Oriented matroids and beyond
complexes, partial cubes, and corners

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November 17, 2021

Abstract

This thesis is an overview of some of my research of the last 9 years, mostly concerned with oriented matroids, their generalizations and links to metric graph theory through tope graphs.

The text is largely based on four of my recent papers on these topics. They are included in the appendix, three of them are published (JCTA, DCG, DM), and one is submitted (Combinatorica).

Contents

1 Prologue

In this section I will briefly walk through some of my research that stands before the content of this thesis, but that still contributed to me ending up doing what I am doing. Some of these projects come up here and there and have had some recent progress that I would like to mention.

I started research in my Diploma thesis [105], with the goal to generalize structure theorems on the set of orientations of a graph with prescribed outdegrees (also known as *α-orientations*) to *oriented matroids*. More precisely, it was known that the set of α-orientations of a *planar graph* carries the structure of a *distributive lattice* induced by flipping directed facial cycles [71]. During my Diploma thesis it became clear, that similar results and methods would not extend beyond the class of regular oriented matroids. By Seymour's structure theorem, regular oriented matroids can be glued from graphic and co-graphic ones (and the exceptional oriented matroid **R10**) [166]. It turns out that the co-graphic setting correspond to orientations with prescribed number of forward arcs on cycles (also known as *c-orientations*) of (general) graphs. In this setting a distributive lattice structure had been obtained by Propp [148].

In my Diploma thesis I showed that any distributive lattice arises this way from a graph, but it comes with extra information, namely as an *embedded sublattice* of $\mathbb{Z}^n$ in the sense of Dilworth [61]. In my PhD thesis [73, 106] these results were generalized considerably to *tensions* (also known as Δ-bonds) of graphs. I showed that these objects carry a polytopal structure that furthermore induces a distributive lattice. These polytopes have been introduced and discovered independently as *alcoved polytopes* [116] and are maybe the most interesting class of *distributive polytopes* [74]. A recent project with Carolina Benedetti and our Master's student Jeronimo Valencia at Universidad de los Andes aims at base polytopes of lattice path matroids [28, 110, 111] and positroids [24, 49, 116], which form a natural subclass of alcoved polytopes. In particular, apart from planar graphic matroids, these yield other classes whose base set carries a natural distributive lattice structure. Alcoved polytopes contain the ubiquitous class of *order polytopes*, who are convex hulls of poset ideals and whose canonical triangulation corresponds to the *linear extension graph*. We aim to push the beautiful interplay of order theory and geometry to this more general class. Part of this satisfying theory has been the topic of my lectures in École de Jeunes Chercheurs at the CIRM 2019 [107] and has been useful in the study of the Ehrhart polynomial of matroid polytopes [111].

Coming back to the original question but now concerning graphic oriented matroids, we encounter ourselves with α-orientations of not necessarily planar graphs. Here, a first (and still open) question is, whether there exists a polynomial set of cycles of a graph, such that by flipping its members (when directed) any two α-orientations can be transformed into each other. While in the plane such a set is given by the faces, for a graph of orientable genus γ at least 2γ cycles apart from the faces are needed. In my Diploma thesis I showed that this suffices for Eulerian orientations of the toroidal square grid. In her Master's thesis at Aix-Marseille under my supervision Hannah Schreiber constructed an explicit set of such cycles of size linear in the lower bound for $(1, 3)$-orientations of the torus grid. In general this question remains open even for the hypercube.

One important class of α-orientations in the plane are Schnyder woods, objects whose importance lies in encoding, drawing, and relations to the order dimension. Several attempts of generalizations of Schnyder woods to orientable surfaces have been proposed, see e.g. [35, 86]. With Benjamin Lévêque and Daniel Gonçalves we give the first such generalization, that again is in bijection with a certain set of α-orientations [85]. However, it even remains open whether any triangulation of an oriented surface admits such a *Schnyder α-orientation*. As a first step into that direction, with Boris Albar and Daniel Gonçalves we verified a conjecture of Barát and Thomassen, namely, we proved that triangulations of any higher genus surfaces admit sink-free orientations with out-degrees divisible by three [4]. The structure of the set of α-orientations remains mysterious.

In a recent work with Oswin Aichholzer, Jean Cardinal, Tony Huynh, Torsten Mütze, Raphael Steiner, and Birgit Vogtenhuber [2], we show that computing the flip-distance between planar α-orientations is NP-complete, when all cycles are allowed to be flipped but if only face flips are allowed the distance can be computed in polynomial time. Again, the latter is based on the lattice structure.

Another related project is with Sarah Blind, whose PhD thesis I co-supervised with Nadia Creignou, Frédéric Olive, and Petru Valicov. We obtained polynomial delay enumeration algorithms for the set of α-orientations of an arbitrary graph [23]. These can be used for enumeration algorithms for all k-arc connected orientations of a graph. This generalizes the case $k = 1$, i.e. strongly connected orientations, where an implementation of an algorithm due to Conte et al [47] with Petru Valicov is available on SageMATH [169]. The general enumeration algorithm has been recently implemented by my Bachelor student Taras Yarema at Universitat de Barcelona.[1]

Even if most of the offspring of my Diploma thesis is about graphs, general oriented matroids have been of central interest for me since then as well [38, 72, 112, 113].

It was during my PhD thesis, that I noticed a common property of many structures that I was eager to find and study: (Upper locally) distributive lattices, antimatroids, linear extension graphs of posets, flip-graphs of strongly connected (or acyclic) orientations of a graph, tope graphs of oriented matroids. Namely, all the above are isometric subgraphs of hypercubes (also known as *partial cubes*) – a beautiful class of graphs that had me ever since, see e.g. [5, 44, 59, 102, 114].

What I discovered again a bit later, is that another more structured subclass of partial cubes captures all these important examples – *tope graphs of complexes of oriented matroids*. These complexes themselves endow the above classes of graphs with a rich underlying structure of cells, minors, carriers and more. This is the point, where the story of the present thesis begins.

[1] `https://github.com/tarasyarema/orientations`

2 What is this thesis about?

This work explores ways of combining methods and tools from combinatorics, metric graph theory and topology in the study of combinatorial systems of *sign vectors*. It is mostly based on results from my papers [16, 42, 108, 109][2], with colleagues Hans-Jürgen Bandelt, Victor Chepoi, and Tilen Marc, who was a PhD-student, that I co-advised with Sandi Klavžar. Our experimental results were obtained with SageMath [169], GAP [82], Bliss [99] and using several repositories of graphs [34, 95, 129, 162].

More recent works of mine on these topics include [43, 44, 55]. Instead of surveying all my research I have chosen to restrict myself to the above papers and even from these I present only a selection of results and in some cases a simplified version. Let us briefly introduce two of the main motivating classes:

Oriented matroids and lopsided sets. One of the main objects we are generalizing here are *oriented matroids (OMs)* [22]. We will define OMs more carefully as special cases of complexes of oriented matroids (COMs). Later on, we will discuss central problems of the theory which are symptomatic for the need to improve the interplay of topology and combinatorics in this area and propose new ways to approach these problems. Quoting from [156]:

> The theory of oriented matroids provides a broad setting in which to model, describe, and analyze combinatorial properties of geometric configurations. Mathematical objects of study that appear to be disjoint and independent, such as point and vector configurations, hyperplane arrangements, convex polytopes, directed graphs, and linear programming find a common generalization in the language of oriented matroids.

In a sense, corresponding to the number of different instances of OMs, there are many equivalent (cryptomorphic) axiomatizations of them. One of the corner stones of the theory of OMs is the Topological Representation Theorem that allows to encode OMs as *arrangements of pseudo-spheres* [76] and which establishes a deep connection between OMs and topology. The axiomatization of OMs that will be most used in this text is in terms of *covectors*, i.e., a set $\mathcal{L} \subseteq \{+,-,0\}^E$ for a finite ground set E satisfying certain axioms, see Definition 3.1. The class of systems of sign-vectors that can be described as halfspaces of OMs are *affine oriented matroids* (AOMs). Also this class admits an intrinsic axiomatization, see Definition 3.1. The area of (A)OMs is an active field of research, se e.g. [8, 51, 84, 123, 134, 176].

Besides OMs and AOMs another important class of systems of sign-vectors are *lopsided sets (LOPs)*, which were introduced by Lawrence [120]. Lopsided sets are rich in application and important to the theory of (oriented) set systems. In a work by Bandelt et.al. [14] among many different characterizations it is shown that LOPs were rediscovered in various disguises, e.g. as *extremal for (reverse) Sauer* [27] or *shattering-extremal* [159]. These concepts have found applications in statistics, combinatorics, learning theory and computational geometry, see e.g. [136]. In particular, they were

[2]They can be found in the appendix.

rediscovered as *ample sets* (AMPs) in extremal combinatorics where they appear with respect to clustering techniques [66]. Lopsided sets encode many important classes of combinatorial objects such as, (conditional) antimatroids, diagrams of (upper locally) distributive lattices, median graphs or cubic CAT(0)-complexes, see Definition 3.1 for the formal definition.

OMs and LOPs together embrace a large set of objects crucial to combinatorial theory. It is in the interest of both theories to enrich each other with tools and properties. This is why it is of central importance in the field to find a common generalization of LOPs and OMs.

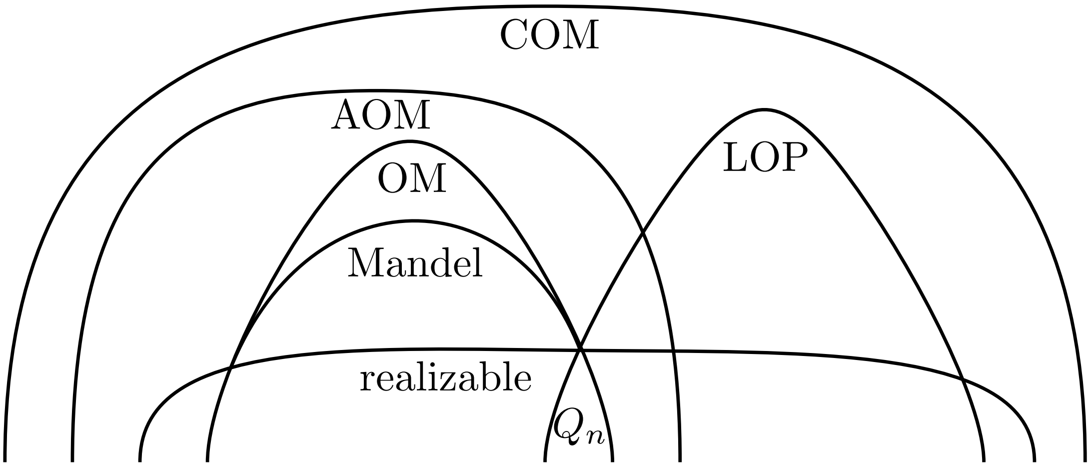


Figure 1: An inclusion diagram of the main classes of COMs discussed in the thesis.

As a main and first objective of this thesis we deliver a satisfactory answer to the above concern:

Complexes of oriented matroids We introduce complexes of oriented matroids (also known as conditional oriented matroids) (COMs) as systems of sign-vectors $\mathcal{L} \in \{+,-,0\}^E$ yielding a common generalization of oriented matroids (OMs), lopsided sets (LOPs), and affine oriented matroids (AOMs). Each of the latter can be defined by three axioms two of which are called *face symmetry* (FS) and *strong elimination* (SE), respectively, and coincide throughout. COMs can be defined by simply relaxing the axiomatization of all the above to (FS) and (SE), only, see Definition 3.1. While this already suggests that COMs are a natural generalization on a formal level, we show that many fundamental properties such as minor-closedness and face structure are preserved. Moreover, the (geometrically) *realizable* setting shows very clearly how the step from OMs to COMs, means to intersect a central hyperplane arrangement with an (open) polyhedron. If the polyhedron is a halfspace, one obtains a realizable AOM, See Figure 1 for an overview.

One of our key results on COMs is that they are indeed complexes whose cells are OMs, and moreover every COM can be built by an *amalgamation* procedure from its cells. In particular we obtain contractibility of the complex.

Furthermore, we present several important and illustrative classes of examples, such as the (realizable) COM of linear extensions of a poset also know as *Ranking COM*, which recently has been related to poset cones [65]. Answering a question of [16] the latter can be seen as a special case of the novel notion of *graphic COMs*, which is a common generalization of graphic OMs and Ranking COMs. As another more involved class of examples, we establish that *CAT(0) Coxeter Complexes*, defined by Haglund and Paulin yield (zonotopally realizable) COMs [90].

COMs have already been spiking research or appeared as the next natural class to attack in different areas such as combinatorial semigroup theory [130], in relation to the Varchenko determinant [94, 151] in OMs, with respect to neural codes [98, 115], sample compression schemes [37, 43, 44], as well as sweeping sequences [142]. One of the main indicators for the "correctness" of the definition of COMs has been that they provide the correct framework of generality to study and characterize:

Tope graphs. To every COM one can associate its tope graph, which is the subgraph of the hypercube $\{+,-\}^E$ induced by the *topes* $\mathcal{T}$ of $\mathcal{L}$, i.e., the sign-vectors without zero-entries. Well-known and important properties of tope graphs of OMs, AOMs, and LOPs generalize to COMs. In particular, the tope graph determines the associated systems of sign vectors uniquely up to isomorphism and therefore is an alternative point of view for the study of these objects. Moreover the tope graph is a *partial cube*, i.e., a subgraph G of a hypercube Q_d, such that for any $u, v \in G$ at least one shortest path from u to v of Q_d is contained in G. This crucial observation creates an important link of the study of COMs to metric graph theory in which (quoting [104]) partial cubes form one of the central classes. Moreover, partial cubes arise in many applications from interconnection networks [87], through media theory [70] to chemical graph theory [69]. They are an active field of research [46, 103, 147].

We give a 3-fold purely graph theoretic characterization of tope graphs of COMs. The first part is in terms of gatedness of antipodal (also known as symmetric-even [19]) subgraphs, combines classical concepts of metric graph theory and is verifiable in polynomial time. The second part is by means of determining the set of excluded *partial cube minors* and stages this novel and natural relation among partial cubes, that can be seen as a metric counterpart of the classical graph minor theory. The last part of our characterization characterizes COMs as those partial cubes, such that all their (iterated) *zone-graphs* are partial cubes as well. It thus connects our research with the classical zone-graph construction in partial cubes [104].

As corollaries, we obtain a new unified proof for characterization theorems of tope sets of LOPs and OMs due to Lawrence [120] and da Silva [50]. Moreover, we obtain a natural extension of a recursive tope set characterization of OMs due to Handa [92].

In particular, we answer a long-standing open question on OMs, i.e., the question for a purely graph theoretical characterization of tope graphs, see [93, Problem 2] that can furthermore be verified in polynomial time, which was posed in [79, Problem 1.2]. Since the tope graph determines a COM, OM, AOM, or LOP up to isomorphism, our results can be seen as identifying the theory of (complexes of) oriented matroids as a part of metric graph theory.

Metric graph classes of COMs. The third part of this text shifts the point of view and instead of using metric graph theory merely as a tool to attack questions in COMs, it features results we obtained in special classes of metric graphs all of which turn out to be COMs. Hypercellular graphs are those partial cubes defined by forbidding the 3-cube minus a vertex Q_3^- as partial cube minor. As median graphs additionally forbid C_6 and bipartite cellular graphs forbid additionally Q_3, hypercellular graphs are a natural generalization, that captures several properties of the above, such as cell-structure.

By definition these classes are strongly related to the notion of *partial cube minors*, which already staged in the characterization of tope graphs of oriented matroids. A well-known super class of hypercellular graphs is the class of Pasch graphs $\mathcal{S}_4$ – a class definable by excluding a set of seven minors [40, 42]. A particular consequence of our tope graph characterization by minors and proving a conjecture from [42] is that $\mathcal{S}_4$ graphs are tope graphs of COMs.

We given an overview of classes of partial cubes determined by a small set of excluded minors, see Figure 2, and show how properties of earlier studied classes such as *median* graph and *bipartite cellular* graph extend to hypercellular graphs.

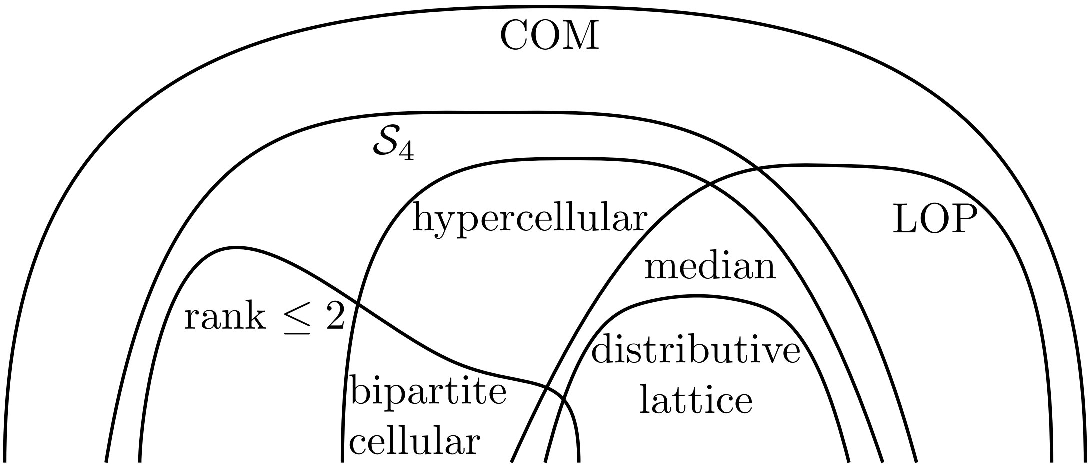


Figure 2: An inclusion diagram of the main classes of metric graph classes discussed in the thesis.

See [128, Table 6.1.] and Table 2 for sizes of classes of partial cubes of given isometric dimension.

Simplicial cells, corners, and corner peelings In the fourth part of this text we will test the theory of tope graphs by phrasing and attacking classical problems of OMs and LOPs in the language of tope graph and by pushing these objects to the world of COMs. More precisely, we chose a central class of problems related to Las Vergnas conjecture on simplicial cells [117], corners, and corner peelings [37]. Simplicial cells are studied in particular in the theory of OMs. Famous Las Vergnas conjecture claims the existence of a simplicial cell in every OM, i.e., a low degree vertex in the graph. The largest class known to satisfy the conjecture are *Mandel OMs*. In fact, in Mandel's thesis [124] it was conjectured as a "wishful thinking statement" that all OMs are Mandel, since this would imply Las Vergnas conjecture.

We are able to show, that Mandel OMs have *many* simplicial vertices, which together with known examples of OMs with few simplicial vertices disproves Mandel's conjecture.

Corners and corner peelings are studied in learning theory and in relation with LOPs and here the generalization to COMs builds a bridge between this area and more topological concepts. Choosing definitions correctly simplicial cells in LOPs are just corners. We extend these notions to COMs introducing corners in COMs. Moreover, we generalize results from LOPs [37, 170], such as corner peelings for rank 2 (also known as two-dimensional) COMs and realizable COMs. Furthermore, we show that hypercellular graphs have corner-peelings, generalizing earlier results for bipartite cellular graphs [11]. Another consequence is a counterexample about realizability of zonotopally realizable COMs from [16].

Conclusions, current and future work. In this last part we will survey the rich body of future work to be explored – in many formats from Master's and PhD theses towards larger projects involving bigger groups and simply my own future research. Currently, I co-advise our PhD-student Manon Philibert with Victor Chepoi in Aix-Marseille and just closed a Master project with Gil Puig in Barcelona on such topics. I will give some indications about these including ongoing projects, partial results and untouched territories. Some questions of big importance remain open in the theory of COMs and also with respect to corners and simpliciality, e.g., the mutation graphs of uniform oriented matroids and cocircuit graphs of OMs. In this part we take the opportunity to go beyond COMs. For instance, we resolved Las Vergnas conjecture on rank 3 antipodal partial cubes, which is a much broader class than rank 3 OMs for which the conjecture was already known to hold, so we wonder if Las Vergnas conjecture could hold for general antipodal partial cubes.

Also apart from this, there are many question concerning classes of partial cubes beyond COMs, e.g., partial cubes of bounded VC-dimension with connections to the sample compression conjecture of Littlestone and Warmuth, or planar partial cubes and topological representation theorems for general partial cubes, and Cayley graphs that are partial cubes. Finally, we also mention recent developments beyond COMs from the perspective of left-regular bands.

3 Complexes of Oriented Matroids (COMs)

In this section we propose a common generalization of oriented matroids and lopsided sets which is so natural that it is surprising that it was not discovered much earlier. It also generalizes such well-known and useful structures as convex geometries and CAT(0) cube (and zonotopal) complexes. In this generalization, global symmetry and the existence of the zero sign vector, required for oriented matroids, are replaced by local relative conditions. Analogous to conditional lattices (see [77, p. 93]) and conditional antimatroids (which are particular lopsided sets [14]), this motivates the name "conditional oriented matroids" (abbreviated: COMs) for these new structures. Furthermore, COMs can be viewed as complexes whose cells are oriented matroids and which are glued together in a lopsided fashion.

In the present section we describe which properties of COMs are a unifying generalization of LOPs, OMs and AOMs and carry over to COMs, what new operations can be applied, and try to give a rich justification of the name *Complex of Oriented Matroids.*

More precisely we describe COMs as complexes whose cells are OMs, and that can be erected via an amalgamation procedure, much as in the *constructibility* of a cell-complex, see [18, 89]. In particular we obtain contractibility. The study of this complex is new and particular to COMs, only in the affine case a couple of questions have been studied [63, 64].

Finally, we describe a vast class of examples of COMs: CAT(0) Coxeter complexes and ranking and graphic COMs.

We will first turn to the motivating model for COMs, LOPs, OMs, and AOMs, coming from real hyperplane arrangements:

3.1 The realizable case

Let us begin by considering the following scenario of hyperplane arrangements and realizable oriented matroids; compare with [22, Sections 2.1, 4.5] or [178, p. 212]. Given a *central arrangement of hyperplanes* of $\mathbb{R}^d$ (i.e., a finite set E of $(d-1)$–dimensional linear subspaces of $\mathbb{R}^d$), the space $\mathbb{R}^d$ is partitioned into open regions and recursively into regions of the intersections of some of the given hyperplanes. Specifically, we may encode the location of any point from all these regions relative to this arrangement when for each hyperplane one of the corresponding halfspaces is regarded as positive and the other one as negative. Zero designates location on that hyperplane. Then the set $\mathcal{L}$ of all sign vectors representing the different regions relative to E is the set of covectors of the oriented matroid of the arrangement E. For $X \in \mathcal{L}$, and $e \in E$ let X_e be the value of X at the coordinate e. The oriented matroids obtained in this way are called *realizable.* If instead of a central arrangement one considers finite arrangements E of affine hyperplanes (an affine hyperplane is the translation of a (linear) hyperplane by a vector), then the sets of sign vectors of regions defined by E are known as *realizable affine oriented matroids* [100] and [14, p.186]. Since an affine arrangement on $\mathbb{R}^d$ can be viewed as the intersection of a central arrangement of $\mathbb{R}^{d+1}$ with a translate of a coordinate hyperplane, each realizable affine oriented matroid can be embedded into a larger realizable oriented matroid.

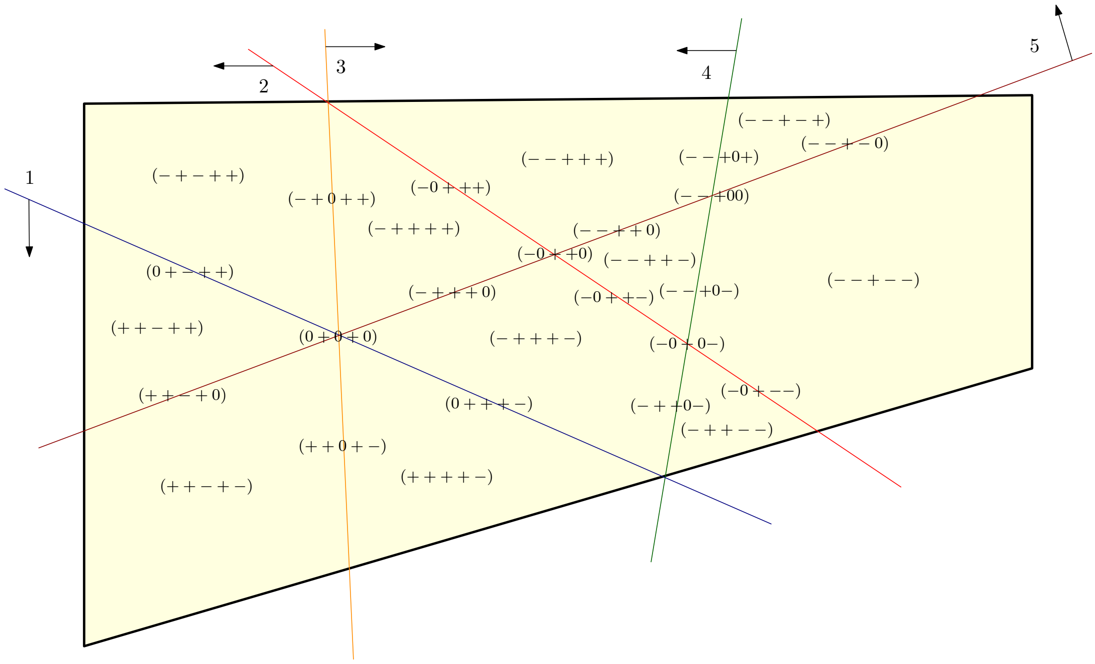


Figure 3: A COM realized by a hyperplane arrangement and an open polyhedron in $\mathbb{R}^2$. The arrows on the hyperplanes indicate their positive side.

Now suppose that E is a central or affine arrangement of hyperplanes of $\mathbb{R}^d$ and C is an open convex set, which may be assumed to intersect all hyperplanes of E in order to avoid redundancy. Restrict the arrangement pattern to C, that is, remove all sign vectors which represent the open regions disjoint from C. Denote the resulting set of sign vectors by $\mathcal{L}(E, C)$ and call it a *realizable COM*. Figure 3 displays an example.

Our model of realizable COMs generalizes realizability of oriented and affine oriented matroids on the one hand and realizability of lopsided sets on the other hand. In the case of a central arrangement E with C being any open convex set containing the origin (e.g., the open unit ball or the entire space $\mathbb{R}^d$), the resulting set $\mathcal{L}(E, C)$ of sign vectors coincides with the realizable oriented matroid of E. If the arrangement E is affine and C is the entire space, then $\mathcal{L}(E, C)$ coincides with the realizable affine oriented matroid of E. The realizable lopsided sets arise by taking the (central) arrangement E of all coordinate hyperplanes E restricted to arbitrary open convex sets C of $\mathbb{R}^d$. In fact, the original definition of realizable lopsided sets by Lawrence [120] is similar but used instead an arbitrary (not necessarily open) convex set K and as regions the closed orthants. Clearly, K can be assumed to be a polytope, namely the convex hull of points representing the closed orthants meeting K. Whenever the polytope K does not meet a closed orthant then some open neighborhood of K does not meet that orthant either. Since there are only finitely many orthants, the intersection of these open neighborhoods results in an open set C which has the same intersection pattern with the closed orthants as K. Now, if an open set meets a closed orthant it will also meet the corresponding open orthant, showing that both concepts of realizable lopsided sets coincide.

Since in all these constructions the choice of positive and negative sides for the hyperplanes is combinatorially rather irrelevant one introduces the following operation. For a subset $A \subseteq E$ and $X \in \mathcal{L}$ the *reorientation* of X with respect to A is the sign-vector defined by

$$({}_AX)_e := \begin{cases} -X_e & \text{if } e \in A \\ X_e & \text{otherwise.} \end{cases}$$

In particular $-X :=_E X$. The *reorientation* of $\mathcal{L}$ with respect to A is defined as ${}_A\mathcal{L} := \{{}_AX \mid X \in \mathcal{L}\}$. In particular, $-\mathcal{L} :=_E \mathcal{L}$.

A important subset of $\mathcal{L}$ are the *topes*, i.e., those covectors without 0-entries. In the realizable setting they correspond to the maximal cells. For us, one of the central objects of study is the *tope graph*, that corresponds to the incidence graph of maximal cells, i.e., two cells share an edge if their intersection is of codimension 1 or alternatively, we can consider the subgraph induced by the topes in the hypercube graph $\{+,-\}^E$. Tope graphs in the realizable setting have been considered recently in the framework of neural codes, see [98]. See Figure 4 for the tope graph of the example from Figure 3.

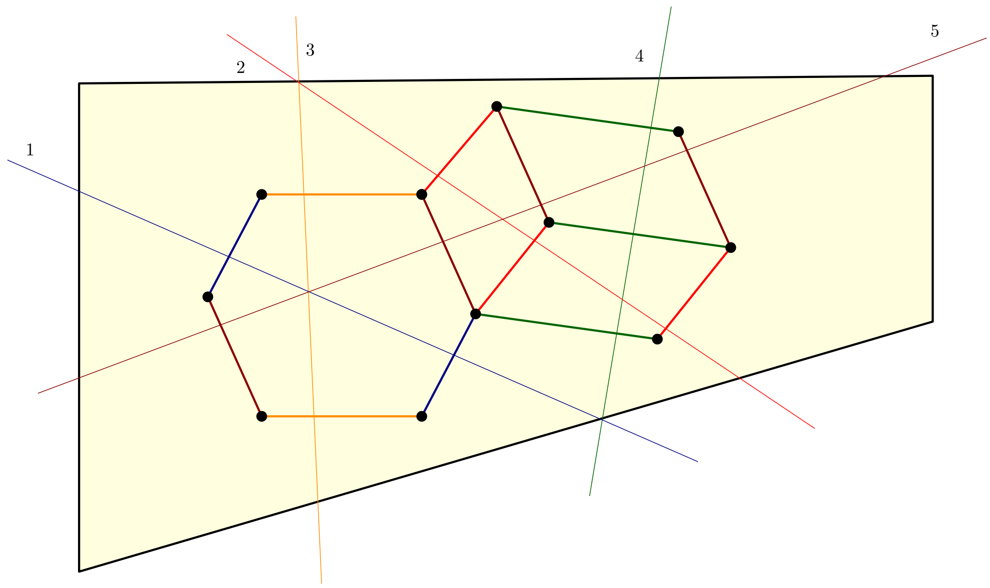


Figure 4: The tope graph of the COM from Figure 3. The color classes of edges giving the embedding into the cube correspond to the hyperplanes of the arrangement in $\mathbb{R}^2$.

3.2 Axioms from Geometry

For the scenario of realizable COMs, we can identify their basic properties in terms of sign-vectors. These will then be used to introduce general COMs, OMs, LOPs, and AOMs axiomatically. Let X and Y be sign vectors belonging to $\mathcal{L}$, thus designating regions represented by two points x and y within C relative to the arrangement E; see Figure 5 (compare with Fig. 4.1.1 of [22]). Connect the two points by a line segment and choose $\epsilon > 0$ small enough so that the open ball of radius ϵ around x intersects only those hyperplanes from E on which x lies. Pick any point w from the intersection of this ϵ-ball with the open line segment between x and y. Then the corresponding sign vector W is the *composition* $X \circ Y$ as defined by

$$(X \circ Y)_e = X_e \text{ if } X_e \neq 0 \text{ and } (X \circ Y)_e = Y_e \text{ if } X_e = 0.$$

Hence the following rule is fulfilled:

(C) $X \circ Y$ belongs to $\mathcal{L}$ for all sign vectors X and Y from $\mathcal{L}$. *(composition)*

Note that the composition operation is associative but not commutative. This is, if (C) holds, then $(\mathcal{L}, \circ)$ forms a semigroup and more specifically a *left regular band*. Recently, COMs and other similar structures have been featured from this point of view, see [130].

If we select instead a point u on the ray from y via x within the ϵ-ball but beyond x, then the corresponding sign vector U has the opposite signs relative to W at the coordinates corresponding to the hyperplanes from E on which x is located and which do not include the ray from y via x. Therefore the following property holds:

(FS) $X \circ -Y$ belongs to $\mathcal{L}$ for all X, Y in $\mathcal{L}$. *(face symmetry).*

Note that face symmetry implies (C): for $X, Y \in \mathcal{L}$ we can transform $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y)$ and the latter is contained in $\mathcal{L}$ by (FS).

Next assume that the hyperplane e from E separates x and y, that is, the line segment between x and y crosses e at some point z. The corresponding sign vector Z is then zero at e and equals the composition $X \circ Y$ except on their *separator* $S(X, Y) = \{e \in E \mid X_e Y_e = -1\}$, i.e., on the coordinates where both have opposite signs:

(SE) for each pair X, Y in $\mathcal{L}$ and for each $e \in S(X, Y)$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X, Y)$. (strong elimination)

Now, the single property of oriented matroids that we have missed in the general scenario is the existence of the zero sign vector, which would correspond to a non-empty intersection of all hyperplanes from E within the open convex set C:

(Z) the zero sign vector $\mathbf{0}$ belongs to $\mathcal{L}$. (zero vector)

On the other hand, if the hyperplanes from E happen to be the coordinate hyperplanes, then wherever a sign vector X has zero coordinates, the composition of X with any sign vector from $\{\pm 1, 0\}^E$ is a sign vector belonging to $\mathcal{L}$. This rule, which clearly implies face symmetry and thus composition, holds in lopsided systems, for which the "tope" sets are exactly the lopsided sets of Lawrence [120]:

(I) $X \circ Y \in \mathcal{L}$ for all $X \in \mathcal{L}$ and all sign vectors $Y \in \{+, -, 0\}^E$. (ideal composition)

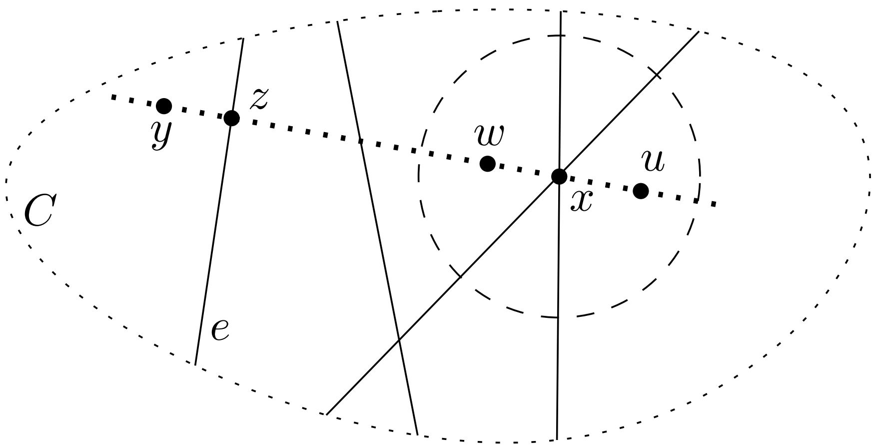


Figure 5: Motivating model for the three axioms.

We will not really use of the axiomatization of AOMs developed in [17,55,100] apart from illustrating that AOMs are a natural subclass of COMs, see Definition 3.1. Let us however briefly introduce an operation on sign-vectors needed to axiomatize AOMs:

$$(X \oplus Y)_e := \begin{cases} 0 & \text{if } e \in S(X,Y) \\ (X \circ Y)_e & \text{otherwise.} \end{cases}$$

Affinity:

(A) Let $X, Y \in \mathcal{L}$ such that for all $e \in S(X,-Y)$ and $W \in \mathcal{L}$ with $W_e = 0$ there are $f, g \in E \setminus S(X,-Y)$ such that $W_f \neq (X \circ -Y)_f$ and $W_g \neq (-X \circ Y)_g$. We have $(X \oplus -Y) \circ Z \in \mathcal{L}$ for all $Z \in \mathcal{L}$.

We are now ready to define the main objects of our study:

Definition 3.1. *A system of sign vectors* $(E, \mathcal{L})$ *is called a:*

- complex of oriented matroids (COM) *if* $\mathcal{L}$ *satisfies (FS) and (SE),*
- affine oriented matroid (AOM) *if* $\mathcal{L}$ *satisfies (A), (FS), and (SE),*
- oriented matroid (OM) *if* $\mathcal{L}$ *satisfies (Z), (FS) and (SE),*
- lopsided system (LOP) *if* $\mathcal{L}$ *satisfies (IC) and (SE).*

In the model of hyperplane arrangements we can retrieve the cells which constitute oriented matroids. Indeed, consider all non-empty intersections of hyperplanes from E that are minimal with respect to inclusion. Select any sufficiently small open ball around some point from each intersection. Then the subarrangement of hyperplanes through each of these points determines regions within these open balls which yield an oriented matroid. Taken together all these constituents form a complex of oriented matroids, where their intersections are either empty or are faces of the oriented matroids involved. These complexes are quite special as they conform to global strong elimination. The latter feature is a not guaranteed in general complexes whose cells are oriented matroids, which were called *bouquets of oriented matroids* [60]. Furthermore, the openness of the convex in the realizable set is essential for having face symmetry. Objects obtained without that assumption yield so-called *oriented matroid polyhedra* [22, p. 420]. The latter form special instances of *strong elimination systems*, as introduced in [16]. They satisfy (SE) and (C) and yield left regular bands, as well [130].

3.3 Minors, fibers, and faces

In the present section we show that the class of COMs is closed under taking minors, defined as for oriented matroids. We use this to establish that simplifications of COMs are minors of COMs and therefore COMs. We also introduce fibers and faces of COMs, which are of importance for the structural study of COMs.

Let $(E, \mathcal{L})$ be a COM and $A \subseteq E$. Given a sign vector $X \in \{\pm 1, 0\}^E$ by $X \setminus A$ we refer to the *restriction* of X to $E \setminus A$, that is $X \setminus A \in \{\pm 1, 0\}^{E \setminus A}$ with $(X \setminus A)_e = X_e$ for all $e \in E \setminus A$. The *deletion* of A is defined as $(E \setminus A, \mathcal{L} \setminus A)$, where $\mathcal{L} \setminus A := \{X \setminus A : X \in \mathcal{L}\}$. The *contraction* of A is defined as $(E \setminus A, \mathcal{L}/A)$, where $\mathcal{L}/A := \{X \setminus A : X \in \mathcal{L} \text{ and } \underline{X} \cap A = \varnothing\}$. If a system of sign vectors arises by deletions and contractions from another one it is said to be *minor* of it. In the realizable setting deletion means removing hyperplanes and contraction means restricting to (intersections of) hyperplanes. The first and easy observations are

Lemma 3.2 ([16, Lemma 1 and 2])**.** *COMs are closed under contractions and deletions and both operations commute.*

For the purpose of this text it is sufficient to restrict our attentions to simple systems of sign vectors, while in the more general treatment of [16] also semisimple systems and general ones have been considered. Simplicity is motivated by the hyperplane model for COMs, that possesses additional properties, reflecting that we have a set of hyperplanes rather than a multiset and that the given convex set is full-dimensional. This is also motivated by the requirement of defining systems of sign vectors not containing coloops and parallel elements, which is relevant, for example, for the identifications of topes.

A *coloop* of $(E, \mathcal{L})$ is an element $e \in E$ such that $X_e = 0$ for all $X \in \mathcal{L}$. Hence $(E, \mathcal{L})$ does not have coloops if and only if for each element e, there exists a covector X with $X_e \neq 0$. Two elements $e, e' \in E$ of $(E, \mathcal{L})$ are *parallel*, denoted $e \parallel e'$, if either $X_e = X_{e'}$ for all $X \in \mathcal{L}$ or $X_e = -X_{e'}$ for all $X \in \mathcal{L}$. It is easy to see that $\parallel$ is an equivalence relation. The condition that $(E, \mathcal{L})$ does not contain parallel elements can be expressed by the requirement that for each pair $e \neq f$ in E, there exist $X, Y \in \mathcal{L}$ with $X_e \neq X_f$ and $Y_e \neq -Y_f$.

Simple systems are defined by two axioms which are slightly stronger than those which ensure the absence of coloops and parallel elements, see [16]. Another property of simple systems is, that the topes are the covectors with full support. One can identify the elements that violate simplicity delete them (or all but one per parallel class). This yields a new COM, called the *simplification*:

Lemma 3.3. *The simplification of a COM $(E, \mathcal{L})$ is a simple minor, unique up to sign reversal, and in particular a COM.*

We will from now on consider only simple COMs. For a COM $(E, \mathcal{L})$ and $X \in \{0, +, -\}^E$ the *fiber* (also known as *supertope* [94]) relative to X is the set of sign vectors defined by

$$\{Y \in \mathcal{L} : Y \setminus X^0 = X\} = \{Y \in \mathcal{L} : Y \geq X\}.$$

A fiber is a *face* if $X \in \mathcal{L}$, in this case it can be expressed in the form:

$$F(X) := \{X \circ Y : Y \in \mathcal{L}\}.$$

Faces of COMs are important for their description as complexes. If $S(X, Y)$ is non-empty for $X, Y \in \mathcal{L}$, then the corresponding faces $F(X)$ and $F(Y)$ are disjoint. Else, if X and Y are sign-consistent, then $F(X) \cap F(Y) = F(X \circ Y)$. In particular $F(X) \subseteq F(Y)$ is equivalent to $X \in F(Y)$, that is, $Y \leq X$. The ordering of faces by inclusion thus reverses the sign ordering, which when endowed with an artificial global maximum $\hat{1}$ is called the *big face lattice* in OMs and in the case of COMs forms *big face (upper) semilattice* $\mathcal{F}_{\text{big}}(\mathcal{L})$. Thus, we can express the fiber (or face) of X as the principal filter (or upset) $\uparrow X$ of X in this semilattice, where in the case of a general fiber $X \notin \mathcal{L}$.

Since fibers are basically isomorphic to restrictions, and in $F(X)$ the covector X plays the role of $\mathbf{0}$, the following observations are straightforward and are the first step into arguing why COMs are indeed complexes of oriented matroids:

Lemma 3.4. *The fibers of a (simple) COM are (simple) COMs and its faces are (simple) OMs. Considering the empty set a face, the set of faces is closed under intersections and its inclusion-order is the lower semilattice, $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$.*

See Figure 6 for an illustration of the (opposite) face semilattice of a COM. We will analyze the semilattice $\mathcal{F}_{\text{big}}(\mathcal{L})$ further in Section 3.5.

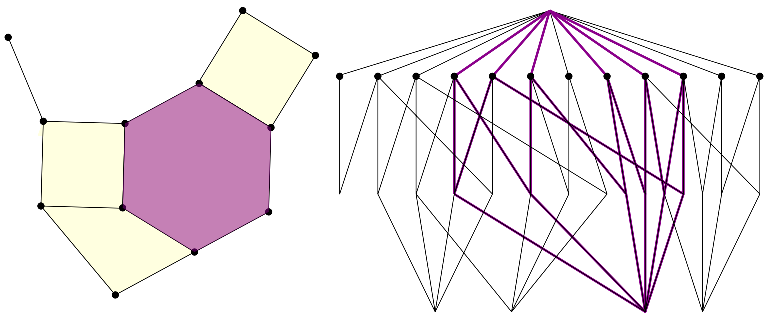


Figure 6: A COM and its face semilattice.

3.4 Hyperplanes, carriers, halfspaces, and amalgamation

We introduce several further operations on COMs, that will be essential to our amalgamation procedure. These come mostly from metric graph theory and apart from hyperplanes do not have a counterpart in OMs.

For a system $(E, \mathcal{L})$ of sign vectors, a *hyperplane* of $\mathcal{L}$ is the set

$$\mathcal{L}_e^0 := \{X \in \mathcal{L} : X_e = 0\} \text{ for some } e \in E.$$

Note that the hyperplane $\mathcal{L}_e^0$ is isomorphic to the contraction $\mathcal{L}/e$. The *carrier* $N(\mathcal{L}_e^0)$ of the hyperplane $\mathcal{L}_e^0$ is the union of all faces $F(X')$ of $\mathcal{L}$ with $X' \in \mathcal{L}_e^0$, that is,

$$N(\mathcal{L}_e^0) := \{X \in \mathcal{L} : W \leq X \text{ for some } W \in \mathcal{L}_e^0\}.$$

The *positive and negative ("open") halfspaces* supported by the hyperplane $\mathcal{L}_e^0$ are

$$\mathcal{L}_e^+ := \{X \in \mathcal{L} : X_e = +1\},$$

$$\mathcal{L}_e^- := \{X \in \mathcal{L} : X_e = -1\}.$$

The carrier $N(\mathcal{L}_e^0)$ minus $\mathcal{L}_e^0$ splits into its positive and negative parts:

$$N^+(\mathcal{L}_e^0) := \mathcal{L}_e^+ \cap N(\mathcal{L}_e^0),$$

$$N^-(\mathcal{L}_e^0) := \mathcal{L}_e^- \cap N(\mathcal{L}_e^0).$$

The closure of the disjoint halfspaces $\mathcal{L}_e^+$ and $\mathcal{L}_e^-$ just adds the corresponding carrier:

$$\overline{\mathcal{L}_e^+} := \mathcal{L}_e^+ \cup N(\mathcal{L}_e^0) = \mathcal{L}_e^+ \cup \mathcal{L}_e^0 \cup N^-(\mathcal{L}_e^0),$$

$$\overline{\mathcal{L}_e^-} := \mathcal{L}_e^- \cup N(\mathcal{L}_e^0) = \mathcal{L}_e^- \cup \mathcal{L}_e^0 \cup N^+(\mathcal{L}_e^0).$$

The former is called the *closed positive halfspace* supported by $\mathcal{L}_e^0$, and the latter is the corresponding *closed negative halfspace*. Both overlap exactly in the carrier.

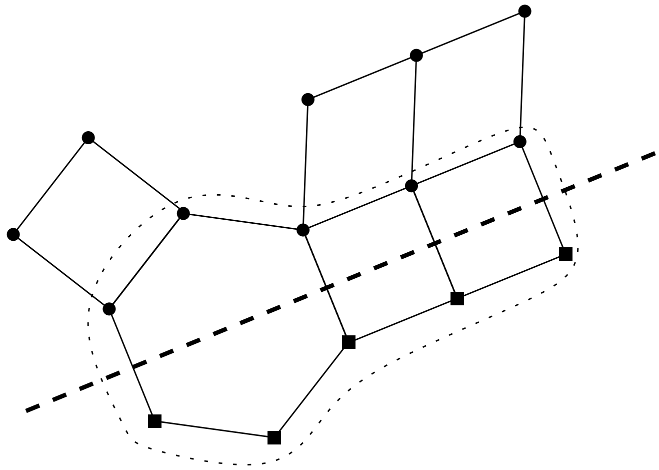


Figure 7: A hyperplane (dashed), its associated open halfspaces (square and round vertices, respectively) and the associated carrier (dotted) in a COM.

Proposition 3.5. *Hyperplanes, carriers, their positive and negative parts, open and closed halfspaces of COMs are COMs. OMs are closed under taking hyperplanes and carriers.*

Proposition 3.5 provides the necessary ingredients for a decomposition of a COM, which is not an OM, into smaller COM constituents. Assume that $(E, \mathcal{L})$ is a simple COM that is not an OM. Put $\mathcal{L}' := \mathcal{L}_e^-$ and $\mathcal{L}'' := \overline{\mathcal{L}_e^+}$. Then $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ and $\mathcal{L}' \cap \mathcal{L}'' = N^-(\mathcal{L}_e^0)$. Since X determines a maximal face not included in $\mathcal{L}_e^0$, we infer that $\mathcal{L}' \setminus \mathcal{L}'' \neq \varnothing$ and trivially $\mathcal{L}'' \setminus \mathcal{L}' \neq \varnothing$. By Proposition 3.5, all three systems $(E, \mathcal{L}')$, $(E, \mathcal{L}'')$, and $(E, \mathcal{L}' \cap \mathcal{L}'')$ are COMs, which are easily seen to be simple.

Moreover, $\mathcal{L}' \circ \mathcal{L}'' \subseteq \mathcal{L}'$ holds trivially. If $W \in \mathcal{L}_e^0$ and $X \in \mathcal{L}_e^-$, then $W \circ X \in F(W) \subseteq N(\mathcal{L}_e^0)$, whence $\mathcal{L}'' \circ \mathcal{L}' \subseteq \mathcal{L}''$. This motivates the following amalgamation process which in a way reverses this decomposition procedure.

We say that a system $(E, \mathcal{L})$ of sign vectors is a *COM amalgam* of two simple COMs $(E, \mathcal{L}')$ and $(E, \mathcal{L}'')$ if the following conditions are satisfied:

(1) $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ with $\mathcal{L}' \setminus \mathcal{L}'', \mathcal{L}'' \setminus \mathcal{L}', \mathcal{L}' \cap \mathcal{L}'' \neq \varnothing$;

(2) $(E, \mathcal{L}' \cap \mathcal{L}'')$ is a simple COM;

(3) $\mathcal{L}' \circ \mathcal{L}'' \subseteq \mathcal{L}'$ and $\mathcal{L}'' \circ \mathcal{L}' \subseteq \mathcal{L}''$;

(4) for $X \in \mathcal{L}' \setminus \mathcal{L}''$ and $Y \in \mathcal{L}'' \setminus \mathcal{L}'$ with $X^0 = Y^0$ there exists a shortest path in the graphical hypercube on $\{\pm 1\}^{E \setminus X^0}$ for which all its vertices and barycenters of its edges belong to $\mathcal{L} \setminus X^0$.

Summarizing the previous discussion and results, we obtain the following corollary, that resembles *constructibility*, also see Figure 8 for an example.

Corollary 3.6. *Simple COMs are obtained via successive COM amalgamations from their maximal faces (that can be contracted to OMs).*

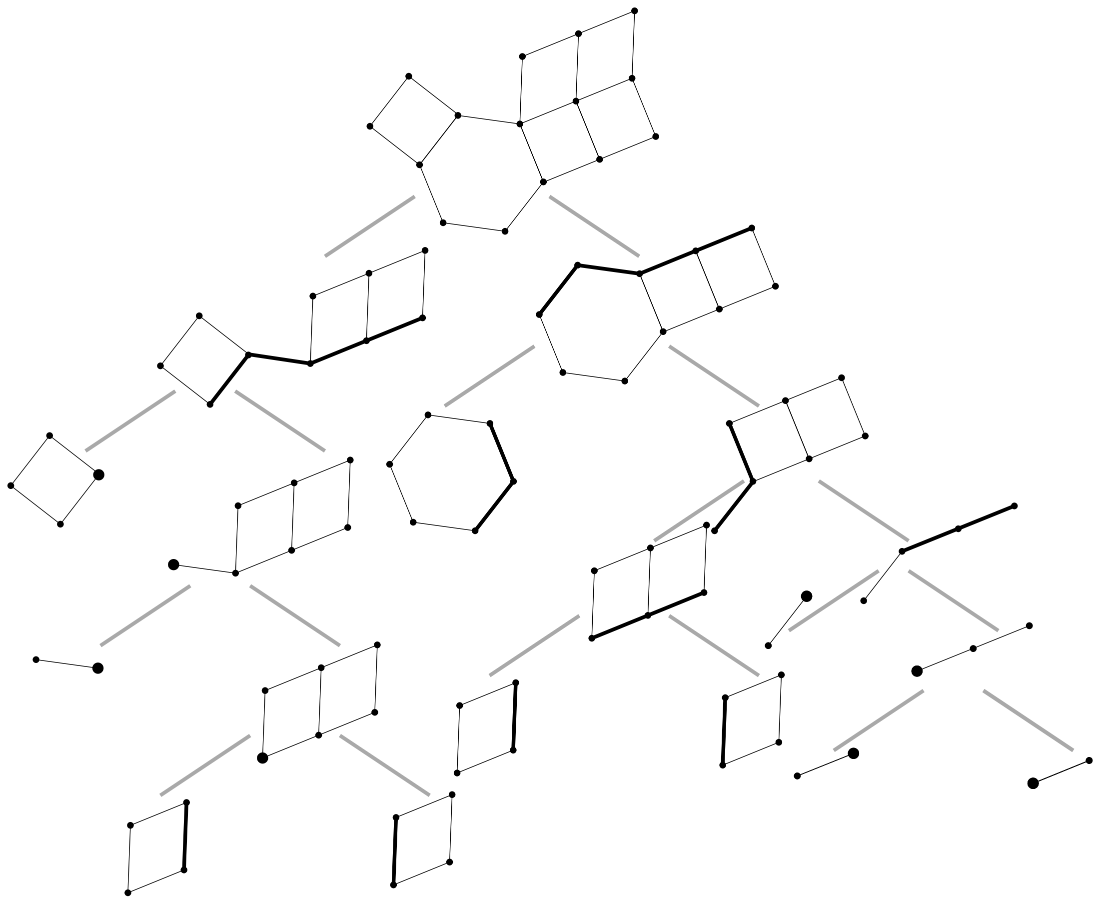


Figure 8: Constructing the COM from Figure 7 by amalgamations.

3.5 COMs as complexes of oriented matroids

In this section we study the complex of a COM from a more topological point of view. In the subsequent definitions, notations, and results we closely follow [22, Section 4] (some missing definitions can be also found there). Let $B^d = \{x \in \mathbb{R}^d : ||x|| \leq 1\}$ be the *standard d-ball* and its boundary $S^{d-1} = \partial B^d = \{x \in \mathbb{R}^d : ||x|| = 1\}$ be the *standard $(d-1)$-sphere*. When saying that a topological space T is a "ball" or a "sphere", it is meant that T is homeomorphic to B^d or S^{d-1} for some d, respectively.

3.5.1 Regular cell complexes

A *(regular) cell complex* Δ constitutes of a covering of a Hausdorff space $||\Delta|| = \bigcup_{\sigma\in\Delta} \sigma$ with finitely many subspaces σ homeomorphic with (open) balls such that

(i) the interiors of $\sigma \in \Delta$ partition $||\Delta||$ (i.e., every $x \in ||\Delta||$ lies in the interior of a single $\sigma \in \Delta$),

(ii) the boundary $\partial\sigma$ of each ball $\sigma \in \Delta$ is a union of some members of Δ [22, Definition 4.7.4].

(iii) Additionally, we will assume that Δ obeys the *intersection property* whenever $\sigma, \tau \in \Delta$ have non-empty intersection then $\sigma \cap \tau \in \Delta$.

The balls $\sigma \in \Delta$ are called *cells* of Δ and the space $||\Delta||$ is called the *underlying space* of Δ. If T is homeomorphic to $||\Delta||$ (notation $T \cong ||\Delta||$), then Δ is said to provide a *regular cell decomposition* of the space T. We will say that a regular cell complex Δ is *contractible* if the topological space $||\Delta||$ is contractible. If $\sigma, \tau \in \Delta$ and $\tau \subseteq \sigma$, then τ is said to be a *face* of σ. We say that $\Delta' \subseteq \Delta$ is a *subcomplex* of Δ if $\tau \in \Delta'$ implies that every face of τ also belongs to Δ'. The 0-cells and 1-cells of Δ are called *vertices* and *edges*. The *1-skeleton* of Δ is encoded by the graph $G(\Delta)$ consisting of the vertices of Δ and graph edges corresponding to the edges of Δ. The set of cells of Δ ordered by containment is denoted by $\mathcal{F}(\Delta)$ (in [52], $\mathcal{F}(\Delta)$ is also called an *abstract cell complex*). Two cell complexes Δ and Δ' are *combinatorially equivalent* if their ordered sets $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta')$ are isomorphic. We continue by recalling several results relating regular cell complexes.

The *order complex* of a finite ordered set P is an abstract simplicial complex $\Delta_{ord}(P)$ whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \cdots < x_k$ of P. For an element x of P let $P_{<x} = \{y \in P : y < x\}$ and $P_{\leq x} = P_{<x} \cup \{x\}$. Some of these notions we will re-encounter in Section 6.2. The following fact expresses that a regular cell complex is homeomorphic to the order complex of its ordered set of faces.

Proposition 3.7 ([22, Proposition 4.7.8]). *Let Δ be a regular cell complex. Then $||\Delta|| \cong ||\Delta_{ord}(\mathcal{F}(\Delta)||$. Moreover, this homeomorphism can be chosen to be* cellular, *i.e., it restricts to a homeomorphism between σ and $||\Delta_{ord}(\mathcal{F}_{\leq\sigma})||$, for all $\sigma \in \Delta$.*

The preceding result says that when considering regular cell complexes (up to isomorphism), we can restrict ourselves to their order complex, i.e., the study of regular cell complexes is the study of a particular class of posets. Interestingly, there is an intrinsic characterization of these posets. The *geometric realization* $|||\Delta|||$ of a complex Δ basically consists of simultaneously replacing all abstract simplices by geometric simplices, see [33] for a formal definition. The ordered sets of faces of regular cell complexes can be characterized in the following way:

Proposition 3.8. *[22, Proposition 4.7.23] Let P be an ordered set. Then $P \cong \mathcal{F}(\Delta)$ for some regular cell complex Δ if and only if $|||\Delta_{ord}(P_{<x})|||$ is homeomorphic to a sphere for all $x \in P$. Furthermore, Δ is uniquely determined by P up to a cellular homeomorphism.*

3.5.2 Cell complexes of COMs

In order to study the topology of COMs we first need to comment on their cells, i.e., OMs. Thus, let $\mathcal{L} \subseteq \{\pm1, 0\}^E$ be the set of covectors of an oriented matroid. Then $(\mathcal{L}, \leq)$ is a semilattice with least element $\mathbf{0}$ (where $\leq$ is the product ordering on $\{\pm1, 0\}^E$ defined above). The semilattice $(\mathcal{L} \cup \{\hat{1}\}, \leq)$, i.e., the semilattice $\mathcal{L}$ with a largest element $\hat{1}$ adjoined, is a lattice, called the *big face lattice* of $\mathcal{L}$ and denoted by $\mathcal{F}_{\text{big}}(\mathcal{L})$. Let $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$ denote the opposite of $\mathcal{F}_{\text{big}}(\mathcal{L})$.

Proposition 3.9. *[22, Corollary 4.3.4 & Lemma 4.4.1] Let $(E, \mathcal{L})$ be an oriented matroid of rank r. Then $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$ is isomorphic to the face lattice of a PL (Piecewise Linear) regular cell decomposition of the $(r-1)$-sphere, denoted by $\Delta(\mathcal{L})$. The tope graph of $\mathcal{L}$ encodes the 1-skeleton of $\Delta(\mathcal{L})$.*

We collected all ingredients necessary to associate to each COM a regular cell complex. Let $\mathcal{L} \subseteq \{\pm 1, 0\}^E$ be the set of covectors of a COM. Analogously to oriented matroids, let $\mathcal{F}_{\text{big}}(\mathcal{L}) := (\mathcal{L} \cup \{\hat{1}\}, \leq)$ denote the ordered set $\mathcal{L}$ with a top element $\hat{1}$ adjoined and call $\mathcal{F}_{\text{big}}(\mathcal{L})$ the *big face semilattice* of $\mathcal{L}$. Let $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$ denote the opposite of $\mathcal{F}_{\text{big}}(\mathcal{L})$. By Lemma 3.4, $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$ is isomorphic to the semilattice comprising the empty set and the faces of $\mathcal{L}$ ordered by inclusion. Recall that for any $X \in \mathcal{L}$, the deletion $(E \setminus \underline{X}, F(X) \setminus \underline{X})$ corresponding to the face $F(X)$ is an oriented matroid, which we will denote by $\mathcal{L}(X)$. Since $F(Y) \subseteq F(X)$ if and only if $Y \in F(X)$, the order ideal $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}_{\leq X}$ coincides with the interval $[\hat{1}, X]$ of $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$ and is isomorphic to the opposite big face lattice $\mathcal{F}_{\text{big}}(\mathcal{L}(X))^{\text{op}}$ of $\mathcal{L}(X)$. By Proposition 3.9, if r is the rank of $\mathcal{L}(X)$, then $\mathcal{F}_{\text{big}}(\mathcal{L}(X))^{\text{op}}$ is isomorphic to the face lattice of a PL cell decomposition $\Delta(\mathcal{L}(X))$ of the $(r-1)$-sphere. Additionally, the tope graph of $\mathcal{L}(X)$ encodes the 1-skeleton of $\Delta(\mathcal{L}(X))$. Denote by $\sigma(\mathcal{L}(X))$ the open PL ball whose boundary is the $(r-1)$-sphere occurring in the definition of $\Delta(\mathcal{L}(X))$. We will call the cells of $\Delta(\mathcal{L}(X))$ *faces* of $\sigma(\mathcal{L}(X))$. The faces of $\Delta(\mathcal{L}(X))$ correspond to the elements of $\mathcal{L}(X) \cup \{\hat{1}\}$. Notice in particular that the adjoined element $\hat{1}$ corresponds to the empty face in $\Delta(\mathcal{L}(X))$ and $\mathbf{0} \in F(X) \setminus \underline{X}$ corresponds to the unique maximal face $\sigma(\mathcal{L}(X))$.

