}$ the out-degree and in-degree of j in $\mathbb{D}_2$. We define the random sets $S^+_0 = \{j \in [n] \mid d^+_{2,j} \le r - 3r^{1/2}\ln r\}$ and $S^-_0 = \{j \in [n] \mid d^-_{2,j} \le r - 3r^{1/2}\ln r\}$.

Claim 9.6.6. *There is a constant $c > 0$ such that, for every $0 \le \theta \le 1$,*

$$\mathbb{P}\left[|S^+_0| + |S^-_0| \ge \frac{n}{r^{1+\theta}}\right] \le \exp\left(\frac{-cn}{r^{5+2\theta}}\right).$$

Proof. For any $k \in S^+_0$, either $k \in S^+_1 := \{j \in [n] \mid d^+_{1,j} \le r - 2r^{1/2}\ln r\}$ or $k \in S^+_2 := \{j \in [n] \mid d^+_{1,j} - d^+_{2,j} \ge r^{1/2}\ln r\}$. Let us bound the size of these sets separately.

We note that $d^+_{1,k}$ has a binomial distribution with parameters m_1 and $1/n$. Applying Chernoff's inequality (Lemma A.2) with $\beta = r_1^{-1/2}\ln r$, we get that

$$\mathbb{P}\left[k \in S^+_1\right] = \mathbb{P}\left[d^+_{1,k} \le r - 2r^{1/2}\ln r\right] \le \mathbb{P}\left[d^+_{1,k} \le (1-\beta)r_1\right] \le 2e^{-\ln^2 r/3}.$$

Hence, $\mathbb{E}(|S^+_1|) \le 2ne^{-\ln^2 r/3}$. We now apply the simple concentration bound (Lemma A.3) to $|S^+_1|$. Since $|S^+_1|$ is determined by $x^+_1, \dots, x^+_{m_1}$, and changing the outcome of any of them affects $|S^+_1|$ by at most 1,

$$\mathbb{P}\left[|S^+_1| \ge \frac{2n}{e^{\ln^2 r/3}} + \lambda\right] \le 2\exp\left(\frac{-\lambda^2}{2m_1}\right) \le 2\exp\left(\frac{-\lambda^2}{2rn}\right)$$

for any $\lambda \ge 0$.

Now we turn to S^+_2. Given $t \in [m_1]$ and $j \in [n]$, we define

$$\eta^+_{t,j} = \begin{cases} 1 & \text{if } x^+_t = j \text{ and } \max\left\{d^+_{1,x^+_t}, d^-_{1,-x^-_t}\right\} > r \\ 0 & \text{otherwise.}\end{cases}$$

Let $\tilde{S}^+_2 = \{j \in [n] \mid \sum_{t=1}^{m_1} \eta^+_{t,j} \ge r^{1/2}\ln r\}$. We note that $S^+_2 \subseteq \tilde{S}^+_2$. Indeed, if $j \in S^+_2$ then there exists a set $T_j \subseteq [m_1]$ of size at least $r^{1/2}\ln r$ such that $\ell(p^+_t) = x^+_t = j$ and $\max\{\rho(p^+_t), \rho(p^-_t)\} > r$ for every $t \in T_j$. Since

$$\max\{\rho(p^+_t), \rho(p^-_t)\} \le \max\left\{\left|Y^+_{x^+_t}\right|, \left|Y^-_{-x^-_t}\right|\right\} = \max\left\{d^+_{1,x^+_t}, d^-_{1,-x^-_t}\right\},$$

we have that $\eta^+_{t,j} = 1$ for every $t \in T_j$, which implies that $j \in \tilde{S}^+_2$.

Now, for every $t \in [m_1]$ and $j \in [n]$, we have:

$$\begin{aligned}\mathbb{P}\left[\eta^+_{t,j} = 1\right] &= \frac{1}{n}\mathbb{P}\left[\max\left\{d^+_{1,x^+_t}, d^-_{1,-x^-_t}\right\} > r \,\middle|\, x^+_t = j\right] \\ &\le \frac{1}{n}\left(\mathbb{P}\left[d^+_{1,x^+_t} > r \,\middle|\, x^+_t = j\right] + \mathbb{P}\left[d^-_{1,-x^-_t} > r \,\middle|\, x^+_t = j\right]\right) \\ &= \frac{2}{n}\mathbb{P}\left[B_{m_1-1,\frac{1}{n}} \ge r\right],\end{aligned}$$

where $B_{m_1-1,\frac{1}{n}}$ is a binomial random variable with parameters $m_1 - 1$ and $1/n$. By Chernoff's inequality (Lemma A.2), $\mathbb{P}[\eta^+_{t,j} = 1] \leq \frac{4}{n} e^{-\ln^2 r/3}$. This implies that

$$\mathbb{E}\left(\sum_{t=1}^{m_1} \eta^+_{t,j}\right) \leq 4re^{-\ln^2 r/3} \quad \text{and} \quad \mathbb{P}\left[\sum_{t=1}^{m_1} \eta^+_{t,j} \geq r^{1/2} \ln r\right] \leq \frac{4r^{1/2}}{e^{\ln^2 r/3} \ln r},$$

by Markov's inequality (Lemma A.1). Hence,

$$\mathbb{E}\left(|\tilde{S}^+_2|\right) \leq \frac{4r^{1/2} n}{e^{\ln^2 r/3} \ln r}.$$

We now apply the simple concentration bound to $|\tilde{S}^+_2|$. This random variable is determined by $x^+_1, \ldots, x^+_{m_1}, x^-_1, \ldots, x^-_{m_1}$. Moreover, changing the outcome of any of these can affect $|\tilde{S}^+_2|$ by at most $r+2$. Indeed, let $\mathbf{x}^+ \in [n]^{m_1}$ and $\mathbf{x}^- \in -[n]^{m_1}$, and let E be the event that $(x^+_1 \ldots, x^+_{m_1}) = \mathbf{x}^+$ and $(x^-_1 \ldots, x^-_{m_1}) = \mathbf{x}^-$. Let $* \in \{+, -\}$ and $u \in [m_1]$, and assume that $x^*_u = *i$ under E. Let E' be the event that $x^+_1, x^-_1, \ldots, x^+_{m_1}, x^-_{m_1}$ are as in E, except x^*_u, which takes the value $*i' \neq *i$. Given $j \in [n]$, suppose that $j \in \tilde{S}^+_2$ holds under E, but not under E'. This means that, for some $t \in [m_1]$, $\eta^+_{t,j} = 1$ under E, but $\eta^+_{t,j} = 0$ under E'. Two situations can occur.

Case 1: *under E', $x^+_t \neq j$.*

Since $x^+_t = j$ under E, we have that $* = +$, $t = u$ and $j = i$.

Case 2: *under E', $x^+_t = j$.*

Then, d^+_{1,x^+_t} and $d^-_{1,-x^-_t}$ are at most r under E', so one of them decreases when changing from E to E'. The only in- or out-degree of $\mathbb{D}_1$ that can behave like that is $d^*_{1,i}$, which decreases from $r+1$ to r. Thus, under E, either $* = +$ and $x^+_t = j = i$, or $* = -$ and $-x^-_t = i$. In the latter case, (j, i) is an arc of $\mathbb{D}_1$. Since under E there are precisely $r+1$ arcs of this form, j can take only the $r+1$ values determined by these arcs.

In any case, j can take at most $r+2$ values. Thus, changing the outcome of x^*_u affects $|\tilde{S}^+_2|$ by at most $r+2$. By Lemma A.3,

$$\mathbb{P}\left[|\tilde{S}^+_2| \geq \frac{4r^{1/2} n}{e^{\ln^2 r/3} \ln r} + \lambda\right] \leq 2\exp\left(\frac{-\lambda^2}{4(r+2)^2 m_1}\right) \leq 2\exp\left(\frac{-\lambda^2}{4r^3 n}\right)$$

for any $\lambda \geq 0$.

Combining the bounds on $|S_1^+|$ and $|\tilde{S}_2^+|$ with $\lambda = \frac{n}{5r^{1+\theta}}$, we get that

$$\begin{aligned}\mathbb{P}\left[|S_0^+| \geq \frac{n}{2r^{1+\theta}}\right] &\leq \mathbb{P}\left[|S_1^+| \geq \frac{n}{4r^{1+\theta}}\right] + \mathbb{P}\left[|S_2^+| \geq \frac{n}{4r^{1+\theta}}\right] \\ &\leq \mathbb{P}\left[|S_1^+| \geq \frac{2n}{e^{\ln^2 r/3}} + \lambda\right] + \mathbb{P}\left[|S_2^+| \geq \frac{4r^{1/2}n}{e^{\ln^2 r/3}\ln r} + \lambda\right] \\ &\leq 4\exp\left(\frac{n}{100r^{5+2\theta}}\right).\end{aligned}$$

A similar bound holds for $|S_0^-|$. The claim follows. ■

From now on, we denote by $\preceq$ the random element from $\{\leq, \geq\}$ such that $\mathbb{G}_1 = \underline{\mathbb{D}_1}^{\preceq}$, and by Φ the set of directed configurations

$$\bigcup_{\substack{Z^+\subseteq W^+,\, Z^-\subseteq W^-,\\ |Z^+|=|Z^-|}} \Phi_{Z^+,Z^-}.$$

Claim 9.6.7. *Let $U \subseteq [n]$ be a function of $\mathbb{G}_1$ such that U is an independent set of $\mathbb{G}_1$ of size at most $\frac{4n\ln r}{r}$. Let $0 \leq \theta \leq 1/3$, and let E_θ be the event that $|S_0^+| + |S_0^-| \leq n/r^{1+\theta}$. Then,*

$$\mathbb{P}\left[\underline{\mathbb{D}}^{\preceq}[U] \text{ has at least } \frac{200n\ln r}{r^{1+\theta}} \text{ edges} \,\middle|\, E_\theta\right] \leq \exp\left(\frac{-n\ln r}{r^{1+\theta}}\right).$$

Proof. To ease the notation, we denote by $\mathbb{P}_\theta$ the conditional probability with respect to E_θ and we set $b = 200n\ln r/r^{1+\theta}$. An arc (j,k) of a digraph D with vertex set $[n]$ is *bad* if $\{j,k\}$ is an edge of $D^{\preceq}[U]$. Let γ be the random variable counting the number of bad arcs of $\underline{\mathbb{D}}$. It is enough to show that

$$\mathbb{P}_\theta\left[\gamma \geq b\right] \leq \exp\left(\frac{-n\ln r}{r^{1+\theta}}\right).$$

We denote F_2 by $F^{(0)}$, and, for $1 \leq t \leq s$, we denote by $F^{(t)}$ the random directed configuration ADD($F^{(t-1)}$, w_t^+, w_t^-) resulting from the t-th iteration of the loop in the algorithm FINISH. We define the two following events:

- $E_1^{(t)}$, the event that $F^{(t)} \setminus F^{(t-1)} = \{(w_t^+, w_t^-)\}$;
- $E_2^{(t)}$, the event that the pair $(z^+, z^-) \in F^{(t-1)}$ chosen during the t-th iteration of the loop in FINISH satisfies that either $\psi(z^+) \in U$ or $\psi(z^-) \in U$.