By Proposition 3.7, for any $X \in \mathcal{L}$ we have:

$$||\Delta(\mathcal{L}(X))|| \cong ||\Delta_{ord}(\mathcal{F}_{\text{big}}(\mathcal{L}(X))^{\text{op}})||.$$

Furthermore, since $\mathcal{F}_{\text{big}}(\mathcal{L}(X))^{\text{op}}$ is isomorphic to $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}_{\leq X}$ we also have:

$$||\Delta_{ord}(\mathcal{F}_{\text{big}}(\mathcal{L}(X))^{\text{op}})|| \cong ||\Delta_{ord}(\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}_{\leq X})||.$$

Thus, for each $X \in \mathcal{L}$, $||\Delta_{ord}(\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}_{<X})||$ is homeomorphic to $||\Delta(\mathcal{L}(X)) \setminus \sigma(\mathcal{L}(X))||$, which is a sphere by Proposition 3.9. Now, by Proposition 3.8, $\mathcal{F}_{\text{big}}(\mathcal{L})^{\text{op}}$ is the face semilattice of a regular cell complex $\Delta(\mathcal{L})$. Moreover, from the proof of Proposition 3.8 it follows that $\Delta(\mathcal{L})$ can be chosen so that its cells are the balls $\sigma(\mathcal{L}(X)), X \in \mathcal{L}$, whose boundary spheres are decomposed by $\Delta(\mathcal{L}(X))$. Since $F(X) \cap F(Y) = F(X \circ Y)$ for any two covectors $X, Y \in \mathcal{L}$ such that $F(X)$ and $F(Y)$ intersect, $\mathcal{F}_{\text{big}}(\mathcal{L}(X \circ Y))^{\text{op}}$ is isomorphic to a sublattice of $\mathcal{F}_{\text{big}}(\mathcal{L}(X))^{\text{op}}$ and to a sublattice of $\mathcal{F}_{\text{big}}(\mathcal{L}(Y))^{\text{op}}$. Therefore the cells $\Delta(\mathcal{L}(X))$ and $\Delta(\mathcal{L}(Y))$ are glued in $\Delta(\mathcal{L})$ along $\Delta(\mathcal{L}(X \circ Y))$, whence $\Delta(\mathcal{L})$ also satisfies the intersection property (iii). Notice also that since the 1-skeleton of each $\Delta(\mathcal{L}(X))$ yields the tope graph of $\mathcal{L}(X)$ and $\Delta(\mathcal{L})$ satisfies (iii), the 1-skeleton of $\Delta(\mathcal{L})$ encodes the tope graph of $\mathcal{L}$. We summarize this in the following proposition, in which we also establish that $\Delta(\mathcal{L})$ is contractible. More precisely, we use the so-called gluing lemma [20, Lemma 10.3] together with our amalgamation procedure from Corollary 3.6 to prove the following inductively:

Proposition 3.10. *If $(E, \mathcal{L})$ is a COM, then $\Delta(\mathcal{L})$ is a contractible regular cell complex and the tope graph of $\mathcal{L}$ is realized by the 1-skeleton of $\Delta(\mathcal{L})$.*

3.6 Locally and zonotopally realizable COMs

As in Section 3.1, let E be a central arrangement of n hyperplanes of $\mathbb{R}^d$ and $\mathcal{L}$ be the oriented matroid corresponding to the regions of $\mathbb{R}^d$ defined by this arrangement. Let $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a set of unit vectors each normal to a different hyperplane of E. The *zonotope* $\mathcal{Z} := \mathcal{Z}(\mathbf{X})$ of $\mathbf{X}$ is the convex polytope of $\mathbb{R}^d$ which can be expressed as the Minkowski sum of n line segments

$$\mathcal{Z} = [-\mathbf{x}_1, \mathbf{x}_1] + [-\mathbf{x}_2, \mathbf{x}_2] + \ldots + [-\mathbf{x}_n, \mathbf{x}_n].$$

Equivalently, $\mathcal{Z}$ is the projection of the n-cube $C_n := \{\sum_{i=1}^n \lambda_i \mathbf{e}_i : -1 \leq \lambda_i \leq +1\} \subset \mathbb{R}^n$ under $\mathbf{X}$ (where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^n$), which sends $\mathbf{e}_i$ to $\mathbf{x}_i$, $i = 1, \ldots, n$:

$$\mathcal{Z} = \{\sum_{i=1}^{n} \lambda_i \mathbf{x}_i : -1 \leq \lambda_i \leq +1\} \subset \mathbb{R}^d.$$

The hyperplane arrangement E is *geometrically polar* to $\mathcal{Z}$: the regions of the arrangement are the cones of outer normals at the faces of $\mathcal{Z}$. The face lattice of $\mathcal{Z}$ is opposite (anti-isomorphic) to the big face lattice of the oriented matroid $\mathcal{L}$ of $\mathbf{X}$, that is, $\mathcal{F}(\mathcal{Z}) \simeq \mathcal{F}_{\mathrm{big}}(\mathcal{L})^{\mathrm{op}}$; for this and other results, see [22, Section 2.2]. Therefore the zonotopes together with their faces can be viewed as the cell complexes associated to realizable oriented matroids.

A COM $(E, \mathcal{L})$ is called *locally realizable* if $\mathcal{L}(X)$ is a realizable oriented matroid for any $X \in \mathcal{L}$. In this case each cell $\Delta(\mathcal{L}(X)), X \in \mathcal{L}$, is combinatorially equivalent to a zonotope and therefore $\Delta(\mathcal{L})$ is what is called a zonotopal complex in [52].

A locally realizable COM $(E, \mathcal{L})$ is called *zonotopally realizable* if for each (maximal) face of $\mathcal{L}$ a realization as a geometric zonotope can be chosen, such that the realizations of the intersections of any two faces are (linearly) isometric. In other words we have a piecewise Euclidean complex (also known as *geometric* complex).

This property is shared by all cube complexes, since cubes can be chosen to be realized as unit cubes. In particular, the cell complex $\Delta(\mathcal{L})$ associated to a lopsided set $(E, \mathcal{L})$ such, see [15]. Therefore LOPs are zonotopally realizable COMs.

Clearly, zonotopally realizable COMs are locally realizable. Moreover, it is easy to see that realizable implies zonotopally realizable:

Proposition 3.11. *If $\mathcal{L}$ is a realizable COM, then $\mathcal{L}$ is zonotopally realizable (and thus locally realizable). In particular, each ranking COM is zonotopally realizable.*

Proof. Since $\mathcal{L}$ is realizable there is a set of oriented affine hyperplanes of $\mathbb{R}^d$ and an open convex set C, such that $\mathcal{L} = \mathcal{L}(E, C)$. Without loss of generality we can assume that C is the interior of a full-dimensional polyhedron P. Let F be the set of supporting hyperplanes of P. Consider the central hyperplane arrangement A resulting from lifting the affine arrangement $E \cup F$ to $\mathbb{R}^{d+1}$. The associated OM $\mathcal{L}'$ is realizable and therefore zonotopally realizable. Since $\Delta(\mathcal{L})$ is a subcomplex of $\Delta(\mathcal{L}')$, also $\mathcal{L}$ is zonotopally realizable. □

While in Section 6.2 we will prove that zonotopally realizable does not imply realizable (and therefore refute a question from [16]), the following remains open, even if the answer is likely to be "no".

Question 1. *Is any locally realizable COM zonotopally realizable?*

3.6.1 CAT(0) Coxeter COMs

We conclude this section by presenting another class of zonotopally realizable COMs. Namely, we prove that the CAT(0) Coxeter (zonotopal) complexes introduced in [90] are COMs. They represent a common generalization of benzenoid systems [91], 2-dimensional cell complexes obtained from bipartite cellular graphs [11], and CAT(0) cube complexes (cube complexes arising from median structures) [13]. One can say that CAT(0) zonotopal complexes generalize CAT(0) cube complexes in the same way as COMs generalize lopsided sets. It remains open but likely, that hypercellular graphs, see Section 5, are tope graphs of CAT(0) Coxeter COMs, see Conjecture 5.

A zonotope $\mathcal{Z}$ is called a *Coxeter zonotope* (an *even polyhedron* [90] or a *Coxeter cell* [53]) if $\mathcal{Z}$ is symmetric with respect to the mid-hyperplane H_f of each edge f of $\mathcal{Z}$, i.e., to the hyperplane perpendicular to f and passing through the middle of f. A cell complex Δ is called a *Coxeter complex* if Δ is a geometric zonotopal complex in which each cell is isometric to a Coxeter zonotope. Throughout this subsection, by Δ we denote a Coxeter complex and by $||\Delta||$ the underlying metric space of Δ.

If $\mathcal{Z}$ is a Coxeter zonotope and f, f' are two parallel edges of $\mathcal{Z}$, then one can easily see that the mid-hyperplanes H_f and $H_{f'}$ coincide. If $\mathcal{Z} = [-\mathbf{x}_1, \mathbf{x}_1] + [-\mathbf{x}_2, \mathbf{x}_2] + \ldots + [-\mathbf{x}_n, \mathbf{x}_n]$, denote by H_i the mid-hyperplane to all edges of $\mathcal{Z}$ parallel to the segment $[-\mathbf{x}_i, \mathbf{x}_i]$, $i = 1, \ldots, n$. Then $\mathcal{Z}$ is the zonotope of the regions defined by the arrangement $\{H_1, \ldots, H_n\}$. It is well-known [53, Definition 7.3.1] (and is also noticed in [90, p.184]) that Coxeter zonotopes are exactly the zonotopes associated to *reflection arrangements* (called also *Coxeter arrangements*) of hyperplanes, i.e., to arrangements of hyperplanes of a finite reflection group [22, Section 2.3]. For each $i = 1, \ldots, n$, denote by $\mathcal{Z}_i$ the intersection of $\mathcal{Z}$ with the hyperplane H_i and call it a *mid-section of* $\mathcal{Z}$. The mid-sections $\mathcal{Z}_i$ of a Coxeter zonotope $\mathcal{Z}$ of dimension d are Coxeter zonotopes of dimension $d-1$.

We continue with the definition of CAT(0) metric spaces and CAT(0) Coxeter complexes. The underlying space (polyhedron) $||\Delta||$ of a geometric zonotopal complex (and, more generally, of a cell complex with Euclidean convex polytopes as cells) Δ can be endowed with an intrinsic l_2-metric in the following way. Assume that inside every maximal face of $||\Delta||$ the distance is measured by the l_2-metric. The *intrinsic l_2-metric* d_2 of $||\Delta||$ is defined by letting the distance between two points $x, y \in ||\Delta||$ be equal to the greatest lower bound on the length of the paths joining them; here a *path* in $||\Delta||$ from x to y is a sequence $x = x_0, x_1, \ldots, x_m = y$ of points in $||\Delta||$ such that for each $i = 0, \ldots, m-1$ there exists a face σ_i containing x_i and x_{i+1}, and the *length* of the path equals $\sum_{i=0}^{m-1} d(x_i, x_{i+1})$, where $d(x_i, x_{i+1})$ is computed inside σ_i according to the respective l_2-metric. The resulting metric space is *geodesic*, i.e., every pair of points in $||\Delta||$ can be joined by a geodesic; see [33].

A *geodesic triangle* $T := T(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of T) and a geodesic between each pair of vertices (the edges of T). A *comparison triangle* for $T(x_1, x_2, x_3)$ is a triangle $T(x_1', x_2', x_3')$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(x_i', x_j') = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is a *CAT(0) space* [88] if all geodesic triangles $T(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov: *If y is a point on the side of $T(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x_1', x_2']$ of the comparison triangle $T(x_1', x_2', x_3')$ such that $d_{\mathbb{R}^2}(x_i', y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{R}^2}(x_3', y')$.*

CAT(0) spaces can be characterized in several different natural ways and have numerous properties (for a full account of this theory consult the book [33]). For instance, a cell complex endowed with a piecewise Euclidean metric is CAT(0) if and only if any two points can be joined by a unique geodesic. Moreover, CAT(0) spaces are contractible.

A *CAT(0) Coxeter complex* is a Coxeter complex Δ for which $||\Delta||$ endowed with the intrinsic l_2-metric d_2 is a CAT(0) space. See Figure 9 for illustrations. In this case, the parallelism relation on edges of cells of Δ induces a parallelism relation on

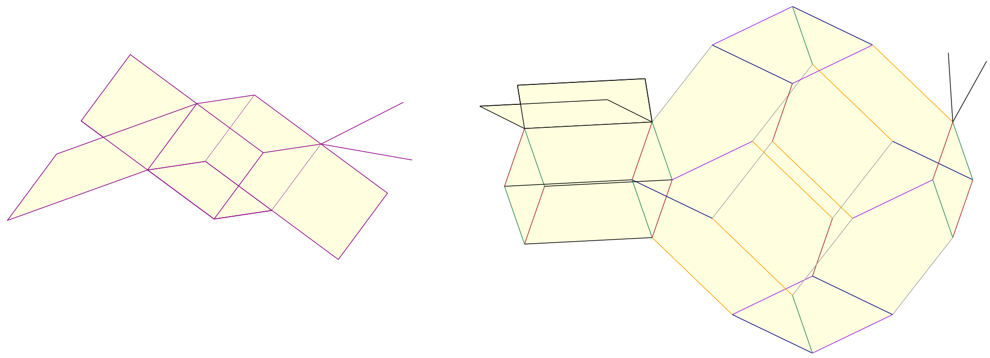


Figure 9: A CAT(0) cube complex and a CAT(0) Coxeter complex.

all edges of Δ: two edges f, f' of Δ are *parallel* if there exists a sequence of edges $f_0 = f, f_1, \ldots, f_{k-1}, f_k = f'$ such that any two consecutive edges f_{i-1}, f_i are parallel edges of a common cell of Δ. Parallelism is an equivalence relation on the edges of Δ. Denote by E the equivalence classes of this parallelism relation. For $e \in E$, we denote by Δ_e the union of all mid-sections of the form $\mathcal{Z}_e$ for cells $\mathcal{Z} \in \Delta$ which contain edges from the equivalence class e (let $||\Delta_e||$ be the underlying space of Δ_e). We call each $||\Delta_e||$ (or Δ_e), $e \in E$, a *mid-hyperplane* (or a *wall* as in [90]) of $||\Delta||$. Since each mid-section included in Δ_e is a Coxeter zonotope, each mid-hyperplane of a Coxeter complex is a Coxeter complex as well. CAT(0) Coxeter complexes have additional nice and strong properties, which have been established in [90].

Lemma 3.12. *[90, Lemme 4.4] Let Δ be a CAT(0) Coxeter complex and Δ_e be a mid-hyperplane of Δ. Then $||\Delta_e||$ is a convex subset of $||\Delta||$ and $||\Delta_e||$ partitions $||\Delta||$ in two connected components $||\Delta_e^-||$ and $||\Delta_e^+||$ (called halfspaces of $||\Delta||$).*

If $x \in ||\Delta_e^-||$ and $y \in ||\Delta_e^+||$, then x and y are said to be *separated* by the mid-hyperplane (wall) $||\Delta_e||$. A path P in $||\Delta||$ *traverses* a mid-hyperplane $||\Delta_e||$ if P contains an edge xy such that x and y are separated by $||\Delta_e||$. Two distinct mid-hyperplanes $||\Delta_e||$ and $||\Delta_f||$ are called *parallel* if $||\Delta_e|| \cap ||\Delta_f|| = \varnothing$ and *crossing* if $||\Delta_e|| \cap ||\Delta_f|| \neq \varnothing$.

Lemma 3.13. *[90, Corollaire 4.10] Two vertices u, v of Δ are adjacent in $G(\Delta)$ if and only if u and v are separated by a single mid-hyperplane of $||\Delta||$.*

Lemma 3.14. *[90, Proposition 4.11] A path P of $G(\Delta)$ between two vertices u, v is a shortest (u, v)-path in $G(\Delta)$ if and only if P traverses each mid-hyperplane of $||\Delta||$ at most once.*

These three results imply that the arrangement of mid-hyperplanes of a CAT(0) Coxeter complex Δ defines a wall system sensu [90], which in turn provides us with a system $\mathcal{L}(\Delta)$ of sign vectors. Define the mapping $\varphi : \Delta \to \{\pm 1, 0\}^E$ in the following way. First, for $e \in E$ and $x \in \Delta$, set

$$\varphi_e(x) := \begin{cases} -1 & \text{if } x \in ||\Delta_e^-||, \\ 0 & \text{if } x \in ||\Delta_e||, \\ +1 & \text{if } x \in ||\Delta_e^+||. \end{cases}$$

Let $\varphi(x) = (\varphi_e(x) : e \in E)$. Denote by $\mathcal{L}(\Delta)$ the set of all sign vectors of the form $\varphi(x), x \in ||\Delta||$. Notice that if a point x of $||\Delta||$ does not belong to any mid-hyperplane of $||\Delta||$, then $\varphi(x) \in \{\pm1\}^E$; in particular, this is the case for the vertices of $G(\Delta)$. Moreover, Lemma 3.14 implies that φ defines an isometric embedding of $G(\Delta)$ into the hypercube $\{\pm1\}^E$.

Theorem 3.15 ([16, Theorem 5]). *Let Δ be a CAT(0) Coxeter complex, E be the classes of parallel edges of Δ, and $\mathcal{L}(\Delta) := \cup\{\varphi(x) : x \in ||\Delta||\} \subseteq \{\pm1, 0\}^E$. Then $(E, \mathcal{L}(\Delta))$ is a simple COM and $G(\Delta)$ is its tope graph.*

Let us to the end of this section mention that the relation of the CAT(0) property and realizability seems of interest to us. Are all CAT(0) zonotopally realizable COMs also realizable? The converse clearly does not hold, since Q_3^- is realizable but not CAT(0).

3.7 Illustration: Rankings and graphic COMs

Particular COMs naturally arise in order theory. For the entire subsection, let $(P, \leq)$ denote an ordered set (alias poset), that is, a finite set P endowed with an order (relation) $\leq$. A *ranking* (alias weak order) is an order for which incomparability is transitive. Equivalently, an order $\leq$ on P is a ranking exactly when P can be partitioned into antichains (where an *antichain* is a set of mutually incomparable elements) $A_1, \ldots, A_k$, such that $x \in A_i$ is below $y \in A_j$ whenever $i < j$. An order $\leq$ on P is *linear* if any two elements of P are comparable, that is, all antichains are trivial (i.e., of size < 2). An order $\leq'$ *extends* an order $\leq$ on P if $x \leq y$ implies $x \leq' y$ for all $x, y \in P$. Of particular interest are the linear extensions and, more generally, the ranking extensions of a given order $\leq$ on P.

Let us now see how to associate a set of sign vectors to an order $\leq$ on $P = \{1, 2, \ldots, n\}$. For this purpose take E to be the set of all 2-subsets of P and encode $\leq$ by its characteristic sign vector $X^{\leq} \in \{0, \pm1\}^E$, which to each 2-subset $e = \{i, j\}$ assigns $X_e^{\leq} = 0$ if i and j are incomparable, $X_e^{\leq} = +1$ if the order agrees with the natural order on the 2-subset e, and else $X_e^{\leq} = -1$. In the sign vector representation the different components are ordered with respect to the lexicographic natural order of the 2-subsets of P.

The composition of sign vectors from different orders $\leq$ and $\leq'$ does not necessarily return an order again. Take for instance, $X^{\leq} = +++$ coming from the natural order on P and $X^{\leq'} = 0-0$ coming from the order with the single (nontrivial) comparability $3 \leq' 1$. The composition $X^{\leq'} \circ X^{\leq}$ equals $+-+$, which signifies a directed 3-cycle and thus no order. The obstacle here is that $X^{\leq'}$ encodes an order for which one element is incomparable with a pair of comparable elements. Transitivity of the incomparability relation is therefore a necessary condition for obtaining a COM.

We denote by $\mathcal{R}(P, \leq)$ the simplification of the set of sign vectors associated to all ranking extensions of $(P, \leq)$. Note that the simplification amounts to omitting the pairs of the ground set corresponding to pairs of comparable pairs of P. The following is straightforward, but in particular follows from realizability, that we will explain below:

Theorem 3.16. *Let $(P, \leq)$ be an ordered set. Then $\mathcal{R}(P, \leq)$ is a realizable COM, called the* ranking COM *of $(P, \leq)$.*

To provide an example for a ranking COM and also illustrate the preceding construction, consider the ordered set ("fence") shown in Figure 10(a). In Figure 10(b), the sign vector X encodes the natural order $<$ and Y the ordering $3 <_Y 2 <_Y j = 5 <_Y i = 1 <_Y 4$, while Z encodes the intermediate ranking with $2 <_Z 3 <_Z 1$ and $5 <_Z 4$.

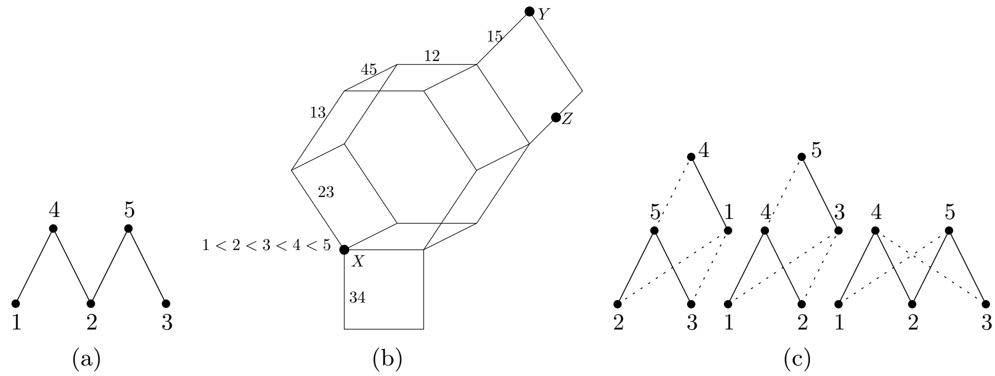


Figure 10: From (a) an ordered set $(P, \leq)$ to (b) the ranking COM $\mathcal{R}(P, \leq)$ comprising three maximal faces determined by (c) the minimal rankings in $\mathcal{R}(P, \leq)$.

The ranking COM $\mathcal{R}(P, \leq)$ is the natural host of all linear extensions of $(P, \leq)$ (as its topes), where the interconnecting rankings signify the cell structure. The *linear extension graph* of an ordered set $(P, \leq)$ is defined on the linear extensions of $(P, \leq)$, where two linear extensions are joined by an edge if and only if they differ on the order of exactly one pair of elements. Thus, the linear extension graph of $(P, \leq)$ is the tope graph of $\mathcal{R}(P, \leq)$. A number of geometric and graph-theoretical features of linear extensions have been studied by various authors [131,140,153], which can be expressed most naturally in the language of COMs.

One such result translates to the fact that ranking COMs are realizable. To see this, first consider the *braid arrangement* of type B_n, i.e., the central hyperplane arrangement $\{H_{ij} : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$, where $H_{ij} = \{x \in \mathbb{R}^n : x_i = x_j\}$ and the position of any point in the corresponding halfspace $\{x \in \mathbb{R}^n : x_i < x_j\}$ is encoded by $+$ with respect to H_{ij}. The resulting OM is known as the *permutahedron* [22]. Given an order $\trianglelefteq$ on $P = \{1, \ldots, n\}$ consider the arrangement $E = \{H_{ij} : i, j \text{ incomparable}\}$ restricted to the open polyhedron $\bigcap_{i \triangleleft j} \{x \in \mathbb{R}^n : x_i < x_j\}$. The closure of the latter intersected with the unit cube $[0, 1]^n$ coincides with the *order polytope* [168] of $(P, \trianglelefteq)$. It is well-known that the maximal cells of the braid arrangement restricted to the order polytope of $(P, \trianglelefteq)$ correspond to the linear extensions of $(P, \trianglelefteq)$. Thus, the COM realized by the order polytope and the braid arrangement has the same set of topes as the ranking COM. Since the topes determine a COM [16], this implies that both COMs coincide. In particular, ranking COMs are realizable.

We will now show how other notions for general COMs translate to ranking COMs. A face $\mathcal{F}$ of $\mathcal{R}(P, \leq)$, as defined in Section 3.3, can be viewed as the set of all rankings that extend some ranking extension $\leq'$ of $\leq$. Hence $\mathcal{F} \cong \mathcal{R}(P, \leq')$, i.e., all faces of a ranking COM are also ranking COMs. The minimal elements of $\mathcal{R}(P, \leq)$ with respect to sign ordering (being the improper cocircuits of $\mathcal{R}(P, \leq)$) are the minimal ranking extensions of $(P, \leq)$, see Figure 10(c).

A hyperplane $\mathcal{R}_e^0$ of $\mathcal{R} := \mathcal{R}(P, \leq)$ relative to $e = \{i, j\} \in E$ corresponds to those ranking extensions of $(P, \leq)$ leaving i, j incomparable. Thus, $\mathcal{R}_e^0$ can be seen as the ranking COM of the ordered set obtained from $(P, \leq)$ by identifying i and j. The open halfspace $\mathcal{R}_e^+$ corresponds to those ranking extensions fixing the natural order on i, j

and is therefore the ranking COM of the ordered set $(P, \leq)$ extended with the natural order on i, j. The analogous statement holds for $\mathcal{R}_e^-$. Similarly, the carrier of $\mathcal{R}$ relative to e can be seen as the ranking COM of the ordered set arising as the intersection of all minimal rankings of $(P, \leq)$ not fixing an order on i, j. So, in all three cases the resulting COMs are again ranking COMs.

One may wonder which are the ordered sets whose ranking COM is an OM or a lopsided system. The maximal cells in Figure 10(b) are symmetric and therefore correspond to OMs. We can find:

Proposition 3.17. *Let $\mathcal{R}(P, \leq)$ be a ranking COM.*

1. *$\mathcal{R}(P, \leq)$ is an OM if and only if $\leq$ is a ranking. In this case, $\mathcal{R}(P, \leq)$ and its proper faces are products of permutahedra.*
2. *$\mathcal{R}(P, \leq)$ is an LOP if and only if $(P, \leq)$ has width at most 2. In this case, the tope graph of $\mathcal{R}(P, \leq)$ is the covering graph (i.e., undirected Hasse diagram) of a distributive lattice.*

Note that the of the distributive lattice in the second part is a bit different, than described [16]: If P is of width 2, then by Dilworth's Theorem [61] $P = C \cup D$ is the union of two chains. Add subdivision elements D' for all elements on D and also one on top and one below, i.e., $|D'| = |D|+1$. Now for any cover relation $c \prec d$ add $c \prec d^-$, where $d^- \in D'$ is the element below d d and for $d \prec c$ add $d^+ \prec c$, where $d^+ \in D'$ is the element above d. Now take the subposet induced by C and D'. The linear extension correspond to the retracts to D'. This is a distributive lattice because it is closed under componentwise maxima and minima.

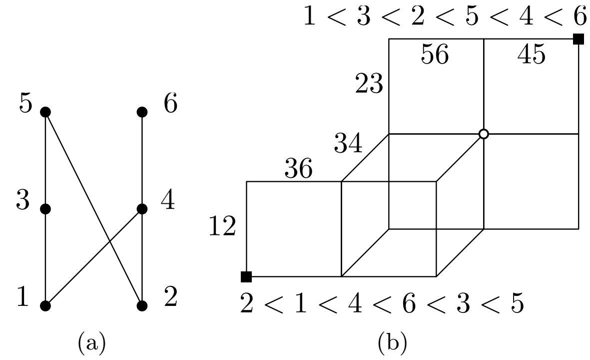


Figure 11: From (a) the Hasse diagram of $(P, \leq)$ having width 2 to (b) the tope graph of the lopsided system $\mathcal{R}(P, \leq)$ oriented as a distributive lattice.

In Figure 11(a), an ordered set $(P, \leq)$ of width 2 is displayed, which has the natural order on $\{1, \ldots, 6\}$ among its linear extensions. Figure 11(b) shows the tope graph of the lopsided system $\mathcal{R}(P, \leq)$ and highlights the pair of diametrical vertices that determine the distributive lattice orientation (and its opposite); the natural order is associated with the (median) vertex (indicated by a small open circle). If we added the compatibility $3 < 6$ to the Hasse diagram, then the tope graph shrinks by collapsing the (two) edges corresponding to $\{3, 6\}$. The resulting graph with $F_7 = 13$ vertices is known as the "Fibonacci cube of order 5", see [101] for a recent survey on these graphs.

Similarly, to the fact that hyperplanes, carriers, and open halfspaces of a ranking COM are also ranking COMs, the class of ranking COMs is also closed with respect to contractions. On the other hand deleting an element in a ranking COM may give a COM which is not a ranking COM.

To give a small example, consider the minor $\mathcal{R}(P, \leq) \setminus \{5, 6\}$ of the lopsided system $\mathcal{R}(P, \leq)$ of Figure 11. Suppose by way of contradiction, that $\mathcal{R}(P, \leq) \setminus \{5, 6\}$ could be represented by some $\mathcal{R}(Q, \leq)$. The ordered set $(Q, \leq)$ must be of width 2 since the tope graph of $\mathcal{R}(Q, \leq)$ is obtained from the graph in Figure 11(b) by contracting the five edges labeled 56 and thus includes no 3-cube. We can keep the current labeling without loss of generality to the point that Q must include four antichains $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{3, 6\}$ of size 2 with exactly this intersection pattern. But then the fifth antichain must be disjoint from $\{1, 2\}$ and $\{2, 3\}$ but intersecting both $\{3, 4\}$ and $\{3, 6\}$, whence it must be $\{4, 6\}$, which however yields a contradiction as $\{3, 4, 6\}$ cannot be an antichain in $(Q, \leq)$. Furthermore, $\mathcal{R}(P, \leq) \setminus \{5, 6\}$ is easily seen to be the COM amalgam of ranking COMs, i.e., the class is also not closed under COM amalgamations.

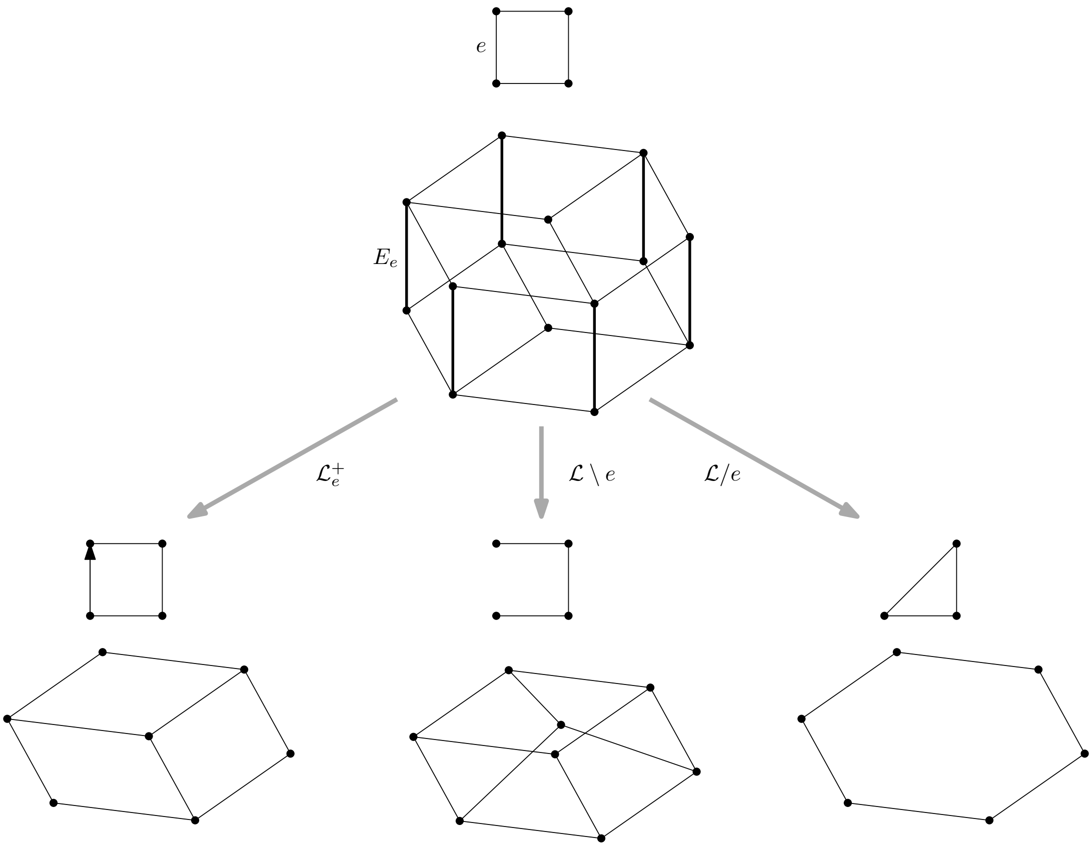


Figure 12: Halfplanes, deletions and contractions of the graphic COM associated to C_4.

Towards the end of this section, let us suggest a natural generalization of ranking COMs, that we will call simply *graphic*. See Figure 12 for an illustration. Given a mixed graph $G = (V, E \cup A)$ such that the directed part $D = (V, A)$ is an induced transitive acyclic subdigraph, we can define a realizable COM much like the ranking COM by considering the open polyhedron $\bigcap_{(i,j) \in A} \{x \in \mathbb{R}^n \mid x_i < x_j\}$ and intersect it with the arrangement $E = \{H_{ij} \mid \{i, j\} \in E\}$. Clearly ranking COMs are graphic, corresponding to the mixed graph, that is a partial orientation of the complete graph, where arcs correspond to comparabilities in the poset. Graphic OMs are also graphic COMs, where $A = \emptyset$. The tope set of the graphic COM of $G = (V, E \cup A)$ can be identified with the acyclic orientations of G and the tope graph is their *flip graph*, i.e. two orientations are linked if they differ in exactly one edge. See e.g. [139] for a graph theoretical study of theses objects.

Graphic COMs are closed under minors and under taking halfspaces. In particular, it is easy to see that they are the closure under these operations of ranking COMs as well as of graphic OMs. Are they closed under amalgamations? Similarly, to Propositions 3.17 one can see that a graphic COM is an OM if and only it has $A = \emptyset$, i.e., it is a graphic OM. What are the graphic LOPs? We believe that graphic COMs serve as a good class to understand COM duality. It seems that in a co-graphic COM, topes correspond to strongly connected orientations of a mixed graph and the tope graph is the corresponding flip graph.

Flip graphs endow the large set of orientations at concern with a structure that can be used for proofs but also lies at the heart of enumeration algorithms, see [23]. In this context it would be of interest to study the set of topes of a co-graphic COM that correspond to orientations of higher strong connectivity. We will now turn now our attention to the general study of tope graphs of COMs.

4 Tope graphs

In this section we focus on COMs and their tope graphs. We present two characterizations of tope graphs and thus two graph theoretical characterizations of COMs. The first characterization is in terms of its complete (infinite) list of excluded pc-minors.

As corollaries we obtain excluded pc-minor characterizations for tope graphs of OMs, AOMs, and LOPs. Moreover, in the case of bounded rank the list of excluded pc-minors is finite. We devise a polynomial time algorithm for checking if a given partial cube has another one as pc-minor, leading to polynomial time recognition algorithms for the classes with a finite list of excluded pc-minors. Another consequence is a third characterization of tope graphs of COMs in terms of iterated zone graphs, which generalizes a result of Handa [92] about tope sets of OMs.

The second characterization of tope graphs of COMs is in terms of the metric behavior of certain subgraphs. More precisely, we prove that a partial cube is the tope graph of a COM if and only if all of its antipodal (also known as symmetric-even [19]) subgraphs are gated. As corollaries, this theorem specializes to tope graphs of OMs, AOMs, and LOPs. In particular, we obtain a new unified proof for characterization theorems of tope sets of LOPs and OMs due to Lawrence [120] and da Silva [50], respectively.

Moreover, this characterization allows to prove that Pasch graphs are COMs, confirming a conjecture of Chepoi, Knauer, and Marc [42]. Finally, our characterization is verifiable in polynomial time, hence gives polynomial time recognition algorithms for tope graphs of COMs, OMs, AOMs, and LOPs, even without bounding the rank. Note that a polynomial time recognition algorithm for tope graphs of OMs was known before, see [81]. This algorithm however works without a characterization of the graphs, but constructs the set of cocircuits from the topes and there verifies the cocircuit axioms.

In particular, we answer a long-standing open question on OMs, i.e., the question for a purely graph theoretical characterization of tope graphs, see [93, Problem 2] that can furthermore be verified in polynomial, which was posed in [79, Problem 1.2]. Since the tope graph determines a COM, OM, AOM, or LOP up to isomorphism, see [16], our results can be seen as identifying the theory of (complexes of) oriented matroids as a part of metric graph theory.

In order to present the ingredients of our characterization of tope graphs we need to give an account of some elements of metric graph theory. In particular we will proceed to discuss partial cubes in greater detail next.

4.1 Partial cubes

A graph $G = (V, E)$ is a *partial cube* if it is (isomorphic to) an isometric subgraph of a hypercube graph Q_n, i.e., $d_G(u, v) = d_{Q_n}(u, v)$ for all $u, v \in V$, where d denotes the distance function of the respective graphs. Partial cubes were introduced by Graham and Pollak [87] in the study of interconnection networks. They form an important graph class in media theory [70], frequently appear in chemical graph theory [69], and quoting [104] present one of the central and most studied classes in metric graph theory.

As mentioned above, important subclasses of partial cubes include median graphs, bipartite cellular graphs, hypercellular graphs, Pasch graphs, and netlike partial cubes.

Partial cubes also capture several important graph classes not directly coming from metric graph theory, such as region graphs of hyperplane arrangements, diagrams of distributive lattices, linear extension graphs of posets. Clearly, for us the most interesting subclasses of partial cubes are tope graphs of oriented matroids (OMs), affine oriented matroids (AOMs), lopsided systems (LOPs), and complexes of oriented matroids (COMs). As it turns out, all of the above mentioned classes of partial cubes are indeed tope graphs of COMs. Note however, that there are also important classes of partial cubs that are not COMs, e.g., middle layer graphs and transitive partial cubes [126], full subdivisions of cliques and two-dimensional partial cubes [44], planar partial cubes [59], or almost median graphs [30, 96].

Partial cubes admit a natural minor-relation (pc-minors for short) consisting of restrictions and contractions. Several of the above classes including tope graphs of COMs, two-dimensional and planar partial cubes are pc-minor closed. Complete (finite) lists of excluded pc-minors are known for median graphs, bipartite cellular graphs, hypercellular graphs and Pasch graphs, and two-dimensional partial cubes see [39, 40, 42, 44]. Another well-known construction of a smaller graph from a partial cube is the zone graph [104]. In tope graphs of COMs these operations correspond can be translated to operations in COMs, i.e., taking halfplanes in a COM are elementary restrictions in the graph, deletions in the COM are contractions in the graph, and contracting in the COM is taking zone graphs in the tope graph.

In the present section we will give an introduction to the theory of partial cubes and its elements that are important to our characterizations of tope graphs of COMs. This will contain many definitions and lemmas, that illustrate the flavor of this theory. All of them are essential in our tope graph characterization. Of central importance are partial cube minors, expansions, zone graphs, and their interactions with metric subgraphs, such as convex, antipodal, affine, and gated subgraphs.

Note that if G is a partial cube, then there is a minimal n, called the *isometric dimension*, such that G embeds isometrically into Q_n. Moreover the isometric embedding of G is unique up to automorphisms of Q_n, see e.g. [70, Chapter 5]. Note that while for instance the path of length 3 is a non-isometric but induced subgraph of Q_3, it has isometric dimension 4. Figure 13 displays a partial cube and an induced subgraph of Q_4, that is not a partial cube.

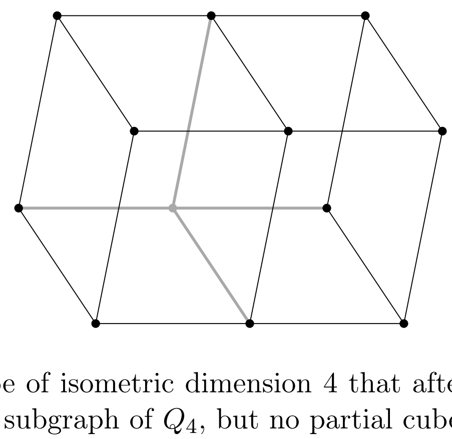


Figure 13: A partial cube of isometric dimension 4 that after removing the Grey elements is still an induced subgraph of Q_4, but no partial cube anymore.

Let us continue by giving an alternative way of characterizing partial cubes. Any isometric embedding of a partial cube into a hypercube leads to the same partition of edges into so-called Θ-classes, where two edges are equivalent, if they correspond to a change in the same coordinate of the hypercube. This can be shown using the Djoković-Winkler-relation Θ which is defined in the graph without reference to an embedding, see [62, 174]. We will describe next, how the relation Θ can be defined independently of an embedding.

A subgraph G' of G is *convex* if for all pairs of vertices in G' all their shortest paths in G stay in G'. For an edge $a = uv$ of G, define the sets $W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$. By a theorem of Djoković [62], a graph G is a partial cube if and only if G is bipartite and for any edge $a = uv$ the sets $W(u, v)$ and $W(v, u)$ are convex. In this case, setting $a\Theta a'$ for $a = uv$ and $a' = u'v'$ if $u' \in W(u, v)$ and $v' \in W(v, u)$ yields Θ.

We index the set of equivalence classes of Θ by a set $\mathcal{E}$. For $f \in E$ we denote the equivalence class by E_f.For an arbitrary (oriented) edge $uv \in E_f$, let $E_f^- := W(u, v)$ and $E_f^+ := W(v, u)$ the pair of complementary convex halfspaces of G. Now, identifying any vertex v of G with $v \in Q_E = \{+, -\}^E$ which for any class of Θ associates the sign of the halfspace containing v gives an isometric embedding of G into Q_E.

4.1.1 Pc-minors, isometric expansions, and zone graphs

We will now introduce the notions of pc-minors (i.e., contraction and restriction), which are methods to obtain smaller partial cubes from bigger ones, as well as, expansions, which are inverses of contractions. We also study zone graphs of partial cubes which are smaller graphs that are not always partial cubes themselves. An important observation in this section is a polynomial time algorithm for checking for a given partial cube minor (Proposition 4.5).

Restrictions Given $f \in E$, an *(elementary) restriction* consists in taking one of the subgraphs $G[E_f^-]$ or $G[E_f^+]$ induced by the complementary halfspaces E_f^- and E_f^+, which we will denote by $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$, respectively. These graphs are isometric subgraphs of the hypercube $Q_{E\setminus\{f\}}$. Now applying two elementary restriction with respect to different coordinates f, g, independently of the order of f and g, we will obtain one of the four (possibly empty) subgraphs induced by $E_f^- \cap E_g^-$, $E_f^- \cap E_g^+$, $E_f^+ \cap E_g^-$, and $E_f^+ \cap E_g^+$. Since the intersection of convex subsets is convex, each of these four sets is convex in G and consequently induces an isometric subgraph of the hypercube $Q_{E\setminus\{f,g\}}$. More generally, a *restriction* is a subgraph of G induced by the intersection of a set of (non-complementary) halfspaces of G. See Figures 14, 17, and 20 for examples of restrictions. We denote restrictions by $\rho_X(G)$, where $X \in \{+, -\}^E$ is a signed set of halfspaces of G. For subset S of the vertices of G and $f \in E$, we denote $\rho_{f^+}(S) := \rho_{f^+}(G) \cap S$ and $\rho_{f^-}(S) := \rho_{f^-}(G) \cap S$, respectively. We will say that E_f *crosses* a subset of vertices S of G if $\rho_{f^+}(S) \neq \emptyset$ and $\rho_{f^-}(S) \neq \emptyset$.

The smallest convex subgraph of G containing V' is called the *convex hull* of V' and denoted by $\text{conv}(V')$. The following is well-known:

Lemma 4.1 ([5, 10, 39]). *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. Indeed, for any subset of vertices V' we have that* $\text{conv}(V')$ *is the intersection of all halfspaces containing V'. In particular, the class of partial cubes is closed under taking restrictions.*

Note that this description was used in [5] to characterize the lattices of convex subgraphs of partial cubes, also see Figure 14

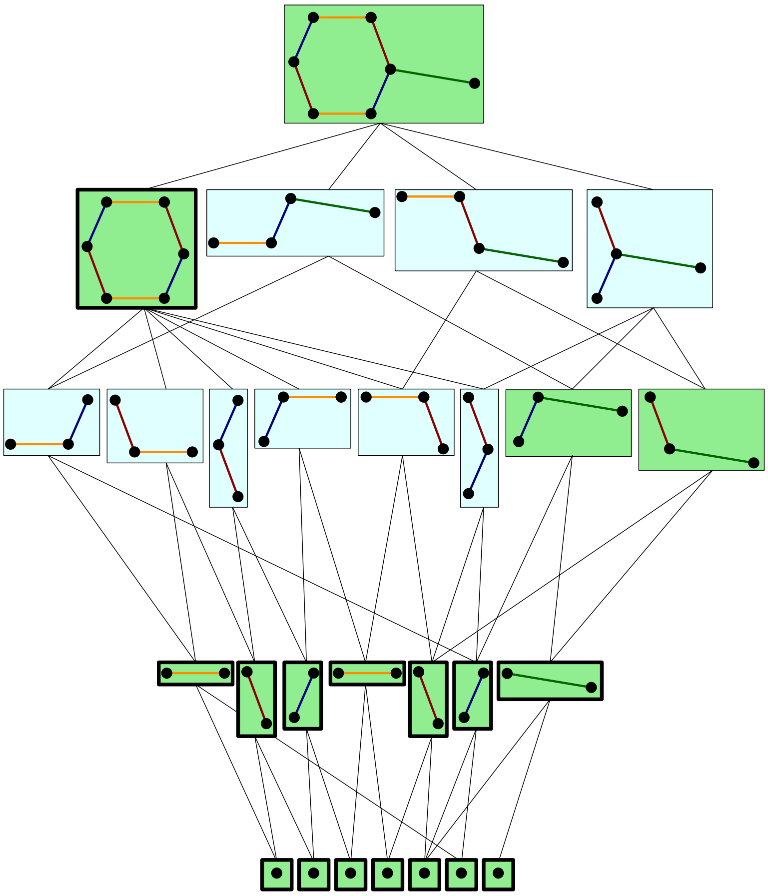


Figure 14: The convex subgraphs of a partial cube ordered by inclusion. Green background means gated, thick outline means antipodal.

Lemma 4.2. *In tope graphs of COMs elementary restrictions correspond to taking halfplanes and convex subgraphs correspond to fibers.*

Contractions For $f \in E$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an *(elementary) contraction* of G. For a vertex v of G, we will denote by $\pi_f(v)$ the image of v under the contraction in G/E_f, i.e. if uv is an edge of E_f, then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subseteq V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular we denote the *contraction* of G by $\pi_f(G)$. See Figures 15, 17, and 20 for examples of contractions.

It is well-known and in particular follows from the proof of the first part of [45, Theorem 3] that $\pi_f(G)$ is an isometric subgraph of $Q_{E\setminus\{f\}}$. Since edge contractions in graphs commute, i.e. the resulting graph does not depend on the order in which a set of edges is contracted, we have:

Lemma 4.3. *The class of partial cubes is closed under contractions. Moreover, contractions commute in partial cubes, i.e. if $f, g \in E$ and $f \neq g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$.*

Consequently, for a set $A \subseteq E$, we denote by $\pi_A(G)$ the isometric subgraph of $Q(E \setminus A)$ obtained from G by contracting the classes $A \subseteq E$ in G. See Figure 15 for examples.

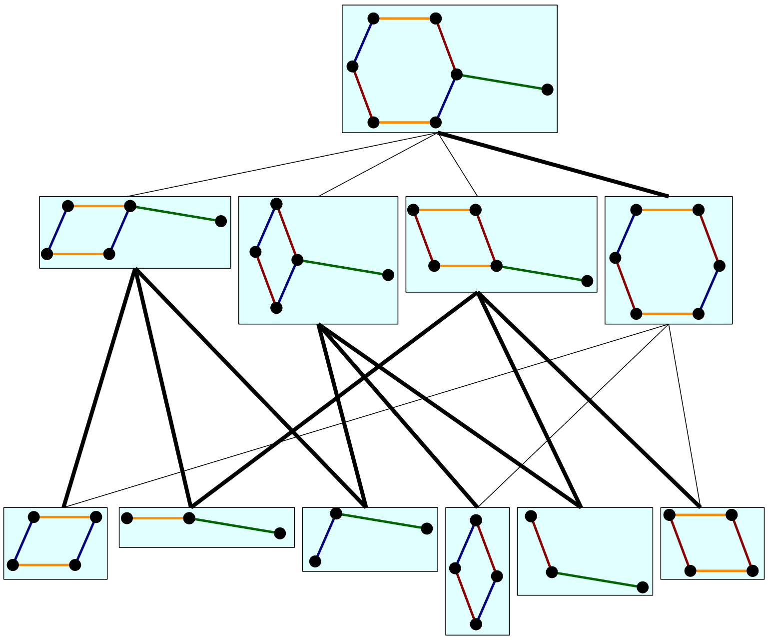


Figure 15: Some contractions of a partial cube. Thick edges mean peripherality.

The following can easily be derived from the definitions, see e.g. [42]:

Lemma 4.4. *Contractions and restrictions commute in partial cubes, i.e. if $f, g \in E$ and $f \neq g$, then $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$.*

The previous lemmas show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and G' is called a *pc-minor* of G. We will have a particular interest in classes of partial cubes that are closed under taking pc-minors. Clearly, any such class has a (possibly infinite) set X of minimal excluded pc-minors. We denote by $\mathcal{F}(X)$ the pc-minor closed class of partial cubes excluding X. It is easy and efficient to search for a given pc-minor G' in G. The algorithm is as follows:

Denote by n' and n the number of vertices of G' and G, respectively, and with k' and k the number of Θ-classes in G' and G.

For every subset V' of at most n' vertices of G do the following: First compute $\text{conv}(V')$ and count the number of Θ-classes of G crossing it, say it equals to k''. Then $k'' \leq k$, and if $k'' < k'$ discard the subgraph. On the other hand, if $k'' \geq k'$, then for every subset S of size $k'' - k'$ of the Θ-classes crossing $\text{conv}(V')$, contract in $\text{conv}(V')$ all the Θ-classes of S. Finally, check if the resulting graph is isomorphic to G'.

Correctness and runtime are easily analyzed and yield:

Proposition 4.5 ([109, Proposition 2.4])**.** *Let X be a finite set of partial cubes. It is decidable in polynomial time if a partial cube G is in $\mathcal{F}(X)$.*

Lemma 4.6. *In tope graphs of COMs (elementary) contractions correspond to (single element) deletions.*

So, a tope graph is a pc-minor of another one if its COM can be obtained by deletions from a fiber of the others COM.

Expansions The inverse operation of contraction is expansion: a partial cube G is an *expansion* of a partial cube G' if $G' = \pi_f(G)$ for some Θ-class f of G. Indeed expansions can be detected within the smaller graph. Let G' be a partial cube containing two isometric subgraphs G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, there are no edges from $V(G'_1 \setminus G'_2)$ to $V(G'_2 \setminus G'_1)$, and denote by $G'_0 := G[V(G'_1) \cap V(G'_2)]$ the subgraph induced by the vertices that are in both G'_1 and G'_2. A graph G is an expansion of G' with respect to G_0 if G is obtained from G' by replacing each vertex v of G'_1 by a vertex v_1 and each vertex v of G'_2 by a vertex v_2 such that u_i and v_i, $i = 1, 2$ are adjacent in G if and only if u and v are adjacent vertices of G'_i, and v_1v_2 is an edge of G if and only if v is a vertex of G'_0. The following is well-known:

Lemma 4.7 ([39,45])**.** *A graph G is a partial cube if and only if G can be obtained by a sequence of expansions from a single vertex.*

We will make use of the following lemma about the interplay of contractions and expansions:

Lemma 4.8. *Assume that we have the following commutative diagram of contractions:*

$$\begin{array}{ccc} G & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(G) \\ \Big\downarrow{\scriptstyle \pi_{f_2}} & & \Big\downarrow{\scriptstyle \pi_{f_2}} \\ \pi_{f_2}(G) & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(\pi_{f_2}(G)) \end{array}$$

If G is expanded from $\pi_{f_1}(G)$ along sets $G_1, G_2 \subseteq \pi_{f_1}(G)$, then $\pi_{f_2}(G)$ is expanded from $\pi_{f_1}(\pi_{f_2}(G))$ along sets $\pi_{f_2}(G_1)$ and $\pi_{f_2}(G_2)$.

Let G be a partial cube and $f \in \mathcal{E}$ indexing one of its Θ-classes E_f. Assume that a halfspace E_f^+ (or E_f^-) is such that all its vertices are incident with edges from E_f. Then we call E_f^+ (or E_f^-) *peripheral*. In such a case we will also call E_f a peripheral Θ-class, and call G a *peripheral expansion* of $\pi_f(G)$. Note that an expansion along sets G_1, G_2 is peripheral if and only if one of the sets G_1, G_2 is the whole graph and the other one an isometric subgraph. An expansion is called *full* if $G_1 = G_2$. Note that in this case, the expanded graph is isomorphic to $G_1 \square K_2$. See Figure 15 for examples of peripheral and non-peripheral expansions.

Lemma 4.9. *In tope graphs of COMs expansions correspond to* single element extensions *in the COM.*

Zone graphs For a partial cube G and $f \in E$ the *zone graph* of G with respect to f is the graph $\zeta_f(G)$ whose vertices correspond to the edges of E_f and two vertices are connected by an edge if the corresponding edges of E_f lie in a convex cycle of G, see [104]. Here, a *convex cycle* is just a convex subgraph that is a cycle. In particular, ζ_f can be seen as a mapping from edges of G that are not in E_f but lie on a convex cycle crossed by E_f to the edges of $\zeta_f(G)$. If $\zeta_f(G)$ is a partial cube, then we say that $\zeta_f(G)$ is *well-embedded* if for two edges a, b of $\zeta_f(G)$ we have $a\Theta b$ if and only if the sets of Θ-classes crossing $\zeta_f^{-1}(a)$ and $\zeta_f^{-1}(b)$ coincide and otherwise they are disjoint. As an example, note that all zone graphs of the graph in Figure 4 are well-embedded paths, while all zone graphs of the graph on top in Figure 17 are triangles. For yet another

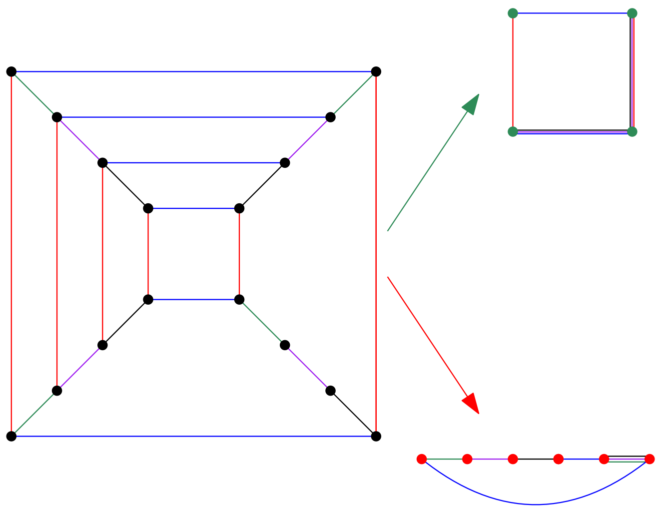


Figure 16: A partial cube all of whose zone graphs are either 4-cycles or 6-cycles, but none of them is well-embedded.

example, see Figure 16. A consequence of Corollary 4.24 will be that out of these three examples only the first one is the tope graph of a COM.

For discussing zone graphs in partial cubes the following will be useful. Let $v_1u_1, v_2u_2 \in E_e$ be edges in a partial cube G with $v_1, v_2 \in E_e^+$. Let $C_1, \ldots, C_n$, $n \geq 1$, be a sequence of convex cycles such that v_1u_1 lies only on C_1, v_2u_2 lies only on C_n, and each pair C_i and C_{i+1}, for $i \in \{1, \ldots, n-1\}$, intersects in exactly one edge and this edge is in E_e, all the other pairs do not intersect. If the shortest path from v_1 to v_2 on the union of $C_1, \ldots, C_n$ is a shortest v_1, v_2-path in G, then we call $C_1, \ldots, C_n$ a *convex traverse* from v_1u_1 to v_2u_2. In [125] it was shown that for every pair of edges v_1u_1, v_2u_2 in relation Θ there exists a convex traverse connecting them.

Lemma 4.10. *Let G be a partial cube and $f \in E$. Then $\zeta_f(G)$ is a well-embedded partial cube if and only if for any two convex cycles C, C' that are crossed by E_f and some E_g both C and C' are crossed by the same set of Θ-classes.*

A well-embedded zone graph $\zeta_f(G)$ thus induces an equivalence relation on the Θ-classes of G except f, that are involved in convex cycles crossed by E_f. Also taking zone graphs harmonizes with contractions and restrictions, see [109] for the details.

Lemma 4.11. *In tope graphs of COMs taking zone graphs corresponds to contraction (and possibly simplification) in the COM.*

4.1.2 Metric subgraphs

In this section we present conditions under which contractions, restrictions, and expansions preserve metric properties of subgraphs, such as convexity, gatedness, antipodality, and affinity. An important result of the section is an intrinsic characterization of affine partial cubes (Proposition 4.17).

Convex subgraphs Let $G = (V, E)$ be an isometric subgraph of the hypercube Q_E and let S be a subset of vertices of G. Let f be any coordinate of E. We will say that E_f is *disjoint* from S if it does not cross S and has no vertices in S. The following three lemmas describe the behavior of convex subgraphs under contractions, restrictions, and expansions. Their (short) proofs can be found in [42].

Lemma 4.12. *If H is a convex subgraph of G and $f \in E$, then $\rho_{f^+}(H)$ is a convex subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H, then also $\pi_f(H)$ is a convex subgraph of $\pi_f(G)$.*

Lemma 4.13. *If S is a subset of vertices of G and $f \in E$, then $\pi_f(\mathrm{conv}(S)) \subseteq \mathrm{conv}(\pi_f(S))$. If E_f crosses S, then $\pi_f(\mathrm{conv}(S)) = \mathrm{conv}(\pi_f(S))$.*

Lemma 4.14. *If H' is a convex subgraph of G' and G is obtained from G' by an isometric expansion, then the expansion of H of H' is a convex subgraph of G.*

Antipodal subgraphs Let H be a subgraph of G. If for a vertex $x \in H$ there is a vertex $-_H x \in H$ such that $\mathrm{conv}(x, -_H x) = H$ we say that $-_H x$ is the *antipode* of x with respect to H and we omit the subscript H if this causes no confusion. Intervals in a partial cube are convex since intervals in hypercubes equal (convex) subhypercubes, therefore $\mathrm{conv}(x, -_H x)$ consists of all the vertices on the shortest paths connecting x and $-_H x$. Hence, it is easy to see, that if a vertex has an antipode, then this antipode is unique. We call a subgraph H of a partial cube $G = (V, E)$ *antipodal* if every vertex x of H has an antipode with respect to H. Note that antipodal graphs are sometimes defined in a different but equivalent way and then are called symmetric-even, see [19]. By definition, antipodal subgraphs are convex. See Figure 14 for examples of antipodal and non-antipodal subgraphs. Their behavior with respect to pc-minors has been described in [42] in the following way:

Lemma 4.15. *Let H be an antipodal subgraph of G and $f \in E$. If E_f is disjoint from H, then $\rho_{f^+}(H)$ is an antipodal subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H, then $\pi_f(H)$ is an antipodal subgraph of $\pi_f(G)$.*

In particular, Lemma 4.15 implies that the class of antipodal partial cubes is closed under contractions. Next we will deduce a characterization of those expansions that generate all antipodal partial cubes from a single vertex, in the same way as a fundamental result of Chepoi [45] characterizes all partial cubes. Let G be an antipodal partial cube and G_1, G_2 two subgraphs corresponding to an isometric expansion. We say that it is an *antipodal expansion* if and only if $-G_1 = G_2$, where $-G_1$ is defined as the set of antipodes of G_1.

Lemma 4.16. *Let G be a partial cube and $\pi_e(G)$ antipodal. Then G is an antipodal expansion of $\pi_e(G)$ if and only if G is antipodal. In particular, all antipodal partial cubes arise from a single vertex by a sequence of antipodal expansion.*

Affine subgraphs We call a partial cube *affine* if it is a halfspace of an antipodal partial cube. All graphs except the one on the top and the $K_{1,3}$ in Figure 14 are affine. We can give the following intrinsic characterization of affine partial cubes, that will play a crucial role in our characterization of tope graphs of AOMs, see Corollary 4.26.

Proposition 4.17. *A partial cube G is affine if and only if for all u, v vertices of G there are $w, -w$ in G such that $\mathrm{conv}(u, w)$ and $\mathrm{conv}(v, -w)$ are crossed by disjoint sets of Θ-classes.*

By Lemma 4.4 a contraction of a halfspace is a halfspace and by Lemma 4.15 antipodal partial cubes are closed under contraction, therefore we immediately get:

Lemma 4.18. *The class of affine partial cubes is closed under contraction.*

Gated subgraphs A subgraph H of G, or just a set of vertices of H, is called *gated* (in G) if for every vertex x outside H there exists a vertex x' in H, the *gate* of x, such that each vertex y of H is connected with x by a shortest path passing through the gate x'. It is easy to see that if x has a gate in H, then it is unique and that gated subgraphs are convex. See [67] for several results on gated sets in metric spaces. See Figure 14 for examples of gated and non-gated subgraphs.

In [42] it was shown that gated subgraphs behave well with respect to pc-minors:

Lemma 4.19. *If H is a gated subgraph of G, then $\rho_{f^+}(H)$ and $\pi_f(H)$ are gated subgraphs of $\rho_{f^+}(G)$ and $\pi_f(G)$, respectively.*

4.2 Characterizations

The main theorem of the section is Theorem 4.23, saying that a graph G is the tope graph of a COM, i.e., $G \in \mathcal{G}_{\text{COM}}$, if and only if G is antipodally gated, i.e., $G \in \text{AG}$, if and only if G does not contain a partial cube minor from a specific set $\mathcal{Q}^-$, i.e., $G \in \mathcal{F}(\mathcal{Q}^-)$. Let us here give an outline of its proof ingredients and its corollaries. The general strategy will be to prove first $\mathcal{G}_{\text{COM}} \subseteq \text{AG}$, then $\mathcal{F}(\mathcal{Q}^-) \subseteq \mathcal{G}_{\text{COM}}$, and finally $\text{AG} \subseteq \mathcal{F}(\mathcal{Q}^-)$. Moreover, we denote $\mathcal{G}_{\text{AOM}}$, $\mathcal{G}_{\text{LOP}}$, and $\mathcal{G}_{\text{OM}}$ the respective classes of tope graphs.

As mentioned earlier, the *tope graph* of a system of sign vectors is the graph induced by its topes, i.e., covectors without zero entries, in the hypercube $\{+,-\}^E$. In case of COMs the tope graph (without labeling of the vertices) determines the COM up to isomorphism, see [16]. Moreover, the tope graph of a COM is a partial cube, see [16]. One way of thinking of being a partial cube is that the edges receive colors corresponding to the dimensions of the hypercube, see Figure 4.

Of particular importance to us are two types of metric subgraphs. An *antipodal* subgraph H of G has the property, that for each vertex v in H there is an antipode $-_H v$, such that the H is smallest convex subgraph of G containing v and $-_H v$. The antipodal subgraphs of the graph in Figure 4 are exactly the vertices, edges, and bounded faces. The second property of a subgraph H is the one of being *gated*. This means, that for every vertex $v \in G$, there is a gate v' in H, such that for every $v'' \in H$ there is a shortest path from v through v' to v''. In partial cubes this amounts to the fact that there is path from v to H that does not use any color that is present on the edges of H. We say that a graph is *antipodally gated* if all of its antipodal subgraphs are gated. The graph in Figure 4 is antipodally gated, but also for instance the subgraph induced by the vertices incident to red or green edges is gated. A non-gated subgraph is given by the path P of length two induced by the three left-most vertices. A vertex that has no gate in P is the degree four vertex v, since all paths from v to P use a color present in P.

Exploring correspondences between axiomatical behavior of sign-vectors and metric subgraphs of partial cubes, we show that tope graphs of COMs are antipodally gated. This is the first part of the proof of Theorem 4.23, i.e., $\mathcal{G}_{\text{COM}} \subseteq \text{AG}$. Much of this least technical part of the proof consists in finding the dictionary between systems of sign-vectors and partial cubes, see Table 1.

Theorem 4.20. *If A is an antipodal subgraph of a tope graph of a COM, then A is gated.*

Clearly, there are partial cubes that do not satisfy Theorem 4.20, i.e., they contain a non-gated antipodal subgraph. Figure 17 shows a partial cube in which the bottom C_6 is an antipodal subgraph, that is not gated, because the red vertex does not have gate. Note that the graph is a full subdivision of K_4.

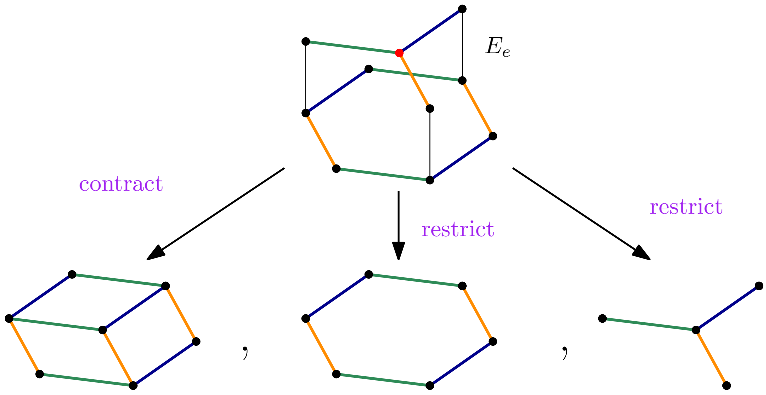


Figure 17: A non antipodally gated partial cube and the partial cube minors obtained by contracting or restricting with respect to the vertical edge class E_e.

Instead of studying all the partial cubes, that have a non-gated antipodal subgraph, we use the notion of partial cube minors, in order to classify only minimal such partial cubes with respect to this operation. Recall that a partial cube minor is either a contraction of a color class or the restriction to one of its sides. Hence, it is a specialization of the standard graph minor notion. The graph in Figure 17 is pc-minor minimal with respect to containing a non-gated antipodal subgraph. The other such partial cubes of isometric dimension at most 4 are shown in Figure 18.

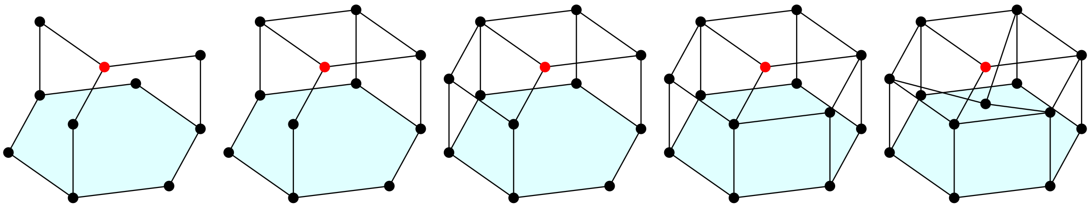


Figure 18: The excluded pc-minors of isometric dimension ≤ 4 for COMs.

The next and second step of our proof is providing a set $\mathcal{Q}^-$ of partial cubes that are minor-minimal with respect to having a non-gated antipodal subgraph. The graph of Figure 17 is the smallest element of $\mathcal{Q}^-$, more of these graphs are depicted in Figure 18 and an idea for a general construction can be seen in Figure 19. Here the light blue Q_d with two antipodal vertices removed (also known as Q_d^{--}) is the antipodal subgraph in which the red vertex has no gate.

It is easy to check that the minors in Figure 17 are antipodally gated, i.e., the graph on top is minimally non antipodally gated. The same holds for the graphs from the general construction of $\mathcal{Q}^-$, see Figure 19.

The class of tope graphs of COMs is closed under pc-minors. This is illustrated by our realizable example in Figure 20 and follows from the correspondence between

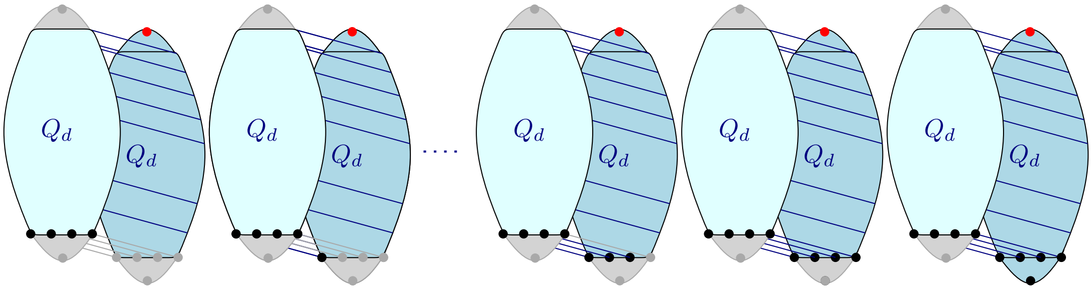


Figure 19: The elements of $\mathcal{Q}^-$ of isometric dimension $d+1$.

pc-minors and COM minors (Lemmas 4.2 and 4.6) and the fact that COMs are closed under COM-minors (Lemma 3.2). We show that, if G is not the tope graph of a COM, then it must have a partial cube minor from $\mathcal{Q}^-$. This concludes the second part of our proof since it means, that if G excludes $\mathcal{Q}^-$, then it is the tope graph of a COM, i.e., $\mathcal{F}(\mathcal{Q}^-) \subseteq \mathcal{G}_{\text{COM}}$.

Theorem 4.21. *If G has no pc-minor from $\mathcal{Q}^-$, then G is the tope graph of a COM.*

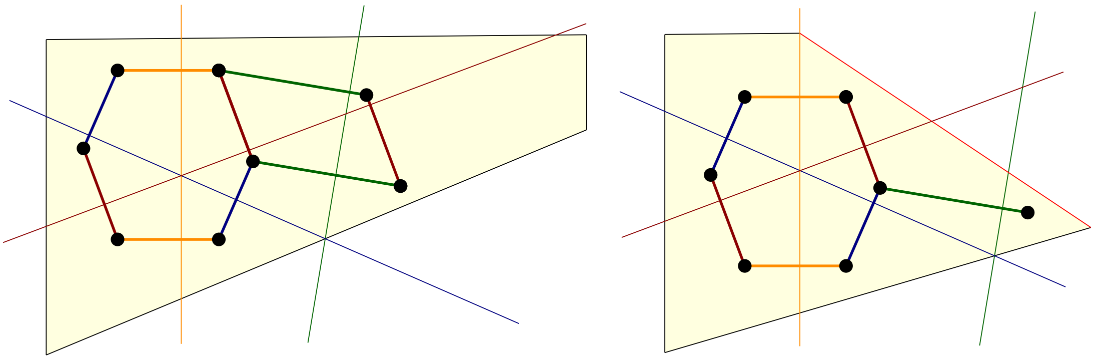


Figure 20: Two partial cube minors obtained from the COM of Figure 4 by contracting and restricting with respect to the red color class.

The last and most technical part of our proof is to show that the class of antipodally gated partial cube does not contain a member of $\mathcal{Q}^-$ as partial cube minor. The class $\mathcal{F}(\mathcal{Q}^-)$ is closed under partial cube minors and graphs in $\mathcal{Q}^-$ are minor-minimal having non-gated antipodal subgraphs. We show that the class of antipodally gated partial cubes is closed under partial cube minors. This implies, that if G has a partial cube minor from $\mathcal{Q}^-$, then it cannot be antipodally gated, i.e., $\text{AG} \subseteq \mathcal{F}(\mathcal{Q}^-)$.

Theorem 4.22. *If all antipodal subgraph of G are gated, then G dos not have a minor from $\mathcal{Q}^-$.*

This concludes the last part of the circular proof of our main theorem, which we will sketch quickly again.

Theorem 4.23. *For a graph G the following conditions are equivalent:*

(i) G is the tope graph of a COM, i.e., $G \in \mathcal{G}_{\text{COM}}$,

(ii) G is an antipodally gated partial cube, i.e., $G \in \text{AG}$,

(iii) *G is a partial cube with no partial cube minor from* $\mathcal{Q}^-$*, i.e.,* $G \in \mathcal{F}(\mathcal{Q}^-)$.

Proof. The implication (i)⇒(ii) is Theorem 4.20. The implication (iii)⇒(i) is Theorem 4.21. Finally, (ii)⇒(iii) follows from the fact that that all the graphs in $\mathcal{Q}^-$ have a non-gated antipodal subgraph and that the class AG is pc-minor closed, see Theorem 4.22. □

We proceed by describing the main implications of the above theorem, i.e., we show a further characterization in terms of zone graphs, give resulting characterizations of tope graphs of OMs, LOPs, and AOMs (of bounded rank) and the polynomial time recognition algorithm. Further applications will appear in Section 5.

4.3 Further characterizations and recognition

In this section we will describe the most important implications of Theorem 4.23. In particular, we give polynomial time recognition algorithm for tope graphs of COMs, specialize our results to tope graphs of OMs, AOMs, and LOPs (of bounded rank).

But first of all, Theorem 4.23 can be used to obtain a characterization of $\mathcal{G}_{\mathrm{COM}}$ in terms of zone graphs that is a generalization of a result of Handa [92].

Corollary 4.24. *A graph G is the tope graph of a COM, i.e.* $G \in \mathcal{G}_{\mathrm{COM}}$*, if and only if G is a partial cube such that all iterated zone graphs are well-embedded partial cubes.*

Theorem 4.23 and Corollary 4.24 specialize to other systems of sign-vectors. Using that OMs are exactly the symmetric COMs we immediately get the following corollary. Recall that for a set X of partial cubes we denote by $\mathcal{F}(X)$ the class of partial cubes that do not have any graph from X as a partial cube minor.

Corollary 4.25. *For a graph G the following conditions are equivalent:*

(i) *G is the tope graph of an OM, i.e.,* $G \in \mathcal{G}_{\mathrm{OM}}$*,*

(ii) *G is an antipodal partial cube and all its antipodal subgraphs are gated,*

(iii) *G is in* $\mathcal{F}(\mathcal{Q}^-)$ *and antipodal,*

(iv) *G is an antipodal partial cube and all its iterated zone graphs are well-embedded partial cubes.*

Note that the equivalence (i)⇔(ii) corresponds to a characterization of tope sets of OMs due to da Silva [50] and (i)⇔(vi) corresponds to a characterization of tope sets of Handa [93].

Let us call an affine subgraph G' of an affine partial cube G *conformal* if for all $v \in G'$ we have $-_{G'}v \in G' \Leftrightarrow -_G v \in G$. We give an intrinsic characterization of $\mathcal{G}_{\mathrm{AOM}}$:

Corollary 4.26. *For a graph G the following conditions are equivalent:*

(i) *G is the tope graph of an AOM, i.e.,* $G \in \mathcal{G}_{\mathrm{AOM}}$*,*

(ii) *G is an affine partial cube and all its antipodal and conformal subgraphs are gated,*

(iii) *G is in* $\mathcal{F}(\mathcal{Q}^-)$ *(or* $G \in \mathcal{G}_{\mathrm{COM}}$*), affine, and all its conformal subgraphs are gated.*

For the next statement denote $\mathcal{Q}^{--} := \{Q_n^{--} \mid n \geq 3\}$.

Corollary 4.27. *For a graph G the following conditions are equivalent:*

(i) G *is the tope graph of a LOP, i.e.,* $G \in \mathcal{G}_{\mathrm{LOP}}$,

(ii) G *is a partial cube and all its antipodal subgraphs are hypercubes,*

(iii) G *is in* $\mathcal{F}(\mathcal{Q}^{--})$.

For lopsided sets the (i)⇔(ii) part of the corollary corresponds to a characterization due to Lawrence [120].

Recall that the *rank* of a COM is the dimension of a maximal hypercube to which its tope graph can be contracted or in other words the VC-dimension of its tope set, when considered as set system, see [43, 44]. Considering COMs of bounded rank, we can reduce the set of excluded pc-minors to a finite list. For any $r \geq 3$ define the following finite sets

$$\mathcal{Q}_r^- := \{Q_n^{-*}, Q_n^{--}(m), Q_{r+2}^{--}(r+2), Q_{r+1} \mid 4 \leq n \leq r+1; 1 \leq m \leq n\} \subset \mathcal{Q}^- \cup \{Q_{r+1}\},$$

and

$$\mathcal{Q}_r^{--} := \{Q_n^{--}, Q_{r+1} \mid 3 \leq n \leq r+1\} \subset \mathcal{Q}^{--} \cup \{Q_{r+1}\}.$$

Corollary 4.28. *For a graph G and an integer $r \geq 3$ we have:*

- $G \in \mathcal{G}_{\mathrm{COM}}$ *of rank at most* $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$.
- $G \in \mathcal{G}_{\mathrm{OM}}$ *of rank at most* $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$ *and G is antipodal.*
- $G \in \mathcal{G}_{\mathrm{AOM}}$ *of rank at most* $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^-)$, *$G$ is affine and all its conformal subgraphs are gated.*
- $G \in \mathcal{G}_{\mathrm{LOP}}$ *of rank at most* $r \Leftrightarrow G \in \mathcal{F}(\mathcal{Q}_r^{--})$.

Note that using Proposition 4.5 can easily be seen to yield a polynomial time recognition algorithm for the recognition of the bounded rank classes above. However, Theorem 4.23 also yields polynomial time recognition algorithms for the unrestricted classes. Since partial cubes can be recognized efficiently [68], it basically suffices to efficiently check for all antipodal subgraphs if they are gated, yielding:

Corollary 4.29. *The recognition of the classes* $\mathcal{G}_{\mathrm{COM}}, \mathcal{G}_{\mathrm{AOM}}, \mathcal{G}_{\mathrm{OM}}, \mathcal{G}_{\mathrm{LOP}}$ *can be done in polynomial time.*

4.4 Dictionary

Let us present a condensed and slightly informal dictionary of notions in partial cubes and systems of sign-vectors. See Table 1.

tope graph G	sign-vectors $\mathcal{L}$	reference
isometric dimension	order	(by definition)
$\max\{d \mid G \in \mathcal{F}(Q_{d+1})\}$	rk($\mathcal{L}$) COM/VC-dim($\mathcal{T}$)	[43, Lemma 13]
cube pc-minors	independent sets	(follows from above)
Q_n^{--} pc-minors	circuits	(follows from above)
contraction $\pi_e(G)$	deletion $\mathcal{L} \setminus e$	Lemma 4.6
restriction $\rho_{e^+}(G)$	halfplane $\mathcal{L}_e^+$	Lemma 4.2
zone graph $\zeta_e(G)$	simplified contraction $\mathcal{L}/e$	[109, Lemma 4.3]
convex subgraph X	sign-vector	[109, Proposition 4.5]
isometric expansion	single-element extension	Lemma 4.9
antipodal	symmetric (Sym)	[109, Lemma 4.7]
gate	composition (C)	[109, Lemma 4.6]
antipodal subgraphs gated	$\mathcal{L}$ COM	Theorem 4.23
antipodal subgraphs cubes	$\mathcal{L}$ LOP	Corollary 4.27
antipodal COM	$\mathcal{L}$ OM	Corollary 4.25
halfspace of OM	$\mathcal{L}$ AOM	Corollary 4.26
antipodal subgraphs	covectors	(follows from above)
maximal antipodal subgraphs	cocircuits	(follows from above)
isometric subgraph	weak map	[22, Prop 7.7.5, Cor 7.7.9]

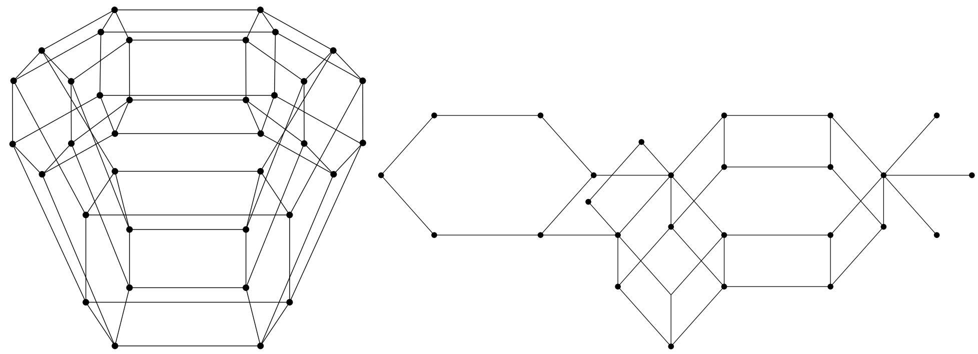
Table 1: A dictionary of notions in sign-vectors and partial cubes.

5 Metric graph classes of COMs

In the same way as COMs, many other important classes of partial cubes are pc-minor closed and can be characterized via (small lists of) forbidden *pc-minors*, see Table 3. In this section we concentrate on results on such classes that are at the same time COMs. The main focus will be on hypercellular graphs. Apart from being minor-closed, hypercellular graphs share another structural property with COMs – they have a cell-structure. Let us start to approach them from that direction first.

Every *median graph* can be obtained by gluing in a specific way cubes of different dimensions. In particular, they give rise not only to contractible but also to CAT(0) cube complexes and form a special class of the CAT(0) Coxeter complexes from Section 3.6.1. Similarly, lopsided sets yield contractible cube complexes, while cellular graphs give contractible polygonal complexes whose cells are regular even polygons. Analogously to median graphs, graphs of CAT(0) Coxeter zonotopal complexes can be viewed as partial cubes obtained by gluing zonotopes. COMs can be viewed as a common generalization of all these notions: their tope graphs are the partial cubes obtained by gluing tope graphs of OMs in a lopsided (and thus contractible) fashion, see Section 3.4.

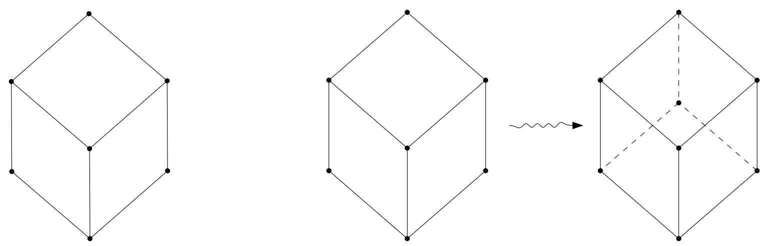
Hypercellular graphs form a subclass of zonotopally realizable COMs, in which all cells are gated subgraphs isomorphic to Cartesian products of edges and even cycles, see Figure 21(a) for such a cell. More precisely, hypercellular partial cubes are those in which all finite convex subgraphs can be obtained from Cartesian products of edges and even cycles by successive gated amalgamations. We show that our graphs share and extend many properties of bipartite cellular graphs of [11]; they can be viewed as high-dimensional analogues of cellular graphs. This is why we call them *hypercellular graphs*, see Figure 21(b) for an example.



(a) (b)

Figure 21: (a) a four-dimensional cell isomorphic to $C_6 \times C_6$. (b) a hypercellular graph with eight maximal cells: C_6, C_4, C_4, $K_2 \times K_2 \times K_2$, $C_6 \times K_2$, and three copies of K_2.

As announced above, there is another way of describing hypercellular graphs in terms of pc-minors. It turns out that the class of hypercellular graphs coincides with the minor-closed class $\mathcal{F}(Q_3^-)$, where Q_3^- denotes the 3-cube minus one vertex, see Figure 22(a).



(a) (b)

Figure 22: (a) Q_3^- – the 3-cube minus one vertex.(b) the 3-cube condition.

In a sense, this is one of the first nontrivial classes one can obtain by forbidding a single pc-minor. Indeed, the class $\mathcal{F}(Q_2)$, is just the class of all trees. $\mathcal{F}(P_3)$ is the class of hypercubes, and $\mathcal{F}(K_2 \Box P_3)$ consists of bipartite cacti [128, page 12]. Other obstructions lead to more interesting classes as implied by our work on $\mathcal{F}(Q_3^-)$ we see that median graphs are $\mathcal{F}(Q_3^-, C_6)$, bipartite cellular graphs are $\mathcal{F}(Q_3^-, Q_3)$, partial cubes which are gated amalgams of even cycles and cubes [145] are $\mathcal{F}(Q_3^-, C_6 \times K_2)$. Furthermore, from Corollary 4.28 we get that rank two COMs are $\mathcal{F}(SK_4, Q_3)$ and rank two LOPs are $\mathcal{F}(C_6, Q_3)$. Here SK_4 denotes the full subdivision of K_4, see the left-most graph in Figure 18. Another class are so-called Pasch graphs. A partial cube is called *Pasch*, their class is denoted by $\mathcal{S}_4$, if any two disjoint convex subgraphs lie in two disjoint halfspaces. They have been characterized in [39, 40] as partial cubes excluding 7 isometric subgraphs of Q_4 as pc-minors, see Figure 23. (Indeed there were some obstructions missing in the original characterization, that we found later in [42].)

In particular, Q_3^- is a pc-minor of all of the graphs from Figure 23. Thus, $\mathcal{F}(Q_3^-) \subseteq$

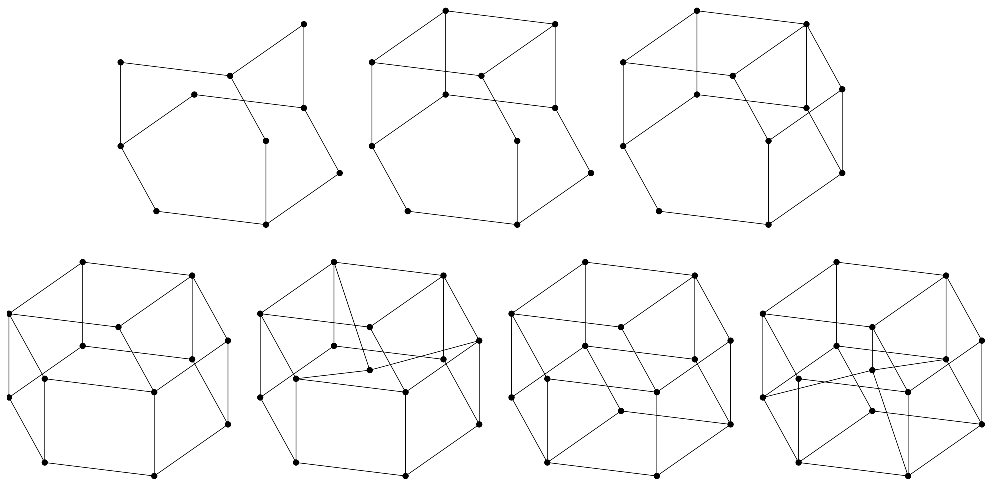


Figure 23: The set of minimal forbidden pc-minors of $\mathcal{S}_4$.

$\mathcal{S}_4$. Similarly, COMs of rank at most 2 are contained in $\mathcal{S}_4$. With the sets of excluded minors we obtain the inclusions claimed in Figure 2.

Combining this with the forbidden minor characterization from Theorem 4.23 we confirm a conjecture from [42]:

Corollary 5.1. *The class $\mathcal{S}_4$ of Pasch graphs are tope graphs of COMs.*

Note that together with a recent paper [146], Corollary 5.1 implies that *netlike* partial cubes are tope graphs of COMs. Moreover, it provides an alternative proof for the fact that hypercellular graphs are tope graphs of COMs, see [42], and therefore also median graphs, and bipartite cellular graphs. However, we will see that hypercellular are even zonotopally realizable later.

The following results mainly concern the cell-structure of graphs from $\mathcal{F}(Q_3^-)$. It is well-known [9] that median graphs are exactly the graphs in which the convex hulls of isometric cycles are hypercubes; these hypercubes are gated subgraphs. Moreover, any finite median graph can be obtained by gated amalgams from cubes [97, 171]. Analogously, it was shown in [11] that any isometric cycle of a bipartite cellular graph is a convex and gated subgraph; moreover, the bipartite cellular graphs are exactly the bipartite graphs which can be obtained by gated amalgams from even cycles. We extend these results in the following way:

Theorem 5.2. *The convex closure of any isometric cycle of a graph $G \in \mathcal{F}(Q_3^-)$ is a gated subgraph isomorphic to a Cartesian product of edges and even cycles. Moreover, the convex closure of any isometric cycle of a graph $G \in \mathcal{S}_4$ is a gated subgraph, which is isomorphic to a Cartesian product of edges and even cycles if it is antipodal.*

In view of Theorem 5.2 we will call a subgraph X of a partial cube G a *cell* if X is a convex subgraph of G which is a Cartesian product of edges and even cycles. Note that since a Cartesian product of edges and even cycles is the convex hull of an isometric cycle, by Theorem 5.2 the cells of $\mathcal{F}(Q_3^-)$ can be equivalently defined as convex hulls of isometric cycles. Notice also that if we replace each cell X of G by a convex polyhedron $[X]$ which is the Cartesian product of segments and regular polygons (a segment for each edge-factor and a regular polygon for each cyclic factor), then we associate with G a cell complex $\mathbf{X}(G)$.

We will say that a partial cube G satisfies the *3-convex cycles condition* (abbreviated, *3CC-condition*) if for any three convex cycles C_1, C_2, C_3 that intersect in a vertex and pairwise intersect in three different edges the convex hull of $C_1 \cup C_2 \cup C_3$ is a cell; see Figure 24 for an example. Notice that the absence of cycles satisfying the preconditions of the 3CC-condition together with the gatedness of isometric cycles characterizes bipartite cellular graphs [11].

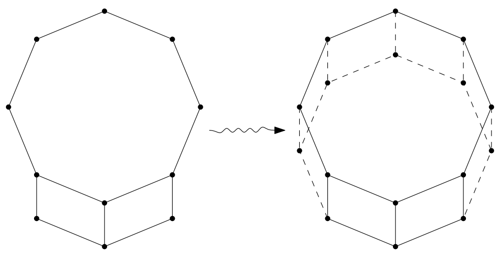


Figure 24: The 3-convex cycles condition.

Defining the dimension of a cell X as the number of edge-factors plus two times the number cyclic factors (which corresponds to the topological dimension of $[X]$) one can give a natural generalization of the 3CC-condition. We say that a partial cube G (or its cell complex $\mathbf{X}(G)$) satisfies the *3-cell condition* (abbreviated, *3C-condition*) if for any three cells X_1, X_2, X_3 of dimension $k+2$ that intersect in a cell of dimension k and pairwise intersect in three different cells of dimension $k+1$ the convex hull of $X_1 \cup X_2 \cup X_3$ is a cell. In case of cubical complexes $\mathbf{X}$, the 3-cell condition coincides with Gromov's flag condition [88] (which can be also called cube condition, see Figure 22(b)), which together with simply connectivity of $\mathbf{X}$ characterize CAT(0) cube complexes. By [41, Theorem 6.1], median graphs are exactly the 1-skeleta of CAT(0) cube complexes (for other generalizations of these two results, see [31, 36]).

The following main characterization of graphs from $\mathcal{F}(Q_3^-)$ establishes those analogies with median and cellular graphs, that lead to the name hypercellular graphs.

Theorem 5.3. *For a partial cube $G = (V, E)$, the following conditions are equivalent:*

(i) $G \in \mathcal{F}(Q_3^-)$, i.e., G is hypercellular;

(ii) any cell of G is gated and G satisfies the 3CC-condition;

(iii) any cell of G is gated and G satisfies the 3C-condition;

(iv) each finite convex subgraph of G can be obtained by gated amalgams from cells.

A further characterization of hypercellular graphs is analogous to median and cellular graphs, see the corresponding properties in Figure 25(a) and 25(b), respectively. We show that hypercellular graphs satisfy the so-called *median-cell property*, which is essentially defined as follows: for any three vertices u, v, w of G there exists a unique gated cell X of G such that if u', v', w' are the gates of u, v, w in X, respectively, then u', v' lie on a common (u, v)-geodesic, v', w' lie on a common (v, w)-geodesic, and w', u' lie on a common (w, u)-geodesic, see Figure 25(c) for an illustration. Namely, we prove:

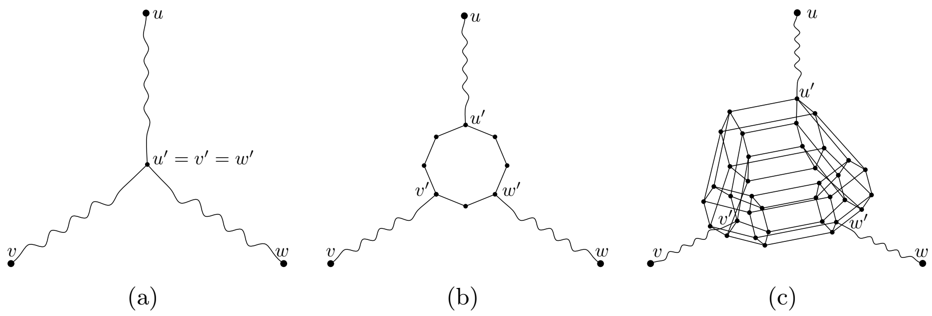


Figure 25: (a) a median-vertex. (b) a median-cycle. (c) a median-cell.

Theorem 5.4. *A partial cube G satisfies the median-cell property if and only if G is hypercellular.*

Theorem 5.3 has several immediate consequences, which we formulate next.

Theorem 5.5. *Let G be a locally finite hypercellular graph. Then $\mathbf{X}(G)$ is a contractible zonotopal complex. Additionally, if G is finite, then G is a tope graph of a zonotopal COM.*

As mentioned above Theorem 5.3 also immediately implies that median graphs and bipartite cellular graphs are hypercellular. Furthermore, a subclass of netlike partial cubes, namely partial cubes which are gated amalgams of even cycles and cubes [145], are hypercellular. In particular, we obtain that these three classes coincide with $\mathcal{F}(Q_3^-, C_6)$, $\mathcal{F}(Q_3^-, Q_3)$, and $\mathcal{F}(Q_3^-, C_6 \times K_2)$, respectively. Other direct consequences of Theorem 5.3 concern convexity invariants (Helly, Caratheodory, Radon, and partition numbers) of hypercellular graphs which are shown to be either a constant or bounded by the topological dimension of $\mathbf{X}(G)$.

6 Corners and simpliciality

The test field that we chose for our approach and the theory of tope graphs are classical problems related to simpliciality in OMs and corners in LOPs. Combining metric graph theory and the theory of oriented matroids we are able to transition and extend results between different classes. More precisely, we can strengthen results of Mandel [124] in OMs in order to finally disprove a conjecture of the latter, that would have implied famous Las Vergnas simplex conjecture [117].

Corners and corner peelings are important for LOPs from the point of view of unlabeled sample compression schemes [37]. We generalize these notions to COMs and obtain generalizations of earlier positive results on realizable LOPs [170], rank 2 LOPs [37], bipartite cellular graphs [11]. Indeed we prove corner peelings for realizable COMs, rank 2 COMs, and hypercellular graphs. A particular consequence of our results is the disproof of a conjecture [16], i.e., we show the existence of a zonotopally realizable COM, that is not realizable.

Recall from Table 1 that the *rank* $\mathrm{rk}(G)$ of a partial cube G is the largest r such that G can be contracted to Q_r. Notice that viewing the vertices of $G \subseteq Q_d$ as a set $\mathcal{S}$ of subsets of $\{1, \ldots, d\}$, $\mathrm{rk}(G)$ coincides with the VC-dimension of $\mathcal{S}$, see [44]. Moreover, the definition of rank in oriented matroid theory is equivalent to this notion of rank when applied to tope graphs, see [43, Lemma 13].

We call a vertex v of an antipodal partial cube G *simplicial* if $\deg(v) = \mathrm{rk}(G)$. In an OM simplicial vertices correspond to simplicial topes via the Topological Representation Theorem. We are ready to formulate one of the two central themes of this section in terms of tope graphs of OMs.

Conjecture 1 (Las Vergnas [117]). *Every OM has a simplicial vertex.*

The conjecture is supported by the fact that it holds for all realizable OMs [167] as well as for all rank 3 OMs, where it can be easily deduced using Euler's Formula, since they are planar [80]. Furthermore the conjecture has been confirmed for uniform OMs on at most 12 elements [26] and very recently for uniform matroid polytopes of rank 4.

The largest class (of unbounded rank) known to satisfy Conjecture 1 was found in [124, Theorem 7]. We call that class *Mandel* here. Realizable OMs and OMs of rank at most 3 are *Euclidean* and the latter are Mandel, but the class is larger. Indeed, Mandel [124, Conjecture 8] conjectured the following as a "wishful thinking statement", since by the above it would imply the conjecture of Las Vergnas:

Conjecture 2 (Mandel [124]). *Every OM is Mandel.*

What is Mandel? Before continuing let us give some definitions in order to explain the notion of Mandel OM. First we need to better explain the structure of antipodal subgraphs of an OM to define the concept of Euclideaness.

Another graph associated to an OM G of rank r is its *cocircuit graph* of G, i.e., the graph G^* whose vertices are the antipodal subgraphs of G of rank $r-1$ and two vertices are adjacent if their intersection in G is an antipodal subgraph of rank $r-2$. We denote the cocircuit graph of G by G^*, since it generalizes planar duality in rank 3, see [80], however in higher rank $(G^*)^*$ is not well-defined, because the cocircuit graph does not uniquely determine the tope graph, see [48]. There has been extensive research on cocircuit graphs [1, 7, 72, 113, 135]. However their characterization and recognition remains open. Cocircuit graphs play a crucial role for the notion of Euclideaness and Mandel as well in the study of corners on COMs.

Consider now a maximal path $A_1, \dots, A_n$ in G^* such that for all $1 < i < n$ the set $A_{i-1} \cap A_i$ is the set of antipodes of $A_i \cap A_{i+1}$ with respect to A_i. It follows from the topological representation of OMs [124], that $A_1, \dots, A_n$ induce a cycle in G^*. Moreover, every $A_{n/2+i}$ is the set of antipodes of A_i with respect to G, for $1 \leq i \leq n/2$ and all intersections $A_i \cap A_{i+1}$ are crossed by the same set F of Θ-classes. Indeed, this cycle can be seen as the line graph of the zone-graph $\zeta_F(G)$. Furthermore, each Θ-class $E_f \notin F$ crosses exactly two pairwise antipodal A_i and $A_{n/2+i}$. The cocircuit graph G^* is the (edge-disjoint) union of such cycles.

Consider a halfspace H of G, i.e. H is an AOM. The induced subsequence $A_k, \dots, A_\ell$ of the above cycle is called a *line* L in H. This name comes from the fact that in the topological representation the sequence corresponds to a pseudo-line, see [124]. Let now $E_e \in \mathcal{E}$ be a Θ-class of H. We say that E_e *crosses* a line L of G, if there exists A_i on L that is crossed by E_e but $A_{i-1} \cap A_i$ or $A_i \cap A_{i+1}$ is not crossed by it. Note that in a line L of an AOM the crossed A_i is unique if it exists. This allows to define *the orientation of L with respect to E_e*: If L is not crossed by E_e we leave its edges undirected. Otherwise, let A_i be the element of L that is crossed by E_e and assume that $A_j \subset E_e^-$ for $j < i$ and $A_j \subset E_e^+$ for $j > i$. We orient all edges of the form A_j, A_{j+1} from A_j to A_{j+1}. This is, the path L is directed from E_i^- to E_i^+.

The edges of the cocircuit graph G^* of an AOM G are partitioned into lines and for every Θ-class E_e we obtain a partial orientation of G^* by orienting every line with respect to E_e. Let us call this mixed graph the orientation of G^* with respect to E_e. Following Mandel [124, Theorem 6], an AOM is *Euclidean* if for every Θ-class E_e the orientation of the cocircuit graph G^* with respect to E_e is *strictly acyclic*, i.e., any directed cycle (following undirected edges or directed edges in the respective orientation) consists of only undirected edges. In other words, any cycle that contains a directed edge contains one into each direction. Euclidean AOMs are important since they allow a generalization of linear programming from realizable AOMs.

Following Fukuda [78], an OM is called *Euclidean* if all of its halfspaces are Euclidean AOMs. Since non-Euclidean AOMs exist, see [78, 124], also non-Euclidean OMs exist. However, there is a larger class of OMs that inherits useful properties of Euclidean AOMs and that was introduced by Mandel [124].

We call an OM *Mandel* if it has an OM-expansion in general position (defined below) such that G_1 and G_2 are Euclidean AOMs. Mandel [124, Theorem 7] proved (and it is up to today the largest class known to have this property) that these OMs satisfy the conjecture of Las Vergnas:

Theorem 6.1 ([124]). *If G is Mandel, then there is a simplicial vertex in G.*

As stated above, Mandel [124, Conjecture 8] conjectured that every OM is Mandel as a "wishful thinking statement", since with Theorem 6.1 it would imply the conjectures of Las Vergnas.

Let us now consider a strengthening of the property claimed by Las Vergnas' conjecture. We say that G is Θ-*Las Vergnas*, if every Θ-class of G contains an edge incident to a simplicial vertex. It is known that in a rank 3 OM every Θ-class is incident to a simplicial vertex [121]. We extend this result to all antipodal partial cubes of rank 3 [108] and more importantly to Mandel OMs. More precisely, we use the ideas from [124] to improve the Theorem 6.1 to showing:

Theorem 6.2. *If G is a Mandel OM, then G is Θ-Las Vergnas.*

On the other hand, uniform OMs of rank 4 violating this property on 21 [154], 17 [26], and 13 [170] elements have been discovered. Figure 26 for the smallest of these examples.

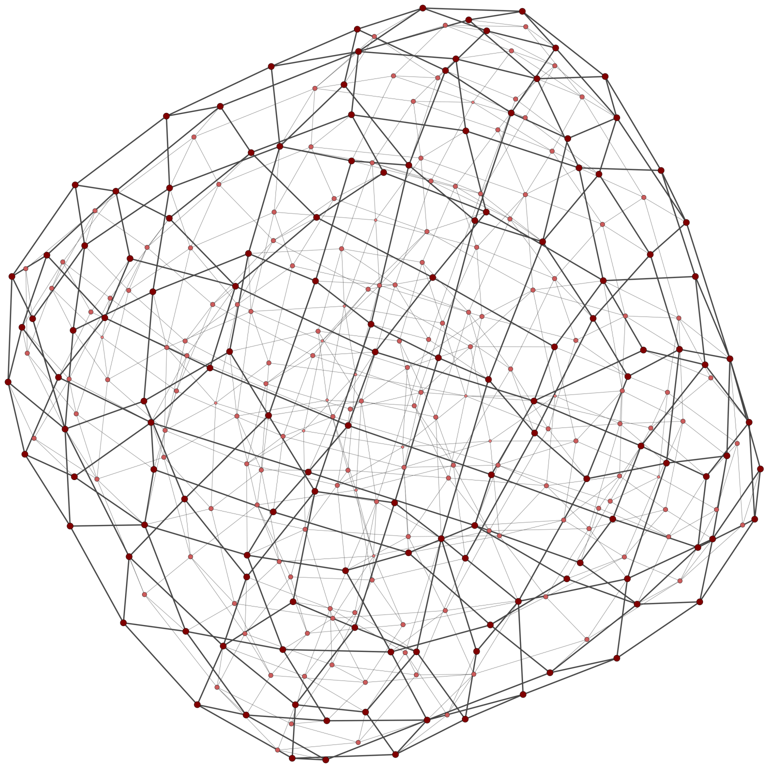


Figure 26: An AOM such that the resulting OM of rank 4 that is not Θ-Las Vergnas. Its boundary are the fat vertices yielding the (planar) tope graph of a rank 3 OM. The example is due to Tracy Hall [170].

Thus, together with Theorem 6.2 this disproves Mandel's conjecture.

Note that we have defined simpliciality only for antipodal partial cubes. In non-antipodal partial cubes the notion of simpliciality is generalized as follows, a vertex $v \in G$ is *simplicial* if it is contained in a unique maximal antipodal subgraph $A \subseteq G$ and $\deg(v) = \mathrm{rk}(A)$. In LOPs simplicial vertices are called *corners*. In particular, if a LOP G is an AOM that embeds into an OM G' along a Θ-class E_e, then G has a corner if and only if G' has a simplicial vertex incident to E_e. Again, by the above examples this proves that there are LOPs without corners, a fact first observed in [37].

6.1 Corners and corner peelings

We propose a common generalization of simpliciality and corners, that furthermore allows us to generalize results of the literature, such as corner peelings for LOPs of rank 2 and realizable LOPs and moreover corner peelings for bipartite cellular graphs. More precisely we show that realizable COMs as well as COMs of rank at most 2 and on the other hand hypercellular graphs all admit corner peelings.

We will approach our general definition of corner of a COM, that generalizes corners on LOPs and simplicial vertices in OMs. The intuitive idea of corner in a COM, is a set of vertices whose removal gives a new (maximal) COM. As a matter of fact it is convenient for us to first define this remaining object and moreover within an OM.

We will say that the subgraph T of an OM H is a *chunk* of H, if H admits an expansion in general position H_1, H_2, such that $T = H_1$. We call the complement $C = H \setminus H_1$ a *corner* of H. In the case that H has rank 1, i.e. H is isomorphic to an edge K_2, then a corner is simply a vertex of H.

This definition extends to COMs by setting C to be a *corner* of a COM G if C is contained in a unique maximal antipodal subgraph H and C is a corner of H.

With these definitions at hand on can prove that chunks and corners as we defined them achieve what we wanted.

Lemma 6.3. *If C is a corner of a COM G, then the chunk $G \setminus C$ is an inclusion maximal proper isometric subgraph of G that is a COM.*

Furthermore, corners in COMs indeed generalize corners of LOPs.

Proposition 6.4. *A subset C of the vertices of a LOP G is a corner if and only if $C = \{v\}$ is contained in a unique maximal cube of G if and only if $C = \{v\}$ such that $G \setminus \{v\}$ is a LOP.*

Note that COMs do not always have corners, e.g., the AOMs mentioned above. However, every OM admits an expansion in general position, see [165, Lemma 1.7] or [22, Proposition 7.2.2]. This yields:

Proposition 6.5. *Every OM has a corner.*

Lemma 6.3 yields the following natural definition. A *corner peeling* in a COM G is an ordered partition $C_1, \ldots, C_k$ of its vertices, such that C_i is a corner in $G - \{C_1, \ldots, C_{i-1}\}$. A subtlety of the above definition is that, when we say that COMs from a certain class have corner peelings, this does not mean that all intermediate COMs are in the same class. Proposition 6.4 however yields that this is the case for LOPs. Another such instance is the class of realizable COMs. Generalizing a results from [170] for realizable LOPs we find:

Proposition 6.6. *Every realizable COM has a corner peeling.*

Proof. We show that a realizable COM G has a realizable chunk T. Represent G as a central hyperplane arrangement $\mathcal{H}$ intersected with an open polyhedron P given by open halfspaces $\mathcal{O}$. Without loss of generality we can assume that the supporting hyperplanes of the halfspaces in $\mathcal{O}$ are in general position with respect to the hyperplanes in $\mathcal{H}$. Now, take some halfspace $O \in \mathcal{O}$ and push it into P until it contains the first minimal dimensional cell C of $\mathcal{H}$. The obtained realizable COM T is a chunk of G, because restricting the OM corresponding to the cell C with respect to O is taking a chunk of C, while no other cells of G are affected and the resulting graph T is a COM. □

In [16, Conjecture 2] it was conjectured that all locally realizable COMs, i.e., those whose cells are realizable OMs, are realizable. Proposition 6.6 yields a disproof of this conjecture, since all cells of a LOP are hypercubes, i.e., LOPs are locally realizable (even zonotopally realizable), but by the examples above there are LOPs that do not have corner peelings. Thus, they cannot be realizable. An open question remains [16, Question 1], i.e., whether every locally realizable COM is zonotopally realizable.

6.2 Corners and corner peelings in further classes

In this section we consider the question of the existence of corners and corner peelings in various classes of graphs. By Proposition 6.4 simplicial vertices in LOPs are corners. Thus, Theorem 6.2 yields:

Corollary 6.7. *Every halfspace of a Mandel UOM has a corner.*

In the following we focus on COMs of rank 2 and hypercellular graphs. In both these proofs we use the zone graph of a partial cube. We start with some necessary observations on cocircuit graphs of COMs.

Cocircuit graphs of COMs

In the following we generalize the concept of orientation of the cocircuit graph introduced above from AOMs to general COMs.

Lemma 6.8. *If G is a COM and a hypercube Q_r a pc-minor of G, then there is an antipodal subgraph H of G that has Q_r a minor. In particular, the rank of a maximal antipodal subgraph of a COM G is the rank of G.*

We define the *cocircuit graph* of a non-antipodal rank r COM as the graph whose vertices are the rank r antipodal subgraphs and two vertices are adjacent if they intersect in a rank $r-1$ antipodal subgraph. By Lemma 6.8 the vertices of the cocircuit graph of a non-antipodal COM G correspond to maximal antipodal subgraphs of G. The cocircuit graph of a COM can be fully disconnected hence we limit ourselves to COMs having all its maximal antipodal subgraphs of the same rank with G^* connected. We call them *pure* COMs. Note that AOMs are pure COMs.

Let G be a pure COM, $\{A_1, A_2\}$ be an edge in G^* and F be the set of Θ-classes crossing $A_1 \cap A_2$. We have seen above that if G is an AOM, then the maximal proper antipodal subgraphs of G crossed by Θ-classes in F induce a path of G^* which we called a line. The following lemma is a generalization of the latter and of general interest with respect to cocircuit graphs of COMs, even if we will use it only in the case of rank 2.

Lemma 6.9. *Let G be a COM that is not an OM, $\{A_1, A_2\}$ an edge in G^*, and F the set of Θ-classes crossing $A_1 \cap A_2$. The maximal proper antipodal subgraphs of G crossed by Θ-classes in F induce a subgraph of G^* isomorphic to the line graph of a tree.*

Lemma 6.9 implies that G^* can be seen as the edge disjoint union of line graphs of trees. We can use this to orient edges of G^*. Similarly as in the settings of AOMs, we will call a *line* in G a maximal path $L = A_1, \dots, A_n$ in the cocircuit graph G^* such that $A_{i-1} \cap A_i$ is the set of antipodes of $A_i \cap A_{i+1}$ with respect to A_i. Let now $E_e \in \mathcal{E}$ be a Θ-class of G. Similarly as before we say that E_e crosses a line L of G^* if there exists A_i on ℓ that is crossed by E_e but $A_{i-1} \cap A_i$ or $A_i \cap A_{i+1}$ is not crossed by it. If A_i exists, it is unique. *The orientation of L with respect to E_e* is the orientation of the path L in G^* from E_e^- to E_e^+ if E_e crosses L and not orienting the edges of L otherwise. Notice that in this way we can orient the edges of G^* with respect to E_e by orienting all the lines simultaneously. The orientation of each edge (if it is oriented) is well defined: If $\{A_j, A_{j+1}\} \in E_e^-$ is an edge in a line graph of a tree that is crossed by E_e in A_i and A_{j+1} is closer to A_i than A_j is, then $\{A_j, A_{j+1}\}$ is oriented from A_j to A_{j+1}. Similarly if $\{A_j, A_{j+1}\} \in E_e^+$ and A_j is closer to A_i than A_{j+1} is, then $\{A_j, A_{j+1}\}$ is oriented from A_j to A_{j+1}. Furthermore, $\{A_j, A_k\}$ is not oriented if A_j, A_k are at the same distance to A_i. See Figure 27 for an illustration.

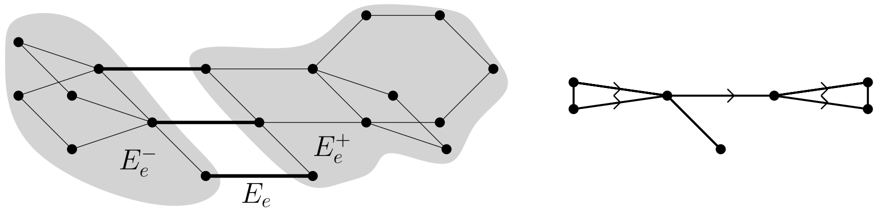


Figure 27: A pure rank 2 COM and its cocircuits graph oriented with respect to E_e.

COMs of rank 2

Mandel proved that every AOM of rank 2 is Euclidean, which by Corollary 6.7 implies that every rank 2 halfspace of a UOM has a corner. We generalize this result.

Let us first consider what corners in rank 2 COMs are. Up to isomorphism the only rank 2 OMs are even cycles. An expansion in general position of an even cycle $G = C_{2n}$ is given by $G_1, G_2 = -G_1$, where G_1 consists of an induced path on $n+1$ vertices. Hence a corner in a rank 2 COM consist of $n-1$ vertices inducing a path, included in a unique antipodal C_{2n}. For example, the COM in Figure 27 has 11 corners. Those contained in a square are single vertices and the ones contained in the C_6 are paths with two vertices.

Proposition 6.10. *Let G be a pure COM of rank 2 and E_e a Θ-class of G. Then G has a corner in E_e^+ and in E_e^-.*

The following is a common generalization of corresponding results for cellular bipartite graphs [11] (being exactly rank 2 hypercellular graphs, which in turn are COMs [42]) and LOPs of rank 2 [37].

Theorem 6.11. *Every rank 2 COM has a corner peeling.*

Proof. Notice that a rank 2 COM is pure if and only if it is 2 connected. Consider the blocks of 2-connectedness of a rank 2 COM G. Then a block corresponding to a leaf in the tree structure of the block graph has 2 corners by Proposition 6.10. This implies that G has a corner. Proposition 6.3 together with the observation that G minus the corner has rank at most 2 yield a corner peeling. □

Hypercellular graphs

Hypercellular graphs were introduced as a natural generalization of median graphs, i.e., skeleta of CAT(0) cube complexes in [42]. They are COMs with many nice properties one of them being that all their antipodal subgraphs are Cartesian products of even cycles and edges, called *cells*. Since median graphs are realizable COMs, see [128], which is also conjectured for hypercellular graphs [42], they have corner peelings by Proposition 6.6. Here, we prove that hypercellular graphs have a corner peeling, which can be seen as a support for their realizability.

The following lemma determines the structure of corners in hypercellular graphs, since the corners of an edge K_2 and an even cycle C_{2n} are simply a vertex and a path P_{n-1}, respectively.

Lemma 6.12. *Let $G = \Box_i A_i$ be the Cartesian product of even cycles and edges. Then the corners of G are precisely sets of the form $\Box_i D_i \subset G$, where D_i is a corner of A_i for every i.*

We shall need the following property about hypercellular graphs.

Lemma 6.13. *Every zone graph $\zeta_f(G)$ of a hypercellular graph G is hypercellular.*

Proof. Every zone graph of the Cartesian product of even cycles and edges is the Cartesian product of even cycles and edges, as it can easily be checked. Let $\zeta_f(G)$ be a zone graph of a hypercellular graph G. Then every cell of rank r in $\zeta_f(G)$ is an image of a cell of rank $r+1$ in G. Hence for every three rank r cells pairwise intersecting in rank $r-1$ cells and sharing a rank $r-2$ cell from $\zeta_f(G)$, there exist three rank $r+1$ cells pairwise intersecting in rank r cells and sharing a rank $r-1$ cell in G. Additionally the latter three cells lie in a common cell H in G. Then the image of H in $\zeta_f(G)$ is a common cell of the three cells from $\zeta_f(G)$. ☐

Let E_e be a Θ-class of a COM G. As usual, see e.g. [16, 42], we call the union of antipodal subgraphs crossed by E_e the *carrier* of E_e. The following is another generalization of the corresponding result for cellular graphs [11] and as mentioned above for median graphs.

Theorem 6.14. *Every hypercellular graph G has a corner peeling.*

Proof. We prove the assertion by induction on the size of G. The technical difficulty of the proof is that removing a corner in a hypercellular graph possibly produces a non-hypercellular graph. Hence we shall prove the above statement for the larger family $\mathcal{F}$ of COMs defined by the following properties:

1. Every antipodal subgraph of $G \in \mathcal{F}$ is a cell.
2. Every carrier of $G \in \mathcal{F}$ is convex.
3. Every zone-graph of $G \in \mathcal{F}$ is in $\mathcal{F}$.

We first prove that hypercellular graph are a part of $\mathcal{F}$. By Lemma 6.13 only the first two properties must be checked. Now, (1) holds by definition of hypercellular graphs. Moreover, (2) follows from the fact that for any Θ-class E_e in a hypercellular graph the carrier of E_e is gated [42, Proposition 7], thus also convex.

We now prove that the graphs in $\mathcal{F}$ have a corner peeling. Let $G \in \mathcal{F}$ and E_e an arbitrary Θ-class in G. Since the carrier of E_e is convex the so-called Convexity Lemma [96] implies that for any edge $g \in E_e^+$ with exactly one endpoint in the carrier its Θ-class E_g does not cross the carrier. Now if the union of cells crossed by E_e does not cover the whole E_e^+, then for any edge g in E_e^+ with exactly one endpoint in the union, one of E_g^+ or E_g^- is completely in E_e^+. Repeating this argument with E_g one can inductively find a Θ-class E_f with the property that the carrier of E_f completely covers E_f^+, without loss of generality.

Let $\zeta_f(G)$ be the zone graph of G with respect to E_f, i.e., the edges of E_f are the vertices of $\zeta_f(G)$ and two such edges are connected if they lie in a common convex cycle. By (3) $\zeta_f(G)$ is in $\mathcal{F}$, thus by induction $\zeta_f(G)$ has a corner D_f. By definition there is a maximal antipodal subgraph A_f in $\zeta_f(G)$ such that the corner D_f is completely in A_f. Moreover, there exists a unique maximal antipodal subgraph A in G whose zone graph is A_f.

We lift the corner D_f from A_f to a corner D of A in the following way. If E_f in A corresponds to an edge factor K_2, then A is simply $K_2 \Box A_f$. In particular we can define $D = \{v\} \Box C_f$ where v is a vertex of K_2 in E_f^+. By Lemma 6.12, this is a corner of A. Since D_f lies only in the maximal antipodal graph A_f, D lies only in A.

Otherwise, assume E_f in A corresponds to a Θ-class of a factor C_{2k} (an even cycle). We can write $A = C_{2k} \Box A'$. Then $A_f = K_2 \Box A'$ with a corner $D_f = \{v\} \Box A'$, by Lemma 6.12. We lift D_f to $D = P_{k-1} \Box A'$. Here P_{k-1} is the path in C_{2k} consisting of the vertices in E_f^+ apart from the one lying on the edge not corresponding to v in the zone graph. As above since D_f lies only in the maximal antipodal graph A_f, D lies only in A.

We have proved that G has a corner D. To prove that it has a corner peeling it suffice to show that $G \backslash D$ is a graph in $\mathcal{F}$. Since removing a corner does not produce any new antipodal subgraph, all the antipodal subgraphs of $G \backslash D$ are cells, showing (1). The latter holds also for all the zone graphs of $G \backslash D$. To prove that (2) holds for $G \backslash D$ consider a Θ-class E_e of $G \backslash D$. By Lemma 6.3, $G \backslash D$ is an isometric subgraph of G, i.e. all the distances between vertices are the same in both graphs. Since the carrier of E_e in G is convex and removing a corner does not produce any new shortest path, the carrier of E_e is convex in $G \backslash D$. The same argument can be repeated in any zone graph of $G \backslash D$. This finishes the proof. $\square$

7 Conclusions, current and future work

In the present section we discuss future directions of research on different levels of depth, difficulty and importance, while also giving some further results.

7.1 COMs and their tope graphs

We have shown how COMs naturally arise as a generalization of (affine) oriented matroids and lopsided sets by relaxing the covector axioms. Also, an axiomatization via cocircuits is available that we chose to omit in the present text, see [16]. Nevertheless, many analogies to the theory of OMs still lack generalization.

Problem 1. *Establish an underlying undirected theory for COMs.*

We expect to obtain set systems instead of oriented sign-vectors, much as the theory of matroids [141, 173] behaves compared to the theory of OMs. Note that there is an unoriented counterpart to AOMs, that is called *semimatroids*, due to Ardila [6]. These objects already find recent applications in [56] and are part of a recent project of mine with Emanuele Delucchi about finitary AOMs [55]. With respect to Problem 1, in what my intuition goes, I believe that a lattice theoretic approach along the lines of bouquets of geometric lattices (also known as *bouquets of matroids*) and ideals in geometric lattices (also known as *wounded matroids*) could be interesting, see [118].

Another important piece of theory missing for COMs is duality. This feature exists for OMs as well as for LOPs, where it simply consists of taking the complement of the tope set in the hypercube $\{+,-\}^E$, see [119]. Possibly Problem 1 could give some hints here, but the same could hold vice versa, so we pose the following independently:

Problem 2. *Investigate COM duality.*

As a natural generalization of ranking COMs we propose *graphic COMs* which are the minimal minor-closed class containing the previous as well as graphic OMs, see the end of Section 3.7. Here the input is a mixed graph $G = (V, A \cup E)$ with edges and arcs. The topes are its acyclic orientations. See Figure 12 for an illustration, which also shows that graphic COMs are not Pasch in general. We believe that graphic COMs serve as a good class to understand COM duality. In particular, the topes of the dual COM associated to $G = (V, A \cup E)$ should be its strongly connected orientations. These realizable COMs are very easy to understand and present themselves as a perfect testing class for Problems 1 and 2.

Graphic COMs are closed under minors and under taking halfspaces. Another interesting question is about the excluded minors of graphic COMs. Is the class closed under amalgamations? Similarly, to Propositions 3.17 one can see that a graphic COM is an OM if and only it has $A = \emptyset$, i.e., it is a graphic OM. What are the graphic LOPs?