Let $T_2 = \{t \in [s] \mid \psi(w_t^+) \in U \text{ or } \psi(w_t^-) \in U\}$. Since $\underline{\mathbb{D}_2}$ has no bad arcs, we can bound γ with a sum of indicators as follows:

$$\gamma \leq \sum_{t=1}^{s} \mathbb{1}_{E_1^{(t)}} + 2\sum_{t\in T_2} \mathbb{1}_{E_2^{(t)}}.$$

Hence, we are interested in bounding the probabilities of these events. More precisely, we are going to bound them conditioning on A_t, where A_t is any event determined by the random choices that take place before the execution of the t-th iteration of the loop in FINISH. We first observe that

$$s \le r \cdot |S_0^+| + 3r^{1/2} \ln r \cdot n,$$

which implies that, under E_θ,

$$|F^{(t)}| \ge |F_2| = rn - s \ge \frac{rn}{2}$$

for every $0 \le t \le s$. Thus, for every $1 \le t \le s$,

$$\mathbb{P}_\theta \left[E_1^{(t)} \,\middle|\, A_t \right] \le \frac{1}{\frac{rn}{2} + 1} \le \frac{2}{rn} =: p_1.$$

Given $G \in \Phi$, we note that the subset $\{(z^+, z^-) \in G \mid \psi(z^+) \in U \text{ or } \psi(z^-) \in U\}$ has size at most $2r|U| \le 8n \ln r$. Using this, we see that, for every $1 \le t \le s$,

$$\begin{aligned}
\mathbb{P}_\theta \left[E_2^{(t)} \,\middle|\, A_t \right] &= \sum_{G \in \Phi} \mathbb{P}_\theta \left[E_2^{(t)} \,\middle|\, F^{(t-1)} = G \text{ and } A_t \right] \mathbb{P}_\theta \left[F^{(t-1)} = G \,\middle|\, A_t \right] \\
&\le \sum_{G \in \Phi} \frac{8n \ln r}{|G|} \mathbb{P}_\theta \left[F^{(t-1)} = G \,\middle|\, A_t \right] \\
&\le \sum_{G \in \Phi} \frac{8n \ln r}{\frac{rn}{2}} \mathbb{P}_\theta \left[F^{(t-1)} = G \,\middle|\, A_t \right] = \frac{16 \ln r}{r} =: p_2.
\end{aligned}$$

Subclaim 9.6.7.1. *Let $T \subseteq [s]$, $\lambda \in \mathbb{R}$ and $i \in \{1, 2\}$. Then,*

$$\mathbb{P}_\theta \left[\sum_{t \in T} \mathbb{1}_{E_i^{(t)}} \ge \lambda \right] \le \mathbb{P}_\theta \left[B_{|T|, p_i} \ge \lambda \right],$$

where, for $N \in \mathbb{N}$ and $0 \le p \le 1$, $B_{N,p}$ is a binomial random variable of parameters N and p, independent of all random choices in the construction of F.

Proof. We can assume that λ is an integer. The proof is by induction on $|T|$. Let $t_0 = \max T$ and $T' = T \setminus \{t_0\}$. When $T = \{t_0\}$,

$$\mathbb{P}_\theta \left[\mathbb{1}_{E_i^{(t_0)}} \ge \lambda \right] \le \left\{ \begin{matrix} 1 & \text{if } \lambda \le 0 \\ p_i & \text{if } 0 < \lambda \le 1 \\ 0 & \text{if } \lambda > 1 \end{matrix} \right\} = \mathbb{P}_\theta \left[B_{1,p_i} \ge \lambda \right].$$

And, when $|T| \geq 2$,

$$\begin{aligned}
&\mathbb{P}_\theta\left[\sum_{t\in T} \mathbb{1}_{E_i^{(t)}} \geq \lambda\right] = \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} \geq \lambda\right] + \mathbb{P}_\theta\left[E_i^{(t_0)} \text{ and } \sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} = \lambda - 1\right] \\
&= \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} \geq \lambda\right] + \mathbb{P}_\theta\left[E_i^{(t_0)} \,\middle|\, \sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} = \lambda - 1\right] \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} = \lambda - 1\right] \\
&\leq \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} \geq \lambda\right] + \mathbb{P}_\theta\left[B_{1,p_i} = 1\right] \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} = \lambda - 1\right] \\
&= \mathbb{P}_\theta\left[B_{1,p_i} = 0\right] \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} \geq \lambda\right] + \mathbb{P}_\theta\left[B_{1,p_i} = 1\right] \mathbb{P}_\theta\left[\sum_{t\in T'} \mathbb{1}_{E_i^{(t)}} \geq \lambda - 1\right] \\
&\leq \mathbb{P}_\theta\left[B_{1,p_i} = 0\right] \mathbb{P}_\theta\left[B_{|T'|,p_i} \geq \lambda\right] + \mathbb{P}_\theta\left[B_{1,p_i} = 1\right] \mathbb{P}_\theta\left[B_{|T'|,p_i} \geq \lambda - 1\right] \\
&= \mathbb{P}_\theta\left[B_{1,p_i} = 0 \text{ and } B_{1,p_i} + B_{|T'|,p_i} \geq \lambda\right] \\
&\quad + \mathbb{P}_\theta\left[B_{1,p_i} = 1 \text{ and } B_{1,p_i} + B_{|T'|,p_i} \geq \lambda\right] \\
&= \mathbb{P}_\theta\left[B_{|T|,p_i} \geq \lambda\right].
\end{aligned}$$

♦

We note that, under E_θ,

$$|T_2| \leq r \cdot \left(|S_0^+| + |S_0^-|\right) + 6r^{1/2} \ln r \cdot |U| \leq \frac{n}{r^\theta} + \frac{24n \ln^2 r}{r^{1/2}} \leq \frac{2n}{r^\theta}.$$

Using this bound and Subclaim 9.6.7.1, letting $N_1 = \lfloor 4r^{1/2} \ln r \rfloor n$, $N_2 = \lceil 2n/r^\theta \rceil$ and $p_1' = N_2 p_2 / N_1 \geq p_1$, and applying Chernoff's inequality (Lemma A.2) with $\beta = \frac{b}{4N_2p_2} - 1$, we get that

$$\begin{aligned}
\mathbb{P}_\theta\left[\gamma \geq b\right] &\leq \mathbb{P}_\theta\left[\sum_{t=1}^{s} \mathbb{1}_{E_1^{(t)}} + 2\sum_{t\in T_2} \mathbb{1}_{E_2^{(t)}} \geq b\right] \\
&\leq \mathbb{P}_\theta\left[\sum_{t=1}^{s} \mathbb{1}_{E_1^{(t)}} \geq \frac{b}{2}\right] + \mathbb{P}_\theta\left[2\sum_{t\in T_2} \mathbb{1}_{E_2^{(t)}} \geq \frac{b}{2}\right] \\
&\leq \mathbb{P}_\theta\left[B_{s,p_1} \geq \frac{b}{2}\right] + \mathbb{P}_\theta\left[B_{|T_2|,p_2} \geq \frac{b}{4}\right] \\
&\leq \mathbb{P}\left[B_{N_1,p_1'} \geq \frac{b}{4}\right] + \mathbb{P}\left[B_{N_2,p_2} \geq \frac{b}{4}\right] \\
&\leq \mathbb{P}\left[|B_{N_1,p_1'} - N_1p_1'| \geq \frac{b}{4} - N_1p_1'\right] + \mathbb{P}\left[|B_{N_2,p_2} - N_2p_2| \geq \frac{b}{4} - N_2p_2\right] \\
&\leq 4e^{-\beta^2 N_2 p_2/3} \leq \exp\left(\frac{-n \ln r}{r^{1+\theta}}\right).
\end{aligned}$$

■

Let $\mathscr{U}$ be the set of $U \subseteq [n]$ satisfying the hypotheses of Claim 9.6.7, and let $U_0 \in \mathscr{U}$ be any one of them of maximum size. Given $\varepsilon > 0$, we take $\delta, \theta > 0$ small enough (in fact, since r is a constant function of n, we can take any $0 < \theta \leq 1/3$). Let E^{e}_θ be the event that $\underline{\mathbb{D}}^{\preceq}[U_0]$ has at most $200n \ln r/r^{1+\theta}$ edges, and E^{i}_δ the event that $\alpha(\mathbb{G}_1) \geq \alpha_\delta$. We note that, under $E^{\mathrm{e}}_\theta \cap E^{\mathrm{i}}_\delta$,

$$\alpha(\underline{\mathbb{D}}^{\preceq}) \geq |U_0| - \frac{200n \ln r}{r^{1+\theta}} \geq \frac{2n}{r}\left(\ln r - \ln\ln r + 1 - \ln 2 - \delta - \frac{100 \ln r}{r^\theta}\right) \geq \alpha_\varepsilon.$$

Thus,

$$\mathbb{P}\left[\vec{\alpha}(\underline{\mathbb{D}}) \geq \alpha_\varepsilon\right] \geq \mathbb{P}\left[E^{\mathrm{e}}_\theta \cap E^{\mathrm{i}}_\delta\right] \geq \mathbb{P}\left[E^{\mathrm{e}}_\theta \mid E_\theta\right]\mathbb{P}\left[E_\theta\right] - \mathbb{P}\left[\overline{E^{\mathrm{i}}_\delta}\right],$$

which, by Claims 9.6.5, 9.6.6 and 9.6.7, tends to 1 as $n \to \infty$. This ends the proof of (i).

(ii) As we have done in the first part, we prove the second part of the statement for $\underline{\mathbb{D}}$. We keep the notation from (i), and the assumptions on r and n. Let $k_0 = \left\lfloor \frac{r}{2\ln r}\left(1 + \frac{30 \ln\ln r}{\ln r}\right)\right\rfloor$.

Claim 9.6.8. *The probability of the event E^{p} that $\mathbb{G}_1$ can be partitioned into k_0 independent sets of size at most $\frac{4n\ln r}{r}$ tends to 1 as $n \to \infty$.*

Proof. We argue as in Claim 9.6.5. Let $p = r/n$. By Theorems 8.2 and 9.5, with probability tending to 1 as $n \to \infty$, $\mathbb{G}(n,p)$ can be partitioned into k_0 independent sets of size at most $\frac{4n\ln r}{r}$. By Lemma B.2, the same thing can be said about $\mathbb{G}(n; \lfloor r(n-1)/2 \rfloor)$. This implies the claim. ■

Let $n_\theta = 400n/r^\theta$ and $k_1 = \lfloor r \ln\ln r / \ln^2 r \rfloor$.