A further direction here comes from the fact that tope graphs of graphic COMs are flip graphs of certain orientations. Thus, they endow a large set with a structure that can be used for proofs but also lies at the heart of enumeration algorithms, see [23]. In this context it would be of interest to study the set of topes of a co-graphic COM that correspond to orientations of higher strong connectivity. What is the structure of the topes corresponding to k-connected orientations? Resuming the above we propose the following as a project suitable for a thesis, maybe even parts of it could be studied in a Master's thesis:

Problem 3. *Study graphic COMs.*

Graphic COMs are realizable and thus by definition fibers (also known as supertopes [94]) of OMs. For this class Problem 1 and 2 may be attacked in a more natural way than in the general situation. The following set of problems comes from the question whether there really *is* a more general situation. It is probably the central open problem in COMs.

We have shown that every COM arises by amalgamation from its OM cells. Conversely, can any COM can be obtained by restricting a larger OM? Let us describe this more precisely. A halfspace of a COM is a COM. Particular examples are the affine oriented matroids, which are halfspaces of OMs. Even stronger, the intersections of halfspaces, i.e., the fibers (also known as supertopes [94]), of a COM are COMs. By definition, realizable COMs are fibers of realizable OMs. How about the general case?

Conjecture 3. *Every COM is a fiber of an OM.*

This generalizes the corresponding conjecture of Lawrence [120] that lopsided sets are fibers of uniform OMs. One way to attack these conjectures would be lattice theoretic via the language of *bouquets of oriented matroids* [60]. Another strategy would be to rephrase Conjecture 3 to tope graphs, where it corresponds to the statement that every tope graph of a COM is a convex subgraph of a tope graph of an OM.

As shown in [109], every partial cube is a convex subgraph of an antipodal partial cube. However, if $G \in \mathcal{G}_{\mathrm{COM}}$ the constructed antipodal graph is not necessary in $\mathcal{G}_{\mathrm{COM}}$.

Not only would Conjecture 3 be a natural generalization of the realizable situation, but using the Topological Representation Theorem of Oriented Matroids [76] it will also give a natural topological representation for COMs. In fact, Conjecture 3 is also equivalent to the following conjecture: For every COM $\mathcal{L}$ there is a number d such that $\mathcal{L}$ can be represented by a set of $(d-2)$-dimensional pseudospheres restricted to the intersection of a set of open $(d-1)$-dimensional pseudo-hemispheres inside a $(d-1)$-sphere. In this sense proving Conjecture 3 would give new insights on attacking Problem 1 and 2.

In an ongoing project with Winfried Hochstaettler and Volkmar Welker we try to push topological results from supertopes to general COMs, thus giving further indications for Conjecture 3 and moreover generalizing earlier results on the (signed) Varchenko determinant of OMs and affine hyperplane arrangements [94, 150].

There are similar, analogues and weakened version of Conjecture 3. In particular there is a refined hierarchy of different types of realizability into *locally realizable*, *zonotopally realizable*, and *CAT(0) zonotopally realizable*, see Section 3.5. A zonotopally realizable version of Conjecture 3 from [16] is wrong, see Section 6.2: there are zonotopally realizable COMs, that are not realizable, i.e., they are not fibers of realizable OMs. We believe that another question from [16], see also Question 1, also has a negative answer:

Conjecture 4. *There are locally realizable COMs, that are not zonotopally realizable.*

To prove Conjecture 4 it would suffice to find a three dimensional zonotope $\mathcal{Z}$ with two incident facets F, F' of same valency, such that in no realization of $\mathcal{Z}$ both F, F' are isometric. Gluing three copies of $\mathcal{Z}$ around the edge separating F, F' would yield an example proving Conjecture 4.

We think that the classes of realizable and zonotopally realizable COMs are interesting and have presented large interesting subclasses, e.g., graphic COMs and CAT(0) Coxeter complexes, respectively. Even if not all zonotopally realizable COMs realizable, it would be interesting to find large classes that are realizable. This, in particular would yield corner peelings. For instance it can be seen, see [128] that median graphs are tope graphs of realizable COMs. How about hypercellular graphs? On the one hand we have shown that they are zonotopally realizable COMs, see Theorem 5.5. Moreover, as an indicator for their realizability we have shown, that they admit corner peelings. On the other hand we have studied CAT(0) Coxeter complexes and shown that their tope graphs generalize median graphs. Since hypercellular graphs are skeleta of Coxeter complexes and generalize median graphs it seems likely, that their complexes are also CAT(0), also see [42, Conjecture 1] and [42, Problem 2]. Indeed, it might be that the median cell property alone (Theorem 5.2) yields the CAT(0)-property for a zonotopally realizable COM and that this in turn yields corner peelings.

Conjecture 5. *Every hypercellular graph is the skeleton of a CAT(0) Coxeter complex.*

Another class containing median graphs are *conditional antimatroids* [14]. These can be seen as *upper locally distributive semilattices* as discussed in [73, 106] and are natural common generalization of antimatroids and median graphs. In an on-going project with Jérémie Chalopin and Victor Chepoi we tackle the question of realizability of this class of LOPs.

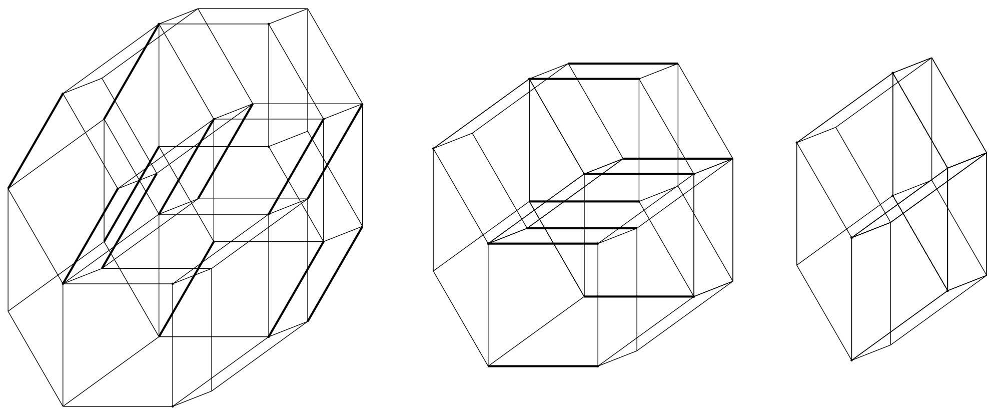


Figure 28: A graph that is apiculate (see Section 7.6), i.e., in particular the convex hull of any isometric cycles is gated, and LOP, i.e., in particular all antipodal subgraphs are hyperprisms. However, it is not in $\mathcal{S}_4$, since as shown it can be contracted to Q_4^-.

Going further, in [42] we have studied the class $\mathcal{S}_4$ of Pasch graphs containing hypercellular graphs. Note that since $Q_3^- \in \mathcal{S}_4$, we know that these graphs cannot be CAT(0)-Coxeter. The class $\mathcal{S}_4$ constitutes another class of COMs. Namely, we prove that if $G \in \mathcal{S}_4$, then the convex hull of any isometric cycle is gated. This implies that Pasch graphs are COMs, since antipodal subgraphs are convex hulls of isometric cycles. Moreover, we show that as in hypercellular graphs, all antipodal subgraphs in $\mathcal{S}_4$ are hyperprisms, i.e., products of even cycles and edges. However, the graph in Figure 28 shows, that $\mathcal{S}_4$ is a strict subclass of the intersection of the other two. We believe that the study of these classes, i.e., COMs in which the convex hull of any isometric cycle is gated and COMs in which all cells are hyperprisms deserve further investigation as generalizations of $\mathcal{S}_4$, but also as generalizations of apiculate and LOPs, respectively.

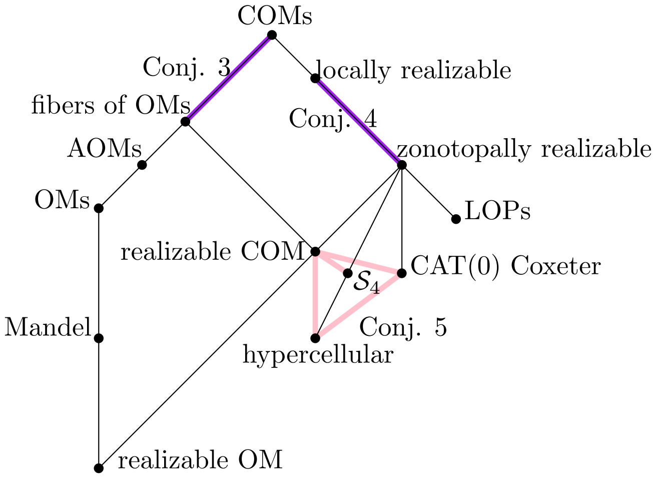


Figure 29: Classes and their (conjectured) relations. Black arcs are strict inclusions, some of which are central results of this thesis, e.g., Mandel$\subsetneq$ OMs, realizable COMs$\subsetneq$ zonotopally realizable COMs, and $\mathcal{S}_4 \subsetneq$ zonotopally realizable COMs. Purple arcs are inclusions not known if strict or not. Pink arcs are likely inclusions (known not to be reverse inclusions).

As mentioned above, the previous conjectures are related to topological representations. Another question concerning topological representation is not only about tope graphs of COMs, but more generally about partial cubes. Our characterization of planar partial cubes as region graphs of non-separating arrangements of Jordan curves in the plane [5] generalizes a tope graph characterization for rank 3 OMs by Fukuda and Handa [80]. Non-separating arrangements therefore generalize topological representations of OMs of rank 3. The property of being *non-separating* can be lifted to pseudo-spheres of higher dimension and the resulting region-graphs constitute partial cubes. The notion of non-separating arrangements generalizes pseudo-sphere arrangements as occurring in the Topological Representation Theorem for OMs. Using topological tools such as the generalized Schönfliess theorem [25] or a lattice theoretic approach, we propose the following:

Problem 4. *Is every partial cube the region graph of a non-separating pseudo-sphere arrangement?*

As mentioned above the answer here is "yes" for planar partial cubes. One of the typical "bad guys" in partial cubes is the Desargues graph, see the left of Figure 37. A first positive result into this direction is that the Desargues graph has a representation in $\mathbb{R}^3$, see Figure 30.

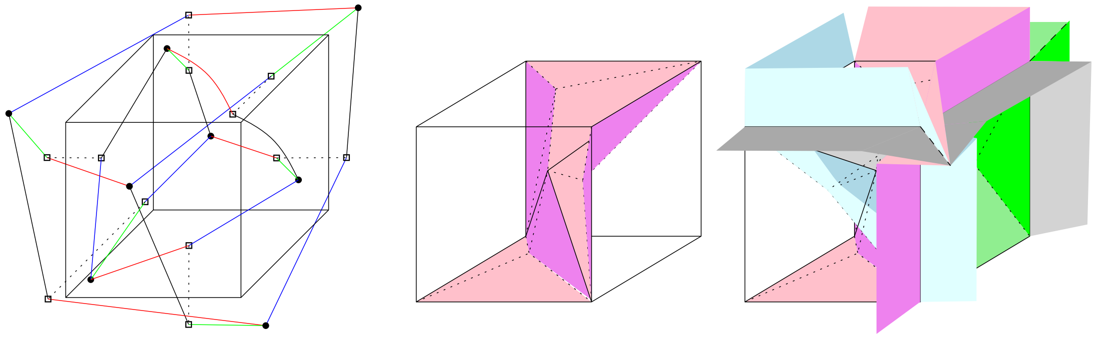


Figure 30: From left to right: A placement of the Desargues graph in $\mathbb{R}^3$, such that one pseudosphere corresponds to the surface of the cube. The part inside the cube of one of the 4 remaining pseudo-spheres. A sketch of a completion to the full arrangement.

7.2 Simpliciality and mutations

One of the motors of our work on corners and simpliciality is Las Vergnas simplex conjecture. It (and as we showed even a strengthening of it) holds for Mandel OMs including realizable and OMs and OMs of rank 3. However it remains one of the big open problems of the area and symptomatic for the still existing lack between topology and combinatorics in OM theory. Recall that it consists in the following:

Problem 5. *Prove that every OM has a mutation, i.e., every OM G has a vertex of degree* $\mathrm{rk}(G)$.

Though this problem has been subject of deep research, e.g. [154,161], up to now Las Vergnas conjecture is only proved for *realizable* OMs [167], OMs of rank at most 3 [121], for rank 4 OMs with few elements [26], rank 4 uniform matroid polytopes [133], and as mentioned above for Euclidean OMs and Mandel OMs [124]. The graph theoretical point of view allows to generalize Las Vergnas conjecture to antipodal partial cubes and we were able to prove it true for antipodal partial cubes of rank at most 3 and by computer for those of isometric dimension at most 7 [108].

n	2	3	4	5	6	7
antipodal	1	2	4	13	115	42257
OM	1	2	4	9	35	381

Table 2: Numbers of antipodal partial cubes and OMs of isometric dimension $n \leq 7$. The latter can also be retrieved from `http://www.om.math.ethz.ch/`

Since already on isometric dimension 6 there are 13488837 partial cubes, instead of filtering those of isometric dimension 7 by antipodality, we filtered those of isometric dimension 6 by affinity. There are 268615 of them. We thus could create all antipodal partial cubes of dimension 7, count them and verify Las Vergnas conjecture also for this set. See Table 2. We extend the prolific Las Vergnas conjecture to a much wider class.

Problem 6. *Does every antipodal partial cube G have a vertex of degree* $\mathrm{rk}(G)$*?*

Let us quickly discuss a related problem. In [160] Roudneff made the following

Conjecture 6. *The tope graph of an OM of rank r and isometric dimension $n \geq 2r-1$ has at most $2\sum_{i=0}^{r-3}\binom{n-1}{i}$ vertices of degree n.*

Here we took the opportunity to rephrase in terms of tope graphs. This bound is tight attained by the point matroid of the cyclic polytope and it is sufficient to prove it for $n = 2r-1$, see [160]. The latter allowed to verify the conjecture for rank at most 4, see [149]. Apart from this it has been shown for so-called Lawrence matroids by Montejano and Ramírez-Alfonsín [134]. A computerized case analysis for $n = 7$ and rank $r = 4$ showed the veracity of Conjecture 6 for antipodal partial cubes with these parameters. We wonder:

Problem 7. *Does every antipodal partial cube G of isometric dimension $n \geq 2\mathrm{rk}(G)-1$ have at most $2\sum_{i=0}^{r-3}\binom{n-1}{i}$ vertices of degree n?*

Through the perspective of the Topological Representation Theorem, Problem 5 is a perfect illustration of how topology and combinatorics can be intertwined. Namely, if a OM has a mutation, then one can understand combinatorially what it means to transform the arrangement *pulling one pseudo-sphere bounding the simplicial cell over its opposite vertex.* Now, the set of OMs of fixed rank and size is connected via mutations if and only if there are continuous transformations between all representing arrangements passing only through arrangements representing OMs. Las Vergnas conjecture states that the *mutation graph* on all OMs of fixed rank and size has no vertices of degree 0. In this context the following conjecture is natural and completely out of reach:

Conjecture 7 (Cordovil-Las Vergnas [161]). *For all r, n the mutation graph on the set of* uniform *OMs on n elements and rank r is connected.*

By Ringel's Homotopy Theorem [157, 158] Conjecture 7 holds for rank at most 3. Also for the realizable case [161], the induced subgraph defined by realizable uniform OMs is connected. We verified the conjecture of Cordovil-Las Vergnas for all uniform OMs on at most 9 elements. See [108] for more details.

The largest mutation graph that had to be checked has 9276595 vertices. Figure 31 shows the graph of uniform OMs of rank 4 on 8 elements.

7.3 Corners peelings and unlabeled sample compression

Going back to corners, recall that we have shown corner peelings for COMs or rank 2 and hypercellular graphs. A common generalization are Pasch graphs [40, 42], which form a class of COMs that excludes all known LOPs without corner peelings [11,37,170].

Problem 8. *Does every Pasch graph have a corner peeling?*

This question is in a sense about the next-class-to-tackle and will allow to improve our techniques, since its similarities to the resolved case of hypercellular graphs are plentiful. Another natural class to try would be zonotopally realizable CAT(0) COMs.

From a more general stand point however, recall that the initial motivation for the study in corner-peelings in LOPs is that their existence guarantees unlabeled compression schemes and henceforth creates a link to machine learning [37]. It would be interesting to see how our contributions translate back to this setting and consider related concepts in the context of COMs.

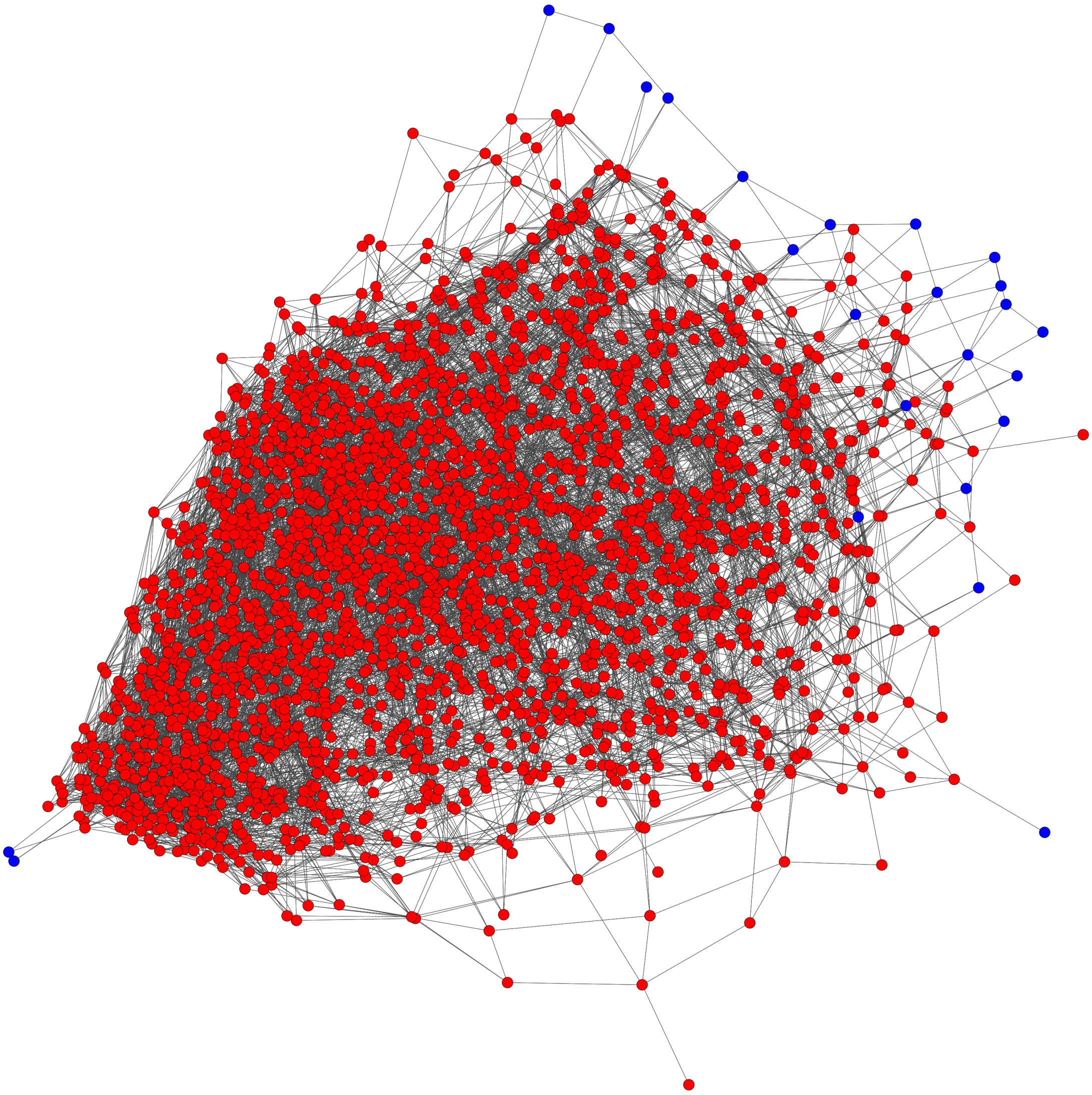


Figure 31: The mutation graph of (isomorphism classes of) uniform OMs of rank 4 on 8 elements. The red vertices are realizable and the blue ones are non-realizable

Problem 9. *Are there unlabeled compression schemes for COMs with corner peeling?*

This problem can probably be solved by adapting the fundamental results showing that corner peelings in LOPs yield unlabeled compression schemes, see [37, 132].

7.4 Lopsided extensions and labeled sample compression

A project with Victor Chepoi and our current PhD student Manon Philibert initiates the systematic study of partial cubes of bonded VC-dimension. We obtain the first structural results on two-dimensional partial cubes, i.e., $\mathcal{F}(Q_3)$, see [44]. Recall that a partial cube G has VC-dimension d if and only if $\mathrm{rk}(G) = d$, i.e., d is the largest integer such that $G \in \mathcal{F}(Q_{d+1})$.

The interpretation in terms of the VC-dimension, that was introduced in statistical learning by Vapnik and Chervonenkis [172], creates links to combinatorics, algorithmics, machine learning, discrete geometry, and combinatorial optimization where it serves as complexity measure and as a combinatorial dimension of a set family. Littlestone and Warmuth [122] introduced the sample compression technique for deriving generalization bounds in machine learning. Floyd and Warmuth [75] asked whether any set family $\mathcal{S}$ of VC-dimension d has a sample compression scheme of size $O(d)$. This question remains one of the oldest open problems in computational machine learning. Labeled compression schemes were designed for lopsided families [137] and for maximum [37] families, respectively.

It was noticed in [137, 163] that the (labeled) sample compression conjecture of [75] would follow from:

Problem 10. *Show that any set family $\mathcal{S}$ of VC-dimension d can be extended to a lopsided partial cube of VC-dimension $O(d)$.*

Problem 10 is already open for set families of VC-dimension 2. As a first non-trivial step we could show that any partial cube of VC-dimension at most 2 can be extended to a LOP even of the same dimension, see [44]. This is best-possible in two senses: On the one hand, there are set systems of VC-dimension 2, that cannot be completed to a partial cube of VC-dimension 2. On the other hand there are partial cubes of VC-dimension 3, that cannot be completed to a LOP of VC-dimension 3, see Figure 32.

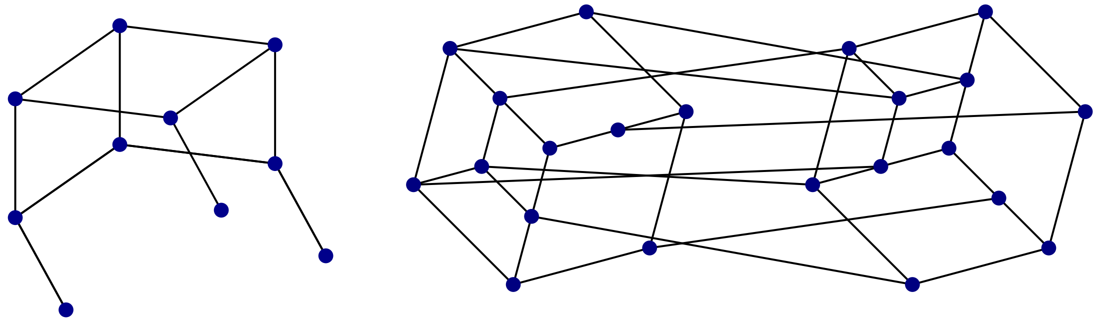


Figure 32: Left: set system of VC-dimension 2 that cannot be completed to a partial cube of VC-dimension 2. Right: partial cube of VC-dimension 3 that cannot be completed to a LOP of VC-dimension 3.

Problem 11. *Show that any partial cube of VC-dimension d can be extended to a lopsided partial cube of VC-dimension $O(d)$.*

In order to attack Problem 11 it would be tempting to proceed as we did for the two-dimensional case, i.e., first extend to a COM and then extend this COM to a LOP. Indeed, we believe the following:

Conjecture 8. *Every COM can be extended to a LOP of same rank.*

We have shown the above conjecture for OMs as well as for CUOMs, i.e., COMs whose cells are uniform oriented matroids [43]. Note that the latter includes COMs of rank 2. Having however in mind the goal of proving labeled sample compression for COMs, one could also consider the proof for LOPs and push it to the more general set-up of COMs.

Conjecture 9. *Every COM of rank d has a labeled compression schemes of size d.*

Conjecture 8 also relates to work with Nicolas Nisse [114], where we wonder if any set family $\mathcal{S} \subseteq Q_n$ can be extended to a partial cube whose number of vertices is polynomial in $|\mathcal{S}| + n$. Making such an extension by keeping just the VC-dimension linear, would be the other missing piece that together with Problem 11 would solve the sample compression conjecture.

7.5 Other Pc-minor-closed classes

We have studied several classes of partial cubes, that can be defined by a simple or small set of excluded minors. See Table 3 for known such classes.

trees	$\mathcal{F}(Q_2)$
hypercubes	$\mathcal{F}(P_3)$
bipartite cacti	$\mathcal{F}(K_2 \square P_3)$ [128, page 12]
hypercellular graphs	$\mathcal{F}(Q_3^-)$
median graphs	$\mathcal{F}(Q_3^-, C_6)$
distributive lattices	$\mathcal{F}(Q_3^-, C_6, K_{1,3})$
gated amalgams of even cycles and cubes	$\mathcal{F}(Q_3^-, C_6 \times K_2)$ [145]
two-dimensional	$\mathcal{F}(Q_3)$ [44]
rank two COMs	$\mathcal{F}(SK_4, Q_3)$
rank two LOPs	$\mathcal{F}(C_6, Q_3)$
almost-median graphs	$\mathcal{F}(C_6)$ [128, Theorem 4.4.4]

Table 3: Classes of partial cubes defined by excluding a small (set of) pc-minors. Here, SK_4 denotes the full subdivision of K_4, see the left-most graph in Figure 18.

In general studying $\mathcal{F}(X)$ for a small partial cube X is an interesting project, that can be insightful and a good way to approach a student to this area. A first natural candidates for X include $K_{1,3}$ or P_4 and the intersection of classes defined by the above or similar excluded minors.

The *global hull-number* $\overline{\text{hnr}}(G) = \max\{\text{hnr}(H) : H \subseteq G \text{ convex}\}$, where $\text{hnr}(H)$ is the *hull-number*, i.e., the size of a smallest set of vertices whose convex hull is H. The class $\mathcal{H}_n$ of partial cubes with globe hull-number at most n is pc-minor closed. At first we believed that $\mathcal{H}_2$ would coincide with $\mathcal{F}(K_{1,3})$ but recently Manon Philibert found another excluded minor for this class, namely Q_3^- with a leaf attached to the antipode of the missing vertex.

Problem 12. *Determine the excluded pc-minors of $\mathcal{H}_n$.*

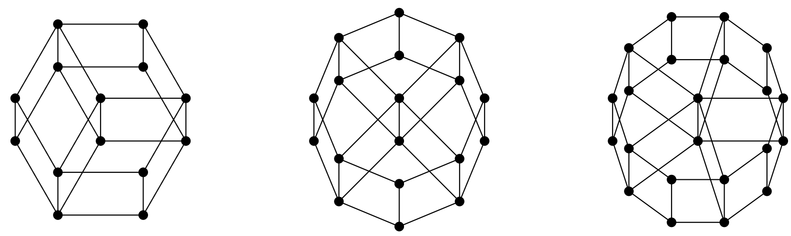
While determining $\text{hnr}(G)$ is NP-complete even for partial cubes [5], we know that given the (large) set of all convex subgraphs, both hnr and $\overline{\text{hnr}}$ can be determined in polynomial time in general, see [114]. Since by Proposition 4.5 pc-minors can be found in polynomial time, providing a finite list in Problem 13 would yield a polynomial time algorithm for determining $\overline{\text{hnr}}$ in partial cubes. A natural (proper) minor-closed subclass of $\mathcal{H}_{2n}$ are those partial cubes isometrically embeddable into $\mathbb{Z}^d$. The set of excluded minors is known only for $d \leq 2$ and will be part of the thesis of Manon Philibert.

Problem 13. *Show that the list of minimal forbidden pc-minors of partial cubes embeddable in $\mathbb{Z}^d$ is finite.*

Another pc-minor closed class of partial cubes are planar partial cubes. Several characterizations of planar partial cubes are known [5, 59], where the latter corrects a characterization from [144]. A particular consequence of Corollary 4.28 is that tope graphs of OMs of rank at most three are characterized by the finite list of excluded pc-minors $\mathcal{Q}_3^-$. By a result of Fukuda and Handa [80] this graph class coincides with the class of antipodal planar partial cubes. Since planar partial cubes are closed under pc-minors, we wonder about an extension of this result to general planar partial cubes:

Problem 14. *Find the list of minimal excluded pc-minors for planar partial cubes.*

It is easy to see, that any answer here will be an infinite list, since a (strict) subfamily is given by the set $\{G_n \Box K_2 \mid n \geq 3\}$, where G_n denotes the *gear graph* (also known as *cogwheel*) on $2n+1$ vertices, see Figure 33.



$G_3 \Box K_2$ $G_4 \Box K_2$ $G_5 \Box K_2$

Figure 33: The first three members of an infinite family of minimal obstructions for planar partial cubes.

However, determining the list for planar partial cubes seems hard. I have directed an ENS student on a research scholarship through this project without obtaining a candidate for a complete list. In view of the results of [80], maybe the characterizing planar COMs by excluded minors would be interesting. We spent quite some time with Jean-Florent Raymond trying to determine the class of excluded minors of planar median graphs (which are COMs), but also here the structure remains mysterious. On the other hand, the exclusions for outerplanar partial cubes are probably easy to determine - a topic for a young student. Indeed, I believe that outerplanar partial cubes can be characterized by excluding the gear graphs, Q_3, and the *book* $K_{1,3} \Box K_2$.

In particular we have the following, if a pc-minor closed class is contained in an (ordinary) minor closed graph class it can still have an infinite list of excluded minors. A tempting probably quite hard problem is the following:

Problem 15. *Give a criterion for when a pc-minor closed class has a finite list of excluded pc-minors.*

Note that classes of the above type as well as those embeddable into $\mathbb{Z}^d$ are also closed under isometric subgraphs. It would in this sense be sufficient to determine the list of obstruction that are minimal with respect to this latter relation. Indeed, one can see that if this smaller set is finite, then the set of minimal forbidden pc-minors will be finite as well.

7.6 Order

As explained in the prologue to this thesis, the first partial cubes I came in contact with were diagrams of distributive lattices. From this point of view it is natural to -

given a partial cube G and a vertex v - consider the poset $P(G, v)$ with base point v. This is, for $u, w \in P(G, v)$ we define $u \leq w$ if there is shortest (v, w)-path through u. This poset is crucial for the proof of shellability of OMs and AOMs, see [22]. With Carolina Benedetti and our Master student Jeronimo Valencia we are using $P(G, v)$ to extend shellability to certain realizable COMs. How about poset properties of $P(G, v)$?

Following Bandelt and Chepoi [12] a partial cube G is called *apiculate* if $P(G, v)$ is a meet-semilattice for all $v \in G$. There it is moreover shown that Pasch graphs are apiculate. See also Figure 28 for some other examples. In [21] the poset $P(G, v)$ is studied for G being an OM, in which case apiculate is equivalent to $P(G, v)$ being a lattice for all v. It is shown that the apiculate OMs are exactly the simplicial ones, i.e., those where $\deg(v) = \mathrm{rk}(G)$ for all vertices v. In case of Coxeter arrangements, this leads to the so-called weak order on a Coxeter group - an important lattice structure that we used recently in [83]. We believe that the following generalization holds:

Conjecture 10. *An antipodal partial cube G is apiculate if and only if G is the tope graph of a simplicial OM.*

Note that it is not true that any regular partial cube is apiculate, since there are regular non-antipodal partial cubes, see Figure 34. However, it might be true that regular COMs are antipodal.

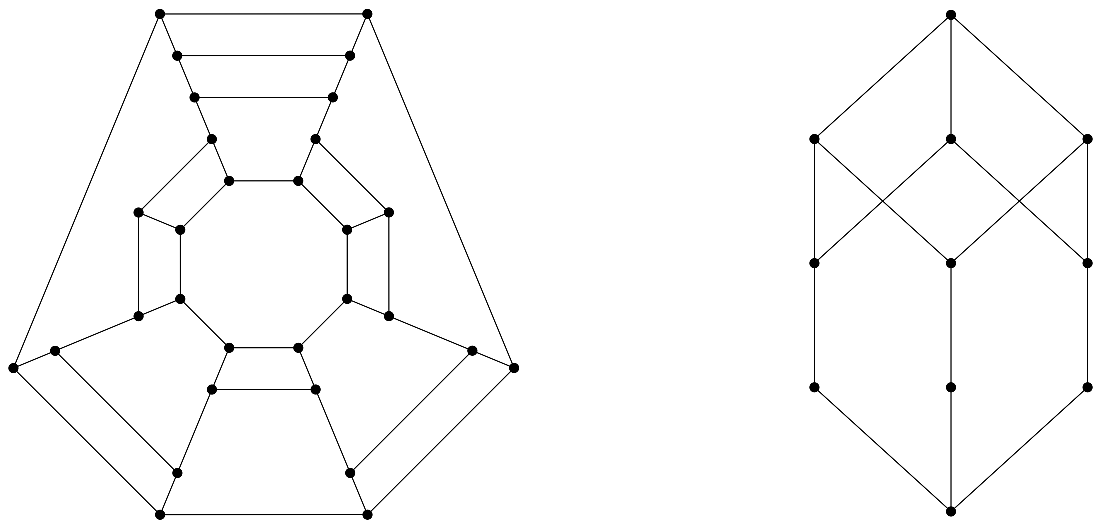


Figure 34: Left: a cubic non-antipodal partial cube. Right: a partial cube lattice that is not a COM.

A wider class can be obtained by saying that a poset or (meet-semi)lattice P is *partial cube* or *COM* if there is such G with a vertex v such that $P \cong P(G, v)$. An example are (lower) distributive (semi) lattices, realted to (conditional) antimatroids. Note that the lattice property of P does not imply that G is a COM, see the right of Figure 34. In a recent series of papers [29, 57, 58, 143] the weak face lattice of realizable OM-lattices was studied. This can be seen as the order on the covectors $\mathcal{L}$ where the product order $(- \leq 0 \leq +)^E$ is imposed. We believe the following to be natural:

Problem 16. *Study the weak face order of COMs.*

A last question concerning order comes from an analogy to distributive lattices and linear extension graphs, both being partial cubes. It might be suitable for a student.

Indeed, given the lattice of ideals P of a poset, its linear extension graph can be obtained from P by taking all maximal chains as vertices and connect two with an edge if they differ on a convex cycle. This construction - called *second graph* - has also been considered for realizable OMs, see [152]. One can thus wonder if given a partial cube lattice $P(G, v)$ its second graph is also a partial cube. The permutahedron does not get a partial cube, see Figure 35. We believe the following:

Conjecture 11. *If P is an upper locally distributive lattice, then its second graph is a partial cube.*

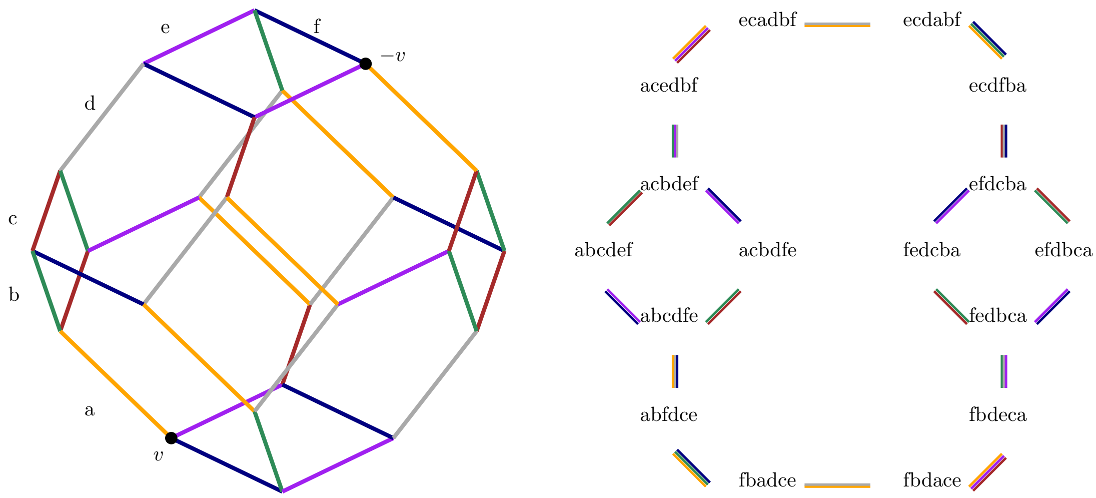


Figure 35: The permutahedron with base vertex v and the corresponding second graph, which is not a partial cube.

7.7 Symmetry

Let us turn our attention to symmetry properties. Given a group Γ and an inverse-closed subset C its *(right) Cayley graph* $\text{Cay}(\Gamma, C)$ has a vertex for each element of Γ and $\{g, h\}$ is an edge if there is $c \in C$ such that $gc = h$. An important special case is when Γ is a finite Coxeter group and C its canonical set of generators, i.e., (Γ, C) is a Coxeter system. In this case $\text{Cay}(\Gamma, C)$ is the dual graph of the Coxeter arrangement associated to Γ and thus in particular the tope graph of a simplicial OM. We say that the corresponding partial cube is *Coxeter*, also know as *mirror graph*, see [32, 127]. Indeed, all known Cayley COMs arise this way, leading to the following:

Conjecture 12. *All Cayley COMs are Coxeter.*

The conjecture holds for all cubic partial cubes [126] and all median graphs [138]. Computer experiments confirm Conjecture 12 for antipodal partial cubes of isometric dimension at most 8 and for cuartic partial cubes on up to 226 vertices. Indeed, the conjecture could even hold for vertex-transitive partial cubes. Moreover, it is known that vertex-transitive zonotopes, i.e., tope graphs of realizable OMs, such that the isometry group of Euclidean space acts transitively on them are exactly the tope graphs of Coxeter arrangements, see [175]. On the other hand there are zonotopes whose graph has an automorphism that cannot be realized by an isometry of Euclidean space, see [155] and Figure 36. Hence, *vertex-transitivity* from a graph point of view, i.e., the automorphism group acts transitively on the graph, is strictly less restrictive.

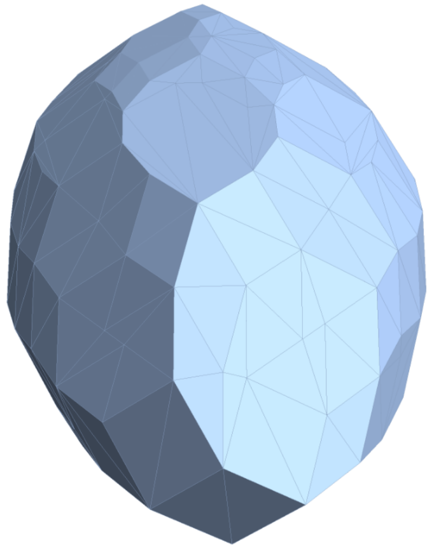


Figure 36: A zonotope whose graph has an automorphism that cannot be realized by an isometry of Euclidean space. The drawing was done by Martin Winter.

However, one difficulty of the above conjecture arises from the fact, that even if a partial cube is Coxeter it might also be the Cayley graph of a non-Coxeter group, e.g., the even cyclic groups. Also, the permutahedron $\mathrm{Cay}(S_5, \{(12), (23), (34), 45)\})$ is isomorphic to $\mathrm{Cay}(S_5, \{(1253), (13)(245)\})$, where the latter is not a Coxeter system. A yet unexplored tool for this question seem to be zone-graphs which preserves symmetry properties and thus allows for inductive proofs on COMs.

On the other hand, recent collaboration with Tilen Marc yields the first Cayley partial cube, that is not Coxeter: the 112 vertex *Dejter graph* [54] is a non-OM antipodal partial cube of isometric dimension 7 and a Cayley graph of $(\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7) \times \mathbb{Z}_2$ with respect to three generators of order 14.

Concerning the wider class of vertex-transitive partial cubes, the only known vertex-transitive Cayley partial cubes of girth 6 are the middle level graphs and the following has been stated:

Conjecture 13. *[126] All vertex-transitive partial cubes of girth* 6 *are middle level graphs.*

Conjecture 13 has been verified for antipodal partial cubes of isometric dimension at most 8.

The restriction to antipodal partial cubes in most of our experiments has computational benefits, but we think that it is actually justified by the following:

Conjecture 14. *All vertex-transitive partial cubes are antipodal.*

Conjectures 13 and 14 hold for cubic partial cubes and median graphs [126, 138]. Furthermore, they have been confirmed for partial cubes of order at most 46.

Various further notions of graph symmetry restricted to partial cubes were considered in the recent Master's thesis of my student Gil Puig. In particular, his results suggest that the only cuartic symmetric (aka arc-transitive) partial cubes might be products of cycles. Again, the Dejter graph shows that for higher degree the situation looks more complicated.

7.8 Cocircuits and the cocircuit graph

The *covectors* of a COM G are its antipodal subgraph. The *cocircuits* are the inclusion-maximal proper antipodal subgraphs, see Table 1. We have not spoken much about cocircuits and the cocircuit graph in this text. There are axiomatizations for COMs available in terms of cocircuits [16] and when speaking about corners and corner peelings cocircuits and the cocircuit graph of pure COMs play an important role [108]. There are some intriguing and hard open problems in this area, mostly concerning OMs.

An important open problem in OMs is to find a generalization of the *Sylvester-Gallai Theorem*, i.e., for every set of points in the plane that does not lie on a single line there is a line, that contains only two points. In [79] this is generalized to OMs as: *Every rank r OM has a point contained in exactly $r-1$ of the $(r-2)$-dimensional spheres.*

This statement is known to hold for rank 3 OMs [22]. The reformulation of the above in terms of metric graph theory reads: *If a tope graph has a cube Q_d as partial cube minor, then it contains a convex Q_{d-1}.*

However, this statement is false. Indeed, any orientation of a product $(U_{2,3})^k$ has rank $2k$ but only contains a convex Q_k since its tope graph is the Cartesian product $(C_6)^k$. The "right" conjecture can be found in Mandel's thesis [124], where it is attributed to Murty. In terms of partial cubes it reads:

Conjecture 15 (Murty)**.** *Every OM of rank r contains a convex subgraph that is the product of an edge and an antipodal graph of rank $r-2$.*

The realizable case of Murty's conjecture is shown by [167] and more generally holds for *Mandel* OMs [124]. Already, for rank 4 OMs it is interesting and claims the existence of a convex prism in the tope graph.

Still Murty's conjecture in general seems out of reach. We propose a reasonable weaker statement to attack:

Problem 17. *Prove that in every OM of rank r there is a convex $Q_{\lceil \frac{r}{2} \rceil}$.*

Note that Problem 17 is tight by the above counterexample to the first generalization of Sylvester-Gallai. Moreover, a generalization of Problem 17 beyond OMs is false already for rank 3 antipodal partial cubes: the Desargues graph has rank 3 but does not contain a Q_2, see the left of Figure 37. More generally, middle level graphs of hypercubes are partial cubes of unbounded rank, but without convex Q_2.

As already observed by Mandel, Conjecture 15 indeed claims a property of the underlying matroid and can hence be reformulated:

Conjecture 16. *Every orientable matroid has a hyperplane with a coloop.*

This point of view offers yet another interesting weakening of Murty's Conjecture:

Conjecture 17. *Every orientable matroid has a disconnected hyperplane.*

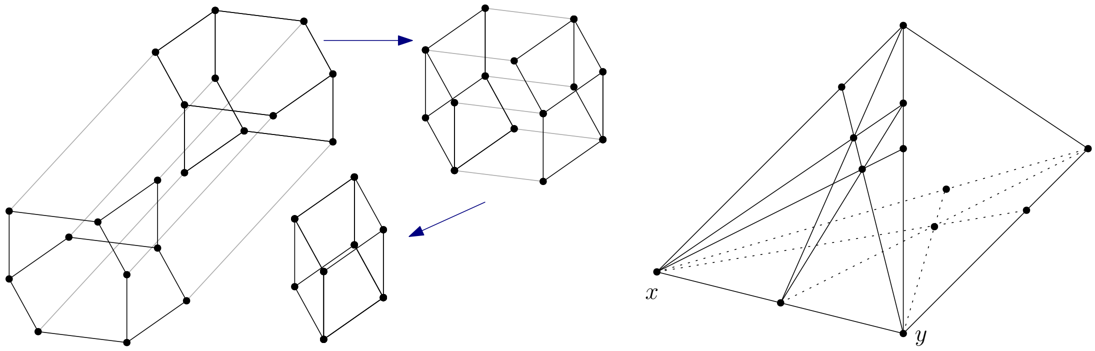


Figure 37: Left: The Desargues graph and a Q_3 minor. Right: Affine representation of the OM $\mathcal{M}$. All $\mathcal{M}$, $\mathcal{M} \setminus x$, and $\mathcal{M} \setminus y$ have isomorphic cocircuit graphs, but the one of $\mathcal{M} \setminus \{x, y\}$ is different.

Note that this property does not hold for general (non-orientable) matroids. For instance, in the Fano matroid all hyperplanes are connected.

Let us now consider not only cocircuits but cocircuit graphs, that we already introduced in Section 6.2 for pure COMs for the treatment of Mandel OMs and COMs of rank 2. The *cocircuit graph* G^* of an OM G can also be seen as the 1-skeleton of the arrangement representing it and therefore can be seen as a dual graph to the tope graph. Graph theoretic descriptions and recognition algorithms of cocircuit graphs have been subjects of my PhD thesis and a series of papers [3, 72, 106, 113]. Contrarily to tope graphs no purely graph theoretic characterization is known. The following would be of interest in particular with respect to the recognition complexity of cocircuit graphs.

Problem 18. *Find a graph theoretical characterization of cocircuit graphs.*

In contrast to the tope graph, the cocircuit graph does not uniquely determining an OM [48]. It is interesting to study the set of OMs with the same cocircuit graph G^*. Indeed, G^* does not even determine the number of elements of the OM. New observations include that there is no unique deletion minimal OM represented by a given cocircuit graph. This is proved by the example in the left of Figure 37 found with Ricardo Strausz and Juan José Montellano-Ballesteros. A natural question in this context can be seen as another weakening of Murty's conjecture:

Problem 19. *Does the cocircuit graph of an OM determine its rank?*

Related questions concern the diameter of G^*. The oldest and most famous question in this context is:

Conjecture 18. *For the cocircuit graph G^* of an OM with n elements and rank r, we have* $\mathrm{diam}(G^*) \leq n - r + 2$.

This is known to hold for $r \leq 3$ [7,72] and has been related to the Hirsch conjecture. Recently, in [1] Conjecture 18 has been reduced to the uniform case. Together with computer experiments this confirms Conjecture 18 for $n \leq 9$ and $n - 4 \leq r$. While it is easy to see that the diameter of a cocircuit graph is in $O(rn)$, a weaker question due to Fukuda asks:

Problem 20. *Is the diameter of the cocircuit graph of an OM on n elements in $O(n)$ (independently of the rank)?*

7.9 Beyond COMs

One might think that already OMs are rather abstract and COMs are then maybe the last layer of generality one might be interested in, but I think (and literature confirms this), that there is further objects to be explored. Indeed, as illustrated throughout this thesis, one fruitful way of extending COMs are partial cubes, which are not exotic to the community at all.

However, I wish to briefly present a different extension of the theory of COMs, due to Margolis, Saliola, and Steinberg [130]. A *left regular band (LRB)* is a semigroup $(\mathcal{L}, \circ)$ with $X \circ Y \circ X = X \circ Y$ and $X \circ X = X$. This leads to a poset structure $\mathcal{F}$: $X \leq Y$ if $X \circ Y = Y$. An LRB is called *CW* if the principal filters in $\mathcal{F}$ are CW-posets.

As shown in [130] this class contains COMs, Björner-Ziegler complex OMs [177], and recently developed oriented interval greedoids [164]. We believe that extending the notions of minors, tope graphs, and further properties such as (FS) and (SE) to this more general context might be very valuable. Let us resume the subsection with a vague and vast goal.

Problem 21. *Extend the theory of CW left regular bands.*

Acknowledgments Mathematics is collaborative and for me a big part of the fun and productivity of it comes from the discussion and work with other people. In the following let me give an incomplete list of my recent and most important mathematical co-authors, students, colleagues, friends, family and hosts. I am grateful for the good times and good work: Adriana Hansberg, Amanda Montejano, Anatolii V. Zhuchok, Anna de Mier, Ararat Harutyunyan, Arnau Padrol, Bartłomiej Bosek, Bartosz Walczak, Benjamin Lévêque, Birgit Vogtenhuber, Boris Albar, Carolina Benedetti, Clément Rambaud, Daniel Gonçalves, Daniel Heldt, Deborah Oliveros, Dominique Poulalhon, Dömötör Pálvölgyi, Emanuele Delucchi, Emeric Gioan, Éric Fusy, Federico Ardila, Felix Schröder, Francisco Santos, František Kardos, Frédéric Havet, Gabriela Araujo, George B. Mertzios, Gil Puig, Gordon Royle, Guillaume Guégan, Guillaume Mercui-Voyant, Günter Rote, Gwen Joret, Grzegorz Matecki, Hannah Schreiber, Hans-Jürgen Bandelt, Ignacio García-Marco, Ilda da Silva, Irene Marquez, Jan Goedgebeur, Jan Philipp Pade, Jean Cardinal, Jean-Florent Raymond, Jérémie Chalopin, Jeronimo Valencia, Jesus de Loera, Jonathan Chappelon, Jonathan Rollin, Jorge Ramírez Alfonsín, Juan José Montellano-Ballesteros, Judith Stumpp, Leonardo Martinez-Sandoval, Leonhard Lücken, Louis Esperet, Luis Montejano, Luis P. Montejano, Manon Philibert, Mareike Massow, Maria A. Axenovich, Marie Albenque, Martin Winter, Matěj Stehlík, Matjaž Kovše, Michael Hoffmann, Michał Lasoń, Nathann Cohen, Nicolas Nisse, Noga Alon, Oswin Aichholzer, Paul S. Wenger, Petru Valicov, Pierre Aboulker, Piotr Micek, Raman Sanyal, Raphael Steiner, Rémi Desgranges, Reza Naserasr, Ricardo Gómez Aiza, Ricardo Strausz, Roland Grappe, Sandi Klavžar, Sarah Blind, Sonja Riedel, Stefan Felsner, Stefan Langerman, Stuart Margolis, Tilen Marc, Tom Trotter, Tony Huynh, Torsten Mütze, Torsten Ueckerdt, Ulrich Knauer, Victor Chepoi, Vincent Pilaud, Winfried Hochstättler, Yann Vaxès, Yida Zhu.

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COMs: Complexes of Oriented Matroids

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Abstract. In his seminal 1983 paper, Jim Lawrence introduced lopsided sets and featured them as asymmetric counterparts of oriented matroids, both sharing the key property of strong elimination. Moreover, symmetry of faces holds in both structures as well as in the so-called affine oriented matroids. These two fundamental properties (formulated for covectors) together lead to the natural notion of "conditional oriented matroid" (abbreviated COM). These novel structures can be characterized in terms of three cocircuits axioms, generalizing the familiar characterization for oriented matroids. We describe a binary composition scheme by which every COM can successively be erected as a certain complex of oriented matroids, in essentially the same way as a lopsided set can be glued together from its maximal hypercube faces. A realizable COM is represented by a hyperplane arrangement restricted to an open convex set. Among these are the examples formed by linear extensions of ordered sets, generalizing the oriented matroids corresponding to the permutohedra. Relaxing realizability to local realizability, we capture a wider class of combinatorial objects: we show that non-positively curved Coxeter zonotopal complexes give rise to locally realizable COMs.

Keywords: oriented matroid, lopsided set, cell complex, tope graph, cocircuit, Coxeter zonotope.

Contents

1. Introduction

1.1. **Avant-propos.** Co-invented by Bland & Las Vergnas [11] and Folkman & Lawrence [21], and further investigated by Edmonds & Mandel [20] and many other authors, oriented matroids represent a unified combinatorial theory of orientations of ordinary matroids, which simultaneously captures the basic properties of sign vectors representing the regions in a hyperplane arrangement in $\mathbb{R}^n$ and of sign vectors of the circuits in a directed graph. Furthermore, oriented matroids find applications in point and vector configurations, convex polytopes, and linear programming. Just as ordinary matroids, oriented matroids may be defined in a multitude of distinct but equivalent ways: in terms of cocircuits, covectors, topes, duality, basis orientations, face lattices, and arrangements of pseudospheres. A full account of the theory of oriented matroids is provided in the book by Björner, Las Vergnas, White, and Ziegler [10] and an introduction to this rich theory is given in the textbook by Ziegler [39].

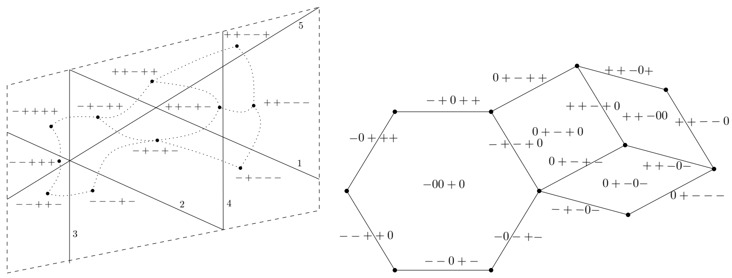
Lopsided sets of sign vectors defined by Lawrence [29] in order to capture the intersection patterns of convex sets with the orthants of $\mathbb{R}^d$ (and further investigated in [3, 4]) have found numerous applications in statistics, combinatorics, learning theory, and computational geometry, see e.g. [34] for further details. Lopsided sets represent an "asymmetric offshoot" of oriented matroid theory. According to the topological representation theorem, oriented matroids can be viewed as regular CW cell complexes decomposing the $(d-1)$-sphere. Lopsided sets on the other hand can be regarded as particular contractible cubical complexes.

In this paper we propose a common generalization of oriented matroids and lopsided sets which is so natural that it is surprising that it was not discovered much earlier. It also generalizes such well-known and useful structures as convex geometries and CAT(0) cube (and zonotopal) complexes. In this generalization, global symmetry and the existence of the zero sign vector, required for oriented matroids, are replaced by local relative conditions. Analogous to conditional lattices (see [22, p. 93]) and conditional antimatroids (which are particular lopsided sets [3]), this motivates the name "conditional oriented matroids" (abbreviated: COMs) for these new structures. Furthermore, COMs can be viewed as complexes whose cells are oriented matroids and which are glued together in a lopsided fashion. To illustrate the concept of a COM and compare it with similar notions of oriented matroids and lopsided sets, we continue by describing the geometric model of realizable COMs.

1.2. **Realizable COMs: motivating example.** Let us begin by considering the following familiar scenario of hyperplane arrangements and realizable oriented matroids; compare with [10, Sections 2.1, 4.5] or [39, p. 212]. Given a *central arrangement of hyperplanes* of $\mathbb{R}^d$ (i.e., a finite set E of $(d-1)$–dimensional linear subspaces of $\mathbb{R}^d$), the space $\mathbb{R}^d$ is partitioned into open regions and recursively into regions of the intersections of some of the given hyperplanes. Specifically, we may encode the location of any point from all these regions relative to this arrangement when for each hyperplane one of the corresponding halfspaces is regarded as positive and the other one as negative. Zero designates location on that hyperplane. Then the set $\mathcal{L}$ of all sign vectors representing the different regions relative to E is the set of covectors of the oriented matroid of the arrangement E. The oriented matroids obtained in this way are called *realizable*. If instead of a central arrangement one considers finite arrangements E of affine hyperplanes (an affine hyperplane is the translation of a (linear) hyperplane by a vector), then the sets of sign vectors of regions defined by E are known as *realizable affine oriented matroids* [27] and [3, p.186]. Since an affine arrangement on $\mathbb{R}^d$ can be viewed as the intersection of a central arrangement of $\mathbb{R}^{d+1}$ with a translate of a coordinate hyperplane, each realizable affine oriented matroid can be embedded into a larger realizable oriented matroid.

Now suppose that E is a central or affine arrangement of hyperplanes of $\mathbb{R}^d$ and C is an open convex set, which may be assumed to intersect all hyperplanes of E in order to avoid redundancy. Restrict the arrangement pattern to C, that is, remove all sign vectors which represent the open regions disjoint from C. Denote the resulting set of sign vectors by $\mathcal{L}(E, C)$ and call it a *realizable COM*. Figure 1(a) displays an arrangement comprising two pairs of parallel lines and a fifth line intersecting the former four lines within the open 4-gon. Three lines (nos. 2, 3, and 5) intersect in a common point. The line arrangement defines 11 open regions within the open 4-gon, which are represented by their topes, viz. ± 1 covectors. The dotted lines connect adjacent topes and thus determine the tope graph of the arrangement. This graph is shown in Figure 1(b) unlabeled, but augmented by the covectors of the 14 one-dimensional and 4 two-dimensional faces.

Our model of realizable COMs generalizes realizability of oriented and affine oriented matroids on the one hand and realizability of lopsided sets on the other hand. In the case of a central arrangement E with C being any open convex set containing the origin (e.g., the open unit ball or the entire space $\mathbb{R}^d$), the resulting set $\mathcal{L}(E, C)$ of sign vectors coincides with the realizable oriented matroid of E. If the arrangement E is affine and C is the entire space, then $\mathcal{L}(E, C)$ coincides with the realizable affine oriented matroid of E. The realizable lopsided sets arise by taking the (central) arrangement E of all coordinate hyperplanes E restricted to arbitrary open convex sets C of $\mathbb{R}^d$. In fact, the original definition of realizable lopsided sets by Lawrence [29] is similar but used instead an arbitrary (not necessarily open) convex set K and as regions the closed orthants. Clearly, K can be assumed to be a polytope, namely the convex hull of points representing the closed orthants meeting K. Whenever the polytope K does not meet a closed orthant then some open neighborhood of K does not meet that orthant either. Since there are only finitely many orthants, the intersection of



(a) (b)

FIGURE 1. (a) An arrangement of five lines and its tope graph. (b) Faces and edges of the tope graph are labeled with corresponding covectors. Sign vectors are abbreviated as strings of $+$, $-$, and 0 and to be read from left to right.

these open neighborhoods results in an open set C which has the same intersection pattern with the closed orthants as K. Now, if an open set meets a closed orthant it will also meet the corresponding open orthant, showing that both concepts of realizable lopsided sets coincide.

1.3. **Properties of realizable COMs.** For the general scenario of realizable COMs, we can attempt to identify its basic properties that are known to hold in oriented matroids. Let X and Y be sign vectors belonging to $\mathcal{L}$, thus designating regions represented by two points x and y within C relative to the arrangement E; see Figure 2 (compare with Fig. 4.1.1 of [10]). Connect the two points by a line segment and choose $\epsilon > 0$ small enough so that the open ball of radius ϵ around x intersects only those hyperplanes from E on which x lies. Pick any point w from the intersection of this ϵ-ball with the open line segment between x and y. Then the corresponding sign vector W is the *composition* $X \circ Y$ as defined by

$$(X \circ Y)_e = X_e \text{ if } X_e \neq 0 \text{ and } (X \circ Y)_e = Y_e \text{ if } X_e = 0.$$

Hence the following rule is fulfilled:

(Composition) $X \circ Y$ belongs to $\mathcal{L}$ for all sign vectors X and Y from $\mathcal{L}$.

If we select instead a point u on the ray from y via x within the ϵ-ball but beyond x, then the corresponding sign vector U has the opposite signs relative to W at the coordinates corresponding to the hyperplanes from E on which x is located and which do not include the ray from y via x. Therefore the following property holds:

(Face symmetry) $X \circ -Y$ belongs to $\mathcal{L}$ for all X, Y in $\mathcal{L}$.

Next assume that the hyperplane e from E separates x and y, that is, the line segment between x and y crosses e at some point z. The corresponding sign vector Z is then zero at e and equals the composition $X \circ Y$ at all coordinates where X and Y are *sign-consistent*, that is, do not have opposite signs:

(Strong elimination) for each pair X, Y in $\mathcal{L}$ and for each $e \in E$ with $X_e Y_e = -1$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E$ with $X_f Y_f \neq -1$.

Now, the single property of oriented matroids that we have missed in the general scenario is the existence of the zero sign vector, which would correspond to a non-empty intersection of all hyperplanes from E within the open convex set C:

(Zero vector) the zero sign vector $\mathbf{0}$ belongs to $\mathcal{L}$.

On the other hand, if the hyperplanes from E happen to be the coordinate hyperplanes, then wherever a sign vector X has zero coordinates, the composition of X with any sign vector from $\{\pm 1, 0\}^E$ is a sign vector belonging to $\mathcal{L}$. This rule, which is stronger than composition and face symmetry, holds in lopsided systems, for which the "tope" sets are exactly the lopsided sets sensu Lawrence [29]:

(Ideal composition) $X \circ Y \in \mathcal{L}$ for all $X \in \mathcal{L}$ and all sign vectors Y, that is, substituting any zero coordinate of a sign vector from $\mathcal{L}$ by any other sign yields a sign vector of $\mathcal{L}$.

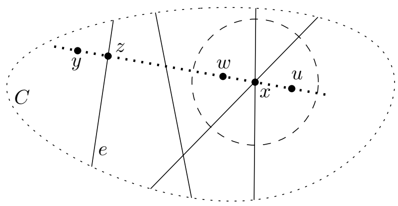


FIGURE 2. Motivating model for the three axioms.

In the model of hyperplane arrangements we can retrieve the cells which constitute oriented matroids. Indeed, consider all non-empty intersections of hyperplanes from E that are minimal with respect to inclusion. Select any sufficiently small open ball around some point from each intersection. Then the subarrangement of hyperplanes through each of these points determines regions within these open balls which yield an oriented matroid of sign vectors. Taken together all these constituents form a complex of oriented matroids, where their intersections are either empty or are faces of the oriented matroids involved. These complexes are still quite special as they conform to global strong elimination. The latter feature is not guaranteed in general complexes of oriented matroids, which were called "bouquets of oriented matroids" [15].

It is somewhat surprising that the generalization of oriented matroids defined by the three fundamental properties of composition, face symmetry, and strong elimination have apparently not yet been studied systematically. On the other hand, the preceding discussion shows that the properties of composition and strong elimination hold whenever C is an arbitrary convex set. We used the hypothesis that the set C be open only for deriving face symmetry. The following example shows that indeed face symmetry may be lost when C is closed: take two distinct lines in the Euclidean plane, intersecting in some point x and choose as C a closed halfspace which includes x and the entire $++$ region but is disjoint from the $--$ region. Then $+-, +0, ++, 0+, -+$, and 00 comprise the sign vectors of the regions within C, thus violating face symmetry. Indeed, the obtained system can be regarded as a lopsided system with an artificial zero added. On the other hand, one can see that objects obtained this way are realizable *oriented matroid polyhedra* [10, p. 420].

1.4. **Structure of the paper.** In Section 2 we will continue by formally introducing the systems of sign vectors considered in this paper. In Section 3 we prove that COMs are closed under minors and simplification, thus sharing this fundamental property with oriented matroids. We also introduce the fundamental concepts of fibers and faces of COMs, and show that faces of COMs are OMs. Section 4 is dedicated to topes and tope graphs of COMs and we show that both these objects uniquely determine a COM. Section 5 is devoted to characterizations of minimal systems of sign-vectors which generate a given COM by composition. In Section 6 we extend these characterizations and, analogously to oriented matroids, obtain a characterization of COMs in terms of cocircuits. In Section 7 we define carriers, hyperplanes, and halfspaces, all being COMs naturally contained in a given COM. We present a characterization of COMs in terms of these substructures. In Section 8 we study decomposition and amalgamation procedures for COMs and show that every COM can be obtained by successive amalgamation of oriented matroids. In Section 9, we extend the Euler-Poincaré formula from OMs to COMs and characterize lopsided sets in terms of a particular variant of it. In Section 10 as a resuming example we study the COMs provided by the ranking extensions – aka weak extensions – of a partially ordered set and illustrate the operations and the results of the paper on them. In Section 11 we consider a topological approach to COMs and study them as complexes of oriented matroids. In particular, we show that non-positively curved Coxeter zonotopal complexes give rise to COMs. We close the paper with several concluding remarks and two conjectures in Section 12.

2. Basic axioms

We follow the standard oriented matroids notation from [10]. Let E be a non-empty finite (ground) set and let $\mathcal{L}$ be a non-empty *set of sign vectors*, i.e., maps from E to $\{\pm 1, 0\} = \{-1, 0, +1\}$. The elements of $\mathcal{L}$ are also referred to as *covectors* and denoted by capital letters X, Y, Z, etc. For $X \in \mathcal{L}$, the subset $\underline{X} = \{e \in E : X_e \neq 0\}$ is called the *support* of X and its complement $X^0 = E \setminus \underline{X} = \{e \in E : X_e = 0\}$ the *zero set* of X (alias the *kernel* of X). Simlaly, we denote $X^+ = \{e \in E : X_e = +\}$ and $X^- = \{e \in E : X_e = -\}$. We can regard a sign vector X as the incidence vector of a ± 1 signed subset $\underline{X}$ of E such that to each element of E one element of the signs $\{\pm 1, 0\}$ is assigned. We denote by $\leq$ the product ordering on $\{\pm 1, 0\}^E$ relative to the standard ordering of signs with $0 \leq -1$ (sic!) and $0 \leq +1$.

For $X, Y \in \mathcal{L}$, we call $S(X,Y) = \{f \in E : X_f Y_f = -1\}$ the *separator* of X and Y. The elements of $S(X,Y)$ are said to *separate* X and Y. If the separator is empty, then X and Y are said to be *sign-consistent*. In particular, this is the case when X is below Y, that is, $X \leq Y$ holds. The *composition* of X and Y is the sign vector $X \circ Y$, where $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ if $X_e = 0$. Note that $X \leq X \circ Y$ for all sign vectors X, Y.

Given a set of sign vectors $\mathcal{L}$, its *topes* are the maximal elements of $\mathcal{L}$ with respect to $\leq$. Further let

$$\uparrow\mathcal{L} := \{Y \in \{\pm 1, 0\}^E : X \leq Y \text{ for some } X \in \mathcal{L}\} = \{X \circ W : X \in \mathcal{L} \text{ and } W \in \{\pm 1, 0\}^E\}$$

be the *upset* of $\mathcal{L}$ in the ordered set $(\{\pm 1, 0\}^E, \leq)$.

If a set of sign vectors is closed with respect to $\circ$, then the resulting idempotent semigroup (indeed a left regular band alias skew semilattice [31, 32]) is called the *braid semigroup*, see e.g. [8]. The composition operation naturally occurs also elsewhere: for a single coordinate, the composite $x \circ y$ on $\{\pm 1, 0\}$ is actually derived as the term $t(x, 0, y)$ (using 0 as a constant) from the ternary discriminator t on $\{\pm 1, 0\}$, which is defined by $t(a, b, c) = a$ if $a \neq b$ and $t(a, b, c) = c$ otherwise. Then in this context of algebra and logic, "composition" on the product $\{\pm 1, 0\}^E$ would rather be referred to as a "skew Boolean join" [6].

We continue with the formal definition of the main axioms as motivated and discussed in the previous section.

Composition:

(C) $X \circ Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

Condition (C) is taken from the list of axioms for oriented matroids. Since $\circ$ is associative, arbitrary finite compositions can be written without bracketing $X_1 \circ \ldots \circ X_k$ so that (C) entails that they all belong to $\mathcal{L}$. Note that contrary to a convention sometimes made in oriented matroids we do not consider compositions over an empty index set, since this would imply that the zero sign vector belonged to $\mathcal{L}$. We highlight condition (C) here although it will turn out to be a consequence of another axiom specific in this context. The reason is that we will later use several weaker forms of the axioms which are no longer consequences from one another.

Strong elimination:

(SE) for each pair $X, Y \in \mathcal{L}$ and for each $e \in S(X,Y)$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X,Y)$.

Note that $(X \circ Y)_f = (Y \circ X)_f$ holds exactly when $f \in E \setminus S(X,Y)$. Therefore the sign vector Z provided by (SE) serves both ordered pairs X, Y and Y, X.

Condition (SE) is one of the axioms for covectors of oriented matroids and is implied by the property of route systems in lopsided sets, see Theorem 5 of [29].

Symmetry:

(Sym) $-\mathcal{L} = \{-X : X \in \mathcal{L}\} = \mathcal{L}$, that is, $\mathcal{L}$ is closed under sign reversal.

Symmetry restricted to zero sets of covectors (where corresponding supports are retained) is dubbed:

Face symmetry:

(FS) $X \circ -Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

This condition can also be expressed by requiring that for each pair X, Y in $\mathcal{L}$ there exists $Z \in \mathcal{L}$ with $X \circ Z = Z$ such that $X = \frac{1}{2}(X \circ Y + X \circ Z)$. Face symmetry trivially implies (C) because by (FS) we first get $X \circ -Y \in \mathcal{L}$ and then $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y)$ for all $X, Y \in \mathcal{L}$.

Ideal composition:

(IC) $\uparrow\mathcal{L} = \mathcal{L}$.

Notice that (IC) implies (C) and (FS). We are now ready to define the main objects of our study:

Definition 1. A system of sign vectors $(E, \mathcal{L})$ is called a:

- *strong elimination system* if $\mathcal{L}$ satisfies (C) and (SE),
- *conditional oriented matroid (COM)* if $\mathcal{L}$ satisfies (FS) and (SE),
- *oriented matroid (OM)* if $\mathcal{L}$ satisfies (C), (Sym), and (SE),
- *lopsided system* if $\mathcal{L}$ satisfies (IC) and (SE).

For oriented matroids one can replace (C) and (Sym) by (FS) and

Zero vector:

(Z) the zero sign vector $\mathbf{0}$ belongs to $\mathcal{L}$.

Notice that the axiom (SE) can be somewhat weakened in the presence of (C), i.e., in particular in Definition 1. If (C) is true in the system $(E, \mathcal{L})$, then for $X, Y \in \mathcal{L}$ we have $X \circ Y = (X \circ Y) \circ (Y \circ X)$, $\underline{X \circ Y} = \underline{Y \circ X} = \underline{X} \cup \underline{Y}$, and also for the separators we have $S(X \circ Y, Y \circ X) = S(X, Y)$.

Therefore, if (C) holds, we may substitute (SE) by

(SE$^=$) for each pair $X, Y \in \mathcal{L}$ with $\underline{X} = \underline{Y}$ and for each $e \in S(X, Y)$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X, Y)$,

The axioms (C), (FS), (SE$^=$) (plus a fourth condition) were used by Karlander [27] in his study of affine oriented matroids that are embedded as "halfspaces" (see Section 7 below) of oriented matroids.

3. Minors, fibers, and faces

In the present technical section we show that the class of COMs is closed under taking minors, defined as for oriented matroids. We use this to establish that simplifications and semisimplifications of COMs are minors of COMs and therefore COMs. We also introduce fibers and faces of COMs, which will be of importance for the rest of the paper.

Let $(E, \mathcal{L})$ be a COM and $A \subseteq E$. Given a sign vector $X \in \{\pm 1, 0\}^E$ by $X \setminus A$ we refer to the *restriction* of X to $E \setminus A$, that is $X \setminus A \in \{\pm 1, 0\}^{E \setminus A}$ with $(X \setminus A)_e = X_e$ for all $e \in E \setminus A$. The *deletion* of A is defined as $(E \setminus A, \mathcal{L} \setminus A)$, where $\mathcal{L} \setminus A := \{X \setminus A : X \in \mathcal{L}\}$. The *contraction* of A is defined as $(E \setminus A, \mathcal{L}/A)$, where $\mathcal{L}/A := \{X \setminus A : X \in \mathcal{L} \text{ and } \underline{X} \cap A = \varnothing\}$. If a system of sign vectors arises by deletions and contractions from another one it is said to be *minor* of it.

Lemma 1. *The properties* (C), (FS), *and* (SE) *are all closed under taking minors. In particular, if $(E, \mathcal{L})$ is a COM and $A \subseteq E$, then $(E \setminus A, \mathcal{L} \setminus A)$ and $(E \setminus A, \mathcal{L}/A)$ are COMs as well.*

Proof. We first prove that $(E \setminus A, \mathcal{L} \setminus A)$ is a COM. To see (C) and (FS) let $X \setminus A, Y \setminus A \in \mathcal{L} \setminus A$. Then $X \circ (\pm Y) \in \mathcal{L}$ and $(X \circ (\pm Y)) \setminus A = X \setminus A \circ (\pm Y \setminus A) \in \mathcal{L} \setminus A$.

To see (SE) let $X \setminus A, Y \setminus A \in \mathcal{L} \setminus A$ and e an element separating $X \setminus A$ and $Y \setminus A$. Then there is $Z \in \mathcal{L}$ with $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X, Y)$. Clearly, $Z \setminus A \in \mathcal{L} \setminus A$ satisfies (SE) with respect to $X \setminus A, Y \setminus A$.

Now, we prove that $(E \setminus A, \mathcal{L}/A)$ is a COM. Let $X \setminus A, Y \setminus A \in \mathcal{L}/A$, i.e., $\underline{X} \cap A = \underline{Y} \cap A = \varnothing$. Hence $\underline{X \circ (\pm Y)} \cap A = \varnothing$ and therefore $X \setminus A \circ (\pm Y \setminus A) \in \mathcal{L}/A$, proving (C) and (FS).

To see (SE) let $X \setminus A, Y \setminus A \in \mathcal{L}/A$ and e an element separating $X \setminus A$ and $Y \setminus A$. Then there is $Z \in \mathcal{L}$ with $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X, Y)$. In particular, if $X_f = Y_f = 0$, then $Z_f = 0$. Therefore, $Z \setminus A \in \mathcal{L}/A$ and it satisfies (SE). □

Lemma 2. *If $(E, \mathcal{L})$ is a system of sign vectors and $A, B \subseteq E$ with $A \cap B = \varnothing$, then $(E \setminus (A \cup B), (\mathcal{L} \setminus A)/B) = (E \setminus (A \cup B), (\mathcal{L}/B) \setminus A)$.*

Proof. It suffices to prove this for A and B consisting of single elements e and f, respectively. Now $X \setminus \{e, f\} \in (\mathcal{L} \setminus \{e\})/\{f\}$ if and only if $X \in \mathcal{L}$ with $X_f = 0$ which is equivalent to $X \setminus \{e\} \in \mathcal{L} \setminus \{e\}$ with $(X \setminus \{e\})_f = 0$. This is, $X \setminus \{e, f\} \in (\mathcal{L}/\{f\}) \setminus \{e\}$. □

Next, we will define simple and semisimple systems of sign vectors. These are motivated by the hyperplane model for COMs, that possesses additional properties, reflecting that we have a set of hyperplanes rather than a multiset and that the given convex set is open. This is also motivated by the requirement of defining systems of sign vectors not containing coloops and parallel elements, which is relevant, for example, for the identifications of topes.

A *coloop* of $(E, \mathcal{L})$ is an element $e \in E$ such that $X_e = 0$ for all $X \in \mathcal{L}$. Hence $(E, \mathcal{L})$ does not have coloops if and only if for each element e, there exists a covector X with $X_e \neq 0$. Two elements $e, e' \in E$ of $(E, \mathcal{L})$ are *parallel*, denoted $e \parallel e'$, if either $X_e = X_{e'}$ for all $X \in \mathcal{L}$ or $X_e = -X_{e'}$ for all $X \in \mathcal{L}$. It is easy to see that $\parallel$ is an equivalence relation. The condition that $(E, \mathcal{L})$ does not contain parallel elements can be expressed by the requirement that for each pair $e \neq f$ in E, there exist $X, Y \in \mathcal{L}$ with $X_e \neq X_f$ and $Y_e \neq -Y_f$.

Simple systems are defined by two axioms which are slightly stronger than those which ensure that the absence of coloops and parallel elements. We call the system $(E, \mathcal{L})$ *simple* if it satisfies the following non-redundancy axioms:

Simplicity:

(N1) for each $e \in E$, $\{\pm 1, 0\} = \{X_e : X \in \mathcal{L}\}$;
(N2) for each pair $e \neq f$ in E, there exist $X, Y \in \mathcal{L}$ with $\{X_e X_f, Y_e Y_f\} = \{\pm 1\}$.

We will also consider the weaker notion of semisimple systems, which are the simple systems when restricted to the set

$$E_\pm := \{e \in E : \{X_e : X \in \mathcal{L}\} \neq \{+1\}, \{-1\}\}$$

of those elements e at which $\mathcal{L}$ is not non-zero constant. We call the system $(E, \mathcal{L})$ *semisimple* if it satisfies the following restricted non-redundancy axioms:

Semisimplicity:

(RN1) for each $e \in E_\pm$, $\{\pm 1, 0\} = \{X_e : X \in \mathcal{L}\}$;
(RN2) for each pair $e \neq f$ in $E_\pm$, there exist $X, Y \in \mathcal{L}$ with $\{X_e X_f, Y_e Y_f\} = \{\pm 1\}$.

Let $E_0 := \{e \in E : X_e = 0$ for all $X \in \mathcal{L}\}$ be the set of all coloops. Condition (RN1) yields that $E_0 = \varnothing$. The condition that there are no coloops is relevant for the identification of topes. Recall that a tope of $\mathcal{L}$ is any covector X that is maximal with respect to the standard sign ordering defined above. In the presence of (C), the covector X is a tope precisely when $X \circ Y = X$ for all $Y \in \mathcal{L}$, that is, for each $e \in E$ either $X_e \in \{\pm 1\}$ or $Y_e = 0$ for all $Y \in \mathcal{L}$. In particular, if both (C) and $E_0 = \varnothing$ hold, then the topes are exactly the covectors with full support E. Notice also that condition (N2) yields that there are no parallel elements. Consequently, simple systems do not contain coloops, parallel elements, and their topes are the covectors with full support (while semisimple systems satisfy the first and the third conditions).

Further put

$$E_1 := \{e \in E : \#\{X_e : X \in \mathcal{L}\} = 1\} = E_0 \cup (E \setminus E_\pm),$$

$$E_2 := \{e \in E : \#\{X_e : X \in \mathcal{L}\} = 2\}.$$

The sets $E_0 \cup E_2$ and $E_1 \cup E_2$ comprise the positions at which $\mathcal{L}$ violates (RN1) or (N1), respectively. Hence the deletions $(E \setminus (E_0 \cup E_2), \mathcal{L} \setminus (E_0 \cup E_2))$ and $(E \setminus (E_1 \cup E_2), \mathcal{L} \setminus (E_1 \cup E_2))$ constitute the canonical transforms of $(E, \mathcal{L})$ satisfying (RN1) and (N1), respectively. One equivalence class of the parallel relation $\|$ restricted to $\mathcal{L} \setminus (E_0 \cup E_2)$ is $E \setminus E_\pm$, i.e., it comprises the (non-zero) constant positions. Exactly this class gets removed when one restricts $\|$ further to $\mathcal{L} \setminus (E_1 \cup E_2)$. Selecting a transversal for the classes of $\|$ on $\mathcal{L} \setminus (E_1 \cup E_2)$, which arbitrarily picks exactly one element from each class and deletes all others, results in a simple system. Restoring the entire class $E \setminus E_\pm$ then yields a semisimple system. We refer to these canonical transforms as to the *simplification* and *semisimplification* of $(E, \mathcal{L})$, respectively. Then from Lemma 1 we obtain:

Lemma 3. *The semisimplification of a system $(E, \mathcal{L})$ of sign vectors is a semisimple minor and the simplification is a simple minor, unique up to sign reversal on subsets of $E_\pm$. Either system is a COM whenever $(E, \mathcal{L})$ is.*

For a system $(E, \mathcal{L})$, a *fiber* relative to some $X \in \mathcal{L}$ and $A \subseteq E$ is a set of sign vectors defined by

$$\mathcal{R} = \{Y \in \mathcal{L} : Y \setminus A = X \setminus A\}.$$

We say that such a fiber is *topal* if $(X \setminus A)^0 = \varnothing$, that is, $X \setminus A$ is a tope of the minor $(E \setminus A, \mathcal{L} \setminus A)$. $\mathcal{R}$ is a *face* if X can be chosen so that $X^0 = A$, whence faces are topal fibers. Note that the entire system $(E, \mathcal{L})$ can be regarded both as a topal fiber and as the result of an empty deletion or contraction. If $(E, \mathcal{L})$ satisfies (C), then the fiber relative to $A := X^0$, alias *X-face*, associated with a sign vector X can be expressed in the form

$$F(X) := \{X \circ Y : Y \in \mathcal{L}\} = \mathcal{L} \cap \uparrow\{X\}.$$

If $S(V, W)$ is non-empty for $V, W \in \mathcal{L}$, then the corresponding faces $F(V)$ and $F(W)$ are disjoint. Else, if V and W are sign-consistent, then $F(V) \cap F(W) = F(V \circ W)$. In particular $F(V) \subseteq F(W)$ is equivalent to $V \in F(W)$, that is, $W \leq V$. The ordering of faces by inclusion thus reverses the sign ordering. The following observations are straightforward and recorded here for later use:

Lemma 4. *If $(E, \mathcal{L})$ is a strong elimination system or a COM, respectively, then so are all fibers of $(E, \mathcal{L})$. If $(E, \mathcal{L})$ is semisimple, then so is every topal fiber. If $(E, \mathcal{L})$ is a COM, then for any $X \in \mathcal{L}$ the minor $(E \setminus \underline{X}, F(X) \setminus \underline{X})$ corresponding to the face $F(X)$ is an OM, which is simple whenever $(E, \mathcal{L})$ is semisimple.*

4. Tope graphs

One may wonder whether and how the topes of a semisimple COM $(E, \mathcal{L})$ determine and generate $\mathcal{L}$. We cannot avoid using face symmetry because one can turn every COM which is not an OM into a strong elimination system by adding the zero vector to the system, without affecting the topes. The following result for simple oriented matroids was first observed by Mandel (unpublished), see [10, Theorem 4.2.13].

Proposition 1. *Every semisimple COM $(E, \mathcal{L})$ is uniquely determined by its set of topes.*

Proof. We proceed by induction on $\#E$. For a single position the assertion is trivial. So assume $\#E \geq 2$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two COMs on E sharing the same set of topes. Then deletion of any $g \in E$ results in two COMs with equal tope sets, whence $\mathcal{L}' \setminus g = \mathcal{L} \setminus g$ by the induction hypothesis. Suppose that there exists some $W \in \mathcal{L}' \setminus \mathcal{L}$ chosen with W^0 as small as possible. Then $\#W^0 > 0$. Take any $e \in W^0$. Then as $\mathcal{L}' \setminus e = \mathcal{L} \setminus e$ by semisimplicity there exists a sign vector V in $\mathcal{L}$ such that $V \setminus e = W \setminus e$ and $V_e \neq 0$. Since $V^0 \subset W^0$, we infer that $V \in \mathcal{L}'$ by the minimality choice of W. Then, by (FS) applied to W and V in $\mathcal{L}'$, we get $W \circ -V \in \mathcal{L}'$. This sign vector also belongs to $\mathcal{L}$ because $\#(W \circ -V)^0 = \#V^0 < \#W^0$. Finally, apply (SE) to the pair $V, W \circ -V$ in $\mathcal{L}$ relative to e and obtain $Z = W \in \mathcal{L}$, in conflict with the initial assumption. □

The *tope graph* of a semisimple COM on E is the graph with all topes as its vertices where two topes are adjacent exactly when they differ in exactly one coordinate. In other words, the tope graph is the subgraph of the $\#E$-dimensional hypercube with vertex set $\{\pm 1\}^E$ induced by the tope set. Isometry means that the internal distance in the subgraph is the same as in the hypercube. Isometric subgraphs of the hypercube are often referred to as a *partial cubes* [25]. For tope graphs of oriented matroids the next result was first proved in [28]

Proposition 2. *The tope graph of a semisimple strong elimination system $(E, \mathcal{L})$ is a partial cube in which the edges correspond to the sign vectors of $\mathcal{L}$ with singleton zero sets.*

Proof. If X and Y are two adjacent topes, say, differing at position $e \in E$, then the vector $Z \in \mathcal{L}$ provided by (SE) for this pair relative to e has 0 at e and coincides with X and Y at all other positions. By way of contradiction assume that now X and Y are two topes which cannot be connected by a path in the tope graph of length $\#S(X,Y) = k > 1$ such that k is as small as possible. Then the interval $[X,Y]$ consisting of all topes on shortest paths between X and Y in the tope graph comprises only X and Y. For $e \in S(X,Y)$ we find some $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_g = X_g$ for all $g \in E \setminus S(X,Y)$ by (SE). If there exists $f \in S(X,Y) \setminus \{e\}$ with $Z_f \neq 0$, then $Z \circ X$ or $Z \circ Y$ is a tope different from X and Y, but contained in $[X,Y]$, a contradiction.

If $Z_f = 0$ for all $f \in S(X,Y) \setminus \{e\}$, then by (RN2) there is $W \in \mathcal{L}$ with $0 \neq W_e W_f \neq X_e X_f \neq 0$. We conclude that $Z \circ W \circ Y$ is a tope different from X and Y but contained in $[X,Y]$, a contradiction. This concludes the proof. □

Isometric embeddings of partial cubes into hypercubes are unique up to automorphisms of the hosting hypercube [16, Proposition 19.1.2] (and addition of superfluous dimensions). Hence, Propositions 1 and 2 together imply the following result, which generalizes a similar result of [9] for tope graphs of OMs:

Proposition 3. *A simple COM is determined by its tope graph up to reorientation.*

5. Minimal generators of strong elimination systems

We have seen in the preceding section that a COM is determined by its tope set. There is a more straightforward way to generate any strong elimination system from bottom to top by taking suprema. This generation process involves only some weaker forms of the axioms (C) and (SE).

Let $(E, \mathcal{L})$ be a system of sign vectors. Given $X, Y \in \mathcal{L}$ consider the following set of sign vectors which partially "conform" to X relative to subsets $A \subseteq S(X,Y)$:

$$\begin{aligned}\mathcal{W}_A(X,Y) &= \{Z \in \mathcal{L} : Z^+ \subseteq X^+ \cup Y^+, Z^- \subseteq X^- \cup Y^-, \text{ and } S(X,Z) \subseteq E \setminus A\} \\ &= \{Z \in \mathcal{L} : Z_g \in \{0, X_g, Y_g\} \text{ for all } g \in E, \text{ and } Z_h \in \{0, X_h\} \text{ for all } h \in A\}.\end{aligned}$$

For $A = \varnothing$ we use the short-hand $\mathcal{W}(X,Y)$, i.e.,

$$\mathcal{W}(X,Y) = \{Z \in \mathcal{L} : Z^+ \subseteq X^+ \cup Y^+, Z^- \subseteq X^- \cup Y^-\}$$

and for the maximum choice $A \supseteq S(X,Y)$ we write $\mathcal{W}_\infty(X,Y)$, i.e.,

$$\mathcal{W}_\infty(X,Y) = \{Z \in \mathcal{W}(X,Y) : S(X,Z) = \varnothing\}.$$

Trivially, $X, X \circ Y \in \mathcal{W}_A(X,Y)$ and $\mathcal{W}_B(X,Y) \subseteq \mathcal{W}_A(X,Y)$ for $A \subseteq B \subseteq E$. Note that $S(X,Z) \subseteq S(X,Y)$ for all $Z \in \mathcal{W}(X,Y)$. Each set $\mathcal{W}_A(X,Y)$ is closed under composition (and trivially is a downset with respect to the sign ordering). For, if $V, W \in \mathcal{W}_A(X,Y)$, then $(V \circ W)^+ \subseteq V^+ \cup W^+$ and $(V \circ W)^- \subseteq V^- \cup W^-$ holds trivially, and further, if $e \in S(X, V \circ W)$, say, $e \in X^+$ and $e \in (V \circ W)^- \subseteq V^- \cup W^-$, then $e \in S(X,V)$ or $e \in S(X,W)$, that is,

$$S(X, V \circ W) \subseteq S(X,V) \cup S(X,W) \subseteq E \setminus A.$$

Since each of the sets $\mathcal{W}_A(X,Y)$ is closed under composition, we may take the composition of all sign vectors in $\mathcal{W}_A(X,Y)$. The result may depend on the order of the constituents.

Some features of strong elimination are captured by weak elimination:

(WE) for each pair $X, Y \in \mathcal{L}$ and $e \in S(X,Y)$ there exists $Z \in \mathcal{W}(X,Y)$ with $Z_e = 0$.

Condition (WE) is in general weaker than (SE): consider, e.g., the four sign vectors $++, +-, --, 00$; the zero vector Z would serve all pairs X, Y for (WE) but for $X = ++$ and $Y = +-$ (SE) would require the existence of $+0$ rather than 00. In the presence of (IC), the strong and the weak versions of elimination are equivalent, that is, lopsided systems are characterized by (IC) and (WE) [4]. With systems satisfying (WE) one can generate lopsided systems by taking the upper sets:

Proposition 4 ([4]). *If $(E, \mathcal{K})$ is a system of sign vectors which satisfies* (WE)*, then $(E, \uparrow\mathcal{K})$ is a lopsided system.*

Proof. We have to show that (WE) holds for $(E, \uparrow\mathcal{K})$. For $X, Y \in \uparrow\mathcal{K}$ and some element e in $S(X,Y)$, pick $V, W \in \mathcal{K}$ with $V \leq X$ and $W \leq Y$. If $e \in S(V,W)$, then by (WE) in $\mathcal{K}$ one obtains some $U \in \uparrow\mathcal{K}$ such that $U_e = 0$ and $U_f \leq V_f \circ W_f \leq X_f \circ Y_f$ for all $f \in E \setminus S(X,Y)$. Then the sign vector Z defined by $Z_g := U_g$ for all $g \in S(X,Y)$ and $Z_f := X_f \circ Y_f$ for all $f \in E \setminus S(X,Y)$ satisfies $U \leq Z$ and hence belongs to $\uparrow\mathcal{K}$. If $e \notin S(V,W)$, then $V_e = 0$, say. Define a sign vector Z similarly as above: $Z_g := V_g$ for $g \in S(X,Y)$ and $Z_f := X_f \circ Y_f \geq V_f$ for $f \in E \setminus S(X,Y)$. Then $Z \in \uparrow\mathcal{K}$ is as required. □

This proposition applied to a COM $(E, \mathcal{L})$ yields an associated lopsided systems $(E, \uparrow\mathcal{L})$ having the same minimal sign vectors as $(E, \mathcal{L})$. This system is referred to as the *lopsided envelope* of $(E, \mathcal{L})$. In contrast to (SE) and ($\text{SE}^=$), the following variant of strong elimination allows us to treat the positions $f \in E \setminus S(X,Y)$ one at a time:

(SE1) for each pair $X, Y \in \mathcal{L}$ and $e \in S(X,Y)$ and $f \in E \setminus S(X,Y)$ there exists $Z \in \mathcal{W}(X,Y)$ such that $Z_e = 0$, and $Z_f = (X \circ Y)_f$.

Nevertheless, under composition axiom (C), all these variants of (SE) are equivalent:

Lemma 5. *Let $(E, \mathcal{L})$ be a system of sign vectors which satisfies* (C)*. Then all three variants* (SE), ($\text{SE}^=$)*, and* (SE1) *are equivalent.*

Proof. The equivalence of (SE) and (SE$^=$) in the presence of (C) was argued at the end of Section 2. Trivially, (SE) implies (SE1). Conversely, if (SE1) holds, then for every $e \in S(X,Y)$ we obtain a set $\{Z^{(\{e\},f)} : f \in E \setminus S(X,Y)\}$ of solutions, one for each f. Then the composition in any order of these solutions yields a solution Z for (SE), because $Z^{(\{e\},f)} \leq Z$ for all $f \in E \setminus S(X,Y)$ and $Z_e = 0$, whence $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X,Y)$ and $Z_e = 0$. □

Since strong elimination captures some features of composition, one may wonder whether (C) can be somewhat weakened in the presence of (SE) or (SE1). Here suprema alias conformal compositions come into play:

(CC) $X \circ Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$ with $S(X,Y) = \varnothing$.

Recall that X and Y are sign-consistent, that is, $S(X,Y) = \varnothing$ exactly when X and Y commute: $X \circ Y = Y \circ X$. We say that a composition $X^{(1)} \circ \ldots \circ X^{(n)}$ of sign vectors is *conformal* if it constitutes the supremum of $X^{(1)}, \ldots, X^{(n)}$ with respect to the sign ordering. Thus, $X^{(1)}, \ldots, X^{(n)}$ commute exactly when they are bounded from above by some sign vector, which is the case when the set of all $X_e^{(i)}$ $(1 \leq i \leq n)$ includes at most one non-zero sign (where e is any fixed element of E). If we wish to highlight this property we denote the supremum of $X^{(1)}, \ldots, X^{(n)}$ by $\bigodot_{i=1}^n X^{(i)}$ or $X^{(1)} \odot \ldots \odot X^{(n)}$ (instead of $X^{(1)} \circ \ldots \circ X^{(n)}$). Clearly the conformal property is Helly-type in the sense that a set of sign vectors has a supremum if each pair in that set does.

Given any system $\mathcal{K}$ of sign vectors on E define $\bigodot \mathcal{K}$ as the set of all (non-empty) suprema of members from $\mathcal{K}$. We say that a system $(E, \mathcal{K})$ of sign vectors *generates* a system $(E, \mathcal{L})$ if $\bigodot \mathcal{K} = \mathcal{L}$. We call a sign vector $X \in \mathcal{L}$ (*supremum-*)*irreducible* if it does not equal the (non-empty!) conformal composition of any sign vectors from $\mathcal{L}$ different from X. Clearly, the irreducible sign vectors of $\mathcal{L}$ are unavoidable when generating $\mathcal{L}$. We denote the set of irreducibles of $\mathcal{L}$ by $\mathcal{J} = \mathcal{J}(\mathcal{L})$.

Theorem 1. *Let $(E, \mathcal{L})$ be a system of sign vectors. Then the following conditions are equivalent:*

(i) $(E, \mathcal{L})$ *is a strong elimination system;*
(ii) $\mathcal{L}$ *satisfies* (CC) *and* (SE1)*;*
(iii) $\mathcal{L}$ *satisfies* (CC) *and some set $\mathcal{K}$ with $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{L}$ satisfies* (SE1).
(iv) $\mathcal{L}$ *satisfies* (CC) *and its set $\mathcal{J}$ of irreducibles satisfies* (SE1).

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial. Now, to see (ii) $\Longrightarrow$ (iv) let $(E, \mathcal{L})$ satisfy (CC) and (SE1). For $X, Y \in \mathcal{J}$, $e \in S(X,Y)$, and $f \in E \setminus S(X,Y)$ we first obtain $Z \in \mathcal{L}$ with $Z_e = 0$ and $Z_f = (X \circ Y)_f$. Since Z is the supremum of some $Z^{(1)}, \ldots, Z^{(n)}$ from $\mathcal{J}$, there must be an index i for which $Z_f^{(i)} = Z_f$ and trivially $Z_e^{(i)} = 0$ holds. Therefore $\mathcal{J}$ satisfies (SE1). This proves (iv). Furthermore, (iv) $\Longrightarrow$ (iii) is trivial.

As for (iii) $\Longrightarrow$ (i) assume that (SE1) holds in $\mathcal{K}$. The first task is to show that the composite $X \circ Y$ for $X, Y \in \mathcal{K}$ can be obtained as a conformal composite (supremum) of X with members $Y^{(f)}$ of $\mathcal{K}$, one for each $f \in E \setminus S(X,Y)$. Given such a position f, start an iteration with $Z^{(\varnothing,f)} := Y$, and as long as $A \neq S(X,Y)$, apply (SE1) to $X, Z^{(A,f)} \in \mathcal{K}$, which then returns a sign vector

$$Z^{(A\cup\{e\},f)} \in \mathcal{W}_A(X, Z^{(A,f)}) \cap \mathcal{K} \subseteq \mathcal{W}_A(X,Y) \cap \mathcal{K} \text{ with}$$

$$Z_e^{(A\cup\{e\},f)} = 0 \text{ and } Z_f^{(A\cup\{e\},f)} = (X \circ Z^{(A,f)})_f = (X \circ Y)_f.$$

In particular, $Z^{(A\cup\{e\},f)} \in \mathcal{W}_{A\cup\{e\}}(X,Y) \cap \mathcal{K}$. Eventually, the iteration stops with

$$Y^{(f)} := Z^{(S(X,Y),f)} \in \mathcal{W}_\infty(X,Y) \cap \mathcal{K} \text{ satisfying}$$

$$Y_f^{(f)} = (X \circ Y)_f \text{ and } (X \circ Y^{(f)})_e = X_e \text{ for all } e \text{ separating } X \text{ and } Y.$$

Now take the supremum of X and all $Y^{(f)}$: then

$$X \circ Y = X \odot \bigodot_{f \in E \setminus S(X,Y)} Y^{(f)}$$

constitutes the desired representation.

Next consider a composition $X \circ X^{(1)} \circ \ldots \circ X^{(n)}$ of $n+1 \geq 3$ sign vectors from $\mathcal{K}$. By induction on n we may assume that

$$X^{(1)} \circ \ldots \circ X^{(n)} = Y^{(1)} \odot \ldots \odot Y^{(m)}$$

where $Y^{(i)} \in \mathcal{K}$ for all $i = 1, \ldots, m$. Since any supremum in $\{\pm 1, 0\}^E$ needs at most $\#E$ constituents, we may well choose $m = \#E$. Similarly, as the case $n = 1$ has been dealt with, each $X \circ Y^{(i)}$ admits a commutative representation

$$X \circ Y^{(i)} = X \odot Z^{(m(i-1)+1)} \odot Z^{(m(i-1)+2)} \odot \ldots \odot Z^{(mi)} \ (i = 1, \ldots, m).$$

We claim that $Z^{(j)}$ and $Z^{(k)}$ commute for all $j, k \in \{1, \ldots, m^2\}$. Indeed,

$$Z^{(j)} \leq X \circ Y^{(h)} \text{ and } Z^{(k)} \leq X \circ Y^{(i)} \text{ for some } h, i \in \{1, \ldots, m\}.$$

Then

$$Z^{(j)}, Z^{(k)} \leq (X \circ Y^{(h)}) \circ Y^{(i)} = (X \circ Y^{(i)}) \circ Y^{(h)}$$

because $Y^{(h)}$ and $Y^{(i)}$ commute, whence $Z^{(j)}$ and $Z^{(k)}$ commute as well. Therefore

$$X \circ X^{(1)} \circ \ldots \circ X^{(n)} = X \circ Y^{(1)} \circ \ldots \circ Y^{(m)} = (X \circ Y^{(1)}) \circ \ldots \circ (X \circ Y^{(m)})$$
$$= X \odot Z^{(1)} \odot \ldots \odot Z^{(m^2)}$$

gives the required representation. We conclude that $(E, \mathcal{L})$ satisfies (C).

To establish (SE1) for $\mathcal{L}$, let $X = X^{(1)} \odot \cdots \odot X^{(n)}$ and $Y = Y^{(1)} \odot \cdots \odot Y^{(m)}$ with $X^{(i)}, Y^{(j)} \in \mathcal{K}$ for all i, j. Take $e, f \in E$ such that e separates X and Y and f does not. We may assume that $X_e^{(i)} = X_e$ for $1 \leq i \leq h$, $Y_e^{(j)} = Y_e$ for $1 \leq j \leq k$, and equal to zero otherwise (where $h, k \geq 1$). Since $\mathcal{K}$ satisfies (SE1) there exists $Z^{(i,j)} \in \mathcal{W}(X^{(i)}, Y^{(j)}) \cap \mathcal{K}$ such that $Z_e^{(i,j)} = 0$ and $Z_f^{(i,j)} = (X^{(i)} \circ Y^{(j)})_f$ for $i \leq h$ and $j \leq k$. Then the composition of all $Z^{(i,1)}$ for $i \leq h$, $X^{(i)}$ for $i > h$ and all $Z^{(1,j)}$ for $j \leq k$, $Y^{(j)}$ for $j > k$ yields the required sign vector $Z \in \mathcal{W}(X,Y)$ with $Z_e = 0$ and $Z_f = (X \circ Y)_f$. We conclude that $(E, \mathcal{L})$ is indeed a strong elimination system by Lemma 5. □

From the preceding proof of (iv) $\Longrightarrow$ (i) we infer that the (supremum-)irreducibles of a strong elimination system $(E, \mathcal{L})$ are a fortiori irreducible with respect to arbitrary composition.

6. Cocircuits of COMs

In an OM, a cocircuit is a support-minimal non-zero covector, and the cocircuits form the unique minimal generating system for the entire set of covectors provided that composition over an empty index set is allowed. Thus, in our context the zero vector would have to be added to the generating set, i.e., we would regard it as a cocircuit as well. The cocircuits of COMs that we will consider next should on the one hand generate the entire system and on the other hand their restriction to any maximal face should be the set of cocircuits of the oriented matroid corresponding to that face via Lemma 4.

For any $\mathcal{K}$ with $\mathcal{J} = \mathcal{J}(\mathcal{L}) \subseteq \mathcal{K} \subseteq \mathcal{L}$ denote by $\mathrm{Min}(\mathcal{K})$ the set of all minimal sign vectors of $\mathcal{K}$. Clearly, $\mathrm{Min}(\bigodot \mathcal{K}) = \mathrm{Min}(\mathcal{K}) = \mathrm{Min}(\mathcal{J})$. We say that Y *covers* X in $\mathcal{L} = \bigodot \mathcal{J}$ (in symbols: $X \prec Y$) if $X < Y$ holds and there is no sign vector $Z \in \mathcal{L}$ with $X < Z < Y$. The following set $\mathcal{C}$ is intermediate between $\mathcal{J}$ and $\mathcal{L}$:

$$\mathcal{C} = \mathcal{C}(\mathcal{L}) := \mathcal{J}(\mathcal{L}) \cup \{X \in \mathcal{L} : W \prec X \text{ for some } W \in \mathrm{Min}(\mathcal{L})\}.$$

Since $\mathrm{Min}(\mathcal{L}) = \mathrm{Min}(\mathcal{J})$ and every cover $X \notin \mathcal{J}$ of some $W \in \mathrm{Min}(\mathcal{J})$ is above some other $V \in \mathrm{Min}(\mathcal{J})$, we obtain:

$$\mathcal{C} = \mathcal{C}(\mathcal{J}) := \mathcal{J} \cup \{W \odot V : V, W \in \mathrm{Min}(\mathcal{J}) \text{ and } W \prec W \odot V\}.$$

We will make use of the following variant of face symmetry restricted to comparable covectors:

(FS$^{\leq}$) $X \circ -Y \in \mathcal{L}$ for all $X \leq Y$ in $\mathcal{L}$.

Note that (FS) and (FS$^{\leq}$) are equivalent in any system $\mathcal{L}$ satisfying (C), as one can let $X \circ Y$ substitute Y in (FS). We can further weaken face symmetry by restricting it to particular covering pairs $X \prec Y$:

(FS$^{\prec}$) $W \circ -Y \in \mathcal{L}$ for all $W \in \mathrm{Min}(\mathcal{L})$ and $Y \in \mathcal{L}$ with $W \prec Y$ in $\mathcal{L}$, or equivalently,
$W \circ -Y \in \mathcal{C}$ for all $W \in \mathrm{Min}(\mathcal{C})$ and $Y \in \mathcal{C}$ with $W \prec Y$.

Indeed, since sign reversal constitutes an order automorphism of $\{\pm 1, 0\}^E$, we readily infer that in (FS$^{\prec}$) $W \circ -Y$ covers W, for if there was $X \in \mathcal{L}$ with $W \prec X < W \circ -Y$, then $W < W \circ -X < W \circ -(W \circ -Y) = W \circ Y = Y$, a contradiction.

To show that (FS$^{\prec}$) implies (FS$^{\leq}$) takes a little argument, as we will see next.

Proposition 5. *Let $(E, \mathcal{J})$ be a system of sign vectors. Then $(E, \bigodot \mathcal{J})$ is a COM such that $\mathcal{J}$ is its set of irreducibles if and only if $\mathcal{C} = \mathcal{C}(\mathcal{J})$ satisfies* (FS$^{\prec}$) *and $\mathcal{J}$ satisfies* (SE1) *and*

(IRR) *if $X = \bigodot_{i=1}^{n} X_i$ for $X, X_1, \ldots, X_n \in \mathcal{J} (n \geq 2)$, then $X = X_i$ for some $1 \leq i \leq n$.*

Proof. First, assume that $(E, \mathcal{L} = \bigodot \mathcal{J})$ is a COM with $\mathcal{J} = \mathcal{J}(\mathcal{L})$. From Theorem 1 we know that $\mathcal{J}$ satisfies (SE1), while (IRR) just expresses irreducibility. Since $\mathcal{L}$ is the set of covectors of a COM, from the discussion preceding the theorem it follows that $\mathcal{L}$ satisfies (FS$^{\prec}$). Consequently, $\mathcal{C} = \mathcal{C}(\mathcal{J})$ satisfies (FS$^{\prec}$).

Conversely, in the light of Theorem 1, it remains to prove that (FS$^{\prec}$) for $\mathcal{C}$ implies (FS$^{\leq}$) for $(E, \mathcal{L})$. Note that for $W < X < Y$ in $\mathcal{L}$ we have $X \circ -Y = X \circ W \circ -Y$, whence for $W < Y \in \mathcal{L}$ we only need to show $W \circ -Y \in \mathcal{L}$ when W is a minimal sign vector of $\mathcal{L}$ (and thus belonging to $\mathcal{J} \subseteq \mathcal{C}$). Now suppose that $W \circ -Y \notin \mathcal{L}$ for some covector Y such that $\#Y^0$ is as large as possible. Thus as $Y \notin \mathcal{C}$ there exists $X \in \mathcal{L}$ with $W \prec X < Y$. By (FS$^{\prec}$), $W \circ -X \in \mathcal{L}$ holds. Pick any element $e \in S(W \circ -X \circ Y, Y) = W^0 \cap \underline{X}$ and choose some $Z \in \mathcal{L}$ with $Z_e = 0$ and $Z_f = (W \circ -X \circ Y)_f$ for all $f \in E \setminus S(W \circ -X \circ Y, Y)$ by virtue of (SE). In particular, $Y = X \circ Z$. Then necessarily $W < Z$ and $Y^0 \cup \{e\} \subseteq Z^0$, so that $W \circ -Z \in \mathcal{L}$ by the maximality hypothesis. Therefore with Theorem 1 we get

$$W \circ -Y = W \circ -(X \circ Z) = (W \circ -X) \circ (W \circ -Z) \in \mathcal{L},$$

which is a contradiction. This establishes (FS$^{\leq}$) for $\mathcal{L}$ and thus completes the proof of Proposition 5. □

Proposition 5 yields the following alternative axiomatization of COMs in terms of covectors, that is of independent interest:

Corollary 1. *A system $(E, \mathcal{L})$ of sign vectors is a COM if and only if $(E, \mathcal{L})$ satisfies* (CC)*,* (SE1) *and* (FS$^{\prec}$)*.*

Let us now advance towards the axiomatization of COMs in terms of cocircuits. Given a COM $(E, \mathcal{L})$, we call the minimal sign vectors of $\mathcal{L}$ the *improper cocircuits* of $(E, \mathcal{L})$. A *proper cocircuit* is any sign vector $Y \in \mathcal{L}$ which *covers* some improper cocircuit X. *Cocircuit* then refers to either kind, improper or proper. Hence, $\mathcal{C}(\mathcal{L})$ is the set of all cocircuits of $\mathcal{L}$. Note that in oriented matroids the zero vector is the only improper cocircuit and the usual OM cocircuits are the proper cocircuits in our terminology. In lopsided systems $(E, \mathcal{L})$, the improper cocircuits are the barycenters of maximal hypercubes [4]. In a COM improper circuits are irreducible, but not all proper circuits need to be irreducible. Here is the main result of this section.

Theorem 2. *Let $(E, \mathcal{C})$ be a system of sign vectors and let $\mathcal{L} := \bigodot \mathcal{C}$. Then $(E, \mathcal{L})$ is a COM such that $\mathcal{C}$ is its set of cocircuits if and only if $\mathcal{C}$ satisfies* (SE1),(FS$^{\prec}$)*, and*

(COC) $\mathcal{C} = \mathrm{Min}(\mathcal{C}) \cup \{Y \in \bigodot \mathcal{C} : W \prec Y \text{ for some } W \in \mathrm{Min}(\mathcal{C})\}$.

Proof. Let $(E, \mathcal{L})$ be a COM and $\mathcal{C}$ be its set of cocircuits. By Proposition 5, $\mathcal{C}$ satisfies (FS$^{\prec}$). From the proof of Theorem 1, part (ii)⇒(iv), we know that a sign vector Z demanded in (SE1) could always be chosen from the irreducibles, which are particular cocircuits. Therefore $\mathcal{C} = \mathcal{C}(L)$ satisfies (SE1). Finally, (COC) just expresses that $\mathcal{C}$ exactly comprises the cocircuits of the set $\mathcal{L}$ it generates.

Conversely, $\mathcal{L}$ satisfies (CC) by definition. Since $\mathcal{J}(\mathcal{L}) \subseteq \mathcal{C}$ and $\mathcal{C}$ satisfies (SE1), applying Theorem 1 we conclude that $\mathcal{J}$ satisfies (SE1). Consequently, as $\mathcal{C}$ satisfies (FS$^{\prec}$) and $\mathcal{J}$ satisfies (SE1), $\mathcal{L}$ is a COM by virtue of Proposition 5. □

To give a simple class of planar examples, consider the hexagonal grid, realized as the 1-skeleton of the regular tiling of the plane with (unit) hexagons. A *benzenoid graph* is the 2-connected graph formed by the vertices and edges from hexagons lying entirely within the region bounded by some cycle in the hexagonal grid; the given cycle thus constitutes the boundary of the resulting benzenoid graph [25]. A *cut segment* is any minimal (closed) segment of a line perpendicular to some edge and passing through its midpoint such that the removal of all edges cut by the segment results in exactly two connected components, one signed $+$ and the other $-$. The ground set E comprises all these cut segments. The set $\mathcal{L}$ then consists of all sign vectors corresponding to the vertices and the barycenters (midpoints) of edges and 6-cycles (hexagons) of this benzenoid graph. For verifying that $(E, \mathcal{L})$ actually constitutes a COM, it is instructive to apply Proposition 5: the set $\mathcal{J}$ of irreducible members of $\mathcal{L}$ encompasses the barycenter vectors of the boundary edges and of all hexagons of the benzenoid. The barycenter vectors of two hexagons/edges/vertices are sign consistent exactly when they are incident. Therefore $\mathcal{J}$ generates all covectors of $\mathcal{L}$ via (CC). Condition (FS$^{\prec}$) is realized through inversion of an edge at the center of a hexagon it is incident with. Condition (SE1) is easily checked by considering two cases each (depending on whether Z is eventually obtained as a barycenter vector of a hexagon or of an edge) for pairs X, Y of hexagon/edge barycenters.

7. Hyperplanes, carriers, and halfspaces

For a system $(E, \mathcal{L})$ of sign vectors, a *hyperplane* of $\mathcal{L}$ is the set

$$\mathcal{L}_e^0 := \{X \in \mathcal{L} : X_e = 0\} \text{ for some } e \in E.$$

The *carrier* $N(\mathcal{L}_e^0)$ of the hyperplane $\mathcal{L}_e^0$ is the union of all faces $F(X')$ of $\mathcal{L}$ with $X' \in \mathcal{L}_e^0$, that is,

$$N(\mathcal{L}_e^0) := \{X \in \mathcal{L} : W \leq X \text{ for some } W \in \mathcal{L}_e^0\}.$$

The *positive and negative ("open") halfspaces* supported by the hyperplane $\mathcal{L}_e^0$ are

$$\mathcal{L}_e^+ := \{X \in \mathcal{L} : X_e = +1\},$$

$$\mathcal{L}_e^- := \{X \in \mathcal{L} : X_e = -1\}.$$

The carrier $N(\mathcal{L}_e^0)$ minus $\mathcal{L}_e^0$ splits into its positive and negative parts:

$$N^+(\mathcal{L}_e^0) := \mathcal{L}_e^+ \cap N(\mathcal{L}_e^0),$$

$$N^-(\mathcal{L}_e^0) := \mathcal{L}_e^- \cap N(\mathcal{L}_e^0).$$

The closure of the disjoint halfspaces $\mathcal{L}_e^+$ and $\mathcal{L}_e^-$ just adds the corresponding carrier:

$$\overline{\mathcal{L}_e^+} := \mathcal{L}_e^+ \cup N(\mathcal{L}_e^0) = \mathcal{L}_e^+ \cup \mathcal{L}_e^0 \cup N^-(\mathcal{L}_e^0),$$

$$\overline{\mathcal{L}_e^-} := \mathcal{L}_e^- \cup N(\mathcal{L}_e^0) = \mathcal{L}_e^- \cup \mathcal{L}_e^0 \cup N^+(\mathcal{L}_e^0).$$

The former is called the *closed positive halfspace* supported by $\mathcal{L}_e^0$, and the latter is the corresponding *closed negative halfspace*. Both overlap exactly in the carrier.

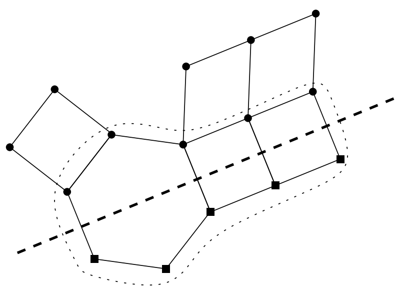


FIGURE 3. A hyperplane (dashed), its associated open halfspaces (square and round vertices, respectively) and the associated carrier (dotted) in a COM.

Proposition 6. *Let $(E, \mathcal{L})$ be a system of sign vectors. Then all its hyperplanes, its carriers and their positive and negative parts, its halfspaces and their closures are strong elimination systems or COMs, respectively, whenever $(E, \mathcal{L})$ is such. If $(E, \mathcal{L})$ is an OM, then so are all its hyperplanes and carriers.*

Proof. We already know that fibers preserve (C),(FS), and (SE). Moreover, intersections preserve the two properties (C) and (FS). Since $X' \leq X$ and $Y' \leq Y$ imply both $X' \leq X \circ Y, X' \leq X \circ (-Y)$ and $Y' \leq Y \circ X, Y' \leq Y \circ (-X)$, we infer that (C) and (FS) carry over from $\mathcal{L}$ to $N(\mathcal{L}_e^0)$.

In what follows let $(E, \mathcal{L})$ be a strong elimination system. We need to show that $N(\mathcal{L}_e^0)$ satisfies (SE). Let $X', Y' \in \mathcal{L}_e^0$ and $X, Y \in \mathcal{L}$ such that $X' \leq X$ and $Y' \leq Y$. Then $S(X', Y') \subseteq S(X, Y)$. Apply (SE) to the pair X, Y in $\mathcal{L}$ relative to some element e' separating X and Y. This yields some $Z \in \mathcal{W}(X, Y)$ with $Z_{e'} = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X, Y)$. If $e' \in S(X', Y')$ as well, then apply (SE) to X', Y' in $\mathcal{L}_e^0$ giving $Z' \in \mathcal{W}(X', Y') \cap \mathcal{L}_e^0$ with $Z'_{e'} = 0$ and $Z'_f = (X' \circ Y')_f$ for all $f \in E \setminus S(X, Y) \subseteq E \setminus S(X', Y')$. If $e' \in \underline{X'} \setminus \underline{Y'}$, then put $Z' := Y'$. Else, if $e' \in E \setminus \underline{X'}$, put $Z' := X'$. Observe that all cases are covered as $S(X', Y') = S(X, Y) \cap \underline{X'} \cap \underline{Y'}$. We claim that in any case $Z' \circ Z$ is the required sign vector fulfilling (SE) for X, Y relative to e'. Indeed, $Z' \circ Z$ belongs to $N(\mathcal{L}_e^0)$ since $Z' \in \mathcal{L}_e^0$ and $Z \in \mathcal{L}$. Then $Z' \circ Z \in \mathcal{W}(X, Y)$ because $\mathcal{W}(X', Y') \subseteq \mathcal{W}(X, Y)$ and $\mathcal{W}(X, Y)$ is closed under composition. Let $f \in E \setminus S(X, Y)$. Then X'_f, X_f, Y'_f, Y_f all commute.

In particular,

$$(Z' \circ Z)_f = Z'_f \circ Z_f = X'_f \circ Y'_f \circ X_f \circ Y_f = X'_f \circ X_f \circ Y'_f \circ Y_f = (X \circ Y)_f$$

whenever both $Y'_{e'} = Y_{e'}$ and $X'_{e'} = X_{e'}$ hold. If however $Y'_e = 0$, then $(Y' \circ Z)_f = Y'_f \circ X_f \circ Y_f = X_f \circ Y'_f \circ Y_f = (X \circ Y)_f$. Else, if $X'_e = 0$, then $(X' \circ Z)_f = X'_f \circ X_f \circ Y_f = (X \circ Y)_f$. This finally shows that the carrier of $\mathcal{L}_e^0$ satisfies (SE).

To prove that $\overline{\mathcal{L}_e^+}$ satisfies (SE) for a pair $X, Y \in \overline{\mathcal{L}_e^+}$ relative to some $e' \in S(X,Y)$, assume that $X \in \mathcal{L}_e^+$ and $Y \in N(\mathcal{L}_e^0) \setminus \mathcal{L}_e^+$ since (SE) has already been established for both $\mathcal{L}_e^+$ and $N(\mathcal{L}_e^0)$ and the required sign vector would equally serve the pair Y, X. Now pick any $Y' \in \mathcal{L}_e^0$ with $Y' \leq Y$. Then two cases can occur for $e' \in S(X,Y)$.

Case 1. $Y'_{e'} = 0$.

Then $(Y' \circ X)_{e'} = X_{e'}, S(Y' \circ X, Y) \subseteq S(X,Y)$, and $Y' \leq Y' \circ X$, whence $Y' \circ X \in N(\mathcal{L}_e^0)$. Applying (SE) to $Y' \circ X, Y$ in $N(\mathcal{L}_e^0)$ relative to e' yields $Z' \in \mathcal{W}(Y' \circ X, Y) \subseteq \mathcal{W}(X,Y)$ with $Z'_{e'} = 0$ and

$$Z'_f = Y'_f \circ X_f \circ Y_f = X_f \circ Y'_f \circ Y_f = (X \circ Y)_f \text{ for all } f \in E \setminus S(X,Y).$$

Case 2. $Y'_{e'} = Y_{e'}$.

As above we can select $Z \in \mathcal{W}(X,Y)$ with $Z_{e'} = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus S(X,Y)$. Analogously choose $Z' \in \mathcal{W}(X,Y')$ with $Z'_{e'} = 0$ and $Z'_f = (X \circ Y')_f$ for all $f \in E \setminus S(X,Y')$. We claim that in this case $Z' \circ Z$ is a sign vector from $\mathcal{L}_e^+$ as required for X, Y relative to e'. Indeed, $Z'_e = (X \circ Y')_e = +1 = (X \circ Y)_e$ because $X_e = +1, Y'_e = 0$ and consequently $e \notin S(X,Y')$. For $f \in E \setminus S(X,Y)$ we have

$$(Z' \circ Z)_f = X_f \circ Y'_f \circ X_f \circ Y_f = X_f \circ X_f \circ Y'_f \circ Y_f = (X \circ Y)_f$$

by commutativity, similarly as above. This proves that $\overline{\mathcal{L}_e^+}$ satisfies (SE).

To show that $N^+(\mathcal{L}_e^0) = \mathcal{L}_e^+ \cap N(\mathcal{L}_e^0)$ satisfies (SE), we can apply (SE) to some pair $X, Y \in N^+(\mathcal{L}_e^0)$ relative to some $e' \in S(X,Y)$ first within $N(\mathcal{L}_e^0)$ and then within $\mathcal{L}_e^+$ to obtain two sign vectors $Z' \in N(\mathcal{L}_e^0) \cap \mathcal{W}(X,Y)$ and $Z \in \mathcal{L}_e^+ \cap \mathcal{W}(X,Y)$ such that $Z'_{e'} = 0 = Z_{e'}$ and $Z'_f = (X \circ Y)_f = Z_f$ for all $f \in E \setminus S(X,Y)$. Then $Z' \leq Z' \circ Z \in N(\mathcal{L}_e^0)$ and $(Z' \circ Z)_e = (X \circ Y)_e = +1$ as $e \notin S(X,Y)$. Moreover, $(Z' \circ Z)_f = (X \circ Y)_f$ for all $f \in E \setminus S(X,Y)$. This establishes (SE) for $N^+(\mathcal{L}_e^0)$. The proofs for $\overline{\mathcal{L}_e^-}$ and $N^-(\mathcal{L}_e^0)$ are completely analogous. The last statement of the proposition is then trivially true because the zero vector, once present in $\mathcal{L}$, is also contained in all hyperplanes (and hence the carriers). □

A particular class of COMs obtained by the above proposition are halfspaces of OMs. These are usually called *affine oriented matroids*, see [27] and [10, p. 154]. Karlander [27] has shown how an OM can be reconstructed from any of its halfspaces. The proof of his intriguing axiomatization of affine oriented matroids, however, has a gap, which has been filled only recently [5]. Only few results exist about the complex given by an affine oriented matroid [18,19].