Claim 9.6.9. *Assuming that $\theta > 0$, the probability that $\underline{\mathbb{D}}^{\preceq}$ has a subgraph with $1 \leq n_1 \leq n_\theta$ vertices and at least $k_1 n_1/2$ edges tends to 0 as $n \to \infty$.*

Proof. Using that $\underline{\mathbb{D}} = \mu(F)$ with F uniformly distributed on Φ_{W^+,W^-}, we see that the probability that there is a set of $n_1 \in [1, n_\theta]$ vertices of $\underline{\mathbb{D}}$ inducing a subgraph with at least $k_1 n_1/2$ arcs is at most

$$\sum_{n_1 = \lceil \frac{k_1}{2} \rceil + 1}^{n_\theta} \binom{n}{n_1} \binom{n_1(n_1-1)}{\lceil \frac{k_1 n_1}{2} \rceil} \prod_{t=1}^{\lceil \frac{k_1 n_1}{2} \rceil} \frac{r^2}{rn - t + 1}$$

$$\leq \sum_{n_1 = \lceil \frac{k_1}{2} \rceil + 1}^{n_\theta} \left(\frac{en}{n_1}\right)^{n_1} \left(\frac{e n_1(n_1-1)}{\lceil \frac{k_1 n_1}{2} \rceil}\right)^{\lceil \frac{k_1 n_1}{2} \rceil} \left(\frac{2r}{n}\right)^{\lceil \frac{k_1 n_1}{2} \rceil}$$

$$\leq \sum_{n_1=\lceil \frac{k_1}{2} \rceil + 1}^{n_\theta} \left(\left(\frac{n_1}{n}\right)^{1-\frac{2}{k_1}} \frac{4e^2 r}{k_1} \right)^{\lceil \frac{k_1 n_1}{2} \rceil}$$

$$\leq n_\theta \max_{\lceil \frac{k_1}{2} \rceil + 1 \leq n_1 \leq n_\theta} \left(\left(\frac{n_1}{n}\right)^{\frac{1}{2}} 4e^2 \ln^2 r \right)^{\lceil \frac{k_1 n_1}{2} \rceil},$$

which tends to 0 as $n \to \infty$. Indeed, for $\lceil k_1/2 \rceil \leq n_1 \leq \ln n$ and $\ln n \leq n_1 \leq n_\theta$ the last expression is respectively at most

$$n \left(\left(\frac{\ln n}{n}\right)^{\frac{1}{2}} 4e^2 \ln^2 r \right)^{\frac{k_1^2}{4}} \quad \text{and at most} \quad n \left(\left(\frac{400}{r^\theta}\right)^{\frac{1}{2}} 4e^2 \ln^2 r \right)^{\frac{k_1 \ln n}{2}}.$$

■

Let $U_1, \ldots, U_{k_0}$ be disjoint independent sets of $\mathbb{G}_1$ of size at most $\frac{4n \ln r}{r}$, such that, under E^{p}, they form a k_0-colouring of $\mathbb{G}_1$. Let O be the set of edges of $\underline{\mathbb{D}}^{\preceq}$ that have both endpoints in one of these sets. Let $V_O \subseteq [n]$ be the set of endpoints of the edges of O, and E^{q} the event that $\chi(\underline{\mathbb{D}}^{\preceq}[V_O]) \leq k_1$.

Claim 9.6.10. *E^{q} holds with probability tending to 1 as $n \to \infty$.*

Proof. Let E_θ^{d} be the event that $\underline{\mathbb{D}}^{\preceq}$ has no subgraph with $1 \leq n_1 \leq 400n/r^\theta$ vertices and at least $k_1 n_1/2$ edges. We have that

$$\mathbb{P}[E^{\mathrm{q}}] \geq \mathbb{P}\left[\underline{\mathbb{D}}^{\preceq}[V_O] \text{ is } (k_1 - 1)\text{-degenerate}\right] \geq \mathbb{P}\left[|V_O| \leq \frac{400n}{r^\theta} \text{ and } E_\theta^{\mathrm{d}}\right]$$

$$\geq \mathbb{P}\left[|V_O| \leq \frac{400n}{r^\theta} \,\middle|\, E_\theta\right] \mathbb{P}[E_\theta] - \mathbb{P}\left[\overline{E_\theta^{\mathrm{d}}}\right].$$

Using Claim 9.6.7, we see that

$$\mathbb{P}\left[|V_O| \geq \frac{400n}{r^\theta} \,\middle|\, E_\theta\right] \leq \mathbb{P}\left[|O| \geq \frac{200n}{r^\theta} \,\middle|\, E_\theta\right] \leq \mathbb{P}\left[|O| \geq \frac{200k_0 n \ln r}{r^{1+\theta}} \,\middle|\, E_\theta\right]$$

$$\leq \mathbb{P}\left[\underline{\mathbb{D}}^{\preceq}[U_i] \text{ has at least } \frac{200n \ln r}{r^{1+\theta}} \text{ edges for some } 1 \leq i \leq k_0 \,\middle|\, E_\theta\right]$$

$$\leq k_0 \exp\left(\frac{-n \ln r}{r^{1+\theta}}\right).$$

Hence, the claim follows from Claims 9.6.6 and 9.6.9. ■

We can now finish the proof. Under $E^{\mathrm{p}} \cap E^{\mathrm{q}}$, we have that

$$\chi(\underline{\mathbb{D}}^{\preceq}) \leq \chi(\underline{\mathbb{D}}^{\preceq}[[n] \setminus V_O]) + \chi(\underline{\mathbb{D}}^{\preceq}[V_O]) \leq k_0 + k_1 \leq \frac{r}{2\ln r}\left(1 + \frac{32 \ln \ln r}{\ln r}\right).$$

Therefore,

$$\mathbb{P}\left[\vec{\chi}(\mathbb{D}) \le \frac{r}{2\ln r}\left(1+\frac{32\ln\ln r}{\ln r}\right)\right] \ge \mathbb{P}\left[E^{\mathrm{p}} \cap E^{\mathrm{q}}\right] \ge \mathbb{P}\left[E^{\mathrm{q}}\right] - \mathbb{P}\left[\overline{E^{\mathrm{p}}}\right],$$

which tends to 1 as $n \to \infty$ by Claims 9.6.8 and 9.6.10. □

Unlike the original result of Frieze and Łuczak for random r-regular graphs (Theorem 9.2), Theorem 9.6 does not tell us what happens when r grows slowly with n. For that, we would need a suitable replacement of Lemma 9.4.

Let us now turn to the other inequalities.

Theorem 9.7. *For every positive integer r, with probability tending to* 1 *as $n \to \infty$,*

$$\vec{\alpha}(\mathbb{O}_{\mathrm{reg}}(n,r)) \le \frac{2n}{r}(\ln r + 2).$$

The same statement holds with $\mathbb{D}_{\mathrm{reg}}(n,r)$ in the place of $\mathbb{O}_{\mathrm{reg}}(n,r)$.

Proof. Let $\mathbb{D}^*_{\mathrm{reg}}(n,r)$ be the random r-regular multidigraph on $[n]$ generated with the directed configuration model. By Lemma 9.4, it suffices to prove that

$$\lim_{n\to\infty} \mathbb{P}\left[\vec{\alpha}(\mathbb{D}^*_{\mathrm{reg}}(n,r)) \ge \frac{2n}{r}(\ln r + 2)\right] = 0.$$

Let k be a positive integer and $0 < \beta < 1$ a real number, for now both of them unspecified. Let ℓ be the integer divisible by k in the interval $[\beta n, \beta n + k)$. Suppose that some set of vertices $A \subseteq [n]$ of size $|A| = \ell$ is acyclic. Then, there is an ordering $\sigma : \{1,\dots,\ell\} \to A$ of A such that each arc of $\mathbb{D}^*_{\mathrm{reg}}(n,r)[A]$ is of the form $(\sigma(i),\sigma(j))$ for some $1 \le i < j \le \ell$. This implies that A can be partitioned into k subsets $A_1,\dots,A_k$ in a way that

(i) $|A_i| = \dfrac{\ell}{k}$;

(ii) for each pair $1 \le i < j \le k$, there is no arc from any element of A_j to any element of A_i.

So, if $\vec{\alpha}(D^*) \ge \ell$, the above condition must hold for one of the ℓ-subsets A of $[n]$. The number of possible ways to choose A is $\binom{n}{\ell}$, and the number of ways to partition such a set into k parts $A_1,\dots,A_k$ is easily at most k^ℓ. Thus, in total there are at most $\binom{n}{\ell}k^\ell \le (ekn/\ell)^\ell$ possible choices.

Now, let us assume that we have fixed $A \subseteq [n]$ with $|A| = \ell$, and a partition $A_1,\dots,A_k$ of A with $|A_i| = \ell/k$. Without loss of generality, we may assume that $A = \{1,\dots,\ell\}$ and that $A_1 = \{1,\dots,\ell/k\},\dots,A_k = \{(k-1)\ell/k+1,\dots,\ell\}$. We would like to compute the probability that there is no *backward* arc, i.e., no arc from A_j to A_i for any $i < j$. Let E_1 be the event that there is no arc from A_k to any of the A_i, for all $i < k$. Clearly,

$$\mathbb{P}[E_1] = \prod_{j=0}^{r\frac{\ell}{k}-1} \frac{rn - r(k-1)\frac{\ell}{k} - j}{rn - j} \leq \left(1 - \frac{(k-1)\ell}{kn}\right)^{\frac{r\ell}{k}}.$$

For $2 \leq i \leq k-1$, let E_i be the event that there is no backward arc leaving A_{k-i+1}. Then,

$$\mathbb{P}[E_i \,|\, E_1 \cap \ldots \cap E_{i-1}] = \prod_{j=0}^{r\frac{\ell}{k}-1} \frac{rn - r(i-1)\frac{\ell}{k} - r(k-i)\frac{\ell}{k} - j}{rn - r(i-1)\frac{\ell}{k} - j}$$

$$\leq \left(1 - \frac{(k-i)\frac{\ell}{k}}{n - (i-1)\frac{\ell}{k}}\right)^{\frac{r\ell}{k}} \leq \left(1 - \frac{(k-i)\ell}{kn}\right)^{\frac{r\ell}{k}} \leq \exp\left(-\frac{r(k-i)\ell^2}{k^2 n}\right).$$

Thus, the probability that A with the partition $A_1, \ldots, A_k$ satisfies (ii) is at most

$$\exp\left(-\sum_{i=1}^{k-1} \frac{r(k-i)\ell^2}{k^2 n}\right) = \exp\left(-\frac{r(1-\frac{1}{k})\ell^2}{2n}\right).$$

Hence, the probability that there is an acyclic set of size ℓ is at most

$$\left(\frac{ekn}{\ell}\right)^{\ell} \exp\left(-\frac{r(1-\frac{1}{k})\ell^2}{2n}\right) \leq \exp\left\{\ell\left(1 + \ln k - \ln\beta - \frac{\beta r}{2}\left(1 - \frac{1}{k}\right)\right)\right\},$$

where we used the facts that $ekn/\ell \leq ek/\beta$ and $r(1-\frac{1}{k})\ell^2 \geq r(1-\frac{1}{k})\beta n\ell$.