We continue with the following recursive characterization of COMs:

Theorem 3. *Let $(E, \mathcal{L})$ be a semisimple system of sign vectors. Then $(E, \mathcal{L})$ is a strong elimination system if and only if the following four requirements are met:*

(1) *the composition rule (C) holds in $(E, \mathcal{L})$,*
(2) *all hyperplanes of $(E, \mathcal{L})$ are strong elimination systems,*
(3) *the tope graph of $(E, \mathcal{L})$ is a partial cube,*

(4) *for each pair X, Y of adjacent topes (i.e., with $\#S(X,Y) = 1$) the barycenter of the corresponding edge, i.e. the sign vector $\frac{1}{2}(X+Y)$, belongs to $\mathcal{L}$.*

Moreover, $(E, \mathcal{L})$ is a COM if and only if it satisfies (1),(3),(4), and

($2'$) *all hyperplanes of $(E, \mathcal{L})$ are COMs,*

In particular, $(E, \mathcal{L})$ is an OM if and only if it satisfies (1),(4), and

($2''$) *all hyperplanes of $(E, \mathcal{L})$ are OMs,*

($3'$) *the tope set of $(E, \mathcal{L})$ is a* simple acycloid, see [26]*, i.e., induces a partial cube and satisfies* (Sym).

Proof. The "if" directions of all three assertions directly follow from Propositions 6 and 2. Conversely, using (1) and Lemma 5, we only need to verify (SE$^{=}$) to prove the first assertion.

To establish (SE$^{=}$), let X and Y be any different sign vectors from $\mathcal{L}$. Assume that $\underline{X} = \underline{Y}$ and $e \in S(X,Y)$. If the supports are not all of E, then we can apply (SE$^{=}$) to the hyperplane associated with a zero coordinate of X and Y according to condition (2) and obtain a sign vector Z as required. Otherwise, both X and Y are topes. Then a shortest path joining X and Y in the tope graph is indexed by the elements of $S(X,Y)$ and thus includes an edge associated with e. Then the corresponding barycenter map Z (that belongs to $\mathcal{L}$ by condition (4)) of this edge does the job. Thus $(E, \mathcal{L})$ is a semisimple strong elimination system.

In order to complete the proof of the second assertion it remains to establish (FS$^{\leq}$). So let X and Y be any different sign vectors from $\mathcal{L}$ with $X \circ Y = Y$. In particular, X is not a tope and Y belongs to the face $F(X)$. If the support $\underline{Y}$ does not equal E, then again we find a common zero coordinate of X and Y, so that we can apply (FS$^{\leq}$) in the corresponding hyperplane to yield the sign vector opposite to Y relative to X. So we may assume that Y is a tope. Since $(E, \mathcal{L})$ is a semisimple strong elimination system, from Proposition 2 we infer that the tope graph of $F(X)$ is a partial cube containing at least two topes. Thus there exists a tope $U \in F(X)$ adjacent to Y in the tope graph, say $S(U,Y) = \{e\}$. Let W be the barycenter map of this edge. Applying (FS$^{\leq}$) for the pair X, W in the hyperplane $\mathcal{L}^0_e$ relative to e we obtain $X \circ (-W) \in \mathcal{L}^0_e$. By (1) we have $X \circ (-W) \circ U \in \mathcal{L}$. Since $X \circ (-W) \circ U = X \circ (-Y)$ this concludes the proof.

As for the third assertion, note that symmetric COMs are OMs and symmetry for non-topes is implied by symmetry for hyperplanes. □

8. Decomposition and amalgamation

Proposition 6 provides the necessary ingredients for a decomposition of a COM, which is not an OM, into smaller COM constituents. Assume that $(E, \mathcal{L})$ is a semisimple COM that is not an OM. Put $\mathcal{L}' := \mathcal{L}^-_e$ and $\mathcal{L}'' := \overline{\mathcal{L}^+_e}$. Then $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ and $\mathcal{L}' \cap \mathcal{L}'' = N^-(\mathcal{L}^0_e)$. Since X determines a maximal face not included in $\mathcal{L}^0_e$, we infer that $\mathcal{L}' \setminus \mathcal{L}'' \neq \varnothing$ and trivially $\mathcal{L}'' \setminus \mathcal{L}' \neq \varnothing$. By Proposition 6, all three systems $(E, \mathcal{L}')$, $(E, \mathcal{L}'')$, and $(E, \mathcal{L}' \cap \mathcal{L}'')$ are COMs, which are easily seen to be semisimple.

Moreover, $\mathcal{L}' \circ \mathcal{L}'' \subseteq \mathcal{L}'$ holds trivially. If $W \in \mathcal{L}^0_e$ and $X \in \mathcal{L}^-_e$, then $W \circ X \in F(W) \subseteq N(\mathcal{L}^0_e)$, whence $\mathcal{L}'' \circ \mathcal{L}' \subseteq \mathcal{L}''$. This motivates the following amalgamation process which in a way reverses this decomposition procedure.

We say that a system $(E, \mathcal{L})$ of sign vectors is a *COM amalgam* of two semisimple COMs $(E, \mathcal{L}')$ and $(E, \mathcal{L}'')$ if the following conditions are satisfied:

(1) $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ with $\mathcal{L}' \setminus \mathcal{L}'', \mathcal{L}'' \setminus \mathcal{L}', \mathcal{L}' \cap \mathcal{L}'' \neq \varnothing$;
(2) $(E, \mathcal{L}' \cap \mathcal{L}'')$ is a semisimple COM;
(3) $\mathcal{L}' \circ \mathcal{L}'' \subseteq \mathcal{L}'$ and $\mathcal{L}'' \circ \mathcal{L}' \subseteq \mathcal{L}''$;
(4) for $X \in \mathcal{L}' \setminus \mathcal{L}''$ and $Y \in \mathcal{L}'' \setminus \mathcal{L}'$ with $X^0 = Y^0$ there exists a shortest path in the graphical hypercube on $\{\pm1\}^{E \setminus X^0}$ for which all its vertices and barycenters of its edges belong to $\mathcal{L} \setminus X^0$.

Proposition 7. *The COM amalgam of semisimple COMs $(E, \mathcal{L}')$ and $(E, \mathcal{L}'')$ constitutes a semisimple COM $(E, \mathcal{L})$ for which every maximal face is a maximal face of at least one of the two constituents.*

Proof. $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ satisfies (C) because $\mathcal{L}'$ and $\mathcal{L}''$ do and for $X \in \mathcal{L}'$ and $Y \in \mathcal{L}''$ one obtains $X \circ Y \in \mathcal{L}' \subseteq \mathcal{L}$ and $Y \circ X \in \mathcal{L}'' \subseteq \mathcal{L}$ by (3). Then $\mathcal{L}$ also satisfies (FS$^{\leq}$) since for $X \leq Y = X \circ Y$ in $\mathcal{L}$ the only nontrivial case is that $X \in \mathcal{L}'$ and $Y \in \mathcal{L}''$, say. Then $Y = X \circ Y \in \mathcal{L}'$ by (3), whence $X \circ -Y \in \mathcal{L}' \subseteq \mathcal{L}$.

Every minimal sign vector $X \in \mathcal{L}$, say $X \in \mathcal{L}'$, yields the face $F(X) = \{X \circ Y : Y \in \mathcal{L}\} \subseteq \mathcal{L}' \circ \mathcal{L} \subseteq \mathcal{L}'$. It is evident that $(E, \mathcal{L})$ is semisimple.

By Lemma 5, it remains to show (SE$^=$) for two sign vectors X and Y of $\mathcal{L}$ with $X^0 = Y^0$, where $X \in \mathcal{L}' \setminus \mathcal{L}''$ and $Y \in \mathcal{L}'' \setminus \mathcal{L}'$. Then let $e \in S(X, Y)$ and $f \in E \setminus S(X, Y)$. Then the barycenter of an e-edge on a shortest path P from $X \setminus X^0$ to $Y \setminus X^0$ between $\mathcal{L}' \setminus X^0$ and $\mathcal{L}'' \setminus X^0$ (guaranteed by condition (4)) yields the desired sign vector $Z \in \mathcal{L}$ with $Z_e = 0$, $X^0 \subseteq Z^0$, and $Z \in \mathcal{W}(X, Y)$. Since $X^0 = Y^0$, we have $X_f = Y_f$ by the choice of f. Since P is shortest, we get $Z_f = (X \circ Y)_f$. □

Summarizing the previous discussion and results, we obtain

Corollary 2. *Semisimple COMs are obtained via successive COM amalgamations from their maximal faces (that can be contracted to OMs).*

9. Euler-Poincaré formulae

In this section, we generalize the Euler-Poincaré formula known for OMs to COMs, which involves the rank function. This is an easy consequence of decomposition and amalgamation. In the case of lopsided systems and their hypercube cells the rank of a cell is simply expressed as the cardinality of the zero set of its associated covector.

Given an OM of rank r, for $0 \leq i \leq r-1$ one defines f_i as the number of cells of dimension f_i of the corresponding decomposition of the $(r-1)$-sphere, see Section 11 for more about this representation. It is well-known (cf. [10, Corollary 4.6.11]) that $\sum_{i=0}^{r-1}(-1)^i f_i = 1 + (-1)^{r-1}$.

Adding the summand $(-1)^{-1} f_{-1} = -1$ here artificially yields $\sum_{i=-1}^{r-1} (-1)^i f_i = (-1)^{r-1}$. Multiplying this equation by $(-1)^{r-1}$ and substituting i by $r-1-j$ yields

$$\sum_{j=0}^{r} (-1)^j f_{r-1-j} = \sum_{i=1}^{r-1} (-1)^{r-1-i} f_i = 1.$$

As f_{r-1-j} gives the number of OM faces of rank j we can restate this formula in covector notation as $\sum_{X\in\mathcal{L}} (-1)^{r(X)} = 1$, where $r(X)$ is the rank of the OM $F(X) \setminus \underline{X}$. We define the rank of the covector of a COM in the same way.

Since COMs arise from OMs by successive COM amalgamations, which do not create new faces, and at a step from $\mathcal{L}'$ and $\mathcal{L}''$ to the amalgamated $\mathcal{L}$ each face in the intersection is counted exactly twice, we obtain

$$\sum_{X\in\mathcal{L}} (-1)^{r(X)} = \sum_{X\in\mathcal{L}'} (-1)^{r(X)} + \sum_{X\in\mathcal{L}''} (-1)^{r(X)} - \sum_{X\in\mathcal{L}'\cap\mathcal{L}''} (-1)^{r(X)} = 1.$$

Proposition 8. *Every COM $(E, \mathcal{L})$ satisfies the Euler-Poincaré formula $\sum_{X\in\mathcal{L}} (-1)^{r(X)} = 1$.*

We now characterize lopsided systems in terms of an Euler-Poincaré formula. A system $(E, \mathcal{L})$ is said to satisfy the *Euler-Poincaré formula for zero sets* if

$$\sum_{X\in\mathcal{L}} (-1)^{\#X^0} = 1.$$

Proposition 9. *The following assertions are equivalent for a system $(E, \mathcal{L})$:*

(i) *$(E, \mathcal{L})$ is lopsided, that is, $(E, \mathcal{L})$ is a COM satisfying (IC);*
(ii) [38] *every topal fiber of $(E, \mathcal{L})$ satisfies the Euler formula for zero sets, and $\mathcal{L}$ is determined by the topes in the following way: for each sign vector $X \in \{\pm1, 0\}^E$, $X \in \mathcal{L} \Rightarrow X \circ Y \in \mathcal{L}$ for all $Y \in \{\pm1\}^E$;*
(iii) *every contraction of a topal fiber of $(E, \mathcal{L})$ satisfies the Euler formula for zero sets in its own right.*

Proof. Deletions, contractions, and fibers of lopsided sets are COMs satisfying (IC) as well, that is, are again lopsided. In case of a lopsided system $(E, \mathcal{L})$ for every $X \in \mathcal{L}$ we have $r(X) = \#X^0$. Therefore by Proposition 8 $(E, \mathcal{L})$ satisfies the Euler formula for zero sets. This proves the implication (i)⇒(iii).

As for (iii)⇒(ii), we proceed by induction on $\#X^0$ for $X \in \mathcal{L}$. Assume that X^0 is not empty. Pick $e \in X^0$ and delete the coordinate subset $X^0 \setminus e$ from X. Consider the topal fiber $\mathcal{R} = \{X' \in \mathcal{L} : X' \setminus \underline{X} = X \setminus \underline{X}\}$ relative to X and $\underline{X}$, and contract $\mathcal{R}$ to $\mathcal{R}/(X^0 \setminus e)$. Let $U^{(e)}$ denote the (unit) sign vector on E with $U_e^{(e)} = +1$ and $U_f^{(e)} = 0$ for $f \neq e$. Since $\mathcal{R}/(X^0 \setminus e)$ satisfies the Euler-Poincaré formula for zero sets, both $X \circ U^{(e)}$ and $X \circ -U^{(e)}$ must belong to $\mathcal{L}$. By the induction hypothesis

$$(X \circ U^{(e)}) \circ Z, (X \circ -U^{(e)}) \circ Z \in \mathcal{L} \text{ for all } Z \in \{\pm1\}^E,$$

whence indeed $X \circ Y \in \mathcal{L}$ for all $Y \in \{\pm1\}^E$.

To prove the final implication (ii)⇒(i), we employ the recursive characterization of Theorem 3. Since (IC) holds by the implication for $X \in \mathcal{L}$ in (ii), property (1) of this theorem is trivially fulfilled. Observe that $\{\pm 1\} \subseteq \{X_e : X \in \mathcal{L}\}$ because the topal fiber relative to $X \in \mathcal{L}$ and $\underline{X} \neq E$ contains all possible $-1, +1$ entries. If X, Y are two topes of $\mathcal{L}$ with $S(X,Y) = \{e\}$, then the topal fiber relative to X and $E \setminus e$ must contain $\frac{1}{2}(X+Y)$ by virtue of the Euler-Poincaré formula for zero sets. This establishes property (4) and (RN1).

Suppose that the topes of $\mathcal{L}$ do not form a partial cube in $\{\pm 1\}^E$. Then choose topes X and Y with $\#S(X,Y) \geq 2$ as small as possible such that the topal fiber $\mathcal{R}$ relative to X and $E \setminus S(X,Y)$ include no other topes than X or Y. The formula for zero sets implies that this topal fiber $\mathcal{R}$ must contain at least some $Z \in \mathcal{L}$ with Z^0 of odd cardinality. Then for $e \neq f$ in $S(X,Y)$ one can select signs for some tope Z' conforming to Z such that $Z'_e Z'_f \neq X_e X_f = Y_e Y_f$. Hence $\mathcal{R}$ contains the tope Z' that is different from X and Y, contrary to the hypothesis. This contradiction establishes that $\mathcal{L}$ fulfills property (3) and is semisimple.

Consider the hyperplane $\mathcal{L}_e^0$ and the corresponding halfspaces $\mathcal{L}_e^+$ and $\mathcal{L}_e^-$(which are two disjoint topal fibers of $\mathcal{L}$). Then the formula

$$\sum_{X \in \mathcal{L}} (-1)^{\#X^0} - \sum_{Y \in \mathcal{L}_e^+} (-1)^{\#Y^0} - \sum_{Z \in \mathcal{L}_e^-} (-1)^{\#Z^0} = -1$$

amounts to

$$\sum_{W \in \mathcal{L}_e^0 \setminus e} (-1)^{\#W^0} = 1,$$

showing that the hyperplane after semisimplification satisfies the Euler-Poincaré formula. The analogous conclusion holds for any topal fiber $\mathcal{L} \setminus A$ of any $X \in \mathcal{L}$ with $A \subseteq \underline{X}$ because taking topal fibers and contractions commute. By induction we conclude that $(E, \mathcal{L})$ is a COM satisfying (IC), that is, a lopsided system. □

Note that the equivalence of (i) and (ii) in Proposition 9 rephrases a result by Wiedemann [38] on lopsided sets. Observe that in condition (iii) one cannot dispense with contractions as the example $\mathcal{L} = \{+00\}$ shows. Neither can one weaken condition (ii) by dismissing topal fibers: consider a path in the 1-skeleton of $[-1,+1]^3$ connecting five vertices of the solid cube, which would yield an induced but non-isometric path of the corresponding graphical 3-cube. Let $\mathcal{L}$ comprise the five vertices and the barycenters of the four edges, being represented by their sign vectors. Then all topal fibers except one satisfy the first statement in (ii), the second one being trivially satisfied.

10. Ranking COMs

Particular COMs naturally arise in order theory. For the entire section, let $(P, \leq)$ denote an ordered set (alias poset), that is, a finite set P endowed with an order (relation) $\leq$. A *ranking* (alias weak order) is an order for which incomparability is transitive. Equivalently, an order $\leq$ on P is a ranking exactly when P can be partitioned into antichains (where an *antichain* is a set of mutually incomparable elements) $A_1, \ldots, A_k$, such that $x \in A_i$ is below

$y \in A_j$ whenever $i < j$. An order $\leq$ on P is *linear* if any two elements of P are comparable, that is, all antichains are trivial (i.e., of size < 2). An order $\leq'$ *extends* an order $\leq$ on P if $x \leq y$ implies $x \leq' y$ for all $x, y \in P$. Of particular interest are the linear extensions and, more generally, the ranking extensions of a given order $\leq$ on P.

Let us now see how to associate a set of sign vectors to an order $\leq$ on $P = \{1, 2, \ldots, n\}$. For this purpose take E to be the set of all 2-subsets of P and encode $\leq$ by its characteristic sign vector $X^{\leq} \in \{0, \pm 1\}^E$, which to each 2-subset $e = \{i, j\}$ assigns $X^{\leq}_e = 0$ if i and j are incomparable, $X^{\leq}_e = +1$ if the order agrees with the natural order on the 2-subset e, and else $X^{\leq}_e = -1$. In the sign vector representation the different components are ordered with respect to the lexicographic natural order of the 2-subsets of P.

The composition of sign vectors from different orders $\leq$ and $\leq'$ does not necessarily return an order again. Take for instance, $X^{\leq} = +++$ coming from the natural order on P and $X^{\leq'} = 0-0$ coming from the order with the single (nontrivial) comparability $3 \leq' 1$. The composition $X^{\leq'} \circ X^{\leq}$ equals $+-+$, which signifies a directed 3-cycle and thus no order. The obstacle here is that $X^{\leq'}$ encodes an order for which one element is incomparable with a pair of comparable elements. Transitivity of the incomparability relation is therefore a necessary condition for obtaining a COM.

We denote by $\mathcal{R}(P, \leq)$ the simplification of the set of sign vectors associated to all ranking extensions of $(P, \leq)$. Note that the simplification amounts to omitting the pairs of the ground set corresponding to pairs of comparable pairs of P.

Theorem 4. *Let $(P, \leq)$ be an ordered set. Then $\mathcal{R}(P, \leq)$ is a realizable COM, called the* ranking COM *of $(P, \leq)$.*

Proof. The composition $X \circ Y$ of two sign vectors X and Y which encode rankings has an immediate order-theoretic interpretation: each (maximal) antichain of the order $\leq_X$ encoded by X gets ordered according to the order $\leq_Y$ corresponding to Y. Similarly, in order to realize $X \circ -Y$ one only needs to reverse the order $\leq_Y$ before imposing it on the antichains of $\leq_X$. This establishes conditions (C) and (FS). To verify strong elimination (SE$^=$), assume that X and Y are given with $\underline{X} = \underline{Y}$, so that the corresponding rankings have the same antichains. These antichains may therefore be contracted (and at the end of the process get restored again). Now, for convenience we may assume that X is the constant $+1$ vector, thus representing the natural linear order on P. Given $e = \{i, j\}$ with $i <_X j$, let $Y_e = -1$, that is, $j <_Y i$. To construct a sign vector Z with $Z_e = 0$ and $Z_f = X_f$ whenever $Y_f = X_f$, take the sign vector of the ranking corresponding to X but place the subchain $\{h : i <_X h <_X j \text{ and } h <_Y j\}$ directly below the newly created antichain $\{i, j\}$, and $\{h : i <_X h <_X j \text{ and } j <_Y h\}$ directly above it, while leaving everything else in the natural order. This establishes that $\mathcal{R}(P, \leq)$ is a COM. Realizability of $\mathcal{R}(P, \leq)$ will be confirmed in the third paragraph below. □

To provide an example for a ranking COM and also illustrate the preceding construction, consider the ordered set ("fence") shown in Figure 4(a). In Figure 4(b), the sign vector X

encodes the natural order $<$ and Y the ordering $3 <_Y 2 <_Y j = 5 <_Y i = 1 <_Y 4$, while Z encodes the intermediate ranking with $2 <_Z 3 <_Z 1$ and $5 <_Z 4$.

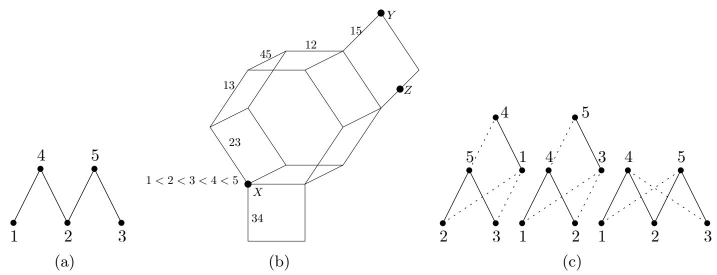


FIGURE 4. From (a) an ordered set $(P, \leq)$ to (b) the ranking COM $\mathcal{R}(P, \leq)$ comprising three maximal faces determined by (c) the minimal rankings in $\mathcal{R}(P, \leq)$.

The ranking COM $\mathcal{R}(P, \leq)$ is the natural host of all linear extensions of $(P, \leq)$ (as its topes), where the interconnecting rankings signify the cell structure. The *linear extension graph* of an ordered set $(P, \leq)$ is defined on the linear extensions of $(P, \leq)$, where two linear extensions are joined by an edge iff they differ on the order of exactly one pair of elements. Thus, the linear extension graph of $(P, \leq)$ is the tope graph of $\mathcal{R}(P, \leq)$. A number of geometric and graph-theoretical features of linear extensions have been studied by various authors [33, 35, 36], which can be expressed most naturally in the language of COMs.

One such result translates to the fact that ranking COMs are realizable. To see this, first consider the *braid arrangement* of type B_n, i.e., the central hyperplane arrangement $\{H_{ij} : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$, where $H_{ij} = \{x \in \mathbb{R}^n : x_i = x_j\}$ and the position of any point in the corresponding halfspace $\{x \in \mathbb{R}^n : x_i < x_j\}$ is encoded by $+$ with respect to H_{ij}. The resulting OM is known as the *permutahedron* [10]. Given an order $\trianglelefteq$ on $P = \{1, \ldots, n\}$ consider the arrangement $E = \{H_{ij} : i, j \text{ incomparable}\}$ restricted to the open polyhedron $\bigcap_{i \triangleleft j} \{x \in \mathbb{R}^n : x_i < x_j\}$. The closure of the latter intersected with the unit cube $[0, 1]^n$ coincides with the *order polytope* [37] of $(P, \trianglelefteq)$. It is well-known that the maximal cells of the braid arrangement restricted to the order polytope of $(P, \trianglelefteq)$ correspond to the linear extensions of $(P, \trianglelefteq)$. Thus, the COM realized by the order polytope and the braid arrangement has the same set of topes as the ranking COM. By the results of Section 4, this implies that both COMs coincide. In particular, ranking COMs are realizable.

We will now show how other notions for general COMs translate to ranking COMs. A face $\mathcal{F}$ of $\mathcal{R}(P, \leq)$, as defined in Section 3, can be viewed as the set of all rankings that extend some ranking extension $\leq'$ of $\leq$. Hence $\mathcal{F} \cong \mathcal{R}(P, \leq')$, i.e., all faces of a ranking COM are also ranking COMs. The minimal elements of $\mathcal{R}(P, \leq)$ with respect to sign ordering (being the improper cocircuits of $\mathcal{R}(P, \leq)$) are the minimal ranking extensions of $(P, \leq)$, see Figure 4(c).

A hyperplane $\mathcal{R}_e^0$ of $\mathcal{R} := \mathcal{R}(P, \leq)$ relative to $e = \{i, j\} \in E$ corresponds to those ranking extensions of $(P, \leq)$ leaving i, j incomparable. Thus, $\mathcal{R}_e^0$ can be seen as the ranking COM of the ordered set obtained from $(P, \leq)$ by identifying i and j. The open halfspace $\mathcal{R}_e^+$ corresponds to those ranking extensions fixing the natural order on i, j and is therefore the ranking COM of the ordered set $(P, \leq)$ extended with the natural order on i, j. The analogous statement holds for $\mathcal{R}_e^-$. Similarly, the carrier of $\mathcal{R}$ relative to e can be seen as the ranking COM of the ordered set arising as the intersection of all minimal rankings of $(P, \leq)$ not fixing an order on i, j. So, in all three cases the resulting COMs are again ranking COMs.

One may wonder which are the ordered sets whose ranking COM is an OM or a lopsided system. The maximal cells in Figure 4(b) are symmetric and therefore correspond to OMs.

Proposition 10. *The ranking COM of $(P, \leq)$ is an OM if and only if $\leq$ is a ranking. In this case, $\mathcal{R}(P, \leq)$ and its proper faces are products of permutohedra.*

Proof. Since any OM has a unique improper cocircuit, $\leq$ needs to be a ranking in order to have that $\mathcal{R}(P, \leq)$ is an OM. On the other hand, it is easy to see that if $\leq$ is a ranking on P, then $\mathcal{R}(P, \leq)$ is a product of permutohedra and, in particular, is symmetric, yielding the claim. □

Proposition 11. *The ranking COM of $(P, \leq)$ is a lopsided system if and only if $(P, \leq)$ has width at most 2. In this case, the tope graph of $\mathcal{R}(P, \leq)$ is the covering graph (i.e., undirected Hasse diagram) of a distributive lattice.*

Proof. If $(P, \leq)$ contains an antichain of size 3, then the corresponding face of $\mathcal{R}(P, \leq)$ does not satisfy ideal composition, so that $\mathcal{R}(P, \leq)$ is not lopsided. Conversely, if all antichains have size at most 2, then the zero entries of a sign vector X encoding a ranking of $(P, \leq)$ correspond to maximal antichains of size 2. Thus, choosing any sign on a zero entry just corresponds to fixing a linear order on the two elements of the antichain. Since these antichains are maximal and X encodes a ranking, the resulting sign vector encodes a ranking, too. This proves ideal composition.

Let us now prove the second part of the claim. By Dilworth's Theorem [17], $(P, \leq)$ can be covered by two disjoint chains, C and D. A linear extension of $(P, \leq)$ corresponds to an order-preserving mapping of C to positions between consecutive elements of D or above or below its maximal or minimal element, respectively. A linear extension $\trianglelefteq$ of $(P, \leq)$ can thus be codified by an order-preserving mapping f from C to the chain $\hat{D} = D \cup \hat{1}$, i.e., D with a new top element $\hat{1}$ added: $f(c) = d \in \hat{D}$ signifies that the subchain $f^{-1}(d)$ of C immediately precedes d in the linear extension $\trianglelefteq$. If there are no comparabilities between C and D in $(P, \leq)$, then the tope graph of $\mathcal{R}(P, \leq)$ is the covering graph of the entire distributive lattice L of order-preserving mappings from C to $\hat{D}$ since covering pairs in L correspond to pairs of linear extensions which are distinguished by a single neighbors swap. Additional covering relations between C and D yield lower and upper bounds for the order-preserving mappings, whence the resulting (distributive) linear-extension lattice of $(P, \leq)$ constitutes some order-interval of L. $\square$

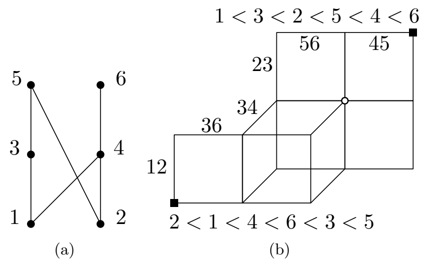


FIGURE 5. From (a) the Hasse diagram of $(P, \leq)$ having width 2 to (b) the tope graph of the lopsided system $\mathcal{R}(P, \leq)$ oriented as a distributive lattice.

In Figure 5(a), an ordered set $(P, \leq)$ of width 2 is displayed, which has the natural order on $\{1, \ldots, 6\}$ among its linear extensions. Figure 5(b) shows the tope graph of the lopsided system $\mathcal{R}(P, \leq)$ and highlights the pair of diametrical vertices that determine the distributive lattice orientation (and its opposite); the natural order is associated with the (median) vertex (indicated by a small open circle). If we added the compatibility $3 < 6$ to the Hasse diagram, then the tope graph shrinks by collapsing the (two) edges corresponding to $\{3, 6\}$. The resulting graph with $F_7 = 13$ vertices is known as the "Fibonacci cube of order 5".

More generally, the *Fibonacci cube of order* $n - 1 \geq 1$ is canonically obtained as the tope graph of $\mathcal{R}(\{1, \ldots n\}, \leq)$, where $(\{1, \ldots n\}, \leq)$ is the ordered set determined by the (cover) comparabilities $1 < 3 < 5 < \ldots < 2\lfloor \frac{n-1}{2} \rfloor + 1$ and $2 < 4 < 6 < \ldots < 2\lfloor \frac{n}{2} \rfloor$ and $k < k+3$ for all $k = 1, \ldots, n-3$. The incomparable pairs thus form the set $E = \{\{i, i+1\} : i = 1, \ldots, n-1\}\}$. The intersection graph of E is a path of length $n-1$, which yields a "fence" when orienting its edges in a zig-zag fashion. This fence and its opposite yield the mutually opposite ordered sets of supremum-irreducibles for the (distributive) "Fibonacci" lattice and its opposite. Recall that the *opposite* $(R, \leq)^{op} = (R, \leq^{op})$ of an ordered set $(R, \leq)$ is defined by switching $\leq$ to $\geq$, that is: $x \leq^{op} y$ if and only if $y \leq x$.

Similarly, to the fact that hyperplanes, carriers, and open halfspaces of a ranking COM are also ranking COMs, the class of ranking COMs is also closed with respect to contractions. On the other hand deleting an element in a ranking COM may give a COM which is not a ranking COM.

To give a small example, consider the minor $\mathcal{R}(P, \leq) \setminus \{5, 6\}$ of the lopsided system $\mathcal{R}(P, \leq)$ of Figure 5. Suppose by way of contradiction, that $\mathcal{R}(P, \leq) \setminus \{5, 6\}$ could be represented by some $\mathcal{R}(Q, \leq)$. The ordered set $(Q, \leq)$ must be of width 2 since the tope graph of $\mathcal{R}(Q, \leq)$ is obtained from the graph in Figure 5(b) by contracting the five edges labeled 56 and thus includes no 3-cube. We can keep the current labeling without loss of generality to the point that Q must include four antichains $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{3,6\}$ of size 2 with exactly this intersection pattern. But then the fifth antichain must be disjoint from $\{1,2\}$ and $\{2,3\}$ but intersecting both $\{3,4\}$ and $\{3,6\}$, whence it must be $\{4,6\}$, which however yields a contradiction as $\{3,4,6\}$ cannot be an antichain in $(Q,\leq)$. Furthermore, $\mathcal{R}(P,\leq)\setminus\{5,6\}$ is easily seen to be the COM amalgam of ranking COMs, i.e., the class is also not closed under COM amalgamations.

It would be interesting to determine the smallest minor-closed class of COMs containing the ranking COMs.

11. COMs as complexes of oriented matroids

In this section we consider a topological approach to COMs. In the subsequent definitions, notations, and results we closely follow Section 4 of [10] (some missing definitions can be also found there). Let $B^d = \{x \in \mathbb{R}^d : ||x|| \leq 1\}$ be the *standard d-ball* and its boundary $S^{d-1} = \partial B^d = \{x \in \mathbb{R}^d : ||x|| = 1\}$ be the *standard $(d-1)$-sphere*. When saying that a topological space T is a "ball" or a "sphere", it is meant that T is homeomorphic to B^d or S^{d-1} for some d, respectively.

11.1. Regular cell complexes.

A *(regular) cell complex* Δ constitutes of a covering of a Hausdorff space $||\Delta|| = \bigcup_{\sigma\in\Delta}\sigma$ with finitely many subspaces σ homeomorphic with (open) balls such that (i) the interiors of $\sigma \in \Delta$ partition $||\Delta||$ (i.e., every $x \in ||\Delta||$ lies in the interior of a single $\sigma \in \Delta$), (ii) the boundary $\partial\sigma$ of each ball $\sigma \in \Delta$ is a union of some members of Δ [10, Definition 4.7.4]. Additionally, we will assume that Δ obeys the *intersection property* (iii) whenever $\sigma, \tau \in \Delta$ have non-empty intersection then $\sigma \cap \tau \in \Delta$. The balls $\sigma \in \Delta$ are called *cells* of Δ and the space $||\Delta||$ is called the *underlying space* of Δ. If T is homeomorphic to $||\Delta||$ (notation $T \cong ||\Delta||$), then Δ is said to provide a *regular cell decomposition* of the space T. We will say that a regular cell complex Δ is *contractible* if the topological space $||\Delta||$ is contractible. If $\sigma, \tau \in \Delta$ and $\tau \subseteq \sigma$, then τ is said to be a *face* of σ. $\Delta' \subseteq \Delta$ is a *subcomplex* of Δ if $\tau \in \Delta'$ implies that every face of τ also belongs to Δ'. The 0-cells and 1-cells of Δ are called *vertices* and *edges*. The *1-skeleton* of Δ is encoded by the graph $G(\Delta)$ consisting of the vertices of Δ and graph edges corresponding to the edges of Δ. The set of cells of Δ ordered by containment is denoted by $\mathcal{F}(\Delta)$ (in [13], $\mathcal{F}(\Delta)$ is also called an *abstract cell complex*). Two cell complexes Δ and Δ' are *combinatorially equivalent* if their ordered sets $\mathcal{F}(\Delta)$ and $\mathcal{F}(\Delta')$ are isomorphic. We continue by recalling several results relating regular cell complexes.

The *order complex* of a finite ordered set P is an abstract simplicial complex $\Delta_{ord}(P)$ whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \cdots < x_k$ of P. The *geometric realization* $||\Delta||$ of a complex Δ basically consists of simultaneously replacing all abstract simplices by geometric simplices, see [12] for a formal definition. In particular, $||\Delta_{ord}(P)||$ is the geometric realization of $\Delta_{ord}(P)$. For an element x of P let

$P_{<x} = \{y \in P : y < x\}$ and $P_{\leq x} = P_{<x} \cup \{x\}$. The following fact expresses that a regular cell complex is homeomorphic to the order complex of its ordered set of faces.

Proposition 12. *[10, Proposition 4.7.8] Let Δ be a regular cell complex. Then $||\Delta|| \cong ||\Delta_{ord}(\mathcal{F}(\Delta)||$. Moreover, this homeomorphism can be chosen to be* cellular*, i.e., it restricts to a homeomorphism between σ and $||\Delta_{ord}(\mathcal{F}_{\leq\sigma})||$, for all $\sigma \in \Delta$.*

The ordered sets of faces of regular cell complexes can be characterized in the following way:

Proposition 13. *[10, Proposition 4.7.23] Let P be an ordered set. Then $P \cong \mathcal{F}(\Delta)$ for some regular cell complex Δ if and only if $||\Delta_{ord}(P_{<x})||$ is homeomorphic to a sphere for all $x \in P$. Furthermore, Δ is uniquely determined by P up to a cellular homeomorphism.*

11.2. **Cell complexes of OMs.** Now, let $\mathcal{L} \subseteq \{\pm 1, 0\}^E$ be the set of covectors of an oriented matroid. Then $(\mathcal{L}, \leq)$ is a semilattice with least element $\mathbf{0}$ (where $\leq$ is the product ordering on $\{\pm 1, 0\}^E$ defined above). The semilattice $(\mathcal{L} \cup \{\hat{1}\}, \leq)$, i.e., the semilattice $\mathcal{L}$ with a largest element $\hat{1}$ adjoined, is a lattice, called the *big face lattice* of $\mathcal{L}$ and denoted by $\mathcal{F}_{big}(\mathcal{L})$. Let $\mathcal{F}_{big}(\mathcal{L})^{op}$ denote the opposite of $\mathcal{F}_{big}(\mathcal{L})$.

Proposition 14. *[10, Corollary 4.3.4 & Lemma 4.4.1] Let $(E, \mathcal{L})$ be an oriented matroid of rank r. Then $\mathcal{F}_{big}(\mathcal{L})^{op}$ is isomorphic to the face lattice of a PL (Piecewise Linear) regular cell decomposition of the $(r-1)$-sphere, denoted by $\Delta(\mathcal{L})$. The tope graph of $\mathcal{L}$ encodes the 1-skeleton of $\Delta(\mathcal{L})$.*

11.3. **Cell complexes of COMs.** We collected all ingredients necessary to associate to each COM a regular cell complex. Let $\mathcal{L} \subseteq \{\pm 1, 0\}^E$ be the set of covectors of a COM. Analogously to oriented matroids, let $\mathcal{F}_{big}(\mathcal{L}) := (\mathcal{L} \cup \{\hat{1}\}, \leq)$ denote the ordered set $\mathcal{L}$ with a top element $\hat{1}$ adjoined and call $\mathcal{F}_{big}(\mathcal{L})$ the *big face semilattice* of $\mathcal{L}$. Let $\mathcal{F}_{big}(\mathcal{L})^{op}$ denote the opposite of $\mathcal{F}_{big}(\mathcal{L})$. $\mathcal{F}_{big}(\mathcal{L})^{op}$ is isomorphic to the semilattice comprising the empty set and the faces of $\mathcal{L}$ ordered by inclusion. Recall that for any $X \in \mathcal{L}$, the deletion $(E \setminus \underline{X}, F(X) \setminus \underline{X})$ corresponding to the face $F(X)$ is an oriented matroid, which we will denote by $\mathcal{L}(X)$. Since $F(Y) \subseteq F(X)$ if and only if $Y \in F(X)$, the order ideal $\mathcal{F}_{big}(\mathcal{L})^{op}_{\leq X}$ coincides with the interval $[\hat{1}, X]$ of $\mathcal{F}_{big}(\mathcal{L})^{op}$ and is isomorphic to the opposite big face lattice $\mathcal{F}_{big}(\mathcal{L}(X))^{op}$ of $\mathcal{L}(X)$. By Proposition 14, if r is the rank of $\mathcal{L}(X)$, then $\mathcal{F}_{big}(\mathcal{L}(X))^{op}$ is isomorphic to the face lattice of a PL cell decomposition $\Delta(\mathcal{L}(X))$ of the $(r-1)$-sphere. Additionally, the tope graph of $\mathcal{L}(X)$ encodes the 1-skeleton of $\Delta(\mathcal{L}(X))$. Denote by $\sigma(\mathcal{L}(X))$ the open PL ball whose boundary is the $(r-1)$-sphere occurring in the definition of $\Delta(\mathcal{L}(X))$. We will call the cells of $\Delta(\mathcal{L}(X))$ *faces* of $\sigma(\mathcal{L}(X))$. The faces of $\Delta(\mathcal{L}(X))$ correspond to the elements of $\mathcal{L}(X) \cup \{\hat{1}\}$. Notice in particular that the adjoined element $\hat{1}$ corresponds to the empty face in $\Delta(\mathcal{L}(X))$ and $\mathbf{0} \in F(X) \setminus \underline{X}$ corresponds to the unique maximal face $\sigma(\mathcal{L}(X))$.

By Proposition 12, for any $X \in \mathcal{L}$ we have $||\Delta(\mathcal{L}(X))|| \cong ||\Delta_{ord}(\mathcal{F}_{big}(\mathcal{L}(X))^{op})||$. Furthermore, since $\mathcal{F}_{big}(\mathcal{L}(X))^{op}$ is isomorphic to $\mathcal{F}_{big}(\mathcal{L})^{op}_{\leq X}$, $||\Delta_{ord}(\mathcal{F}_{big}(\mathcal{L}(X))^{op})|| \cong ||\Delta_{ord}(\mathcal{F}_{big}(\mathcal{L})^{op}_{\leq X})||$. Thus for each $X \in \mathcal{L}$, $||\Delta_{ord}(\mathcal{F}_{big}(\mathcal{L})^{op}_{<X})||$ is homeomorphic to

$||\Delta(\mathcal{L}(X))\backslash\sigma(\mathcal{L}(X))||$, which is a sphere by Proposition 14. Now, by Proposition 13, $\mathcal{F}_{big}(\mathcal{L})^{op}$ is the face semilattice of a regular cell complex $\Delta(\mathcal{L})$. Moreover, from the proof of Proposition 13 it follows that $\Delta(\mathcal{L})$ can be chosen so that its cells are the balls $\sigma(\mathcal{L}(X)), X \in \mathcal{L}$, whose boundary spheres are decomposed by $\Delta(\mathcal{L}(X))$. Since $F(X) \cap F(Y) = F(X \circ Y)$ for any two covectors $X, Y \in \mathcal{L}$ such that $F(X)$ and $F(Y)$ intersect, $\mathcal{F}_{big}(\mathcal{L}(X \circ Y))^{op}$ is isomorphic to a sublattice of $\mathcal{F}_{big}(\mathcal{L}(X))^{op}$ and to a sublattice of $\mathcal{F}_{big}(\mathcal{L}(Y))^{op}$. Therefore the cells $\Delta(\mathcal{L}(X))$ and $\Delta(\mathcal{L}(Y))$ are glued in $\Delta(\mathcal{L})$ along $\Delta(\mathcal{L}(X \circ Y))$, whence $\Delta(\mathcal{L})$ also satisfies the intersection property (iii). Notice also that since the 1-skeleton of each $\Delta(\mathcal{L}(X))$ yields the tope graph of $\mathcal{L}(X)$ and $\Delta(\mathcal{L})$ satisfies (iii), the 1-skeleton of $\Delta(\mathcal{L})$ encodes the tope graph of $\mathcal{L}$. We summarize this in the following proposition, in which we also establish that $\Delta(\mathcal{L})$ is contractible.

Proposition 15. *If $(E, \mathcal{L})$ is a COM, then $\Delta(\mathcal{L})$ is a contractible regular cell complex and the tope graph of $\mathcal{L}$ is realized by the 1-skeleton of $\Delta(\mathcal{L})$.*

Proof. We prove the contractibility of $\Delta(\mathcal{L})$ by induction on the number of maximal cells of $\Delta(\mathcal{L})$ by using the so-called gluing lemma [7, Lemma 10.3] and our decomposition procedure (Proposition 7) for COMs. By the gluing lemma, if Δ is a cell complex which is the union of two contractible cell complexes Δ' and Δ'' such that their intersection $\Delta_0 = \Delta' \cap \Delta''$ is contractible, then Δ is contractible.

If $\Delta(\mathcal{L})$ consists of a single maximal cell $\sigma(\mathcal{L}(X))$, then $(E, \mathcal{L})$ is an OM and therefore is contractible. Otherwise, as shown above there exists an element $e \in E$ such that if we set $\mathcal{L}' := \mathcal{L}_e^-$ and $\mathcal{L}'' := \overline{\mathcal{L}_e^+}$, then $(E, \mathcal{L})$ is the COM amalgam of the COMs $(E, \mathcal{L}')$ and $(E, \mathcal{L}'')$ along the COM $\mathcal{L}' \cap \mathcal{L}'' = N^-(\mathcal{L}_e^0)$. By induction hypothesis, the cell complexes $\Delta(\mathcal{L}'), \Delta(\mathcal{L}'')$, and $\Delta(\mathcal{L}' \cap \mathcal{L}'')$ are contractible. Each maximal cell of $\Delta(\mathcal{L})$ corresponds to a maximal face of $\mathcal{L}$, thus by Proposition 7 it is a maximal cell of $\Delta(\mathcal{L}')$, of $\Delta(\mathcal{L}'')$, or of both (in which case it belongs to $\Delta(\mathcal{L}' \cap \mathcal{L}'')$). Since each cell of $\Delta(\mathcal{L})$ belongs to a maximal cell, this implies that $\Delta(\mathcal{L}) \subseteq \Delta(\mathcal{L}') \cup \Delta(\mathcal{L}'')$. Since $\mathcal{L}' \cup \mathcal{L}'' \subseteq \mathcal{L}$, we also have the converse inclusion $\Delta(\mathcal{L}') \cup \Delta(\mathcal{L}'') \subseteq \Delta(\mathcal{L})$. Finally, since $\mathcal{L}' \cap \mathcal{L}'' = N^-(\mathcal{L}_e^0)$, the definition of carriers implies that $\Delta(\mathcal{L}' \cap \mathcal{L}'') = \Delta(\mathcal{L}') \cap \Delta(\mathcal{L}'')$. □

11.4. **Zonotopal COMs.** As in the introduction, let E be a central arrangement of n hyperplanes of $\mathbb{R}^d$ and $\mathcal{L}$ be the oriented matroid corresponding to the regions of $\mathbb{R}^d$ defined by this arrangement. Let $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a set of unit vectors each normal to a different hyperplane of E. The *zonotope* $\mathcal{Z} := \mathcal{Z}(\mathbf{X})$ of $\mathbf{X}$ is the convex polytope of $\mathbb{R}^d$ which can be expressed as the Minkowski sum of n line segments

$$\mathcal{Z} = [-\mathbf{x}_1, \mathbf{x}_1] + [-\mathbf{x}_2, \mathbf{x}_2] + \ldots + [-\mathbf{x}_n, \mathbf{x}_n].$$

Equivalently, $\mathcal{Z}$ is the projection of the n-cube $C_n := \{\sum_{i=1}^n \lambda_i \mathbf{e}_i : -1 \leq \lambda_i \leq +1\} \subset \mathbb{R}^n$ under $\mathbf{X}$ (where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^n$), which sends $\mathbf{e}_i$ to $\mathbf{x}_i$, $i = 1, \ldots, n$:

$$\mathcal{Z} = \{\sum_{i=1}^n \lambda_i \mathbf{x}_i : -1 \leq \lambda_i \leq +1\} \subset \mathbb{R}^d.$$

The hyperplane arrangement E is *geometrically polar* to $\mathcal{Z}$: the regions of the arrangement are the cones of outer normals at the faces of $\mathcal{Z}$. The face lattice of $\mathcal{Z}$ is opposite (anti-isomorphic) to the big face lattice of the oriented matroid $\mathcal{L}$ of $\mathbf{X}$, that is, $\mathcal{F}(\mathcal{Z}) \simeq \mathcal{F}_{big}(\mathcal{L})^{op}$; for this and other results, see [10, Subsection 2.2]. Therefore the zonotopes together with their faces can be viewed as the cell complexes associated to realizable oriented matroids.

The following properties and examples of zonotopes are well-known:

- any face of a zonotope is a zonotope;
- a polytope P is a zonotope if and only if every 2-dimensional face of P is a zonotope and if and only if every 2-dimensional face of P is centrally symmetric;
- two zonotopes are combinatorially equivalent if and only if their 1-skeletons yield isomorphic graphs;
- the d-cube is the zonotope corresponding to the arrangement of coordinate hyperplanes (called also *Boolean arrangements* [13]) in $\mathbb{R}^d$;
- the permutohedron is the zonotope corresponding to the braid arrangement in $\mathbb{R}^d$.

A regular cell complex Δ is a (combinatorial) *zonotopal complex* if each cell of Δ is combinatorially equivalent to a zonotope [13]. Analogously, Δ is a *cube complex* if each of its cells is a combinatorial cube. A *geometric zonotopal* or *cube complex* is a zonotopal (respectively, cube) complex Δ with a metric such that each face is isometric to a zonotope (respectively, a cube) of the Euclidean space. Moreover, faces are glued together by isometry along their maximal common subfaces. The cell complex $\Delta(\mathcal{L})$ associated to a lopsided set $(E, \mathcal{L})$ is a geometric cube complex: $\Delta(\mathcal{L})$ is the union of all subcubes of the cube $[-1, +1]^E$ whose barycenters are sign vectors from $\mathcal{L}$ [4].

A COM $(E, \mathcal{L})$ is called *locally realizable* (or *zonotopal*) if $\mathcal{L}(X)$ is a realizable oriented matroid for any $X \in \mathcal{L}$. Then $\Delta(\mathcal{L})$ is a zonotopal complex because each cell $\Delta(\mathcal{L}(X)), X \in \mathcal{L}$, is combinatorially equivalent to a zonotope. A zonotopal COM $(E, \mathcal{L})$ is called *zonotopally realizable* if $\Delta(\mathcal{L})$ is a geometric zonotopal complex. Clearly, zonotopally realizable COMs are locally realizable. The converse is the content of the following question:

Question 1. Is any locally realizable COM zonotopally realizable?

Proposition 16. *If $\mathcal{L}$ is a realizable COM, then $\mathcal{L}$ is zonotopally realizable (and thus locally realizable). In particular, each ranking COM is zonotopally realizable.*

Proof. Since $\mathcal{L}$ is realizable there is a set of oriented affine hyperplanes of $\mathbb{R}^d$ and an open convex set C, such that $\mathcal{L} = \mathcal{L}(E, C)$. Without loss of generality we can assume that C is the interior of a full-dimensional polyhedron P. Let F be the set of supporting hyperplanes of P. Consider the central hyperplane arrangement A resulting from lifting the affine arrangement $E \cup F$ to $\mathbb{R}^{d+1}$. The associated OM $\mathcal{L}'$ is realizable and therefore zonotopally realizable. Since $\Delta(\mathcal{L})$ is a subcomplex of $\Delta(\mathcal{L}')$, also $\mathcal{L}$ is zonotopally realizable. □

11.5. **CAT(0) Coxeter COMs.** We conclude this section by presenting another class of zonotopally realizable COMs. Namely, we prove that the CAT(0) Coxeter (zonotopal) complexes introduced in [24] arise from COMs. They represent a common generalization of benzenoid systems (used for illustration in Section 6), 2-dimensional cell complexes obtained from bipartite cellular graphs [1], and CAT(0) cube complexes (cube complexes arising from median structures) [2]. One can say that CAT(0) zonotopal complexes generalize CAT(0) cube complexes in the same way as COMs generalize lopsided sets.

A zonotope $\mathcal{Z}$ is called a *Coxeter zonotope* (an *even polyhedron* [24] or a *Coxeter cell* [14]) if $\mathcal{Z}$ is symmetric with respect to the mid-hyperplane H_f of each edge f of $\mathcal{Z}$, i.e., to the hyperplane perpendicular to f and passing via the middle of f. A cell complex Δ is called a *Coxeter complex* if Δ is a geometric zonotopal complex in which each cell is isometric to a Coxeter zonotope. Throughout this subsection, by Δ we denote a Coxeter complex and by $||\Delta||$ the underlying metric space of Δ.

If $\mathcal{Z}$ is a Coxeter zonotope and f, f' are two parallel edges of $\mathcal{Z}$, then one can easily see that the mid-hyperplanes H_f and $H_{f'}$ coincide. If $\mathcal{Z} = [-\mathbf{x}_1, \mathbf{x}_1] + [-\mathbf{x}_2, \mathbf{x}_2] + \ldots + [-\mathbf{x}_n, \mathbf{x}_n]$, denote by H_i the mid-hyperplane to all edges of $\mathcal{Z}$ parallel to the segment $[-\mathbf{x}_i, \mathbf{x}_i]$, $i = 1, \ldots, n$. Then $\mathcal{Z}$ is the zonotope of the regions defined by the arrangement $\{H_1, \ldots, H_n\}$. It is well-known [14, Definition 7.3.1] (and is also noticed in [24, p.184]) that Coxeter zonotopes are exactly the zonotopes associated to *reflection arrangements* (called also *Coxeter arrangements*) of hyperplanes, i.e., to arrangements of hyperplanes of a finite reflection group [10, Subsection 2.3]. For each $i = 1, \ldots, n$, denote by $\mathcal{Z}_i$ the intersection of $\mathcal{Z}$ with the hyperplane H_i and call it a *mid-section of* $\mathcal{Z}$. The mid-sections $\mathcal{Z}_i$ of a Coxeter zonotope $\mathcal{Z}$ of dimension d are Coxeter zonotopes of dimension $d-1$.

We continue with the definition of CAT(0) metric spaces and CAT(0) Coxeter complexes. The underlying space (polyhedron) $||\Delta||$ of a geometric zonotopal complex (and, more generally, of a cell complex with Euclidean convex polytopes as cells) Δ can be endowed with an intrinsic l_2-metric in the following way. Assume that inside every maximal face of $||\Delta||$ the distance is measured by the l_2-metric. The *intrinsic* l_2*-metric* d_2 of $||\Delta||$ is defined by letting the distance between two points $x, y \in ||\Delta||$ be equal to the greatest lower bound on the length of the paths joining them; here a *path* in $||\Delta||$ from x to y is a sequence $x = x_0, x_1, \ldots, x_m = y$ of points in $||\Delta||$ such that for each $i = 0, \ldots, m-1$ there exists a face σ_i containing x_i and x_{i+1}, and the *length* of the path equals $\sum_{i=0}^{m-1} d(x_i, x_{i+1})$, where $d(x_i, x_{i+1})$ is computed inside σ_i according to the respective l_2-metric. The resulting metric space is *geodesic*, i.e., every pair of points in $||\Delta||$ can be joined by a geodesic; see [12].

A *geodesic triangle* $T := T(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of T) and a geodesic between each pair of vertices (the edges of T). A *comparison triangle* for $T(x_1, x_2, x_3)$ is a triangle $T(x'_1, x'_2, x'_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is a *CAT(0) space* [23] if all geodesic triangles $T(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov: *If y is a point on the side of $T(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $T(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{R}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{R}^2}(x'_3, y')$.*

CAT(0) spaces can be characterized in several different natural ways and have numerous properties (for a full account of this theory consult the book [12]). For instance, a cell complex endowed with a piecewise Euclidean metric is CAT(0) if and only if any two points can be joined by a unique geodesic. Moreover, CAT(0) spaces are contractible.

A *CAT(0) Coxeter complex* is a Coxeter complex Δ for which $||\Delta||$ endowed with the intrinsic l_2-metric d_2 is a CAT(0) space. In this case, the parallelism relation on edges of cells of Δ induces a parallelism relation on all edges of Δ: two edges f, f' of Δ are *parallel* if there exists a sequence of edges $f_0 = f, f_1, \ldots, f_{k-1}, f_k = f'$ such that any two consecutive edges f_{i-1}, f_i are parallel edges of a common cell of Δ. Parallelism is an equivalence relation on the edges of Δ. Denote by E the equivalence classes of this parallelism relation. For $e \in E$, we denote by Δ_e the union of all mid-sections of the form $\mathcal{Z}_e$ for cells $\mathcal{Z} \in \Delta$ which contain edges from the equivalence class e (let $||\Delta_e||$ be the underlying space of Δ_e). We call each $||\Delta_e||$ (or Δ_e), $e \in E$, a *mid-hyperplane* (or a *wall* as in [24]) of $||\Delta||$. Since each mid-section included in Δ_e is a Coxeter zonotope, each mid-hyperplane of a Coxeter complex is a Coxeter complex as well. CAT(0) Coxeter complexes have additional nice and strong properties, which have been established in [24].

Lemma 6. *[24, Lemme 4.4] Let Δ be a CAT(0) Coxeter complex and Δ_e be a mid-hyperplane of Δ. Then $||\Delta_e||$ is a convex subset of $||\Delta||$ and $||\Delta_e||$ partitions $||\Delta||$ in two connected components $||\Delta_e^-||$ and $||\Delta_e^+||$ (called halfspaces of $||\Delta||$).*

If $x \in ||\Delta_e^-||$ and $y \in ||\Delta_e^+||$, then x and y are said to be *separated* by the mid-hyperplane (wall) $||\Delta_e||$. A path P in $||\Delta||$ *traverses* a mid-hyperplane $||\Delta_e||$ if P contains an edge xy such that x and y are separated by $||\Delta_e||$. Two distinct mid-hyperplanes $||\Delta_e||$ and $||\Delta_f||$ are called *parallel* if $||\Delta_e|| \cap ||\Delta_f|| = \varnothing$ and *crossing* if $||\Delta_e|| \cap ||\Delta_f|| \neq \varnothing$.

Lemma 7. *[24, Corollaire 4.10] Two vertices u, v of Δ are adjacent in $G(\Delta)$ if and only if u and v are separated by a single mid-hyperplane of $||\Delta||$.*

Lemma 8. *[24, Proposition 4.11] A path P of $G(\Delta)$ between two vertices u, v is a shortest (u, v)-path in $G(\Delta)$ if and only if P traverses each mid-hyperplane of $||\Delta||$ at most once.*

These three results imply that the arrangement of mid-hyperplanes of a CAT(0) Coxeter complex Δ defines a wall system sensu [24], which in turn provides us with a system $\mathcal{L}(\Delta)$ of sign vectors. Define the mapping $\varphi : \Delta \to \{\pm 1, 0\}^E$ in the following way. First, for $e \in E$ and $x \in \Delta$, set

$$\varphi_e(x) := \begin{cases} -1 & \text{if } x \in ||\Delta_e^-||, \\ 0 & \text{if } x \in ||\Delta_e||, \\ +1 & \text{if } x \in ||\Delta_e^+||. \end{cases}$$

Let $\varphi(x) = (\varphi_e(x) : e \in E)$. Denote by $\mathcal{L}(\Delta)$ the set of all sign vectors of the form $\varphi(x), x \in ||\Delta||$. Notice that if a point x of $||\Delta||$ does not belong to any mid-hyperplane of $||\Delta||$, then $\varphi(x) \in \{\pm 1\}^E$; in particular, this is the case for the vertices of $G(\Delta)$. Moreover, Lemma 8 implies that φ defines an isometric embedding of $G(\Delta)$ into the hypercube $\{\pm 1\}^E$.

Theorem 5. *Let Δ be a CAT(0) Coxeter complex, E be the classes of parallel edges of Δ, and $\mathcal{L}(\Delta) := \cup\{\varphi(x) : x \in ||\Delta||\} \subseteq \{\pm 1, 0\}^E$. Then $(E, \mathcal{L}(\Delta))$ is a simple COM and $G(\Delta)$ is its tope graph.*

Proof. We proceed by induction on the size of E. It suffices to show that $(E, \mathcal{L}(\Delta))$ is simple and satisfies the conditions (1),(3),(4), and (2′) of Theorem 3. That $\mathcal{L}(\Delta)$ satisfies (N1) is evident. To verify the condition (N2), let $e, f \in E$. If the mid-hyperplanes $||\Delta_e||$ and $||\Delta_f||$ cross, then the four intersections $||\Delta_e^-|| \cap ||\Delta_f^-||, ||\Delta_e^-|| \cap ||\Delta_e^+||, ||\Delta_e^+|| \cap ||\Delta_f^-||$, and $||\Delta_e^+|| \cap ||\Delta_f^+||$ are nonempty, and as X and Y with $\{X_eX_f, Y_eY_f\} = \{\pm 1\}$ one can pick the sign vectors $\varphi(x)$ and $\varphi(y)$ of any two points $x \in ||\Delta_e^+|| \cap ||\Delta_f^-||$ and $y \in ||\Delta_e^-|| \cap ||\Delta_f^-||$. Otherwise, if $||\Delta_e||$ and $||\Delta_f||$ are parallel, then one of the four pairwise intersections of halfspaces is empty, say $||\Delta_e^-|| \cap ||\Delta_f^+|| = \varnothing$, and as X and Y one can take the sign vectors $\varphi(x)$ and $\varphi(y)$ of any points $x \in ||\Delta_e^+|| \cap ||\Delta_f^+||$ and $y \in ||\Delta_e^+|| \cap ||\Delta_f^-||$. This establishes that $(E, \mathcal{L}(\Delta))$ is simple.

Notice that the tope graph of $(E, \mathcal{L}(\Delta))$ coincides with $G(\Delta)$. Indeed, let X be a tope of $\mathcal{L}(\Delta)$. Then $X = \varphi(x)$ for some $x \in ||\Delta||$. Let $x \in \mathcal{Z}$ for a cell $\mathcal{Z}$ of $||\Delta||$. The sign maps of $\mathcal{L}(\Delta)$ restricted to $\mathcal{Z}$ define an oriented matroid whose topes are the vertices of $\mathcal{Z}$. Therefore $\mathcal{Z}$ contains a vertex v such that $\varphi(v) = \varphi(x) = X$, whence each tope of $\mathcal{L}(\Delta)$ is a vertex of $G(\Delta)$. Conversely, since each vertex v of $G(\Delta)$ does not belong to any mid-hyperplane of $||\Delta||$, $\varphi(v)$ is a vertex of $\{\pm 1\}^E$, and thus a tope of $\mathcal{L}(\Delta)$. This shows that the tope graph of $\mathcal{L}(\Delta)$ and the 1-skeleton of Δ have the same sets of vertices. Lemma 7 implies that two vertices u and v are adjacent in $G(\Delta)$ if and only if they are adjacent in the tope graph of $\mathcal{L}(\Delta)$. Then Lemma 8 establishes the condition (3). The condition (4) immediately follows from the definition of $\mathcal{L}(\Delta)$.

Now we establish condition (2′) that all hyperplanes $\mathcal{L}_e^0(\Delta)$ of $(E, \mathcal{L}(\Delta))$ are COMs. Notice that $X \in \mathcal{L}_e^0(\Delta)$ if and only if $X = \varphi(x)$ for some $x \in ||\Delta_e||$. Therefore the hyperplane $\mathcal{L}_e^0(\Delta)$ of $\mathcal{L}(\Delta)$ coincides with the restriction of $\mathcal{L}(\Delta)$ to the points of the mid-hyperplane $||\Delta_e||$. Hence $\mathcal{L}_e^0(\Delta)$ can be viewed as $\mathcal{L}(\Delta_e)$, where $\mathcal{L}(\Delta_e)$ is the set of all sign vectors of $\{\pm 1, 0\}^E$ of the form $\varphi(x), x \in ||\Delta_e||$. By Lemma 6, $||\Delta_e||$ is a convex subset of $||\Delta||$, thus Δ_e is a CAT(0) Coxeter complex. Let E' denote the classes of parallel edges of Δ_e; namely, E' consists of all $f \in E \setminus \{e\}$ such that the mid-hyperplanes $||\Delta_e||$ and $||\Delta_f||$ are crossing. Notice that the fth mid-hyperplane $||(\Delta_e)_f||$ of $||\Delta_e||$ is just the intersection $||\Delta_e|| \cap ||\Delta_f||$. Analogously, the halfspaces $||(\Delta_e)_f^-||$ and $||(\Delta_e)_f^+||$ coincide with the intersections $||\Delta_f^-|| \cap ||\Delta_e||$ and $||\Delta_f^+|| \cap ||\Delta_e||$, respectively. Define $\varphi' : ||\Delta_e|| \to \{\pm 1, 0\}^{E'}$ as follows. For $x \in ||\Delta_e||$ and $f \in E'$, set

$$\varphi'_f(x) := \begin{cases} -1 & \text{if } x \in ||(\Delta_e)_f^-||, \\ 0 & \text{if } x \in ||(\Delta_e)_f||, \\ +1 & \text{if } x \in ||(\Delta_e)_f^+||. \end{cases}$$

Let $\varphi'(x) = (\varphi'_f(x) : f \in E')$. Denote by $\mathcal{L}'(\Delta_e)$ the set of all sign vectors of the form $\varphi'(x), x \in ||\Delta_e||$. By the induction hypothesis, $(E', \mathcal{L}'(\Delta_e))$ is a COM. For any point $x \in ||\Delta_e||$, $\varphi(x)$ coincides with $\varphi'(x)$ on E'. Since $\varphi_e(x) = 0$, $\varphi_{e'}(x) = -1$ if $||\Delta_{e'}||$ is parallel to

$||\Delta_e||$ and $||\Delta_e|| \subset ||\Delta^-_{e'}||$, and $\varphi_{e'}(x) = +1$ if $||\Delta_{e'}||$ is parallel to $||\Delta_e||$ and $||\Delta_e|| \subset ||\Delta^+_{e'}||$, $\mathcal{L}(\Delta_e)$ can be obtained from $\mathcal{L}'(\Delta_e)$ by adding to all sign vectors of $\mathcal{L}'(\Delta_e)$ in each coordinate of $E \setminus E'$ a respective constant $0, -1$, or $+1$. Hence $(E, \mathcal{L}(\Delta_e))$ is a COM, thus establishing (2′).

Finally, we show that $\mathcal{L}(\Delta)$ satisfies the condition (1), i.e., the composition rule (C). Let X and Y be two sign vectors of $\mathcal{L}(\Delta)$ and x and y be two points of $||\Delta||$ such that $\varphi(x) = X$ and $\varphi(y) = Y$. As in the case of realizable COMs presented in the introduction, connect the two points x and y by the unique geodesic $\gamma(x, y)$ of $||\Delta||$ and choose $\epsilon > 0$ small enough so that the open ball of radius ϵ around x intersects only those mid-hyperplanes of $||\Delta||$ on which x lies. Pick any point w from the intersection of this ϵ-ball with $\gamma(x, y) \setminus \{x\}$ and let $W = \varphi(w)$. We assert that $W = X \circ Y$. Pick any $e \in E$. First suppose that $X_e \neq 0$. From the choice of w it immediately follows that $W_e = \varphi_e(w) = \varphi_e(x) = X_e$. Now suppose that $X_e = 0$. If $Y_e = 0$, then $x, y \in ||\Delta_e||$. Since by Lemma 6, $||\Delta_e||$ is a convex subset of $||\Delta||$, we have $w \in \gamma(x, y) \subset ||\Delta_e||$, whence $W_e = 0 = X_e \circ Y_e$. Finally, if $Y_e \neq 0$, say $Y_e = +1$, then since the set $||\Delta^+_e|| \cup ||\Delta_e||$ is convex, either $w \in ||\Delta^+_e||$ or $w \in ||\Delta_e||$. In the first case, we have $W_e = +1 = X_e \circ Y_e$ and we are done. On the other hand, we will show below that the case $w \in ||\Delta_e||$ is impossible.

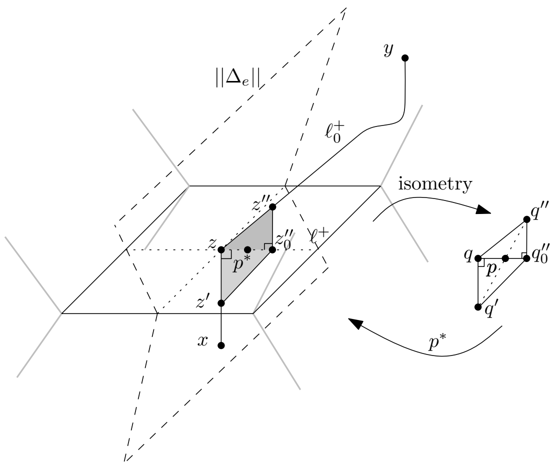


FIGURE 6. To the proof of composition rule in Theorem 5.

Indeed, suppose by way of contradiction that $w \in ||\Delta_e||$. Since $x \in ||\Delta_e||, y \in ||\Delta^+_e||$, and $||\Delta_e||$ is convex, there exists a point $z \in \gamma(x, z)$ such that $\gamma(x, z) \subset ||\Delta_e||$ and $\gamma(z, y) \setminus \{z\} \subset$

36

$||\Delta_e^+||$. Pick points $z' \in \gamma(x,z)$ and $z'' \in \gamma(z,y)$ close enough to z such that each of the couples z', z and z, z'' belongs to a common cell of $||\Delta||$. The choice of z implies that z', z, z'' cannot all belong to a common cell of $||\Delta||$. Denote by $\mathcal{Z}'$ a maximal cell containing z', z and by $\mathcal{Z}''$ a maximal cell containing z, z''. Then z belongs to a common face $\mathcal{Z}_0$ of $\mathcal{Z}'$ and $\mathcal{Z}''$. Let Π' and Π'' denote the supporting Euclidean spaces of $\mathcal{Z}'$ and $\mathcal{Z}''$, respectively. Let ℓ be the line in Π'' passing via the point z and parallel to the edges of $\mathcal{Z}''$ from the equivalence class e and let ℓ^+ be the ray with origin z and containing the points of $\ell \cap ||\Delta_e^+||$ (this intersection is a non-empty half-open interval). Let ℓ_0 be the line in Π'' passing via z and z'' and let ℓ_0^+ be the ray of ℓ_0 with origin z and containing $[z, z'']$. Since ℓ is orthogonal to the supporting plane of the mid-section $\mathcal{Z}'_0 = \mathcal{Z}' \cap \Delta_e$, the angle between ℓ_0^+ and $[z, z']$ is $\frac{\pi}{2}$. Now suppose that z'' is selected so close to z that the orthogonal projection z''_0 of z'' on the line ℓ belongs to the intersection $\mathcal{Z}_0 \cap \ell^+$.

As a result, we obtain two right-angled triangles $T(z, z', z''_0)$ and $T(z, z'', z''_0)$, the first belonging to Π' and the second belonging to Π'' (see Figure 6 for an illustration). Therefore, their union is isometric to a convex quadrilateral $Q = Q(q, q', q''_0, q'')$ in $\mathbb{R}^2$ having the sides $qq', q'q''_0, q''_0q''$, and $q''q$ of lengths $d_2(z, z'), d_2(z', z''_0), d_2(z''_0, z'')$, and $d_2(z'', z)$, respectively. Let p be the intersection of the diagonals $q'q''$ and qq''_0 of Q and let p^* be the point (of Δ) on the segment $[z, z''_0]$ such that $d_2(z, p^*) = d_{\mathbb{R}^2}(q, p)$ and $d_2(z''_0, p^*) = d_{\mathbb{R}^2}(q''_0, p)$. Then

$$\begin{aligned} &d_2(z', z) + d_2(z, z'') = d_{\mathbb{R}^2}(q', q) + d_{\mathbb{R}^2}(q, q'') > d_{\mathbb{R}^2}(q', q'') \\ &= d_{\mathbb{R}^2}(q', p) + d_{\mathbb{R}^2}(p, q'') = d_2(z', p^*) + d_2(p^*, z''), \end{aligned}$$

contrary to the assumption that $z \in \gamma(z', z'') \subset \gamma(x, y)$. This establishes our claim and concludes the proof that $\mathcal{L}(\Delta)$ satisfies the composition rule (C). □

We conclude this section by showing that in fact all COMs having square-free tope graphs arise from 2-dimensional zonotopal COMs:

Proposition 17. *A square-free partial cube G is the tope graph of a COM $\mathcal{L}(G)$ if and only if G does not contain an 8-cycle with two subdivided diagonal chords (graph X in Fig.1 in [30]) as an isometric subgraph. The resulting $\mathcal{L}(G)$ is a zonotopal COM.*

Proof. Notice that a square-free tope graph G of a COM does not contain X as an isometric subgraph. Indeed, since G is square-free, the four 6-cycles of X are convex and, moreover, X must be a convex subgraph of G. Since X isometrically embeds into a 4-cube, it can be directly checked that X is not the tope graph as a COM, consequently, X cannot occur in the tope graph of a COM.

Conversely, let G be a square-free partial cube not containing X as an isometric subgraph. By Proposition 2.6. of [30], any pair of isometric cycles intersect in at most one edge. By replacing each isometric cycle of G with a regular polygon with the same number of edges, we get a 2-dimensional Coxeter zonotopal complex $||\Delta(G)||$. Since isometric cycles of a graph generate the cycle space, this complex is simply connected. Since the sum of angles around any vertex of $||\Delta(G)||$ is at least 2π, by Gromov's result for 2-dimensional complexes [12, p.215],

$||\Delta(G)||$ is CAT(0). Thus, $\Delta(G)$ is a 2-dimensional CAT(0) Coxeter zonotopal complex and by Theorem 5, G can be realized as the tope graph of a COM. □

12. Concluding remarks

In this paper, we show how COMs naturally arise as a generalization of oriented matroids and lopsided sets by relaxing the covector axioms. Furthermore, we give several descriptions of COMs, in particular, in terms of cocircuit axioms. Nevertheless, such important features of the theory of OMs like duality and topological representation still lack generalization. We believe that the following problem, which is well-known for OMs and lopsided sets, is an important next step.

Problem 1. Establish duality theory for COMs.

By Proposition 6, the halfspaces of a COM are COMs. Particular examples are the affine oriented matroids, which are halfspaces of OMs. Even stronger, Lemma 4 shows that the intersections of halfspaces, i.e., the fibers, of a COM are COMs. While the following is true by definition for realizable COMs (see for instance the proof of Proposition 16), we believe that every COM arises this way from an OM:

Conjecture 1. Every COM is a fiber of some OM.

This generalizes the corresponding conjecture of Lawrence [29] that lopsided sets are fibers of uniform OMs. Not only would Conjecture 1 be a natural generalization of the realizable situation, but using the Topological Representation Theorem of Oriented Matroids [21] it will also give a natural topological representation for COMs. In fact, Conjecture 1 is equivalent to the following conjecture: For every COM $\mathcal{L}$ there is a number d such that $\mathcal{L}$ can be represented by a set of $(d-2)$-dimensional pseudospheres restricted to the intersection of a set of open $(d-1)$-dimensional pseudoballs inside a $(d-1)$-sphere.

For locally realizable COMs, the following version of Conjecture 1 would imply a positive answer to Question 1:

Conjecture 2. Every locally realizable COM is a fiber of a realizable OM.

Conjecture 2 can be rephrased as: The tope graph of a locally realizable COM is a convex subgraph of the 1-skeleton of a zonotope. Analogously, Conjecture 1 can be rephrased as: The tope graph of a COM is a convex subgraph of the tope graph of an OM.

Acknowledgements. We would like to acknowledge the referees for useful remarks improving the presentation. Furthermore, we thank Matjaž Kovše (Leipzig) for initial discussions about partial cubes and tope graphs of oriented matroids and are grateful to Andrea Baum, Tabea Beese, and especially Yida Zhu (Hamburg) for careful reading of previous versions and suggesting a simpler argument in the proof of Theorem 1. V.C. and K.K. were partially supported by ANR project TEOMATRO (ANR-10-BLAN-0207). K.K was furthermore supported by PEPS grant EROS and ANR project GATO (ANR-16-CE40-0009-01).

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On tope graphs of complexes of oriented matroids

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May 28, 2019

Abstract

We give two graph theoretical characterizations of tope graphs of (complexes of) oriented matroids. The first is in terms of excluded partial cube minors, the second is that all antipodal subgraphs are gated. A direct consequence is a third characterization in terms of zone graphs of tope graphs.

Further corollaries include a characterization of topes of oriented matroids due to da Silva, another one of Handa, a characterization of lopsided systems due to Lawrence, and an intrinsic characterization of tope graphs of affine oriented matroids. Moreover, we obtain purely graph theoretic polynomial time recognition algorithms for tope graphs of the above and a finite list of excluded partial cube minors for the bounded rank case.

In particular, our results answer a relatively long-standing open question in oriented matroids and can be seen as identifying the theory of (complexes of) oriented matroids as a part of metric graph theory. Another consequence is that all finite Pasch graphs are tope graphs of complexes of oriented matroids, which confirms a conjecture of Chepoi and the two authors.

Contents

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1 Introduction

A graph $G = (V, E)$ is a ***partial cube*** if it is (isomorphic to) an isometric subgraph of a hypercube graph Q_n, i.e., $d_G(u, v) = d_{Q_n}(u, v)$ for all $u, v \in V$, where d denotes the distance function of the respective graphs. Partial cubes were introduced by Graham and Pollak [25] in the study of interconnection networks. They form an important graph class in media theory [21], frequently appear in chemical graph theory [20], and quoting [29] present one of the central and most studied classes in metric graph theory.

Important subclasses of partial cubes include median graphs, bipartite cellular graphs, hypercellular graphs, Pasch graphs, and netlike partial cubes. Partial cubes also capture several important graph classes not directly coming from metric graph theory, such as region graphs of hyperplane arrangements, diagrams of distributive lattices, linear extension graphs of posets, tope graphs of oriented matroids (OMs), tope graphs of affine oriented matroids (AOMs), and lopsided systems (LOPs). A recently introduced unifying generalization of these classes are complexes of oriented matroids (COMs), whose tope graphs are partial cubes as well [3]. As it turns out, all of the above mentioned classes or partial cubes are indeed tope graphs of COMs.

Partial cubes admit a natural minor-relation (pc-minors for short) and several of the above classes including tope graphs of COMs are pc-minor closed. Complete (finite) lists of excluded pc-minors are known for median graphs, bipartite cellular graphs, hypercellular graphs and Pasch graphs, see [9, 10, 11]. Another well-known construction of a smaller graph from a partial cube is the zone graph [29].

In this paper we focus on COMs and their tope graphs. We present two characterizations of the tope graphs and thus two graph theoretical characterizations of COMs. The first characterization is in terms of its complete (infinite) list of excluded pc-minors. As corollaries we obtain excluded pc-minor characterizations for tope graphs of OMs, AOMs, and LOPs. Moreover, in the case of bounded rank the list of excluded pc-minors is finite. We devise a polynomial time algorithm for checking if a given partial cube has another one as pc-minor, leading to polynomial time recognition algorithms for the classes with a finite list of excluded pc-minors. Another consequence is a characterization of tope graphs of COMs in terms of iterated zone graphs, which generalizes a result of Handa [26] about tope sets of OMs.

The second characterization of tope graphs of COMs is in terms of the metric behavior of certain subgraphs. More precisely, we prove that a partial cube is the tope graph of a COM if and only if all of its antipodal (also known as symmetric-even [5]) subgraphs are gated. As corollaries, this theorem specializes to tope graphs of OMs, AOMs, and LOPs. In particular, we obtain a new unified proof for characterization theorems of tope sets of LOPs and OMs due to Lawrence [30] and da Silva [14], respectively. Moreover, this characterization allows to prove that Pasch graphs are COMs, confirming a conjecture of Chepoi, Knauer, and Marc [11]. Finally, our characterization is verifiable in polynomial time, hence gives polynomial time recognition algorithms for tope graphs of COMs, OMs, AOMs, and LOPs, even without bounding the rank. Note that a polynomial time recognition algorithm for tope graphs of OMs was known before, see [24]. However, this algorithm works without a characterization of the graphs, but constructs the set of cocircuits from the topes and there verifies the cocircuit axioms.

In particular, we answer a long-standing open question on OMs, i.e., the question for a purely graph theoretical characterization of tope graphs, see [27, Problem 2] that can furthermore be verified in polynomial time, which was posed in [22, Problem 1.2]. Since the tope graph determines a COM, OM, AOM, or LOP up to isomorphism, see [3], our results can be seen as identifying the theory of (complexes of) oriented matroids as a part of metric graph theory.

Content of the paper: The main theorem of the paper is Theorem 1.1, saying that a graph G is the tope graph of a COM, i.e., $G \in \mathcal{G}_{\text{COM}}$, if and only if G is antipodally gated, i.e., $G \in \text{AG}$, if and only if G

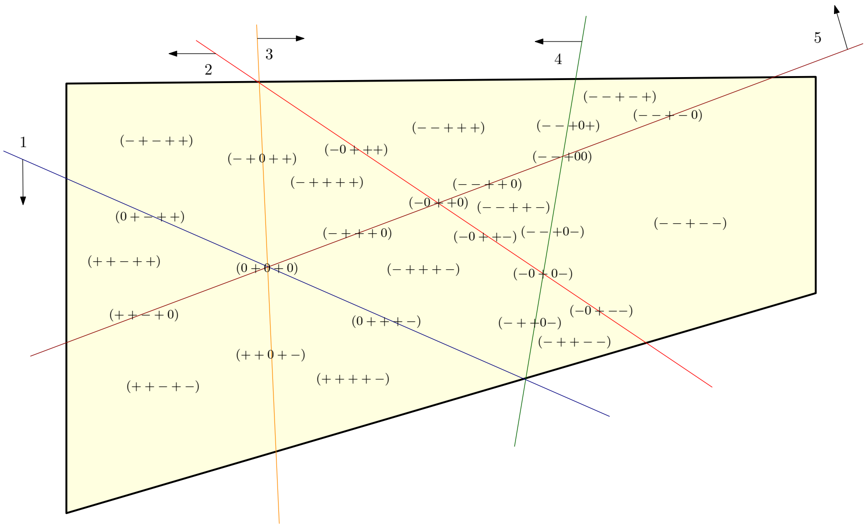

Figure 1: A COM realized by a hyperplane arrangement and an open polyhedron in $\mathbb{R}^2$. The arrows on the hyperplanes indicate their positive side.

does not contain a partial cube minor from a specific set $\mathcal{Q}^-$, i.e., $G \in \mathcal{F}(\mathcal{Q}^-)$. (Generally, for a set X of partial cubes we denote by $\mathcal{F}(X)$ the class of partial cubes that do not have any graph from X as a partial cube minor.) Let us here give an outline of its proof ingredients and its corollaries, before going into the technicalities starting in the next section. The general strategy will be to prove first $\mathcal{G}_{\mathrm{COM}} \subseteq \mathrm{AG}$, then $\mathcal{F}(\mathcal{Q}^-) \subseteq \mathcal{G}_{\mathrm{COM}}$, and finally $\mathrm{AG} \subseteq \mathcal{F}(\mathcal{Q}^-)$.

Let us start by introducing the idea of a COM, which usually is encoded as a system of sign-vectors $\mathcal{L} \in \{+,-,0\}^{\mathcal{E}}$ on a finite ground set $\mathcal{E}$. In the illustrative realizable case one can think of $\mathcal{L}$ as the relative positions of cells in an oriented hyperplane arrangement intersected with an open polyhedron. See Figure 1 for an instructive example. This is a unifying generalization of realizable LOPs, OMs, and AOMs. The definition of LOPs, COMs, OMs, and AOMs can all be given in terms of axiomatics of the set of covectors $\mathcal{L}$ and also from this point of view it is reflected how COMs generalize the other ones naturally, see Definition 3.1.

The ***tope graph*** of a system of sign vectors is the graph induced by its topes, i.e., covectors without zero entries, in the hypercube $\{+,-\}^{\mathcal{E}}$. In case of COMs the tope graph (without labeling of the vertices) determines the COM up to isomorphism, see [3]. Moreover, the tope graph of a COM is a partial cube, see [3]. One way of thinking of being a partial cube is that the edges receive colors corresponding to the dimensions of the hypercube, see Figure 2.

Of particular importance to us are two types of metric subgraphs. An ***antipodal*** subgraph H of G has the property, that for each vertex v in H there is an antipode $-_H v$, such that the H is smallest convex subgraph of G containing v and $-_H v$. The antipodal subgraphs of the graph in Figure 2 are exactly the vertices, edges, and bounded faces. The second property of a subgraph H is the one of being ***gated***. This means, that for every vertex $v \in G$, there is a gate v' in H, such that for every $v'' \in H$ there is a shortest path from v through v' to v''. In partial cubes this amounts to the fact that there is path from v to H that does not use any color that is present on the edges of H. We say that a graph is ***antipodally gated*** if all of its antipodal subgraphs are gated. The graph in Figure 2 is antipodally gated, but also for instance

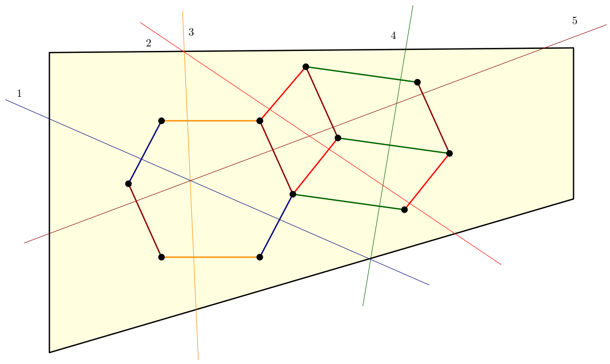

Figure 2: The tope graph of the COM from Figure 1. The color classes of edges giving the embedding into the cube correspond to the hyperplanes of the arrangement in $\mathbb{R}^2$.

the subgraph induced by the vertices incident to red or green edges is gated. A non-gated subgraph is given by the path P of length two induced by the three left-most vertices. A vertex that has no gate in P is the degree four vertex v, since all paths from v to P use a color present in P.

Exploring correspondences between axiomatical behavior of sign-vectors and metric subgraphs of partial cubes, in Theorem 4.9 we show that tope graphs of COMs are antipodally gated. This is the first part of the proof of Theorem 1.1, i.e., $\mathscr{G}_{\mathrm{COM}} \subseteq \mathrm{AG}$.

Clearly, there are partial cubes that do not satisfy Theorem 4.9, i.e., they contain a non-gated antipodal subgraph. Figure 3 shows a partial cube in which the bottom C_6 is an antipodal subgraph, that is not gated, because the red vertex does not have gate.

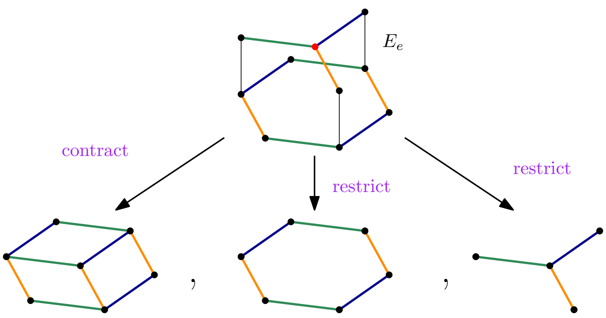

Figure 3: A non antipodally gated partial cube and the partial cube minors obtained by contracting or restricting with respect to the vertical edge class E_e.

Instead of studying all the partial cubes, that have a non-gated antipodal subgraph, we use the notion of partial cube minors, in order to classify only minimal such partial cubes with respect to this operation. A ***partial cube minor*** is either a contraction of a color class or the restriction to one its sides. Hence, it is a specialization of the standard graph minor notion. In Figure 3 we illustrate the partial cube minor on an example.

The next and second step of our proof is providing a set $\mathscr{Q}^-$ of partial cubes that are minor-minimal with respect to having a non-gated antipodal subgraph. The graph of Figure 3 is the smallest element of $\mathcal{Q}^-$, more of these graphs are depicted in Figure 10. It is easy to check that the minors in Figure 3 are antipodally gated, i.e., the graph on top is minimally non antipodally gated. The class of tope graphs of COMs is closed under pc-minors. This is illustrated by our realizable example in Figure 4. By definition the class $\mathcal{F}(\mathcal{Q}^-)$ of partial cubes excludes the minors from $\mathcal{Q}^-$ as well. In Theorem 5.7 we use this to show that, if G is not the tope graph of a COM, then it must have a partial cube minor from $\mathcal{Q}^-$. This concludes the second part of our proof since it means, that if G excludes $\mathcal{Q}^-$, then it is the tope graph of a COM, i.e., $\mathcal{F}(\mathcal{Q}^-) \subseteq \mathcal{G}_{\text{COM}}$.

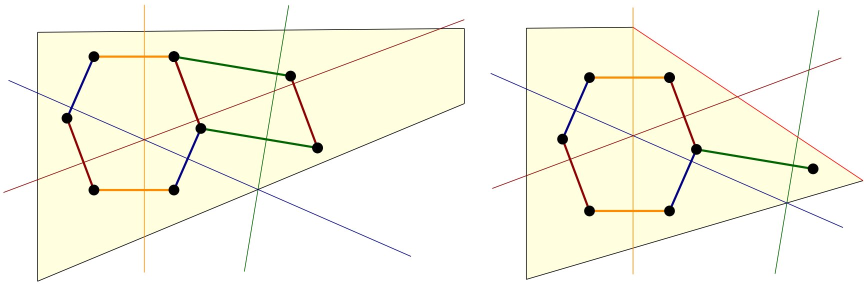

Figure 4: Two partial cube minors obtained from the COM of Figure 2 by contracting and restricting with respect to the red color class.

The last and most technical part of our proof is to show that the class of antipodally gated partial cube does not contain a member of $\mathcal{Q}^-$ as partial cube minor. The class $\mathcal{F}(\mathcal{Q}^-)$ is closed under partial cube minors and graphs in $\mathcal{Q}^-$ are minor-minimal having non-gated antipodal subgraphs. We will thus show that the class of antipodally gated partial cubes is closed under partial cube minors (Theorem 6.1). This implies, that if G has a partial cube minor from $\mathcal{Q}^-$, then it cannot be antipodally gated, i.e., $\text{AG} \subseteq \mathcal{F}(\mathcal{Q}^-)$. This concludes the last part of the circular proof of our main theorem, which we will sketch quickly again.

Theorem 1.1. *For a graph G the following conditions are equivalent:*

(i) G is the tope graph of a COM, i.e., $G \in \mathcal{G}_{\text{COM}}$,

(ii) G is an antipodally gated partial cube, i.e., $G \in \text{AG}$,

(iii) G is a partial cube with no partial cube minor from $\mathcal{Q}^-$, i.e., $G \in \mathcal{F}(\mathcal{Q}^-)$.

Proof. The implication (i)⇒(ii) is Theorem 4.9. The implication (iii)⇒(i) is Theorem 5.7. Finally, (ii)⇒(iii) follows from the fact that all the graphs in $\mathcal{Q}^-$ have a non-gated antipodal subgraph asserted in Lemma 5.1 and that the class AG is pc-minor closed, see Theorem 6.1. □

In order to get an idea of the further implications of this theorem without going into the technical details we propose to jump directly to Section 7. The first corollary will establish a generalization of a theorem of Handa and characterize tope graphs of COMs by the fact that all iterated zone graphs are partial cubes. Afterwards resulting characterizations of tope graphs of OMs, LOPs, and AOMs (of bounded rank) are given. Moreover, the polynomial time recognition is shown and answering a conjecture of [11] it is shown that Pasch graphs are tope graphs of COMs.

In the following we survey in more detail the **Structure of the paper:**

The next two sections (2 and 3) are preliminaries dedicated on the one hand to partial cubes and on the other hand to systems of sign vectors. They can be skipped and looked back at when necessary. Apart from this purpose a couple of results of independent interest are given in Section 2.

In Section 2 we introduce partial cubes with some more care, as well as metric subgraphs such as convex, gated, antipodal, and affine subgraphs and we discuss their behavior with respect to pc-minors and expansions. This section is quite technical and heavy in definitions since it introduces all the necessary background and auxiliary lemmas, that are needed for the rest of the paper. We discuss zone graphs of partial cubes, which play a role in part of our proof. We devise a polynomial time algorithm for checking if a given partial cube has another one as a pc-minor (Proposition 2.4). We give an expansion procedure of how to construct all antipodal partial cubes from a single vertex (Lemma 2.14) and provide an intrinsic characterization of affine partial cubes (Proposition 2.16).

Section 3 is dedicated to the introduction of the systems of sign-vectors relevant to this paper, i.e. COMs, OMs, AOMs, and LOPs, and their behavior under the usual minor-relations. Also, here quite some amount of definitions is introduced, that however coincides with the standard such as given in [3].

In Section 4 we bring the content of the first two sections together and explain how systems of sign-vectors lead to partial cubes and vice versa. We show how metric properties of subgraphs and pc-minors correspond to axiomatic properties and minor relations of systems of sign-vectors. In particular we prove that tope graphs of COMs are antipodally gated (Theorem 4.9), and characterize tope graphs of OMs, AOMs, and LOPs as special tope graphs of COMs. Theorem 4.9 gives the first implication for our characterization theorem (Theorem 1.1).

In Section 5 we introduce the (infinite) set of excluded pc-minors of tope graphs of COMs and provide some of its crucial properties, that will be used throughout the proofs in the following sections. In particular, we show that every member of the class has an antipodal subgraph that is not gated (Lemma 5.1). We conclude Section 5 with the the proof that partial cubes excluding all pc-minors from the class are tope graphs of COMs (Theorem 5.7). In particular, Theorem 5.7 gives the second implication of Theorem 1.1.

Finally, in Section 6 we show that the class of antipodally gated partial cubes under pc-minors (Theorem 6.1). Since the members of our set $\mathcal{Q}^-$ of excluded pc-minors have non-gated antipodal subgraphs, this yields the third and last implication of Theorem 1.1.

Section 7 is dedicated to the corollaries of our theorem, that are announced above. In, particular we prove the generalization of Handa's Theorem (Corollary 7.1) and prove a conjecture of [11] (Corollary 7.7). We conclude the paper with several further questions in Section 8.

2 Pc-minors, expansions, zone graphs, and metric subgraphs

In the present section we will give a thorough introduction to the theory of partial cubes and its elements that are important to our characterization. This will contain many definitions and lemmas, that will be used later. Of central importance are partial cube minors, expansions, zone graphs, and their interactions with metric subgraphs, such as convex, antipodal, affine, and gated subgraphs.

Let us start by giving an alternative way of characterizing partial cubes. Any isometric embedding of a partial cube into a hypercube leads to the same partition of edges into so-called Θ-classes, where two edges are equivalent, if they correspond to a change in the same coordinate of the hypercube. This can be shown using the Djoković-Winkler-relation Θ which is defined in the graph without reference to an embedding, see [17, 36]. We will describe next, how the relation Θ can be defined independently of an embedding.

A subgraph G' of G is ***convex*** if for all pairs of vertices in G' all their shortest paths in G stay in G'. For an edge $a = uv$ of G, define the sets $W(u,v) = \{x \in V : d(x,u) < d(x,v)\}$. By a theorem of Djoković [17], a graph G is a partial cube if and only if G is bipartite and for any edge $a = uv$ the sets $W(u,v)$ and $W(v,u)$ are convex. In this case, setting $a\Theta a'$ for $a = uv$ and $a' = u'v'$ if $u' \in W(u,v)$ and $v' \in W(v,u)$ yields Θ.

We index the set of equivalence classes of Θ by a set $\mathscr{E}$. For $f \in \mathscr{E}$ we denote the equivalence class by E_f.For an arbitrary (oriented) edge $uv \in E_f$, let $E_f^- := W(u,v)$ and $E_f^+ := W(v,u)$ the pair of complementary convex halfspaces of G. Now, identifying any vertex v of G with $v \in Q_{\mathscr{E}} = \{+,-\}^{\mathscr{E}}$ which for any class of Θ associates the sign of the halfspace containing v gives an isometric embedding of G into $Q_{\mathscr{E}}$.

2.1 Pc-minors, expansions, and zone graphs

We will now introduce the notions of Pc-minors (i.e., contraction and restriction) and zone graphs, which are methods to obtain smaller partial cubes from bigger ones, as well as, expansions, which are inverses of contractions. An important observation in this section is a polynomial time algorithm for checking for a given partial cube minor (Proposition 2.4).

2.1.1 Restrictions

Given $f \in \mathscr{E}$, an ***(elementary) restriction*** consists in taking one of the subgraphs $G[E_f^-]$ or $G[E_f^+]$ induced by the complementary halfspaces E_f^- and E_f^+, which we will denote by $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$, respectively. These graphs are isometric subgraphs of the hypercube $Q_{\mathscr{E}\setminus\{f\}}$. Now applying two elementary restriction with respect to different coordinates f, g, independently of the order of f and g, we will obtain one of the four (possibly empty) subgraphs induced by $E_f^- \cap E_g^-, E_f^- \cap E_g^+, E_f^+ \cap E_g^-$, and $E_f^+ \cap E_g^+$. Since the intersection of convex subsets is convex, each of these four sets is convex in G and consequently induces an isometric subgraph of the hypercube $Q_{\mathscr{E}\setminus\{f,g\}}$. More generally, a ***restriction*** is a subgraph of G induced by the intersection of a set of (non-complementary) halfspaces of G. See Figures 3, 4, and 5 for examples of restrictions. We denote restrictions by $\rho_X(G)$, where $X \in \{+,-\}^{\mathscr{E}}$ is a signed set of halfspaces of G. For subset S of the vertices of G and $f \in \mathscr{E}$, we denote $\rho_{f^+}(S) := \rho_{f^+}(G) \cap S$ and $\rho_{f^-}(S) := \rho_{f^-}(G) \cap S$, respectively. We will say that E_f ***crosses*** a subset of vertices S of G if $\rho_{f^+}(S) \neq \emptyset$ and $\rho_{f^-}(S) \neq \emptyset$.

The smallest convex subgraph of G containing V' is called the ***convex hull*** of V' and denoted by $\mathrm{conv}(V')$. The following is well-known, also see Figure 5:

Lemma 2.1 ([1, 2, 9])**.** *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. Indeed, for any subset of vertices V' we have that* $\mathrm{conv}(V')$ *is the intersection of all halfspaces containing V'. In particular, the class of partial cubes is closed under taking restrictions.*

2.1.2 Contractions

For $f \in \mathscr{E}$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an ***(elementary) contraction*** of G. For a vertex v of G, we will denote by $\pi_f(v)$ the image of v under the contraction in G/E_f, i.e. if uv is an edge of E_f, then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subseteq V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular we denote the ***contraction*** of G by $\pi_f(G)$. See Figures 3, 4, and 6 for examples of contractions.

It is well-known and in particular follows from the proof of the first part of [12, Theorem 3] that $\pi_f(G)$ is an isometric subgraph of $Q_{\mathscr{E}\setminus\{f\}}$. Since edge contractions in graphs commute, i.e. the resulting graph does not depend on the order in which a set of edges is contracted, we have:

Lemma 2.2. *The class of partial cubes is closed under contractions. Moreover, contractions commute in partial cubes, i.e. if $f, g \in \mathscr{E}$ and $f \neq g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$.*

Consequently, for a set $A \subseteq \mathscr{E}$, we denote by $\pi_A(G)$ the isometric subgraph of $Q(\mathscr{E} \setminus A)$ obtained from G by contracting the classes $A \subseteq \mathscr{E}$ in G. See Figure 6 for examples.

The following can easily be derived from the definitions, see e.g. [11]:

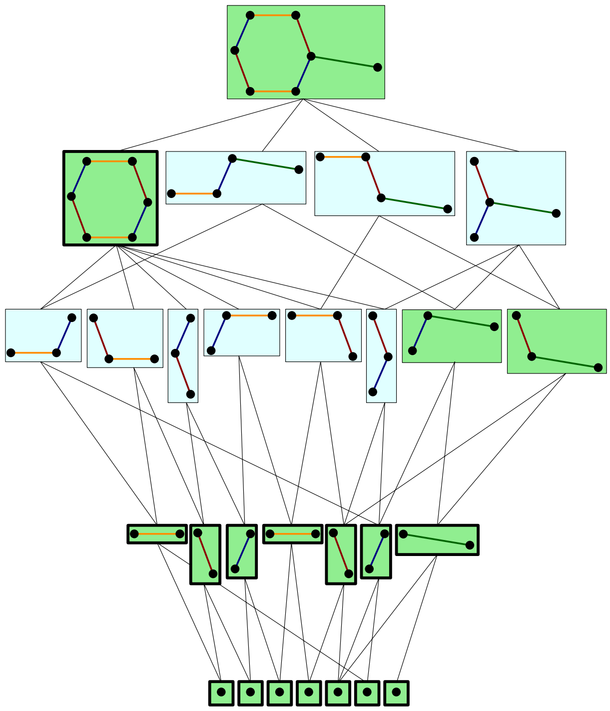

Figure 5: The convex subgraphs of a partial cube ordered by inclusion. Green background means gated, thick outline means antipodal.

Lemma 2.3. *Contractions and restrictions commute in partial cubes, i.e. if* $f, g \in \mathcal{E}$ *and* $f \neq g$*, then* $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$*.*

The previous lemmas show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and G' is called a ***pc-minor*** of G. In this paper we will study classes of partial cubes that are closed under taking pc-minors. Clearly, any such class has a (possibly infinite) set X of minimal excluded pc-minors. We denote by $\mathcal{F}(X)$ the pc-minor closed class of partial cubes excluding X.

Proposition 2.4. *Let X be a finite set of partial cubes. It is decidable in polynomial time if a partial cube G is in $\mathcal{F}(X)$.*

Proof. Let G', G be partial cubes. Denote by n' and n the number of vertices of G' and G, respectively, and with k' and k the number of Θ-classes in G' and G. We will show that testing if G' is a pc-minor of G can be done in polynomial time with respect to n. This clearly implies the result.

For every subset V' of at most n' vertices of G do the following: First compute conv(V') and count the number of Θ-classes of G crossing it, say it equals to k''. Then $k'' \leq k$, and if $k'' < k'$ discard the subgraph. On the other hand, if $k'' \geq k'$, then for every subset S of size $k'' - k'$ of the Θ-classes crossing

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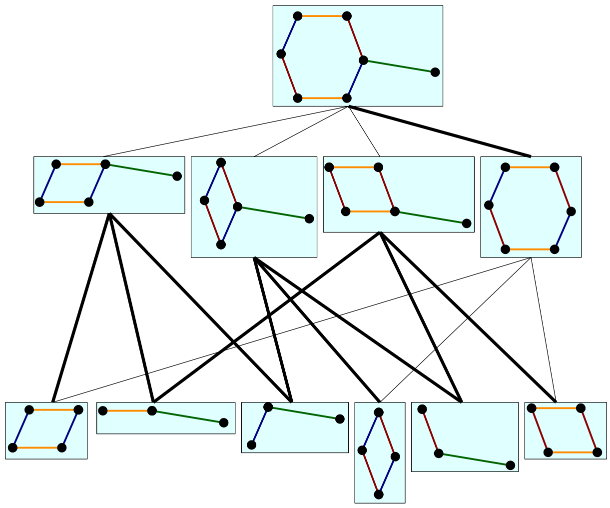


Figure 6: Some contractions of a partial cube. Thick edges mean peripherality.

conv(V'), contract in conv(V') all the Θ-classes of S. Finally, check if the resulting graph is isomorphic to G'.

Using Lemma 2.3, we know that G' is a pc-minor if and only if it can be obtained by first restricting and then contracting, and by Lemma 2.1, taking restrictions coincides with taking convex hulls. We need to prove that one can take a convex hull of exactly n' vertices. For that assume that G' can be obtained from G by first restricting to G'' and then contracting. For each vertex v' of G' pick a vertex v'' in G'' such that v'' maps to v' under contraction. Then the set of all such v'' is also a subset in G, call it S. We claim that taking a convex hull of S and then contracting gives G'. In fact, the obtained graph must be a subgraph of G' since the convex hull of S is a subset of G''. On the other hand, every vertex of G' is obtained in this way since S includes a representative of pre-image of every vertex of G'. This gives the correctness of the algorithm.

For the running time, we have a loop of length $\mathcal{O}(n^{n'})$. In each execution we compute conv(V') which via Lemma 2.1 can be easily done by intersecting all the halfspaces containing V'. Then we have $\mathcal{O}(\binom{k''}{k''-k'}) = \mathcal{O}(\binom{k''}{k'}) \leq \mathcal{O}(\binom{n}{k'}) = \mathcal{O}(n^{k'})$ choices for the contractions of the Θ-classes, each of which can clearly be done in polynomial time, too. Note that $k' < n'$. Finally, we check if the obtained graph is isomorphic to G', which only depends on n'. $\square$

2.1.3 Expansions

Later on we will also consider the inverse operation of contraction: a partial cube G is an ***expansion*** of a partial cube G' if $G' = \pi_f(G)$ for some Θ-class f of G. Indeed expansions can be detected within the smaller graph. Let G' be a partial cube containing two isometric subgraphs G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, there are no edges from $V(G'_1 \setminus G'_2)$ to $V(G'_2 \setminus G'_1)$, and denote by $G'_0 := G[V(G'_1) \cap V(G'_2)]$ the subgraph induced by the vertices that are in both G'_1 and G'_2. A graph G is an expansion of G' with respect to G_0 if G is obtained from G' by replacing each vertex v of G'_1 by a vertex v_1 and each vertex v of G'_2 by a vertex v_2 such that u_i and v_i, $i = 1, 2$ are adjacent in G if and only if u and v are adjacent vertices of G'_i, and $v_1 v_2$ is an edge of G if and only if v is a vertex of G'_0. The following is well-known:

Lemma 2.5 ([9, 12])**.** *A graph G is a partial cube if and only if G can be obtained by a sequence of expansions from a single vertex.*

We will make use of the following lemma about the interplay of contractions and expansions:

Lemma 2.6. *Assume that we have the following commutative diagram of contractions:*

$$\begin{array}{ccc} G & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(G) \\ \downarrow{\scriptstyle \pi_{f_2}} & & \downarrow{\scriptstyle \pi_{f_2}} \\ \pi_{f_2}(G) & \xrightarrow{\pi_{f_1}} & \pi_{f_1}(\pi_{f_2}(G)) \end{array}$$

If G is expanded from $\pi_{f_1}(G)$ along sets $G_1, G_2 \subseteq \pi_{f_1}(G)$, then $\pi_{f_2}(G)$ is expanded from $\pi_{f_1}(\pi_{f_2}(G))$ along sets $\pi_{f_2}(G_1)$ and $\pi_{f_2}(G_2)$.

Proof. Let $\pi_{f_2}(G)$ be expanded from $\pi_{f_1}(\pi_{f_2}(G))$ along sets H_1, H_2. Consider $v \in \pi_{f_1}(\pi_{f_2}(G))$. Vertex v is in $H_1 \cap H_2$ if and only if its preimage in $\pi_{f_2}(G)$ is an edge $a \in E_{f_1}$. This is equivalent to $\pi_{f_1}^{-1}(a)$ being intersected by $E_{f_1}^+$ and $E_{f_1}^-$ in G. But this means, that the image of $\pi_{f_2}^{-1}(a)$ in $\pi_{f_1}(G)$, say $I := \pi_{f_1}(\pi_{f_2}^{-1}(a))$, has at least one vertex in $G_1 \cap G_2$. The image I is contracted to v by π_{f_2}, thus I is an edge or a vertex. Since every edge of $\pi_{f_1}(G)$ must have both its endpoints in G_1 or both its endpoints in G_2, we deduce that I has a vertex in $G_1 \cap G_2$ if and only if v in $\pi_{f_2}(G_1) \cap \pi_{f_2}(G_2)$. This proves that $H_1 \cap H_2 = \pi_{f_2}(G_1) \cap \pi_{f_2}(G_2)$.

Removing $H_1 \cap H_2 = \pi_{f_2}(G_1) \cap \pi_{f_2}(G_2)$ from $\pi_{f_1}(\pi_{f_2}(G))$ cuts it into two connected components, one a subset of H_1, one a subset of H_2. On the other hand, removing $G_1 \cap G_2$ from $\pi_{f_1}(G)$ also cuts it into two connected components, one in G_1 and one in G_2. Since π_{f_2} maps connected subgraphs to connected subgraphs, we see that $H_1 = \pi_{f_2}(G_1)$ and $H_2 = \pi_{f_2}(G_2)$, or the other way around. □

Let G be a partial cube and $f \in \mathscr{E}$ indexing one of its Θ-classes E_f. Assume that a halfspace E_f^+ (or E_f^-) is such that all its vertices are incident with edges from E_f. Then we call E_f^+ (or E_f^-) ***peripheral***. In such a case we will also call E_f a peripheral Θ-class, and call G a ***peripheral expansion*** of $\pi_f(G)$. Note that an expansion along sets G_1, G_2 is peripheral if and only if one of the sets G_1, G_2 is the whole graph and the other one an isometric subgraph. An expansion is called ***full*** if $G_1 = G_2$. Note that in this case, the expanded graph is isomorphic to $G_1 \square K_2$. See Figure 6 for examples of peripheral and non-peripheral expansions.

2.1.4 Zone graphs

For a partial cube G and $f \in \mathscr{E}$ the ***zone graph*** of G with respect to f is the graph $\zeta_f(G)$ whose vertices correspond to the edges of E_f and two vertices are connected by an edge if the corresponding edges of E_f lie in a convex cycle of G, see [29]. Here, a ***convex cycle*** is just a convex subgraph that is a cycle. In particular, ζ_f can be seen as a mapping from edges of G that are not in E_f but lie on a convex cycle crossed by E_f to the edges of $\zeta_f(G)$. If $\zeta_f(G)$ is a partial cube, then we say that $\zeta_f(G)$ is ***well-embedded*** if for two edges a, b of $\zeta_f(G)$ we have $a \Theta b$ if and only if the sets of Θ-classes crossing $\zeta_f^{-1}(a)$ and $\zeta_f^{-1}(b)$ coincide and otherwise they are disjoint. As an example, note that all zone graphs of the graph in Figure 2 are well-embedded paths, while all zone graphs of the graph on top in Figure 3 are triangles. For yet another example, see Figure 7. A consequence of Corollary 7.1 will be that out of these three examples only the first one is the tope graph of a COM.

For discussing zone graphs in partial cubes the following will be useful. Let $v_1u_1, v_2u_2 \in E_e$ be edges in a partial cube G with $v_1, v_2 \in E_e^+$. Let $C_1, \dots, C_n$, $n \geq 1$, be a sequence of convex cycles such that v_1u_1 lies only on C_1, v_2u_2 lies only on C_n, and each pair C_i and C_{i+1}, for $i \in \{1, \dots, n-1\}$, intersects in exactly one edge and this edge is in E_e, all the other pairs do not intersect. If the shortest path from v_1 to v_2 on the union of $C_1, \dots, C_n$ is a shortest v_1, v_2-path in G, then we call $C_1, \dots, C_n$ a ***convex traverse*** from v_1u_1

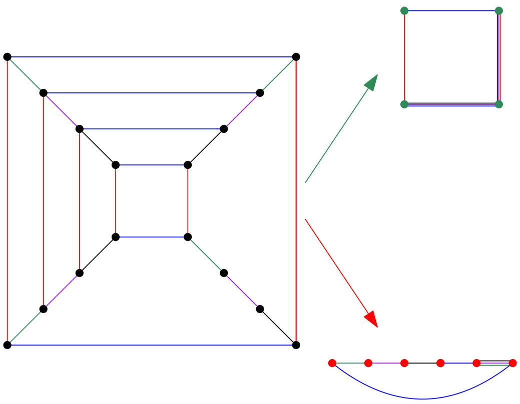


Figure 7: A partial cube all of whose zone graphs are either 4-cycles or 6-cycles, but none of them are well-embedded.

to v_2u_2. In [31] it was shown that for every pair of edges v_1u_1, v_2u_2 in relation Θ there exists a convex traverse connecting them.

Lemma 2.7. *Let G be a partial cube and $f \in \mathscr{E}$. Then $\zeta_f(G)$ is a well-embedded partial cube if and only if for any two convex cycles C, C' that are crossed by E_f and some E_g both C and C' are crossed by the same set of Θ-classes.*

Proof. The direction "⇒" follows immediately from the definition of well-embedded.

For "⇐" let G satisfy the property that for any two convex cycles C, C' that are crossed by E_f and some E_g both C, C' are crossed by the same set of Θ-classes.

Define an equivalence relation on the edges of $\zeta_f(G)$ by $a \sim b$ if and only if $\zeta_f^{-1}(a)$ and $\zeta_f^{-1}(b)$ are crossed by the same set of Θ-classes. Let $a', b' \in E_f$ be two edges of G corresponding to vertices of $\zeta_f(G)$. Then, there exists a convex traverse T from a' to b', i.e., no two cycles in T share Θ-classes apart from f. By the property on convex cycles in G all such paths from a' to b' in $\zeta_f(G)$ are crossed by the same set of equivalence classes and each exactly once. Furthermore, if there was a path in $\zeta_f(G)$ not corresponding to a traverse, its cycles would repeat Θ-classes of G, thus cross several times equivalence classes of $\zeta_f(G)$. Thus, every equivalence class of $\sim$ cuts $\zeta_f(G)$ into two convex subgraphs. We have that $\zeta_f(G)$ is a partial cube and the embedding we defined shows that it is well-embedded. □

A well-embedded zone graph $\zeta_f(G)$ thus induces an equivalence relation on the Θ-classes of G except f, that are involved in convex cycles crossed by E_f. We denote by $\bar{e}$ the class of Θ-classes containing E_e. Note that $\bar{e}$ corresponds to a Θ-class of $\zeta_f(G)$ and vice versa.

The following will be useful:

Lemma 2.8. *Let $\zeta_f(G)$ be a well-embedded partial cube and g, h two equivalent Θ-classes of G and C a convex cycle crossed by E_f, E_g, E_h. If $a \in E_f$ is an edge such that $a = C \cap E_f \cap E_g^+ = C \cap E_f \cap E_h^+$, then each edge of E_f is either in $E_g^+ \cap E_h^+$ or in $E_g^- \cap E_h^-$.*

Proof. Suppose otherwise, that there is an edge b in $E_f \cap E_g^+ \cap E_h^-$. The interval from a to b is crossed by E_h but not by E_g. Let T be a convex traverse from a to b. Then there exists a convex cycle on T crossed by E_f and E_h but not by E_g contradicting Lemma 2.7. □

Lemma 2.8 justifies that if $\zeta_f(G)$ is a well-embedded partial cube and $g \in \mathscr{E} \setminus \{f\}$, then we can orient $\overline{g}$ in $\zeta_f(G)$ such that $\rho_{\overline{g}^+}(\zeta_f(G)) = \zeta_f(\rho_{\overline{g}^+}(G))$.

Lemma 2.9. *Let G be a partial cube, $f \in \mathscr{E}$ such that $\zeta_f(G)$ is a well-embedded partial cube, $A \subseteq \mathscr{E} \setminus \{f\}$ and $X \in \{+,-\}^A$. We have $\pi_{\{\overline{e} \mid e \subseteq A\}}(\zeta_f(G)) = \zeta_f(\pi_A(G))$ and $\rho_{\{\overline{e}^{X_e} \mid e \in A\}}(\zeta_f(G)) = \zeta_f(\rho_X(G))$.*

Proof. For the contractions, clearly any contraction in $\zeta_f(G)$ corresponds to contracting the corresponding equivalence classes in G. Conversely, if some Θ-classes A are contracted in G, this affects only the classes of $\zeta_f(G)$ such that all the corresponding edges in G are contained in A.

Taking a restriction in $\zeta_f(G)$ can be modeled by restricting to the respective sides of all the elements of the corresponding class of Θ-classes of G.

By Lemma 2.8, if a set of restrictions in G leads to a non-empty zone graph, there is an orientation for all the elements of the classes of Θ-classes containing them leading to the same result. □

2.2 Pc-minors and expansions versus metric subgraphs

In this section we present conditions under which contractions, restrictions, and expansions preserve metric properties of subgraphs, such as convexity, gatedness, antipodality, and affinity. An important result of the section is an intrinsic characterization of affine partial cubes (Proposition 2.16).

2.2.1 Convex subgraphs

Let $G = (V, E)$ be an isometric subgraph of the hypercube $Q_{\mathscr{E}}$ and let S be a subset of vertices of G. Let f be any coordinate of $\mathscr{E}$. We will say that E_f is ***disjoint*** from S if it does not cross S and has no vertices in S. Note that a non-crossing class E_f can have vertices in S, e.g., $S \cap E_f^+ \neq \emptyset$. Thus, disjointness is stronger than to be non-crossing. The following three lemmas describe the behavior of convex subgraphs under contractions, restrictions, and expansions. Their (short) proofs can be found in [11].

Lemma 2.10. *If H is a convex subgraph of G and $f \in \mathscr{E}$, then $\rho_{f^+}(H)$ is a convex subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H, then also $\pi_f(H)$ is a convex subgraph of $\pi_f(G)$.*

Lemma 2.11. *If S is a subset of vertices of G and $f \in \mathscr{E}$, then $\pi_f(\mathrm{conv}(S)) \subseteq \mathrm{conv}(\pi_f(S))$. If E_f crosses S, then $\pi_f(\mathrm{conv}(S)) = \mathrm{conv}(\pi_f(S))$.*

Lemma 2.12. *If H' is a convex subgraph of G' and G is obtained from G' by an isometric expansion, then the expansion of H of H' is a convex subgraph of G.*

2.2.2 Antipodal subgraphs

Let H be a subgraph of G. If for a vertex $x \in H$ there is a vertex $-_H x \in H$ such that $\mathrm{conv}(x, -_H x) = H$ we say that $-_H x$ is the ***antipode*** of x with respect to H and we omit the subscript H if this causes no confusion. Intervals in a partial cube are convex since intervals in hypercubes equal (convex) subhypercubes, therefore $\mathrm{conv}(x, -_H x)$ consists of all the vertices on the shortest paths connecting x and $-_H x$. Then it is easy to see, that if a vertex has an antipode, it is unique. We call a subgraph H of a partial cube $G = (V, E)$ ***antipodal*** if every vertex x of H has an antipode with respect to H. Note that antipodal graphs are sometimes defined in a different but equivalent way and then are called symmetric-even, see [5]. By definition, antipodal subgraphs are convex. See Figure 5 for examples of antipodal and non-antipodal subgraphs. Their behavior with respect to pc-minors has been described in [11] in the following way:

Lemma 2.13. *Let H be an antipodal subgraph of G and $f \in \mathscr{E}$. If E_f is disjoint from H, then $\rho_{f^+}(H)$ is an antipodal subgraph of $\rho_{f^+}(G)$. If E_f crosses H or is disjoint from H, then $\pi_f(H)$ is an antipodal subgraph of $\pi_f(G)$.*

In particular, Lemma 2.13 implies that the class of antipodal partial cubes is closed under contractions. Next we will deduce a characterization of those expansions that generate all antipodal partial cubes from a single vertex, in the same way as Lemma 2.5 characterizes all partial cubes. Let G be an antipodal partial cube and G_1, G_2 two subgraphs corresponding to an isometric expansion. We say that it is an ***antipodal expansion*** if and only if $-G_1 = G_2$, where $-G_1$ is defined as the set of antipodes of G_1.

Lemma 2.14. *Let G be a partial cube and $\pi_e(G)$ antipodal. Then G is an antipodal expansion of $\pi_e(G)$ if and only if G is antipodal. In particular, all antipodal partial cubes arise from a single vertex by a sequence of antipodal expansion.*

Proof. Say $\pi_e(G)$ is expanded to G along sets G_1, G_2. Let $v \in G_1$ and $v' \in G$ a vertex with $\pi_e(v') = v$.

If G is antipodal, there exists a vertex $-v'$ whose distance to v' is equal to the number of Θ-classes of G. In particular, the shortest path must cross E_e, proving that $\pi_e(-v') \in G_2$. But $\pi_e(-v') = -v$ proving that $-v \in G_2$.

Conversely, if $-G_1 = G_2$ it is easy to see, that the antipode of v' is in $\pi_e^{-1}(-v)$. □

A further useful property of antipodal subgraphs of partial cubes proved in [11] is the following:

Lemma 2.15. *Let H be an antipodal subgraph of G and $u, v \in H$, then H contains an isometric cycle C through $v, u -_H v$ such that* $\mathrm{conv}(C) = H$.

2.2.3 Affine subgraphs

We call a partial cube ***affine*** if it is a halfspace of an antipodal partial cube. All graphs except the one on the top and the $K_{1,3}$ in Figure 5 are affine. We can give the following intrinsic characterization of affine partial cubes, that will play a crucial role in our characterization of tope graphs of AOMs, see Corollary 7.3.

Proposition 2.16. *A partial cube G is affine if and only if for all u, v vertices of G there are $w, -w$ in G such that* $\mathrm{conv}(u, w)$ *and* $\mathrm{conv}(v, -w)$ *are crossed by disjoint sets of Θ-classes.*

Proof. Let $G = E_f^+(\widetilde{G})$ be a halfspace of an antipodal partial cube $\widetilde{G}$. For $u, v \in G$ consider the antipode $-_{\widetilde{G}} v$ of v in $E_f^-(\widetilde{G})$. By Lemma 2.15, we can consider an isometric cycle C through $v, u, -_{\widetilde{G}} v$ such that $\mathrm{conv}(C) = \widetilde{G}$. The two vertices w, z on $C \cap E_f^+(\widetilde{G})$ that are incident with edges from $E_f(\widetilde{G})$ are connected on C by a shortest path crossing all the Θ-classes of G, i.e. $z = -_G w$. By symmetry of $w, -_G w$, we can assume that v appears before u on a shortest path from w to $-_G w$. Thus $w, -_G w \in G$ are such that $\mathrm{conv}(u, w)$ and $\mathrm{conv}(v, -_G w)$ are crossed by disjoint sets of Θ-classes.

Conversely, let G be such that for all $u, v \in G$ there are $w, -w \in G$ such that $\mathrm{conv}(u, w)$ and $\mathrm{conv}(v, -w)$ are crossed by disjoint sets of Θ-classes. We construct $\widetilde{G}$ by taking a copy G' of G and join w with an edge to $(-w)'$ for each pair $w, -w \in G$. Associating all these new edges to a new coordinate of the hypercube we get an embedding into a hypercube of dimension one higher. First we show that $\widetilde{G}$ is a partial cube. Since G and its copy on their own are partial cubes, suppose now that $u \in G$ and $v' \in G'$. In G we can take $w, -_G w \in G$ such that $\mathrm{conv}_G(u, w)$ and $\mathrm{conv}_G(v, -_G w)$ are crossed by disjoint sets of Θ-classes. Consider a shortest path from u to w, then the edge to $(-w)'$, and finally a shortest path from $(-w)'$ to v'. Since none of the original Θ-classes was crossed twice, this is a shortest path of the hypercube that $\widetilde{G}$ is embedded in.

It remains to show that $\widetilde{G}$ is antipodal. For every vertex $v \in G$ there exists $w, -w \in G$ such that $\mathrm{conv}(v, w)$ and $\mathrm{conv}(v, -w)$ are crossed by disjoint sets of Θ-classes. In fact, in this case $\mathrm{conv}(v, w)$ and $\mathrm{conv}(v, -w)$ together cross all Θ-classes of G. Hence taking a shortest path from v to w, then the edge to $(-w)'$ and from there a shortest path to v' yields a path from v to v' crossing each Θ-class of $\widetilde{G}$ exactly once. This implies that v' is an antipode of v. □

By Lemma 2.3 a contraction of a halfspace is a halfspace and by Lemma 2.13 antipodal partial cubes are closed under contraction, therefore we immediately get:

Lemma 2.17. *The class of affine partial cubes is closed under contraction.*

2.2.4 Gated subgraphs

A subgraph H of G, or just a set of vertices of H, is called ***gated*** (in G) if for every vertex x outside H there exists a vertex x' in H, the ***gate*** of x, such that each vertex y of H is connected with x by a shortest path passing through the gate x'. It is easy to see that if x has a gate in H, then it is unique and that gated subgraphs are convex. See [18] for several results on gated sets in metric spaces. See Figure 5 for examples of gated and non-gated subgraphs.