Now, we fix $\beta = \frac{2}{r}(\ln\frac{3r}{4} + 2)$ and $k = \lceil \beta r/2 \rceil$. Clearly, we can assume that $r \geq 2$. This implies that $\beta > \frac{4}{r}$, so we have the bound $k < \frac{\beta r}{2} + 1 < \frac{3\beta r}{4}$. Let us denote by c_r the quantity $1 + \ln k - \ln\beta - \frac{\beta r}{2}(1 - \frac{1}{k})$ and observe that $c_r < 1 + \ln\frac{3r}{4} - \frac{\beta r}{2} + 1 = 0$. Since c_r is independent of n, we have that, for n large enough,

$$\mathbb{P}\left[\vec{\alpha}(\mathbb{D}^*_{\text{reg}}(n,r)) \geq \frac{2n}{r}(\ln r + 2)\right] \leq \mathbb{P}\left[\vec{\alpha}(\mathbb{D}^*_{\text{reg}}(n,r)) \geq \beta n + k\right]$$

$$\leq \mathbb{P}\left[\vec{\alpha}(\mathbb{D}^*_{\text{reg}}(n,r)) \geq \ell\right] \leq e^{c_r \ell} \leq e^{c_r \beta n}.$$

This completes the proof. □

Unfortunately, the bound of Theorem 9.7 is meaningless for small r. It makes sense to push the analysis above to try to find a constant $c < 1$ such that $\vec{\alpha}(\mathbb{D}^*_{\text{reg}}(n,r)) \leq cn$ with high probability. We note that we cannot expect that to work for $r = 1$. Indeed, it is well-known that the number of cycles of the uniform random permutation $\pi \in S_n$ is concentrated around its mean $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ [68, Example III.4]. It follows that, when $r = 1$, $\vec{\alpha}(\mathbb{D}^*_{\text{reg}}(n,r)) = n - (1 + o(1))\ln n$ with probability tending to 1 as $n \to \infty$. However, for $r \geq 2$, one can already take $c = 99/100$.

Remark 9.8. *For every integer $r \geq 2$, $\vec{\alpha}(\mathbb{O}_{\text{reg}}(n,r)) \leq 99n/100$ with probability tending to 1 as $n \to \infty$. The same statement holds with $\mathbb{D}_{\text{reg}}(n,r)$ in the place of $\mathbb{O}_{\text{reg}}(n,r)$.*

Proof. Going over the proof of Theorem 9.7, we now bound the probability of the event E_i conditioned on $E_1 \cap \ldots \cap E_{i-1}$ as follows:

$$\mathbb{P}\left[E_i \,|\, E_1 \cap \ldots \cap E_{i-1}\right] = \prod_{j=0}^{r\frac{\ell}{k}-1} \frac{rn + r\frac{\ell}{k} - r\ell - j}{rn - r(i-1)\frac{\ell}{k} - j} \leq \left(\frac{n + \frac{\ell}{k} - \ell}{n - (i-1)\frac{\ell}{k}}\right)^{\frac{r\ell}{k}}$$

$$\leq \left(\frac{\frac{k}{\beta} + 1 - k}{\frac{k}{\beta} + 1 - i}\right)^{\frac{r\ell}{k}},$$

and so the product $\prod_{i=1}^{k-1} \mathbb{P}\left[E_i \,|\, E_1 \cap \ldots \cap E_{i-1}\right]$ is upper-bounded by

$$\left(\left(\frac{k}{\beta} + 1 - k\right)^k \prod_{i=1}^{k} \frac{1}{\frac{k}{\beta} + 1 - i}\right)^{\frac{r\ell}{k}}.$$

We have that

$$\begin{aligned}
\sum_{i=1}^{k} \ln\left(\frac{k}{\beta} + 1 - i\right) &\geq \int_1^k \ln\left(\frac{k}{\beta} + 1 - x\right) dx \\
&= \left[-x - \left(\frac{k}{\beta} + 1 - x\right) \ln\left(\frac{k}{\beta} + 1 - x\right)\right]_1^k \\
&= 1 - k - \left(\frac{k}{\beta} + 1 - k\right) \ln\left(\frac{k}{\beta} + 1 - k\right) + \frac{k}{\beta} \ln \frac{k}{\beta},
\end{aligned}$$

so $\mathbb{P}[\vec{\alpha}(\mathbb{D}^*_{\text{reg}}(n,r)) \geq \ell]$ is at most

$$\exp\left\{\ell\left(1 + r\left(1 - \frac{1}{k}\right) + \left(1 - \frac{r}{\beta}\right) \ln \frac{k}{\beta} + r\left(\frac{1}{\beta} + \frac{1}{k}\right) \ln\left(\frac{k}{\beta} + 1 - k\right)\right)\right\}.$$

Therefore, it is enough to ask that

$$1 + r\left(1 - \frac{1}{k}\right) + \left(1 - \frac{r}{\beta}\right) \ln \frac{k}{\beta} + r\left(\frac{1}{\beta} + \frac{1}{k}\right) \ln\left(\frac{k}{\beta} + 1 - k\right) < 0,$$

which, for $r \geq 2$, is satisfied by $k = 100$ and $\beta = 99/100$. □

9.4 Contiguity

Let $(\Omega_n, \mathcal{F}_n)_{n \in \mathbb{Z}^+}$ be a sequence of measurable spaces and, for each $n \geq 1$, let P_n and Q_n be two probability measures defined on $(\Omega_n, \mathcal{F}_n)$. The sequences $(P_n)_{n\in\mathbb{Z}^+}$ and $(Q_n)_{n\in\mathbb{Z}^+}$ are *contiguous* if, for each sequence of events $(A_n)_{n\in\mathbb{Z}^+}$ with $A_n \in \mathcal{F}_n$ for each $n \geq 1$,

$$\lim_{n\to\infty} P_n(A_n) = 0 \iff \lim_{n\to\infty} Q_n(A_n) = 0.$$

We also say that the probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega_n, \mathcal{F}_n, Q_n)$ are *contiguous* (denoted $(\Omega_n, \mathcal{F}_n, P_n) \approx (\Omega_n, \mathcal{F}_n, Q_n)$) *as* $n \to \infty$. (For brevity, we systematically drop the 'as $n \to \infty$' part, assuming it implicitly.)

Contiguity may be defined in many equivalent ways. Here, for convenience, we restrict to the case where Ω_n is countable and an additional convergence hypothesis is satisfied.

Proposition 9.9. [101, 102] *Let $(\Omega_n, \mathcal{F}_n)_{n\in\mathbb{Z}^+}$ be a sequence of countable measurable spaces. For each $n \geq 1$, let P_n and Q_n be two probability measures defined on $(\Omega_n, \mathcal{F}_n)$, and L_n a random variable on $(\Omega_n, \mathcal{F}_n, P_n)$ such that $L_n(\omega) = \frac{Q_n}{P_n}(\omega)$ for every $\omega \in \Omega_n$ with $P_n(\omega) \neq 0$. Suppose that $(L_n)_{n\in\mathbb{Z}^+}$ converges in distribution to some random variable L. Then, $(P_n)_{n\in\mathbb{Z}^+}$ and $(Q_n)_{n\in\mathbb{Z}^+}$ are contiguous if and only if $L > 0$ almost surely and $\mathbb{E}(L) = 1$.*

To check that the property from Proposition 9.9 is verified, one can use the following result of Janson [101, Theorem 1 and Lemma 1] (see also [102]). In this theorem, 0^0 is interpreted as 1.

Given $p \in \mathbb{R}^+$, a sequence $(W_n)_{n\in\mathbb{Z}^+}$ of random variables *converges in L^p* to a random variable W if and only if $\mathbb{E}(|W_n|^p)$ and $\mathbb{E}(|W|^p)$ are finite for every n, and

$$\lim_{n\to\infty} \mathbb{E}(|W_n - W|^p) = 0.$$

Given two non-negative integers k and j, $(k)_j$ denotes the truncated factorial $k(k-1)\dots(k-j+1)$.

Theorem 9.10. [101, 102] *For each $i \in \mathbb{Z}^+$ let $\lambda_i > 0$ and $\mu_i \geq 0$ be constants, let $\delta_i = \frac{\mu_i}{\lambda_i} - 1$, and suppose that for each $n \in \mathbb{Z}^+$ there are non-negative random variables $X_{i,n}$, Y_n defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$, such that $X_{i,n}$ is integer-valued and $\mathbb{E}(Y_n) \neq 0$ and, furthermore, the following conditions are satisfied:*

(i) $X_{i,n} \xrightarrow{d} X_i$ *as $n \to \infty$, jointly for all i, where $X_i \sim \mathrm{Pois}(\lambda_i)$ are independent Poisson random variables;*

(ii) for every finite sequence $j_1, \dots, j_k$ of non-negative integers,

$$\lim_{n\to\infty} \frac{\mathbb{E}(Y_n (X_{1,n})_{j_1} \dots (X_{k,n})_{j_k})}{\mathbb{E}(Y_n)} = \prod_{i=1}^{k} \mu_i^{j_i};$$

(iii) $\sum_{i=1}^{\infty} \lambda_i \delta_i^2 < \infty$;

(iv) $\lim_{n\to\infty} \frac{\mathbb{E}(Y_n^2)}{(\mathbb{E}(Y_n))^2} = \exp\left(\sum_{i=1}^{\infty} \lambda_i \delta_i^2\right)$.

Then,

$$\frac{Y_n}{\mathbb{E}(Y_n)} \xrightarrow{d} W := \prod_{i=1}^{\infty} (1+\delta_i)^{X_i} e^{-\lambda_i \delta_i} \quad \text{as } n \to \infty; \tag{9.1}$$

moreover, (9.1) *and the convergence in* (i) *hold jointly. The infinite product defining* W *converges almost surely and in* L^2*, with* $\mathbb{E}(W) = 1$ *and* $\mathbb{E}(W^2) = \exp\left(\sum_{i=1}^{\infty} \lambda_i \delta_i^2\right)$*. Furthermore, the event* $W = 0$ *equals, up to a set of probability zero, the event that* $X_i > 0$ *for some* i *with* $\delta_i = -1$*. In particular,* $W > 0$ *almost surely if and only if* $\delta_i > -1$ *for every* i*.*

In what follows, we keep the notation from Section 9.2.

Corollary 9.11. *For every integer* $r \geq 2$*,*

(i) $\mathbb{G}^*_{\text{reg}}(n, 2r) \approx \operatorname{forg} \mathbb{D}^*_{\text{reg}}(n, r)$ *and*

(ii) $\mathbb{G}_{\text{reg}}(n, 2r) \approx \operatorname{forg} \mathbb{O}_{\text{reg}}(n, r)$.

Proof. (i) It is shown in the proof of Theorem 3.50 from [54] that the hypotheses of Theorem 9.10 hold when, for positive integers i and n, $\lambda_i = \frac{(2r-1)^i}{2i}$, $\mu_i = \frac{(2r-1)^i+1}{2i}$, and $X_{i,n}$ and $Y_n = E^*_{n,2r}(\mathbb{G}^*_{\text{reg}}(n, 2r))$ are the random variables that count, respectively, the number of cycles of length i and the number of arc-labelled Eulerian orientations of $\mathbb{G}^*_{\text{reg}}(n, 2r)$. (We note that, in this particular situation, condition (i) from Theorem 9.10 refers to a classical result on the distribution of short cycle counts in random regular graphs; see for instance [102, Theorem 9.5].) Hence, Theorem 9.10 and Lemma 9.3 imply that $\frac{Q_{n,r}}{P_{n,2r}}$ converges in distribution to a random variable W with $\mathbb{E}(W) = 1$. Moreover, since $\delta_i = \frac{\mu_i}{\lambda_i} - 1 = (2r-1)^{-i} > -1$ for every i, W is positive almost surely. The conclusion follows from Proposition 9.9.

(ii) Let $(\mathcal{A}_n)_{n\in\mathbb{Z}^+}$ be a sequence of sets of regular graphs with $\mathcal{A}_n \subseteq \mathscr{G}_{\text{reg}}(n, 2r)$ for each n. We have the following chain of implications.

$$\begin{aligned}
\lim_{n\to\infty} P_{n,2r}(\mathcal{A}_n \mid \mathscr{G}_{\text{reg}}(n, 2r)) = 0 &\implies \lim_{n\to\infty} P_{n,2r}(\mathcal{A}_n) = 0 \\
&\implies \lim_{n\to\infty} Q_{n,r}(\mathcal{A}_n) = 0 \qquad \text{by } (i) \\
&\implies \lim_{n\to\infty} Q_{n,r}(\mathcal{A}_n \mid \mathscr{G}_{\text{reg}}(n, 2r)) = 0,
\end{aligned}$$

where in the last step we used that $Q_{n,r}(\mathscr{G}_{\text{reg}}(n, 2r)) = \mathbb{P}[\operatorname{forg} \mathbb{D}^*_{\text{reg}}(n, r) \in \mathscr{G}_{\text{reg}}(n, 2r)] = \mathbb{P}[\mathbb{D}^*_{\text{reg}}(n, r) \in \mathscr{O}_{\text{reg}}(n, r)]$, in combination with Lemma 9.4. The argument for the other direction is symmetric; it suffices to replace Lemma 9.4 by the analogous, well-known fact [102, Corollary 9.7] that

$$\lim_{n\to\infty} \mathbb{P}\left[\mathbb{G}^*_{\text{reg}}(n, r) \in \mathscr{G}_{\text{reg}}(n, r)\right] = e^{-(r^2-1)/4}.$$

□

Corollary 9.11 cannot be extended to $r = 1$, see [102, Remark 9.45].

9.5 Spectral bounds

Graph theory has for long benefited of the power of linear algebra via the association of matrices to graphs. Their spectrum often carries a good amount of combinatorial information. For instance, there exist several bounds for the independence and chromatic numbers of a graph in terms of their eigenvalues, of which those of Hoffman [81] and Wilf [157] are two classical examples. In this section we provide similar bounds for the size of the largest acyclic set and the dichromatic number of the orientations of a given graph. Working with digraph spectra is also possible (see [110] and [121] for results that link them to digraph colouring), but seems less common. We refer the interested reader to Brualdi's survey [41] and Guo's doctoral dissertation [79] for more information on spectra of digraphs.

The *Laplacian matrix* of a graph G of order n is the $n \times n$ matrix $L(G) = (\ell_{u,v})$, doubly indexed by the set of vertices of G, and defined as

$$\ell_{u,v} = \begin{cases} \deg(u) & \text{if } u = v \\ -1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Since $L(G)$ is symmetric, it is diagonalisable over $\mathbb{R}$. Its eigenvalues $\mu_1 \leq \ldots \leq \mu_n$ are called the *Laplacian eigenvalues* of G. We have that $\mu_n \leq n$, $\mu_1 = 0$, and the multiplicity of 0 as an eigenvalue equals the number of connected components of G. Thus, μ_2, known as the *algebraic connectivity* of G, is positive if and only if G is connected. We also have that $\mu_2 \leq \delta(G) + 1$, and, if G has at least one edge, $\mu_n \geq \Delta(G) + 1$ [40, Section 1.3 and Propositions 1.7.2 and 3.9.3].

The following bound is instrumental for the results in this section. Due to the lack of a reference, we include its proof in Appendix C.

Lemma 9.12. *Let G be a graph on $n \geq 2$ vertices, and let $0 = \mu_1 \leq \ldots \leq \mu_n$ be its Laplacian eigenvalues. If S is a non-empty set of vertices of G and d_S is the average degree of $G[S]$, then*

$$\delta(G) - \left(1 - \frac{|S|}{n}\right)\mu_n \leq d_S \leq \Delta(G) - \left(1 - \frac{|S|}{n}\right)\mu_2.$$

We recall that $\vec{\alpha}(G)$ and $\vec{\chi}(G)$ denote, respectively, the minimum of $\vec{\alpha}(D)$ and the maximum of $\vec{\chi}(D)$ over all the orientations D of the graph G. Our bounds for $\vec{\alpha}(G)$ and $\vec{\chi}(G)$ are an adaptation of analogous results of Hoffman [81] and Alon, Krivelevich and Sudakov [16] for $\alpha(G)$ and $\chi(G)$. Essentially, the same proofs go through in the directed setting; roughly speaking, in some cases we gain a factor of 2 with respect to the undirected version of the bound.

Proposition 9.13. *Let G be a connected graph on $n \geq 2$ vertices, with minimum degree δ and maximum degree Δ, and let $0 = \mu_1 \leq \ldots \leq \mu_n$ be its Laplacian eigenvalues. Then, for any non-empty set S of vertices of G,*

$$f_1(n, |S|, \Delta, \mu_2) < \vec{\alpha}(G[S]) < f_2(n, \delta, \Delta, \mu_2, \mu_n),$$

where

$$f_1(n, |S|, \Delta, \mu_2) = \left(\frac{2n}{\mu_2} - 1\right) \ln\left(\frac{|S|}{n} \frac{\mu_2}{\Delta - \mu_2 + 2} + 1\right),$$

$$f_2(n, \delta, \Delta, \mu_2, \mu_n) = \frac{(\mu_n - \delta)n}{\mu_n} \left(1 + \frac{2}{\mu_n - \delta} \log_2\left(\frac{e\mu_n(\Delta - \mu_2 + 1)}{\mu_n - \delta} + e\mu_2\right)\right).$$

Proof. We first prove the lower bound. Let D be any orientation of G. We construct a sequence of sets $S_0, S_1, \ldots$ as follows. We let $S_0 = S$. Given $k \geq 1$, we let $S_k = \emptyset$ if $S_{k-1} = \emptyset$. Otherwise, we pick a vertex $v_{k-1} \in S_{k-1}$ with at most $\Delta - (1 - |S_{k-1}|/n)\,\mu_2$ neighbours in S_{k-1}, which exists due to Lemma 9.12. Then, we let S_k be the set with largest cardinality among $S_{k-1} \setminus \{v_{k-1}\} \setminus N^-(v_{k-1})$ and $S_{k-1} \setminus \{v_{k-1}\} \setminus N^+(v_{k-1})$, where $N^-(v_{k-1})$ and $N^+(v_{k-1})$ denote the sets of in-neighbours and out-neighbours of v_{k-1}, respectively. We note that, for each $k \geq 0$ such that $S_k \neq \emptyset$, $\{v_0, v_1, \ldots, v_k\}$ is an acyclic set of $D[S]$. Thus, it suffices to show that S_k is non-empty for

$$k = \lfloor f_1(n, |S|, \Delta, \mu_2) \rfloor .$$

Let a denote the quantity $1 - \frac{\mu_2}{2n}$; since G is connected, $\frac{1}{2} \leq a < 1$. We have that

$$|S_k| \geq |S_{k-1}| - 1 - \frac{\Delta}{2} + \left(1 - \frac{|S_{k-1}|}{n}\right) \frac{\mu_2}{2} = a|S_{k-1}| - 1 - \frac{\Delta}{2} + \frac{\mu_2}{2}$$

$$\geq a\left(a|S_{k-2}| - 1 - \frac{\Delta}{2} + \frac{\mu_2}{2}\right) - 1 - \frac{\Delta}{2} + \frac{\mu_2}{2} \geq \ldots$$

$$\geq a^k|S_0| + (1 + a + \ldots + a^{k-1}) \left(-1 - \frac{\Delta}{2} + \frac{\mu_2}{2}\right)$$

$$= a^k|S| + \frac{1 - a^k}{1 - a} \left(-1 - \frac{\Delta}{2} + \frac{\mu_2}{2}\right).$$

We note that $-1 - \Delta/2 + \mu_2/2 \leq (-1 - \Delta + \delta)/2 < 0$. Hence, we need to check that

$$\begin{aligned}
a^k|S| + \frac{1-a^k}{1-a}\left(-1-\frac{\Delta}{2}+\frac{\mu_2}{2}\right) &> 0 \\
a^k|S|\frac{1-a}{1+\frac{\Delta}{2}-\frac{\mu_2}{2}} &> 1-a^k \\
a^k\left(|S|\frac{1-a}{1+\frac{\Delta}{2}-\frac{\mu_2}{2}}+1\right) &> 1 \\
k\ln a + \ln\left(|S|\frac{1-a}{1+\frac{\Delta}{2}-\frac{\mu_2}{2}}+1\right) &> 0 \\
-\frac{\ln\left(|S|\frac{1-a}{1+\frac{\Delta}{2}-\frac{\mu_2}{2}}+1\right)}{\ln a} &> k.
\end{aligned}$$

This follows from the fact that

$$\ln\frac{1}{a} = \ln\left(1+\frac{\mu_2}{2n-\mu_2}\right) < \frac{\mu_2}{2n-\mu_2}.$$

Let us now bound $\vec{\alpha}(G)$ from above. We let D be the random orientation of G obtained by orienting every edge independently with probability $1/2$ in each direction. Let T be a subset of vertices of size

$$t = \lceil f_2(n,\delta,\Delta,\mu_2,\mu_n)\rceil,$$

and let d_T be the average degree of the subgraph induced by T. By Lemma 9.12 and Theorem 2.3, the probability that $D[T]$ is acyclic is at most

$$(d_T+1)^t\cdot 2^{-\frac{1}{2}d_T t} \leq \left(\Delta - \left(1-\frac{t}{n}\right)\mu_2+1\right)^t\cdot 2^{-\frac{t}{2}\left(\delta-\left(1-\frac{t}{n}\right)\mu_n\right)}.$$

Therefore, since there are $\binom{n}{t} < \left(\frac{en}{t}\right)^t$ t-subsets of $V(G)$, the probability that one of them is acyclic is less than

$$\left(\frac{en}{t}\right)^t\cdot\left(\Delta-\left(1-\frac{t}{n}\right)\mu_2+1\right)^t\cdot 2^{-\frac{t}{2}\left(\delta-\left(1-\frac{t}{n}\right)\mu_n\right)}.$$

It will be enough to check that this quantity is at most 1, i.e., that

$$\begin{aligned}
\frac{en}{t}\cdot\left(\Delta-\left(1-\frac{t}{n}\right)\mu_2+1\right)\cdot 2^{-\frac{1}{2}\left(\delta-\left(1-\frac{t}{n}\right)\mu_n\right)} &\leq 1 \\
\log_2\left(\frac{en}{t}(\Delta-\mu_2+1)+e\mu_2\right)-\frac{\delta}{2}+\frac{\mu_n}{2} &\leq \frac{t\mu_n}{2n} \\
\frac{(\mu_n-\delta)n}{\mu_n}\left(\frac{2}{\mu_n-\delta}\log_2\left(\frac{en}{t}(\Delta-\mu_2+1)+e\mu_2\right)+1\right) &\leq t.
\end{aligned}$$

We note that $\mu_n-\delta \geq \Delta+1-\delta > 0$, because G has at least one edge. Together with the fact that $\mu_2 \leq \delta+1 \leq \Delta+1$, this implies that $t \geq (\mu_n-\delta)n/\mu_n$, so the inequality above holds. □

Proposition 9.14. *Let $G = (V, E)$ be a connected graph on $n \geq 2$ vertices, with minimum degree δ and maximum degree Δ, and let $0 = \mu_1 \leq \ldots \leq \mu_n$ be its Laplacian eigenvalues. Then,*

$$\frac{n}{f_2(n, \delta, \Delta, \mu_2, \mu_n)} < \vec{\chi}(G) < f_3(n, \delta, \Delta, \mu_2),$$

where f_2 is the function from Proposition 9.13 and

$$f_3(n, \delta, \Delta, \mu_2) = (e-1)\left(\frac{\Delta - \mu_2}{2} + 1\right) +$$
$$+ \frac{\mu_2}{2 \ln \frac{\Delta+2}{\Delta-\mu_2+2}} \left(\left(1 - \frac{\delta+1}{2n}\right)^{-1} \left(1 - \frac{2 \ln \ln \frac{\Delta+2}{\Delta-\mu_2+2}}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}}\right)^{-1} + \frac{1}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}} \right).$$

Proof. The lower bound follows from Proposition 9.13. For the upper bound, let D be any orientation of G. We distinguish two cases.

We suppose first that $\frac{\Delta+2}{\Delta-\mu_2+2} \leq e$. Using that $(1 - \frac{2\ln x}{x})^{-1} > 0$ for all $x > 0$, we see that the claimed upper bound is larger than

$$(e-1)\left(\frac{\Delta - \mu_2}{2} + 1\right) + \frac{\mu_2}{2 \ln\left(\frac{\mu_2}{\Delta-\mu_2+2} + 1\right)} \frac{1}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}}$$

$$\geq (e-1)\left(\frac{\Delta - \mu_2}{2} + 1\right) + \frac{\Delta - \mu_2 + 2}{2} = e\left(\frac{\Delta - \mu_2}{2} + 1\right) \geq \frac{\Delta}{2} + 1 \geq \vec{\chi}(D).$$

Let us now assume that $\frac{\Delta+2}{\Delta-\mu_2+2} \geq e$. We let $0 < \varepsilon \leq 1$ be a constant that will be specified later, and construct a sequence of sets $S_0, S_1, \ldots$ as follows. We set $S_0 = V$. For $k \geq 1$, if

$$|S_{k-1}| < \frac{\varepsilon n}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}}, \tag{9.2}$$

then we set $S_k = S_{k-1}$. Otherwise, by Proposition 9.13, $D[S_{k-1}]$ has an acyclic set T_{k-1} of size larger than

$$\left(\frac{2n}{\mu_2} - 1\right) \ln\left(\frac{|S_{k-1}|}{n} \frac{\mu_2}{\Delta - \mu_2 + 2} + 1\right)$$
$$\geq \left(\frac{2n}{\mu_2} - 1\right) \ln\left(\frac{\varepsilon}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}} \frac{\mu_2}{\Delta - \mu_2 + 2} + 1\right)$$
$$\geq \left(\frac{2n}{\mu_2} - 1\right) \ln\left(\frac{\varepsilon}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}} \left(\frac{\mu_2}{\Delta - \mu_2 + 2} + 1\right)\right)$$
$$= \left(\frac{2n}{\mu_2} - 1\right) \left(\ln \frac{\Delta+2}{\Delta - \mu_2 + 2} - \ln \ln \frac{\Delta+2}{\Delta - \mu_2 + 2} + \ln \varepsilon\right).$$

In this case, we set $S_k = S_{k-1} \setminus T_{k-1}$. Let $\ell \geq 1$ be the least integer for which (9.2) holds with $k-1=\ell$. We note that $T_0, \dots, T_{\ell-1}$ is a partition of $D[V \setminus S_\ell]$ into ℓ acyclic sets. On the other hand, $D[S_\ell]$ can be partitioned into $\lfloor g/2 \rfloor + 1$ acyclic sets, where g is the degeneracy of $G[S_\ell]$. By Lemma 9.12,

$$g \leq \Delta - \left(1 - \frac{|S_\ell|}{n}\right)\mu_2 < \Delta - \left(1 - \frac{\varepsilon}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}}\right)\mu_2.$$

Hence, we have that

$$\vec{\chi}(D) \leq \ell + \left\lfloor \frac{g}{2} \right\rfloor + 1$$

$$\leq \frac{\mu_2}{2\ln \frac{\Delta+2}{\Delta-\mu_2+2}} \left(\left(1 - \frac{\mu_2}{2n}\right)^{-1} \left(1 + \frac{\ln \varepsilon - \ln\ln \frac{\Delta+2}{\Delta-\mu_2+2}}{\ln \frac{\Delta+2}{\Delta-\mu_2+2}}\right)^{-1} + \varepsilon \right) + \frac{\Delta-\mu_2}{2} + 1.$$

We get the desired conclusion by choosing $\varepsilon = \ln^{-1} \frac{\Delta+2}{\Delta-\mu_2+2}$. □

We now apply Propositions 9.13 and 9.14 to random r-regular digraphs to construct explicit acyclic sets and dicolourings of the optimal order of magnitude. To that end, let us take a look at the spectral properties of $\mathrm{forg}\,\mathbb{O}_{\mathrm{reg}}(n,r)$.

The *adjacency matrix* of a graph G of order n is the $n \times n$ matrix $A(G) = (a_{u,v})$, doubly indexed by the set of vertices of G, and defined as

$$a_{u,v} = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$ of $A(G)$ are called the *eigenvalues* of G. We let $\lambda(G)$ denote the quantity $\max\{|\lambda_2|, |\lambda_n|\}$. It is easy to check that, if G is d-regular, its eigenvalues and its Laplacian eigenvalues $\mu_1 \leq \dots \leq \mu_n$ satisfy $\lambda_1 + \mu_1 = \dots = \lambda_n + \mu_n = d$. In particular, $\lambda_1 = d$.

Alon [13] famously conjectured that, for every $d > 1$ and $\varepsilon > 0$, as $n \to \infty$, the second largest eigenvalue of almost all d-regular n-vertex graphs is at most $2\sqrt{d-1}+\varepsilon$. This has been certified by Friedman [69] (see also [35]) for multiple models of random d-regular graphs. Together with the work of Greenhill, Janson, Kim and Wormald on contiguity [77], Friedman's result has the following immediate consequence.

Theorem 9.15. [69] *For every even $d \geq 4$ and every $\varepsilon > 0$,*

$$\lambda(\mathbb{G}^*_{\mathrm{reg}}(n,d)) \leq 2\sqrt{d-1} + \varepsilon$$

with probability tending to 1 as $n \to \infty$.

Combining the results of this section with those of Section 9.4, we obtain a constructive version of Theorem 9.6. For large degrees, it is, essentially, a factor of 4 away from the optimal bounds. A different constructive proof of Corollary 9.16(i), not reliant on heavy tools, can be found in [88].

Corollary 9.16. *For any fixed integer* $r \geq 2$*, both*

(i) $\vec{\alpha}(\mathbb{O}_{\mathrm{reg}}(n,r)) \geq (1+o_r(1))\dfrac{n\ln r}{2r}$ *and*

(ii) $\vec{\chi}(\mathbb{O}_{\mathrm{reg}}(n,r)) \leq (1+o_r(1))\dfrac{2r}{\ln r}$

hold with probability tending to 1 *as* $n \to \infty$.

Proof. We apply Propositions 9.13 and 9.14 to $\operatorname{forg}\mathbb{O}_{\mathrm{reg}}(n,r)$. Let $d = 2r$ be the degree of $\operatorname{forg}\mathbb{O}_{\mathrm{reg}}(n,r)$, and let μ_2 be its second smallest Laplacian eigenvalue. By Theorem 9.15, Corollary 9.11 and Lemma 9.4, for any $\varepsilon > 0$, we have that $\mu_2 \geq d - 2\sqrt{d-1} - \varepsilon$ whp. We choose ε small enough so that all future claims involving it actually hold. Similarly, $\operatorname{forg}\mathbb{O}_{\mathrm{reg}}(n,r)$ mimics the property of $\mathbb{G}_{\mathrm{reg}}(n,d)$ of being d-connected whp (see [30, Theorem 7.32]). (We recall that a graph G is *k-connected* if at least k vertices have to be removed from G in order to obtain a disconnected graph or the one-vertex graph.)

Let f_1 be as in Proposition 9.13. For n large enough, the function $x \mapsto f_1(n,n,d,x)$ is increasing in the interval $I = [d-2\sqrt{d-1}-\varepsilon, d+1]$. Hence, whp $\vec{\alpha}(\mathbb{O}_{\mathrm{reg}}(n,r))$ is at least

$$\vec{\alpha}(\operatorname{forg}\mathbb{O}_{\mathrm{reg}}(n,r)) > f_1(n,n,2r,2r-2\sqrt{2r-1}-\varepsilon) = (1+o_r(1))\frac{n\ln r}{2r}.$$

Let f_3 be as in Proposition 9.14. For d large enough and $n \geq d+1$, the function $x \mapsto f_3(n,d,d,x)$ is decreasing in I. Hence, whp $\vec{\chi}(\mathbb{O}_{\mathrm{reg}}(n,r))$ is at most

$$\vec{\chi}(\operatorname{forg}\mathbb{O}_{\mathrm{reg}}(n,r)) < f_3(n,2r,2r,2r-2\sqrt{2r-1}-\varepsilon) = (1+o_r(1))\frac{2r}{\ln r}.$$

□

Appendix A

Probabilistic tools

A.1 Concentration inequalities

When analysing random variables, one often wants to control the probability that they differ too much from their expectations. There are several concentration inequalities that can be used to that end; here we collect the ones used in this thesis. Even if some of them can be deduced from some others, we decided to include them all in order to favour the clarity of proofs.

Lemma A.1. (Markov's inequality) [30] *Let X be a non-negative random variable and let $\lambda \in \mathbb{R}^+$. Then,*

$$\mathbb{P}\left[X \geq \lambda \mathbb{E}(X)\right] \leq \frac{1}{\lambda}.$$

If the random variables satisfy some special hypotheses, much stronger bounds can be obtained. A typical example is that of binomial random variables.

Lemma A.2. (Chernoff's inequality) [102, Corollary 2.3] *Let $B_{N,p}$ be the sum of N independent Bernoulli random variables of parameter p, and let $0 \leq \beta \leq 3/2$. Then,*

$$\mathbb{P}\left[|B_{N,p} - Np| \geq \beta Np\right] \leq 2e^{-\beta^2 Np/3}.$$

More generally, this applies to certain "Lipschitz" functions of arbitrary independent random variables.

Lemma A.3. (Simple concentration bound) [102, Corollary 2.27] *Let $Z_1, \ldots, Z_N$ be independent random variables, with Z_k taking values in a set Ω_k. Assume that a measurable function $f : \Omega_1 \times \ldots \times \Omega_N \to \mathbb{R}$ satisfies the following Lipschitz condition for some constant $c > 0$:*

(L) *if two vectors $\omega, \omega' \in \Omega_1 \times \ldots \times \Omega_N$ differ only in one coordinate, then $|f(\omega) - f(\omega')| \leq c$.*

Then, for any real number $\lambda \geq 0$, the random variable $X = f(Z_1, \ldots, Z_N)$ satisfies

$$\mathbb{P}\left[|X - \mathbb{E}(X)| \geq \lambda\right] \leq 2\exp\left(\frac{-\lambda^2}{2c^2 N}\right).$$

A notorious result of this kind is the more general Azuma's inequality. See [117] for an exposition of the topic.

Lemma A.4. (Azuma's inequality) [102, Theorem 2.25] *Let $X_0, X_1, \ldots, X_N$ be a martingale with $X_N = X$ and $X_0 = \mathbb{E}(X)$, and let $c_1, \ldots, c_N > 0$ be constants such that $|X_k - X_{k-1}| \leq c_k$ for every $1 \leq k \leq N$. Then, for every $\lambda \geq 0$,*

$$\mathbb{P}\left[X \geq \mathbb{E}(X) + \lambda\right] \leq \exp\left(\frac{-\lambda^2}{2\sum_{k=1}^N c_k^2}\right) \quad \text{and}$$

$$\mathbb{P}\left[X \leq \mathbb{E}(X) - \lambda\right] \leq \exp\left(\frac{-\lambda^2}{2\sum_{k=1}^N c_k^2}\right).$$

In practice, the following consequence of Azuma's inequality is sometimes more convenient. It can be derived in the same way as [102, Corollary 2.27] (see also [126, Section 11.1]).

Lemma A.5. (Azuma's inequality, combinatorial version) *Let $Z_1, \ldots, Z_N$ be independent random variables, with Z_k taking values in a discrete set Ω_k. Let $f : \Omega_1 \times \ldots \times \Omega_N \to \mathbb{R}$ be a function, and consider the random variable $X = f(Z_1, \ldots, Z_N)$. Let $c_1, \ldots, c_N > 0$ be constants such that, for each $1 \leq k \leq N$ and any $\omega_1 \in \Omega_1, \ldots, \omega_{k-1} \in \Omega_{k-1}$ and $\omega_k, \omega_k' \in \Omega_k$,*

$$\left|\mathbb{E}\left(X \mid Z_1 = \omega_1, \ldots, Z_k = \omega_k\right) - \mathbb{E}\left(X \mid Z_1 = \omega_1, \ldots, Z_{k-1} = \omega_{k-1}, Z_k = \omega_k'\right)\right|$$

is at most c_k. Then, for every $\lambda \geq 0$,

$$\mathbb{P}\left[|X - \mathbb{E}(X)| > \lambda\right] \leq 2\exp\left(\frac{-\lambda^2}{2\sum_{k=1}^N c_k^2}\right).$$

The last concentration inequality is a consequence of Talagrand's inequality. We use an adaptation of a version from [125] to random variables taking values in $\mathbb{N}$.

Lemma A.6. (Talagrand's inequality) [87] *Let $Z_1, \ldots, Z_N$ be independent random variables, with Z_k taking values in a set Ω_k, and let $\Omega = \Omega_1 \times \ldots \times \Omega_N$. Assume that a measurable function $f : \Omega \to \mathbb{N}$ satisfies the following conditions for some positive integers c and r:*

(L) *if two vectors $\omega, \omega' \in \Omega$ differ only in one coordinate, then $|f(\omega) - f(\omega')| \leq c$;*

(C) *for every* $s \in \mathbb{N}$ *and every* $\omega \in \Omega$ *with* $f(\omega) \geq s$*, there is a set of indices* $I \subseteq [N]$ *of size* $|I| \leq rs$ *satisfying that, for every* $\omega' \in \Omega$ *that coincides with* ω *on all coordinates indexed by* I*,* $f(\omega') \geq s$*.*

Then, the random variable $X = f(Z_1, \ldots, Z_N)$ *satisfies*

$$\mathbb{P}\left[|X - \mathbb{E}(X)| > \lambda\right] \leq 4 \exp\left(\frac{-\lambda^2}{32c^2 r(\mathbb{E}(X) + \lambda)}\right)$$

for any real number $\lambda > 126c\sqrt{r\mathbb{E}(X)} + 344c^2 r$*.*

A.2 The Lovász local lemma

Let $\mathcal{E} = \{E_1, \ldots, E_n\}$ be a finite set of events in a probability space. $\mathcal{E}$ is called *mutually independent* (often simply *independent*) if, for any subset $\mathcal{F} \subseteq \mathcal{E}$, $\mathbb{P}[\cap \mathcal{F}] = \prod_{F \in \mathcal{F}} \mathbb{P}[F]$.

When $\mathcal{E}$ is mutually independent and all the events in $\mathcal{E}$ occur with positive probability, the probability that all the events in $\mathcal{E}$ occur simultaneously is positive. The local lemma (Lemma A.7) tells us that this remains true when there are only a few dependencies between the events. We include its proof from [18], rewritten in a way that maybe will be helpful to the reader.

An event E is *mutually independent* of $\mathcal{E}$ if, for any subset $\mathcal{F} \subseteq \mathcal{E}$, $\mathbb{P}[E \cap (\cap \mathcal{F})] = \mathbb{P}[E]\,\mathbb{P}[\cap \mathcal{F}]$. A *dependency digraph* for $\mathcal{E}$ is a digraph D with vertex set $[n]$ such that, for each $i \in [n]$, E_i is mutually independent of $\{E_j \in \mathcal{E} \mid (i,j) \notin A(D)\} \setminus \{E_i\}$.

Lemma A.7. (Local lemma, general case) [18, Theorem 5.1.1] *Let* $E_1, \ldots, E_n$ *be events in an arbitrary probability space. Suppose that* $D = (V, A)$ *is a dependency digraph for these events and that there are real numbers* $x_1, \ldots, x_n$ *such that* $0 \leq x_i < 1$ *and* $\mathbb{P}[E_i] \leq x_i \prod_{(i,j) \in A}(1 - x_j)$ *for all* $1 \leq i \leq n$*. Then*

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{E_i}\right] \geq \prod_{i=1}^{n}(1 - x_i).$$

Proof. Our aim is to prove by induction that, for every $S \subseteq [n]$ and every $i \in S$,

$$\mathbb{P}\left[\bigcap_{j \in S} \overline{E_j}\right] \geq (1 - x_i)\mathbb{P}\left[\bigcap_{j \in S \setminus \{i\}} \overline{E_j}\right]. \tag{A.1}$$

When $|S| = 1$ this is certainly true, as

$$\mathbb{P}\left[\overline{E_i}\right] = 1 - \mathbb{P}[E_i] \geq 1 - x_i \prod_{(i,j) \in A}(1 - x_j) \geq 1 - x_i.$$

For the case $|S| \geq 2$, we suppose that (A.1) holds for every proper subset of S. In particular, $\mathbb{P}\left[\cap_{j \in S \setminus \{i\}} \overline{E_j}\right] > 0$. Let $S_1 = \{j \in S \setminus \{i\} \mid (i,j) \in A\}$ and $S_2 = S \setminus \{i\} \setminus S_1$. Then

$$\mathbb{P}\left[E_i \,\middle|\, \bigcap_{j \in S \setminus \{i\}} \overline{E_j}\right] = \frac{\mathbb{P}\left[E_i \cap \bigcap_{j \in S_1} \overline{E_j} \,\middle|\, \bigcap_{j \in S_2} \overline{E_j}\right]}{\mathbb{P}\left[\bigcap_{j \in S_1} \overline{E_j} \,\middle|\, \bigcap_{j \in S_2} \overline{E_j}\right]}. \tag{A.2}$$

Let us now bound (A.2) from above. The numerator is at most

$$\mathbb{P}\left[E_i \,\middle|\, \bigcap_{j \in S_2} \overline{E_j}\right] = \mathbb{P}\left[E_i\right] \leq x_i \prod_{(i,j) \in A} (1 - x_j).$$

And, using the induction hypothesis, we see that the denominator is at least

$$\frac{\mathbb{P}\left[\bigcap_{j \in S \setminus \{i\}} \overline{E_j}\right]}{\mathbb{P}\left[\bigcap_{j \in S_2} \overline{E_j}\right]} \geq \prod_{j \in S_1} (1 - x_j) \geq \prod_{(i,j) \in A} (1 - x_j).$$

Therefore, $\mathbb{P}\left[E_i \,\middle|\, \cap_{j \in S \setminus \{i\}} \overline{E_j}\right] \leq x_i$. This implies that

$$\mathbb{P}\left[\bigcap_{j \in S} \overline{E_j}\right] = \mathbb{P}\left[\overline{E_i} \,\middle|\, \bigcap_{j \in S \setminus \{i\}} \overline{E_j}\right] \mathbb{P}\left[\bigcap_{j \in S \setminus \{i\}} \overline{E_j}\right] \geq (1 - x_i) \mathbb{P}\left[\bigcap_{j \in S \setminus \{i\}} \overline{E_j}\right].$$

□

As a consequence, the following is obtained (see [18]).

Lemma A.8. (Local lemma, symmetric case) [18, Corollary 5.1.2] *Let $E_1, \ldots, E_n$ be events in an arbitrary probability space. Suppose that each event E_i is mutually independent of a set of all the other events but at most d, and that $\mathbb{P}\left[E_i\right] \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $\mathbb{P}\left[\bigcap_{i=1}^n \overline{E_i}\right] > 0$.*

Appendix B

Some basic properties of random graphs and digraphs

B.1 Monotonicity

We stick to the notation from Section 1.4.

A *property of graphs on* $[n]$ is a subset $\mathcal{Q}$ of $\mathscr{G}(n)$ which is closed under the graph isomorphism relation on $\mathscr{G}(n)$. $\mathcal{Q}$ is *increasing* if, for any $G \in \mathcal{Q}$ and any supergraph $H \in \mathscr{G}(n)$ of G, $H \in \mathcal{Q}$. $\mathcal{Q}$ is *decreasing* if, for any $G \in \mathcal{Q}$ and any subgraph $H \in \mathscr{G}(n)$ of G, $H \in \mathcal{Q}$ (or, equivalently, if $\mathscr{G}(n) \setminus \mathcal{Q}$ is increasing). If $\mathcal{Q}$ is either increasing or decreasing, then it is called *monotone*.

Analogously, we define *property of digraphs on* $[n]$, *property of oriented graphs on* $[n]$, and the associated terminology.

Informally, the following lemma says that increasing properties are more likely to occur in denser random graphs. Parts (i) and (ii) are special cases of a more general statement about random subsets [102, Lemma 1.10], and (iii) can be proved in a similar way.

Lemma B.1. (i) *Let n be a positive integer and $\mathcal{Q} \subseteq \mathscr{G}(n)$ an increasing property of graphs on $[n]$. Given $0 \leq p_1 \leq p_2 \leq 1$ and integers $0 \leq M_1 \leq M_2 \leq \binom{n}{2}$,*

$$\mathbb{P}\left[\mathbb{G}(n, p_1) \in \mathcal{Q}\right] \leq \mathbb{P}\left[\mathbb{G}(n, p_2) \in \mathcal{Q}\right] \quad \text{and}$$

$$\mathbb{P}\left[\mathbb{G}(n; M_1) \in \mathcal{Q}\right] \leq \mathbb{P}\left[\mathbb{G}(n; M_2) \in \mathcal{Q}\right].$$

(ii) *The same statement holds if $\mathcal{Q} \subseteq \mathscr{D}(n)$ is an increasing property of digraphs on $[n]$ and $0 \leq M_1 \leq M_2 \leq n(n-1)$, replacing $\mathbb{G}(n, p_i)$ and $\mathbb{G}(n; M_i)$ by $\mathbb{D}(n, p_i)$ and $\mathbb{D}(n; M_i)$.*

(iii) *The same statement holds if $\mathcal{Q} \subseteq \mathscr{O}(n)$ is an increasing property of oriented graphs on $[n]$, replacing $\mathbb{G}(n, p_i)$ and $\mathbb{G}(n; M_i)$ by $\mathbb{O}(n, p_i/2)$ and $\mathbb{O}(n; M_i)$.*

For M close to $p\binom{n}{2}$ and n large, the behaviour of the random graphs $\mathbb{G}(n,p)$ and $\mathbb{G}(n;M)$ is somewhat similar, see for instance [102, Section 1.4]. For monotone properties, we can use the following useful fact.

Lemma B.2. [102, Corollary 1.16] *Let $\mathcal{Q} \subseteq \mathscr{G}(n)$ be an increasing property of graphs on $[n]$, and let $M = M(n) \to \infty$. Assume that $\delta > 0$ is a fixed constant with $0 \le (1-\delta)M/\binom{n}{2} \le (1+\delta)M/\binom{n}{2} \le 1$.*

(i) If $\mathbb{P}\left[\mathbb{G}(n, M/\binom{n}{2}) \in \mathcal{Q}\right] \to 1$, then $\mathbb{P}\left[\mathbb{G}(n;M) \in \mathcal{Q}\right] \to 1$.

(ii) If $\mathbb{P}\left[\mathbb{G}(n, M/\binom{n}{2}) \in \mathcal{Q}\right] \to 0$, then $\mathbb{P}\left[\mathbb{G}(n;M) \in \mathcal{Q}\right] \to 0$.

(iii) If $\mathbb{P}\left[\mathbb{G}(n;M) \in \mathcal{Q}\right] \to 1$, then $\mathbb{P}\left[\mathbb{G}(n, (1+\delta)M/\binom{n}{2}) \in \mathcal{Q}\right] \to 1$.

(iv) If $\mathbb{P}\left[\mathbb{G}(n;M) \in \mathcal{Q}\right] \to 0$, then $\mathbb{P}\left[\mathbb{G}(n, (1-\delta)M/\binom{n}{2}) \in \mathcal{Q}\right] \to 0$.

(All the limits are for $n \to \infty$.) The same hold if $\mathcal{Q} \subseteq \mathscr{D}(n)$ is an increasing property of digraphs on $[n]$, replacing $\mathbb{G}(n,p)$ and $\mathbb{G}(n;M)$ by $\mathbb{D}(n,p)$ and $\mathbb{D}(n;M)$, and $\binom{n}{2}$ by $n(n-1)$.

B.2 Vertex exposure martingales

The following lemma is a consequence of Azuma's inequality (Lemma A.4); see [18, Section 7.4].

Lemma B.3. [18, Theorem 7.4.2] *Let n be a positive integer, $0 \le p \le 1$, $L : \mathscr{G}(n) \to \mathbb{R}$, and let X be the random variable $L(\mathbb{G}(n,p))$. If L satisfies*

$$\forall G, G' \in \mathscr{G}(n)\ \forall k \in [n]\ G - k = G' - k \ \Rightarrow\ |L(G') - L(G)| \le 1,$$

then, for all $\lambda \ge 0$,

$$\mathbb{P}\left[X \ge \mathbb{E}(X) + \lambda\sqrt{n}\right] \le e^{-\lambda^2/2} \quad \text{and} \quad \mathbb{P}\left[X \le \mathbb{E}(X) - \lambda\sqrt{n}\right] \le e^{-\lambda^2/2}.$$

The same statement holds if $\mathscr{G}(n)$ and $\mathbb{G}(n,p)$ are replaced by $\mathscr{O}(n)$ and $\mathbb{O}(n,p/2)$, or if they are replaced by $\mathscr{D}(n)$ and $\mathbb{D}(n,p)$.

Appendix C

Eigenvalue interlacing

C.1 Proof of Lemma 9.12

Lemma 9.12 is a consequence of eigenvalue interlacing. Let A be a real symmetric $n \times n$ matrix with eigenvalues $\theta_1(A) \geq \ldots \geq \theta_n(A)$ and B a real symmetric $m \times m$ matrix with eigenvalues $\theta_1(B) \geq \ldots \geq \theta_m(B)$, where $m \leq n$. The eigenvalues of B *interlace* the eigenvalues of A if, for every $1 \leq i \leq m$,

$$\theta_{n-m+i}(A) \leq \theta_i(B) \leq \theta_i(A).$$

Suppose that the rows and columns of A are indexed by a set X. Given a partition $\{X_1, \ldots, X_s\}$ of X, we let

$$A = \begin{pmatrix} A_{1,1} & \ldots & A_{1,s} \\ \vdots & & \vdots \\ A_{s,1} & \ldots & A_{s,s} \end{pmatrix}$$

be the expression of A as a block matrix, where the entries of A are grouped according to $\{X_1, \ldots, X_s\}$. That is, for $1 \leq i, j \leq s$, $A_{i,j}$ is the submatrix of A resulting from combining the rows with index in X_i with the columns with index in X_j. Let $c_{i,j}$ denote the average row sum of $A_{i,j}$. The $s \times s$ matrix $C = (c_{i,j})$ is called the *quotient matrix* of A with respect to the partition $\{X_1, \ldots, X_s\}$.

The proof of Lemma 9.12 is based on that of an analogous result concerning the eigenvalues of the adjacency matrix of regular graphs [43, Theorem 4.1]. It relies on the following special case of the so-called 'interlacing theorem'.

Theorem C.1. [40, Corollary 2.5.4] *Let C be the quotient matrix of a real symmetric matrix A with respect to a partition $\{X_1, \ldots, X_s\}$. Then, the eigenvalues of C interlace the eigenvalues of A.*

Lemma 9.12. *Let G be a graph on $n \geq 2$ vertices, and let $0 = \mu_1 \leq \ldots \leq \mu_n$ be its Laplacian eigenvalues. If S is a non-empty set of vertices of G and d_S is the average degree of $G[S]$, then*

$$\delta(G) - \left(1 - \frac{|S|}{n}\right)\mu_n \leq d_S \leq \Delta(G) - \left(1 - \frac{|S|}{n}\right)\mu_2.$$

Proof. Let L be the Laplacian matrix of G, C the quotient matrix of L with respect to the partition $\{S, V \setminus S\}$ of the vertex set V of G, and t the number of edges with an endpoint in S and the other in $V \setminus S$. We have that

$$C = \begin{pmatrix} \frac{t}{|S|} & \frac{-t}{|S|} \\ \frac{-t}{|V \setminus S|} & \frac{t}{|V \setminus S|} \end{pmatrix}.$$

The eigenvalues of C are 0 and $\frac{t}{|S|} + \frac{t}{|V \setminus S|}$. By Theorem C.1, $\frac{t}{|S|} + \frac{t}{|V \setminus S|} \leq \mu_n$.

Let $\overline{G}$ be the complement graph of G, i.e. $V(\overline{G}) = V(G)$, and two vertices are adjacent in $\overline{G}$ if and only if they are distinct and they are not adjacent in G. Let $0 = \overline{\mu}_1 \leq \ldots \leq \overline{\mu}_n$ be the Laplacian eigenvalues of $\overline{G}$, and let $\bar{t}$ be the number of edges between S and $V \setminus S$ in $\overline{G}$. We note that $\overline{\mu}_i = n - \mu_{n+2-i}$ for $2 \leq i \leq n$; this follows from the fact that the Laplacian matrix $\overline{L}$ of $\overline{G}$ can be written as $\overline{L} = nI - J - L$, where I is the $n \times n$ identity matrix and J is the $n \times n$ all-ones matrix, from the fact that there exists an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors of L (because L is symmetric), and the fact that

$$L \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now, the argument above yields that

$$\begin{aligned} \frac{\bar{t}}{|S|} + \frac{\bar{t}}{|V \setminus S|} &\leq \overline{\mu}_n \\ \frac{|S||V \setminus S| - t}{|S|} + \frac{|S||V \setminus S| - t}{|V \setminus S|} &\leq n - \mu_2 \\ \mu_2 &\leq \frac{t}{|S|} + \frac{t}{|V \setminus S|}, \end{aligned}$$

so

$$\frac{|S||V \setminus S|}{n}\mu_2 \leq t \leq \frac{|S||V \setminus S|}{n}\mu_n.$$

This implies that

$$\delta - \left(1 - \frac{|S|}{n}\right)\mu_n \leq d_S = \frac{1}{|S|}\left(\sum_{v \in S} \deg v - t\right) \leq \Delta - \left(1 - \frac{|S|}{n}\right)\mu_2.$$

□

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RÉSUMÉ

On étudie le problème de partitionnement de digraphes en sous-graphes acycliques, et son paramètre associé : le nombre dichromatique. Ce paradigme, conçu par Neumann-Lara vers la fin des années 1970, nous a prodigué un bon nombre de généralisations de théorèmes classiques sur la coloration de graphes.
Nous dédions une attention spéciale aux bornes que l'on peut donner pour plusieurs classes de digraphes, comprenant les tournois locaux, les orientations des graphes de Kneser, les orientations des triangulations 2-planaires extérieures imbriquées, les digraphes aléatoires, et les digraphes aléatoires r-réguliers.
Nous nous intéressons aussi aux relations entre le nombre dichromatique et autres paramètres, comme le degré maximal et le nombre de biclique. En particulier, nous montrons qu'une version orientée de la conjecture de Borodin–Kostochka est valide pour des grands degrés maximaux, ce qui généralise un résultat de Reed.
Finalement, nous ajoutons quelques considérations sur les variantes circulaire, fractionnaire, et de liste du problème.

MOTS CLÉS

digraphe, coloration, ensemble acyclique, nombre dichromatique, digraphe aléatoire

ABSTRACT

We study the problem of partitioning digraphs into acyclic subgraphs, and its associated parameter: the dichromatic number. To this line of research, initiated by Neumann-Lara in the late 1970s, we owe many generalisations of classical results about graph colouring.
Our emphasis is placed on the bounds that can be given for several digraph classes, including local tournaments, orientations of Kneser graphs, orientations of nested 2-outerplanar triangulations, random digraphs, and random r-regular digraphs.
We also concern ourselves with the relationship between the dichromatic number and other digraph parameters, such as the maximum degree and the biclique number. In particular, we show that a directed version of the Borodin–Kostochka conjecture is true for large maximum degrees, thus generalising a result of Reed.
Finally, we include some considerations about the circular, the fractional, and the list variants of the problem.

KEYWORDS

digraph, colouring, acyclic set, dichromatic number, random digraph