In [11] it was shown that gated subgraphs behave well with respect to pc-minors:

Lemma 2.18. *If H is a gated subgraph of G, then $\rho_{f^+}(H)$ and $\pi_f(H)$ are gated subgraphs of $\rho_{f^+}(G)$ and $\pi_f(G)$, respectively.*

In the next lemma we will see that expansions can turn gated graphs into non-gated graphs.

Lemma 2.19. *Let G be an expansion of $\pi_e(G)$ along sets G_1, G_2. Let H be a gated subgraph of $\pi_e(G)$, v a vertex of $\pi_e(G)$ and v' the gate of v in H. If $v \in G_1 \cap G_2$, $v' \notin G_1 \cap G_2$ and there exist $v'' \in H$, $v'' \in G_1 \cap G_2$, then the expansion of H in G is not gated.*

Proof. Let v, v', v'', H be as in the lemma and without loss of generality assume that $v' \in G_1 \backslash G_2$. Let E_e^+ correspond to G_1 and E_e^- to G_2 in G. Since vertex $v \in G_1 \cap G_2$, it is expanded to an edge in G. Let u be the vertex on this edge in E_e^+. Then every shortest path form u to the expansion of H must cross at least the same Θ-classes as a shortest path from v to v'. On the other hand, $v, v' \in G_1$, thus in G_1 there exists a shortest path from v to v'. Then there exists a shortest path from u to the expansion of H, first crossing E_e and then all the Θ-classes in the shortest path from v to v'. Note that the expansion of v' is not the gate of u since there is no shortest path from u to the expansion of v'' in E_e^+ passing this vertex. Thus if u has a gate to the expansion of H, it must be at distance $d(v, v')$ to u and the shortest path connecting them must be crossed by exactly those Θ-classes that cross shortest paths from v to v'. Then this gate must be adjacent to the expansion of v' and be in E_e^+. This is impossible, since $v' \in G_1 \backslash G_2$. □

3 Systems of sign-vectors

In the present sections we will introduce the standard definitions concerning systems of sign-vectors. In particular we will introduce the axiomatics for COMs, AOMs, OMs, and LOPs and operations such as reorientations and minors. Recall that Figure 1 depicts the example of a COM.

We follow the standard OM notation from [8] and concerning COMs we stick to [3]. Let $\mathscr{E}$ be a non-empty finite (ground) set and let $\emptyset \neq \mathscr{L} \subseteq \{+,-,0\}^{\mathscr{E}}$. The elements of $\mathscr{L}$ are referred to as ***covectors***.

For $X \in \mathscr{L}$, and $e \in \mathscr{E}$ let X_e be the value of X at the coordinate e. The subset $\underline{X} = \{e \in \mathscr{E} : X_e \neq 0\}$ is called the ***support*** of X and its complement $X^0 = \mathscr{E} \setminus \underline{X} = \{e \in \mathscr{E} : X_e = 0\}$ the ***zero set*** of X. For $X, Y \in \mathscr{L}$, we call $S(X,Y) = \{f \in \mathscr{E} : X_f Y_f = -\}$ the ***separator*** of X and Y. The ***composition*** of X and Y is the sign-vector $X \circ Y$, where $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ if $X_e = 0$. For a subset $A \subseteq \mathscr{E}$ and $X \in \mathscr{L}$ the ***reorientation*** of X with respect to A is the sign-vector defined by

$$({}_A X)_e := \begin{cases} -X_e & \text{if } e \in A \\ X_e & \text{otherwise.} \end{cases}$$

In particular $-X := {}_{\mathscr{E}} X$. The ***reorientation*** of $\mathscr{L}$ with respect to A is defined as ${}_A\mathscr{L} := \{{}_A X \mid X \in \mathscr{L}\}$. In particular, $-\mathscr{L} := {}_{\mathscr{E}} \mathscr{L}$.

We continue with the formal definition of the main axioms relevant for COMs, AOMs, OMs, and LOPs. All of them are closed under reorientation.

Composition:

(C) $X \circ Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

Since $\circ$ is associative, arbitrary finite compositions can be written without bracketing $X_1 \circ \ldots \circ X_k$ so that (C) entails that they all belong to $\mathcal{L}$. Note that contrary to a convention sometimes made in OMs we do not consider compositions over an empty index set, since this would imply that the zero sign-vector belonged to $\mathcal{L}$. The same consideration applies for the following two strengthenings of (C).

Face symmetry:

(FS) $X \circ -Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

By (FS) we first get $X \circ -Y \in \mathcal{L}$ and then $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y) \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$. Thus, (FS) implies (C).

Ideal composition:

(IC) $X \circ Y \in \mathcal{L}$ for all $X \in \mathcal{L}$ and $Y \in \{+, -, 0\}^{\mathcal{E}}$.

Note that (IC) implies (C) and (FS). The following axiom is part of all the systems of sign-vectors discussed in the paper:

Strong elimination:

(SE) for each pair $X, Y \in \mathcal{L}$ and for each $e \in S(X, Y)$ there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in \mathcal{E} \setminus S(X, Y)$.

An axiom particular to OMs is:

Zero vector:

(Z) the zero sign-vector $\mathbf{0}$ belongs to $\mathcal{L}$.

We will not make proper use of the axiomatization of AOMs due to [4, 28] apart from illustrating that AOMs are a natural subclass of COMs, see Definition 3.1. Let us however briefly introduce an operation on sign-vectors needed to axiomatize AOMs:

$$(X \oplus Y)_e := \begin{cases} 0 & \text{if } e \in S(X, Y) \\ (X \circ Y)_e & \text{otherwise.} \end{cases}$$

Affinity:

(A) Let $X, Y \in \mathcal{L}$ such that for all $e \in S(X, -Y)$ and $W \in \mathcal{L}$ with $W_e = 0$ there are $f, g \in \mathcal{E} \setminus S(X, -Y)$ such that $W_f \neq (X \circ -Y)_f$ and $W_g \neq (-X \circ Y)_g$. We have $(X \oplus -Y) \circ Z \in \mathcal{L}$ for all $Z \in \mathcal{L}$.

We are now ready to define the central systems of sign-vectors of the present paper:

Definition 3.1. *A system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is called a:*

- ***complex of oriented matroids (COM)*** *if $\mathcal{L}$ satisfies (FS) and (SE),*
- ***affine oriented matroid (AOM)*** *if $\mathcal{L}$ satisfies (A), (FS), and (SE),*
- ***oriented matroid (OM)*** *if $\mathcal{L}$ satisfies (Z), (FS), and (SE),*
- ***lopsided system (LOP)*** *if $\mathcal{L}$ satisfies (IC) and (SE).*

Note that for OMs one can replace (Z) and (FS) by (C) and:

Symmetry:

(Sym) $-X \in \mathcal{L}$ for all $X \in \mathcal{L}$.

Let $\mathcal{L} \subseteq \{0,-1,1\}^{\mathcal{E}}$ be a system of sign-vectors and $e \in \mathcal{E}$. For $X \in \mathcal{L}$ let $X\backslash e$ be the element of $\{0,-1,1\}^{\mathcal{E}\backslash\{e\}}$ obtained by deleting the coordinate e from X. Define operations $\mathcal{L}/e = \{X\backslash e \mid X \in \mathcal{L}, X_e = 0\}$ as taking the ***hyperplane*** of e (usually referred to as contraction) and $\mathcal{L}\backslash e = \{X\backslash e \mid X \in \mathcal{L}\}$ as the ***deletion*** of e. A sign-system that arises by deletion and taking hyperplanes from another one is called a ***minor***. Furthermore denote by $\mathcal{L}_e^+ := \{X\backslash e \in\mid X \in \mathcal{L}, X_e = +\}$ and $\mathcal{L}_e^- := \{X\backslash e \mid X \in \mathcal{L}, X_e = -\}$ the positive and negative (open) ***halfspaces*** with respect to e.

A theorem due to Karlander [28] characterizes AOMs as exactly the halfspaces of OMs. However, his proof contains a flaw that has only been observed and fixed recently in [4].

The following is easy to see, see e.g. [3, Lemma 2].

Lemma 3.2. *For any system of sign-vectors the operations of taking halfspaces, hyperplanes and deletion commute.*

Our systems of sign-vectors behave well with respect to the above operations:

Lemma 3.3. *The classes of COMs, AOMs, OMs, and LOPs are minor closed. Moreover, COMs and LOPs are closed under taking halfspaces.*

Proof. The result for OMs is folklore and can be found for instance in [6]. For COMs this was shown in [3, Lemma 1] and [3, Lemma 4]. For LOPs it is easy to see, that (IC) is preserved under minors and taking halfspaces. The minor-closedness of AOMs is a little more involved, when only using the axioms given above, see e.g. [15] but using that they are halfspaces of OMs this follows directly from Lemma 3.2. □

The ***rank*** of a system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is the largest integer r such that there is subset $A \subseteq \mathcal{E}$ of size $|\mathcal{E}| - r$ such that $\mathcal{L}\backslash A = \{+,-,0\}^r$. In other words, the rank of $(\mathcal{E}, \mathcal{L})$ is just the VC-dimension of $\mathcal{L}$, see [35]. Note that this definition of rank coincides with the usual rank definition for OMs, see [14].

A system of sign-vectors $(\mathcal{E}, \mathcal{L})$ is ***simple*** if it satisfies the following two conditions:

(N1*) for each $e \in \mathcal{E}$, $\{+,-,0\} = \{X_e : X \in \mathcal{L}\}$;

(N2*) for each pair $e \neq f$ in $\mathcal{E}$, there exist $X, Y \in \mathcal{L}$ with $\{X_e X_f, Y_e Y_f\} = \{+,-\}$.

An element $e \in \mathcal{E}$ not satisfying (N1*) is called ***redundant***. Note that redundant elements in OMs are zero everywhere and are called ***loops***, see [8], but in COMs also a sign can be present on a redundant element. Two elements $e, f \in \mathcal{E}$ are called ***parallel*** if they do not satisfy (N2*). Note that parallelism is an equivalence relation on $\mathcal{E}$. We denote by $\overline{e}$ the class of elements parallel to e, for $e \in \mathcal{E}$. The notion of parallelism coincides with the one in OMs.

For every COM $(\mathcal{E}, \mathcal{L})$ there exists up to reorientation and relabeling of coordinates a unique simple COM, obtained by successively applying operation $\mathcal{L}\backslash e$ to the redundant coordinates $e \in \mathcal{E}$ and to elements of parallel classes with more than one element. See [3, Proposition 3] for the details. Note that by Lemma 3.2 the order in which these operations are taken is irrelevant and by Lemma 3.3 all the classes of systems of sign-vectors at consideration here, are closed under this operation. We will denote by $\mathcal{S}(\mathcal{E}, \mathcal{L})$ the unique ***simplification*** of $(\mathcal{E}, \mathcal{L})$.

4 Systems of sign-vectors and partial cubes

This section finally builds the link between systems of sign-vectors and partial cubes, that in a sense is the very basis of our results. It is therefore built on properties established in Sections 2 and 3. We begin with a kind of dictionary between axiomatic properties of systems of sign vectors and the behavior of metric subgraphs of partial cubes. In particular we will show in Theorem 4.9 that in tope graphs of COMs all antipodal subgraphs are gated and its corollaries for OMs, AOMs, and LOPs later. Recall that Figure 2 shows the tope graph of a COM.

The ***topes*** of a system of sign-vectors $(\mathscr{E},\mathscr{L})$ are the elements of $\mathscr{T} := \mathscr{L} \cap \{+,-\}^{\mathscr{E}}$. If $(\mathscr{E},\mathscr{L})$ is simple, we define the ***tope graph*** $G(\mathscr{L})$ of $(\mathscr{E},\mathscr{L})$ as the (unlabeled) subgraph of $Q_{\mathscr{E}}$ induced by $\mathscr{T}$. If $(\mathscr{E},\mathscr{L})$ is non-simple, we consider $G(\mathscr{L})$ as the tope graph of its simplification $\mathscr{S}(\mathscr{E},\mathscr{L})$.

In general $G(\mathscr{L})$ is an unlabeled graph and even though it is defined as a subgraph of a hypercube $Q_{\mathscr{E}}$ it could possibly have multiple non-equivalent embeddings in $Q_{\mathscr{E}}$. We call a system $(\mathscr{E},\mathscr{L})$ a ***partial cube system*** if its tope graph $G(\mathscr{L})$ is an isometric subgraph of $Q_{\mathscr{E}}$ in which the edges correspond to sign-vectors of $\mathscr{L}$ with a single 0. It is well-known that partial cubes have a unique embedding in $Q_{\mathscr{E}}$ up to automorphisms of $Q_{\mathscr{E}}$, see e.g. [32, Chapter 5]. In other words, the tope graph of a simple partial cube system is invariant under reorientation. For this reason we will, possibly without an explicit note, identify vertices of a partial cube $G(\mathscr{L})$ with subsets of $\{+,-\}^{\mathscr{E}}$. The following was proved in [3, Proposition 2]:

Lemma 4.1. *Simple COMs are partial cube systems.*

Before presenting basic results regarding partial cube systems, we discuss how the minor operations and taking halfspaces as defined in Section 3 affect tope graphs. So let $(\mathscr{E},\mathscr{L})$ be a simple partial cube system.

First note that deletion does not affect the simplicity of $(\mathscr{E},\mathscr{L})$. Furthermore, since $(\mathscr{E},\mathscr{L})$ is a partial cube system, the tope graph $G(\mathscr{L}\setminus e)$ corresponds to $\pi_e(G(\mathscr{L}))$ obtained from $G(\mathscr{L})$ by contracting all the edges in the Θ-class corresponding to coordinate e, as defined in Section 2. Noticing that $\mathscr{L}(Q_r)$, for a hypercube Q_r, equals $\{+,-,0\}^r$ we immediately get the following lemma from the definition of the rank of a system of sign-vectors.

Lemma 4.2. *The rank of a partial cube system $(\mathscr{E},\mathscr{L})$ is the largest r such that $G(\mathscr{L})$ contracts to Q_r.*

Also, the halfspace $\mathscr{L}_e^+$ is easily seen to be simple and its tope graph corresponds to the restriction $\rho_{e^+}(G(\mathscr{L}))$ to the positive halfspace of E_e for $e \in \mathscr{E}$, as defined in Section 2.

The hyperplane $\mathscr{L}/e$ does not need to be a simple system of sign-vectors nor a partial cube system. However, we can establish the following:

Lemma 4.3. *Let $(\mathscr{E},\mathscr{L})$ be a partial cube system and $e \in \mathscr{E}$. If $\zeta_e(G(\mathscr{L}))$ is a well-embedded partial cube, then $\zeta_e(G(\mathscr{L})) \cong G(\mathscr{L}/e)$.*

Proof. Clearly, both sets of vertices correspond to the set of edges of $G(\mathscr{L})$. If there is an edge in $G(\mathscr{L}/e)$, it corresponds to two edges of $G(\mathscr{L})$ in E_e such that the Θ-classes different than E_e crossing the interval between the two edges form parallel elements of $G(\mathscr{L}/e)$. This means that the interval does not cross E_e in other elements besides the two edges. The interval must include a convex traverse between the two edges, but since there is no other edge in E_e in it, the traverse is a convex cycle. This implies that there is a corresponding edge in $\zeta_e(G(\mathscr{L}))$.

Now, let $\zeta_e(G(\mathscr{L}))$ be a well-embedded partial cube. An edge of it corresponds to a convex cycle C. By Lemma 2.7 all cycles crossing E_e and another Θ-class from C cross all its Θ-classes. By Lemma 2.8, the corresponding elements in $\mathscr{L}/e$ are parallel. Therefore the edge corresponds to an edge of $G(\mathscr{L}/e)$. □

The correspondences before the lemma in particular give that deletions and halfspaces of partial cube systems coincide with pc-minors, which together with Lemma 3.3 gives:

Proposition 4.4. *Let $(\mathscr{E},\mathscr{L})$ and $(\mathscr{E}',\mathscr{L}')$ be simple partial cube systems with tope graphs $G(\mathscr{L})$ and $G(\mathscr{L}')$, respectively. If $(\mathscr{E}',\mathscr{L}')$ arises from $(\mathscr{E},\mathscr{L})$ by deletion and taking halfspaces, then $G(\mathscr{L}')$ is a pc-minor of $G(\mathscr{L})$. Moreover, the families of tope graphs of COMs and LOPs are pc-minor closed.*

In the following, we will describe further how pc-minors and equivalently deletions and halfspaces of partial cube systems translate metric graph properties as introduced in Section 2 into properties of sign-vectors.

For $X \in \mathscr{L}$ we set $\mathscr{T}(X) := \{T \in \mathscr{T} \mid X \circ T = T\}$ and denote by $G(X)$ the subgraph of $G(\mathscr{L})$ induced by $\mathscr{T}(X)$. Note that in OMs the set $\mathscr{T}(X)$ is sometimes denoted as star(X), see [8]. Furthermore, let $\mathscr{H}(\mathscr{L}) = \{G(X) \mid X \in \mathscr{L}\}$ be the set of subgraphs of $G(\mathscr{L})$ obtained by considering $G(X)$ for all $X \in \mathscr{L}$. Conversely, given a convex subgraph G' of a partial cube G with Θ-classes $\mathscr{E}$ denote by $\chi(G') \in \{+,-,0\}^{\mathscr{E}}$ the sign-vector defined by setting for $e \in \mathscr{E}$:

$$\chi(G')_e = \begin{cases} + & \text{if } G' \subseteq E_e^+, \\ - & \text{if } G' \subseteq E_e^-, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each vertex $v \in G(\mathscr{L})$, $\chi(v) = v$. Furthermore, let $\mathscr{L}(\mathscr{H}) = \{\chi(G') \mid G' \in \mathscr{H}\}$ for a set $\mathscr{H}$ of convex subgraphs of G.

Proposition 4.5. *In a simple partial cube system $(\mathscr{E},\mathscr{L})$ for each $X \in \mathscr{L}$ its tope-graph $G(X)$ is a convex subgraph of $G(\mathscr{L})$. Conversely, if $G = (V,E)$ is a partial cube and $\mathscr{H}$ a set of convex subgraphs of G, such that $\mathscr{H}$ includes all the vertices of G, then there is a simple $(\mathscr{E},\mathscr{L})$ such that $G = G(\mathscr{L})$ and $\mathscr{H} = \mathscr{H}(\mathscr{L})$.*

Proof. Let $X \in \mathscr{L}$. Since $G(\mathscr{L})$ is an isometric subgraph of $Q_{\mathscr{E}}$, each $e \in \mathscr{E}$ can be identified with a Θ-class E_e for $e \in \mathscr{E}$ such that the halfspaces E_e^+ and E_e^- of $G(\mathscr{L}) \setminus E_e$ are convex. Now, $\mathscr{T}(X)$ induces the subgraph $\bigcap_{e \in \underline{X}} E_e^{X_e}$, i.e. is a restriction of $G(\mathscr{L})$ and therefore is convex.

For the converse, by Lemma 2.1 any convex subgraph $G' \in \mathscr{H}$ may be described as the intersection of convex halfspaces $E_e^{+,-}$ for some E_e with $e \in \mathscr{E}$. This allows for a correspondence of convex graphs $G' \in \mathscr{H}$ and sign-vector via $\chi(G') \in \{+,-,0\}^{\mathscr{E}}$ and establishes $\mathscr{H} = \mathscr{H}(\mathscr{L})$. Since every vertex is contained in $\mathscr{H}$, we have $G = G(\mathscr{L})$. □

The following establishes a connection between the gates of a convex set and the composition operator. The statement specialized to tope graphs of OMs can be found in [8, Exercise 4.10].

Lemma 4.6. *Let G be a partial cube embedded in a hypercube, G' a convex subgraph of G and v a vertex of G. Then w is the gate for v in G' if and only if $\chi(w) = \chi(G') \circ \chi(v)$. Therefore, a subgraph G' is gated if and only if for all $v \in G$ there is a $w \in G$ such that $\chi(G') \circ \chi(v) = \chi(w)$.*

Proof. First note that $\chi(w) = \chi(G') \circ \chi(v) \Leftrightarrow S(\chi(v),\chi(w)) = S(\chi(v),\chi(G'))$. Thus, if $\chi(w) = \chi(G') \circ \chi(v)$ then $S(\chi(v),\chi(w)) = S(\chi(v),\chi(G'))$. Hence, the concatenation of a shortest (v,w)-path and a shortest (w,w')-path for $w' \in G'$ does not cross any Θ-class twice, since by convexity of G' the Θ-classes crossed by the second part all are from $\chi(G')^0$. Hence it is a shortest path.

Now, let $v \in G$ and w a gate for v in G'. Suppose $e \in S(\chi(v),\chi(w)) \setminus S(\chi(v),\chi(G'))$. This means a Θ-class E_e for $e \in \mathscr{E}$ splits G and also G' into two halfspaces, but v and w do not lie on the same side. This is w cannot lie on a shortest path from v to a vertex in the halfspace of G' not containing w. This contradicts that w is a gate for v in G' and hence $S(\chi(v),\chi(w)) = S(\chi(v),\chi(G'))$, which implies $\chi(w) = \chi(G') \circ \chi(v)$. □

Lemma 4.7. *In an antipodal partial cube G, the **antipodal mapping** $v \mapsto -v$ is a graph automorphism and for every convex subgraph $-\chi(G') = \chi(-G')$.*

Proof. Since every vertex has a unique antipode, $v \mapsto -v$ is indeed a mapping. Since it is an involution it is bijective. To show that it is indeed a homomorphism let vw be an edge of G. Since G is a partial cube and since $\mathrm{conv}(v, -v) = G$ there is a shortest $(v, -v)$-path starting with the edge vw. Analogously, there is a shortest $(w, -w)$-path starting with the edge vw. Say this edge is in Θ-class E_e. Now, $S(\chi(v), \chi(-v)) \setminus S(\chi(v), \chi(-w)) = e$ and hence $S(\chi(-v), \chi(-w)) = e$, i.e. $-v - w$ is an edge.

Now, let G' be a convex subgraph of G. Since $\chi(G')$ just records the signs of the halfspaces of the classes of $\mathscr{E}$ which contain G' and the antipodal mapping sends each vertex v to the vertex sitting on all the other sides, we obtain the result. □

For a partial cube G isometrically embedded in a hypercube $Q_{\mathscr{E}}$ define the set of sign vectors

$$\mathscr{L}(G) = \{X \in \{0,+,-\}^{\mathscr{E}} \mid \text{ for all } v \in G \text{ there exists } w \in G : X \circ (-\chi(v)) = \chi(w)\}.$$

Lemma 4.8. *Let G be a partial cube isometrically embedded in a hypercube $Q_{\mathscr{E}}$. Then $\mathscr{L}(G)$ is a partial cube system that satisfies (FS) (and therefore (C)) and the set $\mathscr{H}(\mathscr{L}(G))$ of corresponding subgraphs coincides with the antipodal gated subgraphs of G.*

Proof. Clearly, the topes of $\mathscr{L}(G)$ correspond to the vertices of G. Moreover, a sign-vector with a single 0-entry is in $\mathscr{L}(G)$ if and only if both of the possibly signings of that entry are topes of $\mathscr{L}(G)$.

Since (FS) implies (C) (also when restricted to topes) we have that $X \in \mathscr{L}(G)$ implies $X \circ \chi(v) \in \mathscr{L}(G)$ for all $v \in G$. To show (FS) for $\mathscr{L}(G)$ let $X, Y \in \mathscr{L}(G)$ and $v \in G$ and note that $(X \circ (-Y)) \circ (-\chi(v)) = X \circ (-(Y \circ \chi(v)))$. Since by (C) we have that $Y \circ \chi(v) \in \mathscr{L}(G)$ it follows that $X \circ (-Y) \in \mathscr{L}(G)$.

We prove now the second part of the statement. Let $X \in \mathscr{L}(G)$ and a vertex $v \in G(X)$. We have that $X \circ -\chi(v) \in \mathscr{T}(X)$ is the antipode of v in $G(X)$. This is, $G(X)$ is antipodal. Furthermore, since (FS) implies (C) we also have that $X \circ \chi(v) \in \mathscr{T}(X)$, for all $v \in G$. By Lemma 4.6 we have that $G(X)$ is gated.

Conversely, if A is an antipodal gated subgraph of G, then by Lemma 4.6, we have that for the gate v' of v in A it holds $\chi(A) \circ \chi(v) = \chi(v')$. Now, the antipode of the gate of v' in A has to correspond to $\chi(A) \circ -\chi(v)$. Thus, $\chi(A) \in \mathscr{L}(G)$.

□

Proposition 4.5 states that in a simple system of sign-vectors there is a correspondence between its vectors and a subset of the set of convex subgraphs of its tope graph. The following proposition determines which convex subgraphs are in the subset if the system is a COM. For its statement denote by $\mathscr{G}_{\mathrm{COM}}$ the class of tope graphs of COMs. Moreover, we call a partial cube ***antipodally gated*** if all its antipodal subgraphs are gated and denote their class by AG.

Theorem 4.9. *For a simple COM $(\mathscr{E}, \mathscr{L})$ with embedded tope graph G we have*

$$\begin{aligned} \mathscr{L} &= \{\chi(G') \mid G' \text{ antipodal subgraph of } G(\mathscr{L})\} \\ &= \{\chi(G') \mid G' \text{ antipodal gated subgraph of } G(\mathscr{L})\} \\ &= \mathscr{L}(G). \end{aligned}$$

In particular, tope graphs of COMs are antipodally gated, i.e, $\mathscr{G}_{\mathrm{COM}} \subseteq \mathrm{AG}$.

Proof. Trivially, we have

$$\{\chi(G') \mid G' \text{ antipodal gated subgraph of } G(\mathscr{L})\} \subseteq \{\chi(G') \mid G' \text{ antipodal subgraph of } G(\mathscr{L})\}.$$

The equality

$$\mathscr{L}(G) = \{\chi(G') \mid G' \text{ antipodal gated subgraph of } G(\mathscr{L})\}$$

is precisely Lemma 4.8, while the inclusion $\mathscr{L} \subseteq \mathscr{L}(G)$ follows immediately from (FS).

Thus, we end by proving $\{\chi(G') \mid G' \text{ antipodal subgraph of } G(\mathscr{L})\} \subseteq \mathscr{L}$. Let G' be an antipodal subgraph of $G(\mathscr{L})$. We show $\chi(G') \in \mathscr{L}$ by induction on $|\chi(G')^0|$. If $|\chi(G')^0| = 0$ then $\chi(G') \in \mathscr{T}$

and we are done. If $|\chi(G')^0| > 0$ then take a maximal proper antipodal subgraph G'' of G'. Since vertices are antipodal subgraphs, G'' exists. By induction hypothesis $\chi(G'') \in \mathcal{L}$. Define G''' to be the subgraph of G' induced by all antipodes of vertices of G'' with respect to G'. By Lemma 4.7, we have $G''' \cong G''$. Moreover, $\chi(G'') = \chi(G') \circ -\chi(G''')$. Since G'' is a proper antipodal subgraph of G', there exists an $e \in \underline{\chi(G'')} \cap \chi(G')^0$. Then $e \in S(\chi(G''), \chi(G'''))$ and we now can apply (SE) to $\chi(G'')$ and $\chi(G''')$ with respect to e. We obtain $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (\chi(G'') \circ \chi(G'''))_f$ for all $f \in \mathcal{E} \setminus S(\chi(G''), \chi(G'''))$. By the inclusions that we have shown in the first two parts of the proof, Z corresponds to an antipodal subgraph $G(Z)$. By (SE), $G(Z)$ strictly contains G'' and is contained in G'. By the maximality of G'' we have $G(Z) = G'$ and therefore $\chi(G') = Z \in \mathcal{L}$. □

As a consequence of Theorem 4.9 we immediately get:

Corollary 4.10. *Every simple COM is uniquely determined by its tope set and up to reorientation by its tope graph.*

Corollary 4.10 had only been proved in a non-constructive way, see [3, Propositions 1 & 3]. The constructive statement here is in fact a generalization of a theorem known for OMs, usually attributed to Mandel, see [13] and the fact that an OM is determined up to reorientation by its tope graph is due to [7].

The following justifies that we can restrict ourselves to simple COMs when studying tope graphs.

Lemma 4.11. *A partial cube G is in $\mathcal{G}_{\text{COM}}$ if and only if $\mathcal{L}(G)$ is a simple COM.*

Proof. If $\mathcal{L}(G)$ is a simple COM, then G is by definition its tope graph, thus G is in $\mathcal{G}_{\text{COM}}$. On the other hand, let G be a graph in $\mathcal{G}_{\text{COM}}$, and $\mathcal{L}$ its up to reorientation unique simple COM. By Theorem 4.9, $\mathcal{L} = \{X \in \{0,-1,1\}^{\mathcal{E}} \mid X \circ -T \in \mathcal{T} \text{ for all } T \in \mathcal{T}\} = \mathcal{L}(G)$. □

Th following will be essential for the Handa-type characterization of tope graphs of COMs in Corollary 7.1. It can be seen as the graph theoretical analogue of the fact the COMs are closed under taking hyperplanes.

Lemma 4.12. *In a COM $(\mathcal{E}, \mathcal{L})$ all zone graphs $\zeta_f(G(\mathcal{L}))$ are well-embedded partial cubes. In particular, $\mathcal{G}_{\text{COM}}$ is closed under taking zone graphs.*

Proof. Suppose there are two convex cycles C, C' in $G(\mathcal{L})$ that contradict Lemma 2.7, i.e., both are crossed by E_f and E_g and C is crossed by E_h but C' is not. By the second equivalence in Theorem 4.9, the cycles are gated. Without loss of generality assume that C' is completely in E_h^+. On the other hand, we can reorient E_f and E_g in a way that the vertices on $C \cap E_h^+$ are in $E_f^+ \cap E_g^+, E_f^+ \cap E_g^-$ and $E_f^- \cap E_g^-$. But then any vertex in C' that is in $E_f^- \cap E_g^+$ has no gate to C, a contradiction.

Now, since COM is closed under taking hyperplanes by Lemma 3.3, Lemma 4.3 gives that $\mathcal{G}_{\text{COM}}$ is closed under taking zone graphs.

□

Let us finally describe how tope graphs of the other systems of sign-vectors from Section 3 specialize tope graphs of COMs. We will denote the classes of tope graphs of OMs, AOMs, and LOPs by $\mathcal{G}_{\text{OM}}$, $\mathcal{G}_{\text{AOM}}$, and $\mathcal{G}_{\text{LOP}}$. The following three propositions will be cast in the proofs of the characterizations of $\mathcal{G}_{\text{OM}}$, $\mathcal{G}_{\text{AOM}}$, and $\mathcal{G}_{\text{LOP}}$, see Corollaries 7.2, 7.3, and 7.4.

A consequence of Lemma 4.7 is:

Proposition 4.13. *A graph is in $\mathcal{G}_{\text{OM}}$ if and only if it is antipodal and in $\mathcal{G}_{\text{COM}}$.*

A not yet intrinsic description of tope graphs of AOMs follows:

Proposition 4.14. *A graph is in $\mathcal{G}_{\text{AOM}}$ if and only if it is a halfspace of a graph in $\mathcal{G}_{\text{OM}}$.*

Interpreting axiom (IC) in the partial cube model we also get:

Proposition 4.15. *A graph is in $\mathcal{G}_{\text{LOP}}$ if and only if all its antipodal subgraphs are hypercubes and it is in $\mathcal{G}_{\text{COM}}$.*

5 The excluded pc-minors

In the present section we will introduce the set $\mathcal{Q}^-$ of minimal excluded pc-minors for tope graphs of COMs. After providing several properties with respect to pc-minors and zone graphs, we give a couple of methods of detecting a member of $\mathcal{Q}^-$ in a given graph. The main result of the section is that partial cubes excluding $\mathcal{Q}^-$ are topes graphs of COMs (Theorem 5.7). The proofs in this section use properties of zone graphs established in Section 2, minor-closedness of COMs seen in Section 3, as well as properties of tope graphs of COMs shown in Section 4

Let Q_n be the hypercube, $v \in Q_n$ any of its vertices and $-v$ its antipode. Let $Q_n^- := Q_n \setminus -v$ be the hypercube minus one vertex. Consider the set of partial cubes arising from Q_n^- by deleting any subset of $N(v) \cup \{v\}$. It is easy to see that if $n \geq 4$ a graph obtained this way from Q_n^- is a partial cube unless v is not deleted but at least two of its neighbors are deleted. Denote by Q_n^{-*} the partial cube obtained by deleting exactly one neighbor of v, and by $Q_n^{--}(m)$ the graph obtained by deleting v and m neighbors of v, respectively, where for $Q_n^{--}(0)$ we sometimes simply write Q_n^{--}. It is easy to see that Q_n^- and Q_n^{--} are tope graphs of (realizable) COMs. For $n \leq 3$ all the partial cubes arising by the above procedure are isomorphic to Q_n^- or Q_n^{--}, thus the interesting graphs appear for $n \geq 4$. Denote their collection by $\mathcal{Q}^- = \{Q_n^{-*}, Q_n^{--}(m) \mid 4 \leq n; 1 \leq m \leq n\}$.

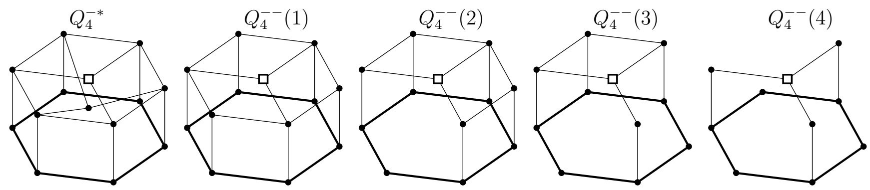


Figure 8: Graphs $Q_4^{-*}, Q_4^{--}(m)$, for $1 \leq m \leq 4$. The square vertex has no gate in the bold C_6.

Lemma 5.1. *The set $\mathcal{Q}^-$ is pc-minor minimal, i.e. any pc-minor of a graph in $\mathcal{Q}^-$ is not in $\mathcal{Q}^-$. Furthermore, any graph in $\mathcal{Q}^-$ contains an antipodal subgraph that is not gated, i.e. $\mathcal{Q}^- \subseteq \overline{\mathrm{AG}}$, where $\overline{\mathrm{AG}}$ denotes the complementary class of* AG.

Proof. For any $G \in \mathcal{Q}^-$ a contraction or a restriction of it is a graph isomorphic to a hypercube, a hypercube minus a vertex, or a hypercube minus two antipodal vertices. All pc-minors of these graphs are isomorphic to a hypercube or a hypercube minus a vertex. Thus, no proper pc-minor of G is in $\mathcal{Q}^-$.

To see that any graph in $\mathcal{Q}^-$ contains an antipodal subgraph that is not gated, let $G \in \mathcal{Q}^-$ and let w be a neighbor of v that was deleted from Q_n^- to obtain G. The convex subgraph $A \subseteq G$ obtained from restricting Q_n^- to the halfspace containing w and not v is isomorphic to Q_{n-1}^{--}. In particular, A is antipodal. But A is not gated, since the neighbor u of $-v$ in $G \setminus A$ has no gate in A. In fact, in can be seen by Lemma 4.6, that the gate of u, if existent, must be of the form $\chi(A) \circ \chi(u) = \chi(-v)$ which is not in G. □

The following lemma will be useful for detecting pc-minors from $\mathcal{Q}^-$. We define H to be the ***full subdivision*** of a graph G if every edge of it is replaced by a path of length 2 to obtain H. The vertices of H that correspond to vertices of G are called its ***original vertices***.

Lemma 5.2. *Let G be a partial cube and H an isometric subgraph isomorphic to a full subdivision of K_m such that*

- *no vertex of G is adjacent to all the original vertices of H,*
- *the convex hull of H is neither isomorphic to Q_m^- nor Q_m^{--},*

- *H is inclusion minimal with this properties.*

Then the convex hull of H is in $\mathcal{Q}^-$.

Proof. Let H be as in the lemma. First note that $m \geq 4$ since for $m = 2$, H is isomorphic to $P_3 = Q_2^-$, and for $m = 3$, H is isomorphic to C_6. Taking into account that there is no vertex of G adjacent to all the original vertices of H, the convex hull in both cases is isomorphic to Q_m^- or to Q_m^{--}.

Now consider H embedded in a hypercube Q_m with $m \geq 4$. Since H is an isometric subgraph of G, it is a partial cube and thus has up to reorientation a unique embedding in Q_m. Note that one (and therefore the only) possible embedding, is such that there exists a vertex v in Q_m adjacent to exactly the original vertices of H. Indeed, embed the original vertices as vectors with precisely one $+$ and the subdivision vertices as vectors with two $+$-signs. Then, v can be chosen to be the vector consisting only of $-$. By the assumption, v is not in G. Let $\{v_1, \dots, v_{m-1}\}$ be any subset of the original vertices of H of size $m-1$. Since H is inclusion minimal, the convex hull of $\{v_1, \dots, v_{m-1}\}$ is a graph isomorphic to Q_{m-1}^- or Q_{m-1}^{--} in which v is a missing vertex. Note that the antipode of v in Q_m is at distance m from v. We have proved that all the vertices of Q_m at distance at most $m-2$ from v are in G while some of the neighbors of the antipode of v in Q_m are possibly not in G. Thus the convex hull of H is a graph in $\mathcal{Q}^-$. □

A second useful lemma, tells us how to find excluded pc-minors in non-antipodal graphs with antipodal contractions.

Lemma 5.3. *Let G be a partial cube with $v \in G$ such that $-v \notin G$, and $E_{e_1}, \dots, E_{e_k}$ be Θ-classes of G such that $\pi_{e_i}(G)$ is antipodal for all $1 \leq i \leq k$. Then either G contains a convex subgraph from $\mathcal{Q}^-$ or G contains a convex Q_k^- or Q_k^{--} crossed by precisely $E_{e_1}, \dots, E_{e_k}$ such that $-v$ is a missing vertex of it. In particular, if the latter holds and $E_{e_1}, \dots, E_{e_k}$ are all the Θ-classes of G, then G is isomorphic to Q_k^-.*

Proof. Let $v \in G$ be a vertex without an antipode in G. Let $E_{e_1}, \dots, E_{e_k}$ be the Θ-classes as above. Every contraction $\pi_{e_i}(G)$ is antipodal, therefore $\pi_{e_i}(v)$ has an antipode in $\pi_{e_i}(G)$. Let $v_i \in G$ be the preimage of the antipode $-_{\pi_{e_i}(G)}\pi_{e_i}(v)$ of $\pi_{e_i}(v)$ in $\pi_{e_i}(G)$ for $1 \leq i \leq k$, respectively. By definition, $v_1, \dots, v_k$ are pairwise at distance 2, thus a part of an isometric subgraph isomorphic to the subdivision of K_k. Moreover, there is no vertex adjacent to all of them, since v has no antipode in G. By Lemma 5.2, the convex hull of $v_1, \dots, v_k$ is isomorphic to one of Q_k^-, Q_k^{--}, or there is an inclusion minimal subdivided $K_{k'}$ for $k' \leq k$ whose convex closure is not isomorphic to $Q_{k'}^-$ or $Q_{k'}^{--}$. In the former case we are done while in the latter case the convex closure of the inclusion minimal subdivided $K_{k'}$ is in $\mathcal{Q}^-$.

Assuming that $E_{e_1}, \dots, E_{e_k}$ are all the classes of G, the convex hull of $v_1, \dots, v_k$ is crossed by all the Θ-classes of G, hence it is G. Then G is isomorphic to Q_k^- or Q_k^{--} if it has no convex subgraph in $\mathcal{Q}^-$. Since G is not antipodal, it must be isomorphic to Q_k^-. □

Lemma 5.4. *Let G be a partial cube. If a zone graph $\zeta_e(G)$ is not a well-embedded partial cube, then G has a pc-minor in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \leq m \leq 4\}$.*

Proof. For the sake of contradiction assume G does not satisfy the conditions of Lemma 2.7, i.e., there are convex cycles C_1, C_2 both crossed by E_e and E_f and C_1 is crossed by E_g but C_2 is not. Without loss of generality assume that C_2 is completely in E_g^+. On the other hand, we can reorient E_e and E_f in a way that the vertices on $C_1 \cap E_g^+$ are in $E_e^+ \cap E_f^+, E_e^+ \cap E_f^-$ and $E_e^- \cap E_f^-$. But then any vertex v in C_2 that is in $E_e^- \cap E_f^+$ has no gate to C_1.

We will now see, that this leads to the existence of a pc-minor in $\{Q_4^{-*}\} \cup \{Q_4^{--}(m) \mid 1 \leq m \leq 4\}$. First contract any Θ-class different from E_e, E_f, E_g but crossing C_1, obtaining graph G'. Then C_1 is contracted to a convex 6-cycle C_1' in G', by Lemma 2.11. It is non-gated since the image of v in G' still is in $E_e^- \cap E_f^+ \cap E_g^+$ while $E_e^- \cap E_f^+ \cap E_g^+ \cap C_1' = \emptyset$. Now consider a maximal sequence S of contractions of Θ-classes different from E_e, E_f, E_g such that for the image C_1'' of C_1' we have that $\operatorname{conv}(C_1'')$ is a 6-cycle. Since v has no gate to C_1', contracting all the Θ-classes different from E_e, E_f, E_g maps v to a vertex in

$\operatorname{conv}(C_1'') \setminus C_1''$ contradicting that $\operatorname{conv}(C_1'')$ is a 6-cycle. Thus S is not equal to all the Θ-classes different from E_e, E_f, E_g.

Pick any Θ-class E_h not in $S \cup \{E_e, E_f, E_g\}$. By maximality $\operatorname{conv}(\pi_h(C_1''))$ is not a 6-cycle thus it must be isomorphic to Q_3^- or Q_3. Let u be a vertex in Q_3^- or Q_3 adjacent to three vertices $\pi_h(u_1), \pi_h(u_2), \pi_h(u_3) \in \pi_h(C_1'')$ with $u_i \in C_1''$ for $i \in \{1,2,3\}$. No preimage u' of u is adjacent to any u_i, for $i \in \{1,2,3\}$, since otherwise u' would be in $\operatorname{conv}(C_1'')$. Thus, u', u_1, u_2, u_3 are pairwise at distance two. Since $\operatorname{conv}(C_1'')$ is a 6-cycle, there is no vertex adjacent to all of them. Together with their connecting 2-paths they form an isometric K_4. Moreover, their convex hull is not isomorphic to Q_4^- or Q_4^{--} since such graphs have no convex 6-cycles. By Lemma 5.2, G has a pc-minor in $\{Q_4^{-*}, Q_4^{--}(m) \mid 1 \le m \le 4\}$. □

Lemma 5.5. *If $G \in \mathcal{Q}^-$, then there is a sequence $(e_1, \dots, e_k)$ of Θ-classes such that $\zeta_{e_1,\dots,e_k}(G)$ is not a partial cube.*

Proof. Let $G \in \{Q_n^{-*}, Q_n^{--}(m) \mid 1 \le m \le n\}$ for some $n \ge 4$. Let $v \in Q_n$ the vertex from the definition of $\mathcal{Q}^-$.

If $n = 4$ one can easily see in Figure 8 that G has a zone graph containing a C_5 if $G \in \{Q_4^{-*}, Q_4^{--}(1), Q_4^{--}(2)\}$ or a C_3 if $G \in \{Q_4^{--}(3), Q_4^{--}(4)\}$ i.e., the zone graph is not a partial cube.

Let now $n > 4$. If $G \in \{Q_n^{-*}, Q_n^{--}(m) \mid 1 \le m \le n-1\}$, let w be one of the neighbors of v that is in G and let e be the Θ-class of G coming from the edge vw in Q_n. It is easy to see that $\zeta_e(G) \in \{Q_{n-1}^{-*}, Q_{n-1}^{--}(m) \mid 1 \le m \le n-1\}$. If otherwise $G = Q_n^{--}(n)$, then let w be any of the neighbors of v. Then w is missing in G and let e be the Θ-class of G coming from the edge vw in Q_n. One can check that $\zeta_e(G) = Q_{n-1}^{--}(n-1)$. The lemma follows by iterating the given zone graphs until arriving at a non partial cube. □

A useful sort of converse of the proof of Lemma 5.5 is the following:

Lemma 5.6. *If G is partial cube and $\zeta_e(G) \in \mathcal{Q}^-$ for some Θ-class e, then G has a pc-minor in $\mathcal{Q}^-$.*

Proof. Assume that $\zeta_e(G)$ is well-embedded since otherwise G has a pc-minor in $\mathcal{Q}^-$, by Lemma 5.4. Let G be pc-minor minimal, without affecting $H := \zeta_e(G)$ and let N be the number of Θ-classes of H. Then by minimality G is the convex hull of E_e and has $N+1$ Θ-classes. There exist two isometric copies of H in B with edges of E_e being a matching of them. Let $v_1, \dots, v_N$ be the original vertices of the subdivided K_N in the first copy and $v_1', \dots, v_N'$ the original vertices of the subdivided K_N in the second copy. Not both subdivisions can have a vertex adjacent to all of the original vertices of the subdivisions since then there would be an edge in E_e between the two vertices, which is not true by the definition of H. Without loss of generality, assume that $v_1, \dots, v_N$ have no common neighbor. Then by Lemma 5.2, either G has a pc-minor in $\mathcal{Q}^-$ and we are done, or the convex hull of $v_1, \dots, v_N$ is isomorphic to Q_N^- or Q_N^{--}. Analogously, if $v_1', \dots, v_N'$ have no common neighbor their convex hull is isomorphic to Q_N^- or Q_N^{--}. But then H is isomorphic to Q_N^- or Q_N^{--} which is not the case. Thus $v_1', \dots, v_N'$ have a common neighbor, say u. Then $u, v_1, \dots, v_N$ are pairwise at distance 2, without a common neighbor. Then by Lemma 5.2, either G has a pc-minor in $\mathcal{Q}^-$ or their convex hull H is isomorphic to Q_{N+1}^- or Q_{N+1}^{--}. The latter cannot be since then H is not in $\mathcal{Q}^-$. Therefore, G has the pc-minor in $\mathcal{Q}^-$. □

We are ready to prove the main theorem of this section.

Theorem 5.7. *A partial cube that has no pc-minor from the set $\mathcal{Q}^-$ is the tope graph of a COM, i.e., $\mathcal{F}(\mathcal{Q}^-) \subseteq \mathcal{G}_{\mathrm{COM}}$.*

Proof. For the sake of contradiction, assume that we can pick G, a smallest graph that is not in $\mathcal{G}_{\mathrm{COM}}$ but in $\mathcal{F}(\mathcal{Q}^-)$. In particular, since $\mathcal{F}(\mathcal{Q}^-)$ is pc-minor closed, every pc-minor of G is in $\mathcal{G}_{\mathrm{COM}}$.

By Lemma 4.11, G is a graph in $\mathcal{G}_{\mathrm{COM}}$ if and only if $\mathcal{L}(G)$ is a COM. Since G is not in $\mathcal{G}_{\mathrm{COM}}$, but G is a partial cube and by Lemma 4.8 $\mathcal{L}(G)$ is a partial cube system that satisfies (C), [3, Theorem 3] gives that $\mathcal{L}(G)$ has a hyperplane $\mathcal{L}(G)/e$ that is not a COM.

Since $G \in \mathcal{F}(\mathcal{Q}^-)$, by Lemma 5.4 we have that $G' := \zeta_e(G)$ is a well-embedded partial cube. By Lemma 4.3 we get $G' \cong G(\mathcal{L}(G)/e)$, i.e., it is the tope graph of the hyperplane.

Claim 5.8. *We have* $\mathcal{S}(\mathcal{L}(G)/e) = \mathcal{L}(G')$*.*

Proof. By Lemma 4.8, the elements of $\mathcal{L}(G')$ correspond to antipodal subgraphs of G'. Furthermore, it is not hard to see that elements of $\mathcal{S}(\mathcal{L}(G)/e)$ correspond to antipodal subgraphs of G that are crossed by E_e, where redundant coordinates have been deleted. If A is an antipodal subgraph of G crossed by E_e, then by Lemma 4.7, each edge $uv \in E_e$ of A has an antipodal edge $-_A u -_A v \in E_e$. Thus the zone graph of A corresponding to E_e is an antipodal subgraph of G' and we get that $\mathcal{S}(\mathcal{L}(G)/e) \subseteq \mathcal{L}(G')$.

Conversely, assume that there is an antipodal graph A' in G' that does not correspond to a zone graph of an antipodal subgraph of G. By definition of G' we can identify its vertices with edges of G in E_e. Let A be the convex hull of those edges in G that correspond to vertices of A', and let $E_e, E_{e_1}, \ldots, E_{e_k}$ be the Θ-classes crossing A. Since A' does not correspond to a zone graph of an antipodal graph, A is not antipodal.

By minimality of G, for every E_{e_j} the contraction $\pi_{e_j}(G)$ is the tope graph of a COM. This is, $\mathcal{L}(\pi_{e_j}(G)) = \mathcal{L}(G)\backslash e_j$ is a COM. Hence, by Lemma 3.3 its hyperplane $(\mathcal{L}(G)\backslash e_j)/e$ is a COM as well. Now Lemma 3.2 gives $\mathcal{S}((\mathcal{L}(G)\backslash e_j)/e) = \mathcal{S}(\mathcal{L}(G)/e)\backslash e_j$. We have proved that $\mathcal{S}(\mathcal{L}(G)/e)\backslash e_j$ is a COM for every $E_{e_j} \in \{E_{e_1}, \ldots, E_{e_k}\}$. Note that $\pi_{e_j}(G')$ is the tope graph of $\mathcal{S}(\mathcal{L}(G)/e)\backslash e_j$ and therefore it is in $\mathcal{G}_{\mathrm{COM}}$.

By Theorem 4.9, the covectors of the COM corresponding to $\pi_{e_j}(G')$ are precisely its antipodal subgraphs. Since $\pi_{e_j}(A')$ is antipodal, it corresponds to a covector. But then this covector is in $\mathcal{S}((\mathcal{L}(G)\backslash e_j)/e)$, i.e. there is an antipodal graph in $\pi_{e_j}(G)$ whose zone graph is $\pi_{e_j}(A')$. By definition this must be $\pi_{e_j}(A)$, proving that $\pi_{e_j}(A)$ is antipodal for every $E_{e_j} \in \{E_{e_1}, \ldots, E_{e_k}\}$.

Let $v \in A$ be a vertex without an antipode in A. Without loss of generality, $v \in E_e^+$. By Lemma 5.3, A either has a pc-minor in $\mathcal{Q}^-$ and we are done, or there is a Q_k^- or Q_k^{--} in A crossed by precisely $E_{e_1}, \ldots, E_{e_k}$ and its missing vertex is the missing antipode of v in A. Then this convex subgraph is precisely $E_e^- \cap A$.

First assume that $E_e^- \cap A$ is isomorphic to Q_k^-. Thus, v has a neighbor in E_e^-, say u. If u has no antipode in A we deduce as above that E_e^+ is isomorphic to Q_k^- or Q_k^{--}. Since $v \in A$ the halfspace E_e^+ must be isomorphic to Q_k^- and thus $A \cong Q_k^- \square K_2$. But then the zone graph of A corresponding to E_e is not antipodal and A' is not antipodal. A contradiction.

Finally, assume that E_e^- is isomorphic to Q_k^{--}. Then there are k vertices in E_e^- at distance 2 from v and pairwise also at distance 2 but there is no vertex adjacent to all of them. By Lemma 5.2, A either has a pc-minor in $\mathcal{Q}^-$ and we are done, or A is isomorphic to Q_{k+1}^- or Q_{k+1}^{--}. In the first case, none of the zone graphs of Q_{k+1}^- is antipodal, while in the second case A is antipodal. A contradiction. This finishes the proof that $\mathcal{L}(G)/e = \mathcal{L}(G')$.

□

Now we can assume that G' is a well-embedded partial cube, but is not in $\mathcal{G}_{\mathrm{COM}}$ since $\mathcal{L}(G') = \mathcal{S}(\mathcal{L}(G)/e)$, and $\mathcal{L}(G)/e$ is not a COM. By minimality of G, $G' = \zeta_e(G)$ has a pc-minor $H' \in \mathcal{Q}^-$, i.e., $H' = \rho_X(\pi_A(\zeta_e(G)))$ for some Θ-classes A and an oriented set X of Θ-classes of G'. By Lemma 2.9 we have $H' = \zeta_e(\rho_{X'}(\pi_{A'}(G)))$ for Θ-classes A' and an oriented set X' of Θ-classes of G. Let H be the graph $\rho_{X'}(\pi_{A'}(G))$. Lemma 5.6 gives that H has a pc-minor in $\mathcal{Q}^-$, contradicting that $G \in \mathcal{F}(\mathcal{Q}^-)$. This concludes the proof of Theorem 5.7.

□

6 Antipodally gated partial cubes are pc-minor closed

The main result of the present section is that if in a partial cube all antipodal subgraphs are gated, then the same holds for all its minors (Theorem 6.1). Recall that the class of these antipodally gated partial cubes is denoted by AG. Since by Lemma 5.1 none of the graphs in $\mathcal{Q}^-$ is in AG, minor-closedness of AG implies that antipodally gated partial cubes exclude minors from $\mathcal{Q}^-$, i.e., AG $\subseteq \mathcal{F}(\mathcal{Q}^-)$. This section is proof wise the hardest one of the paper. It heavily builds on interactions of antipodality and gatedness with respect to pc-minors, expansions, and zone-graphs established in Section 2. We want to prove:

Theorem 6.1. *If G is antipodally gated, then so are all pc-minors of G.*

For this we will show two auxiliary statements. The first one is:

Lemma 6.2. *Let G be an antipodal graph from* AG *such that all its pc-minors are in* AG *as well and G' an expansion of G. Then one of the following occurs:*

1. *G' is antipodal,*
2. *G' is a peripheral expansion of G,*
3. *G' is not in* AG.

The second one shows how to use the first one in order to prove Theorem 6.1.

Lemma 6.3. *If Lemma 6.2 holds for all pairs G, G' where G' is on less than n vertices, then Theorem 6.1 holds for all the partial cubes on at most n vertices.*

Proof of Lemma 6.3. Suppose that Theorem 6.1 does not hold and let G be a minimal counterexample, while Lemma 6.2 holds for all the expansions of size less than the size of G. First, observe that AG is closed under restrictions, since restrictions cannot create new antipodal subgraphs and gated subgraphs remain gated. So, let $\pi_e(G) \notin$ AG be a contraction of G that is not in AG. Let A be a smallest antipodal subgraph of $\pi_e(G)$, that is not gated in $\pi_e(G)$. In particular A is a proper subgraph. By the minimality in the choice of A, itself is in AG. Now, by the minimality in the choice of G, all pc-minors of A are also in AG. Let A' denote the expansion of A with respect to e, that appears as a proper subgraph of G. If E_e does not cross A', then $A' \cong A$ is antipodal subgraph and is non-gated, since otherwise $A = \pi_e(A')$ would be gated as well by Lemma 2.18. This contradicts $G \in$ AG. If E_e crosses A', we can apply Lemma 6.2 to A, since A' has less vertices than G. We get that either A' is antipodal, A' is a peripheral expansion of A, or A' is not in AG. The latter cannot be since G in AG. In the former two cases, either A' is antipodal or has A as a subgraph. In both cases, we have an antipodal subgraph that is contracted to A in $\pi_e(G)$. By Lemma 2.18, the antipodal subgraph in G is non-gated contradicting $G \in$ AG. □

Proof of Lemma 6.2. Suppose that the lemma is false. Let G, G' be a minimal counterexample, i.e. G is an antipodal graph from AG such that all its pc-minors are in AG. Furthermore, G' is an expansion of G that is not antipodal, not peripheral, but in AG, with minimal number of vertices possible. Let E_c be the Θ-class such that $\pi_c(G') = G$.

Claim 6.4. *Any pc-minor of G' is in* AG.

Proof. Let n be the number of vertices in G'. Since G' is a minimal counterexample to Lemma 6.2, the lemma holds for all the graphs on less than n vertices. Then by Lemma 6.3, Theorem 6.1 holds for all graphs on at most n vertices. In particular it holds for G', thus all its pc-minors are in AG. □

We will call a contraction of a partial cube ***antipodal*** if the contracted graph is antipodal. The following claim is immediate since a contraction of an antipodal graph is antipodal and contractions commute.

Claim 6.5. *If $\pi_e(H)$ is an antipodal contraction of H, then $\pi_e(\pi_f(H))$ is an antipodal contraction of $\pi_f(H)$ for all $f \in \mathcal{E}$.*

Every contraction $\pi_e(G')$ of G', $E_e \neq E_c$, makes E_c peripheral or it is antipodal, since otherwise the contraction $\pi_e(G')$ together with the contraction $\pi_c(\pi_e(G'))$ would yield a smaller counterexample to the lemma. We can divide the Θ-classes of G' into two sets: call the index set of the Θ-classes of the antipodal contractions $\mathscr{A}$, and the index set of the remaining Θ-classes $\mathscr{B}$. By the above, a contraction $\pi_e(G')$ of G', for every $e \in \mathscr{B}$, makes E_c peripheral in $\pi_e(G')$. Note that $c \in \mathscr{A}$, i.e. in particular $\mathscr{A}$ is non-empty. Also $\mathscr{B}$ is non-empty, because otherwise, every contraction of G' is antipodal, thus by

Lemma 5.3, G' is isomorphic to Q_n^- or Q_n^{--}. The latter cannot be since G' is not antipodal. The former is impossible since then $G = \pi_c(G') \cong Q_{n-1}$, and G' is a peripheral expansion.

Furthermore, note that peripherality of a Θ-class is preserved under contraction.

Claim 6.6. *For every $e \in \mathscr{B}$ and every $f \in \mathscr{A}$, the Θ-class E_f is peripheral in $\pi_e(G')$.*

Proof. Assume that this is not the case. Then $\pi_e(G')$ is not antipodal by definition, while $\pi_f(\pi_e(G'))$ is antipodal by Claim 6.5. Moreover, E_f is not peripheral in $\pi_e(G')$ by assumption, while all pc-minors $\pi_e(G')$ are in AG by Claim 6.4. Thus $\pi_e(G')$ and $\pi_f(\pi_e(G'))$ are a smaller counterexample to the lemma, a contradiction. □

Now, consider the halfspaces E_c^+ and E_c^- in G'. Contracting any Θ-class e from $\mathscr{B}$ makes E_c peripheral in $\pi_e(G')$, which implies that either $\pi_e(E_c^+)$ or $\pi_e(E_c^-)$ is a peripheral halfspace in $\pi_e(G')$. For this reason denote by $\mathscr{B}^+$ those $e \in \mathscr{B}$ such that $\pi_e(E_c^+)$ is peripheral and let $\mathscr{B}^- = \mathscr{B} \setminus \mathscr{B}^+$, i.e., $\pi_e(E_c^-)$ is peripheral for $e \in \mathscr{B}^-$.

Claim 6.7. *Let $e \in \mathscr{B}^+$ end $f \in \mathscr{B}^-$. Then $\pi_f(\pi_e(G'))$ is antipodal. Moreover, E_f is peripheral in $\pi_e(G')$.*

Proof. Since peripherality is closed under contraction, both $\pi_f(\pi_e(E_c^+))$ and $\pi_f(\pi_e(E_c^-))$ are peripheral in $\pi_f(\pi_e(G'))$. Thus $\pi_f(\pi_e(G')) \cong K_2 \square A$ for some graph A. On one hand contracting E_c in $\pi_f(\pi_e(G'))$ gives an antipodal graph by the choice of E_c, on the other hand it is isomorphic to A. Thus A is an antipodal graph. Then also $\pi_f(\pi_e(G')) \cong K_2 \square A$ is antipodal.

Now consider the pair of graphs $\pi_f(\pi_e(G'))$ and $\pi_e(G')$. The first is antipodal by the above, while the second is its expansion. Since both are pc-minors of G', $\pi_e(G')$ is in AG and $\pi_f(\pi_e(G'))$ and all its pc-minors are in AG as well, by Claim 6.4. Furthermore, $\pi_e(G')$ is not antipodal since $e \in \mathscr{B}$. By the minimality of G' the expansion $\pi_e(G')$ must be peripheral, proving that E_f is peripheral in $\pi_e(G')$. □

Let G'' be obtained from G' by a maximal chain of contractions of Θ-classes $\mathscr{C} = \{e_1, \ldots, e_p\}$, such that G'' is non-antipodal. Note that $\mathscr{C} \subseteq \mathscr{B}$ since every contraction from a Θ-class in $\mathscr{A}$ makes the graph antipodal. Moreover, by Claim 6.7 we have $\mathscr{C} \subseteq \mathscr{B}^+$ or $\mathscr{C} \subseteq \mathscr{B}^-$. Without loss of generality assume that $\mathscr{C} \subseteq \mathscr{B}^+$.

Claim 6.8. *The contraction $\pi_{e_1}(G')$ contains a hypercube that is crossed exactly by $\mathscr{B}^+ \setminus \{e_1\}$.*

Proof. By assumption E_c^+ is not peripheral in G', hence there exists a vertex $v \in E_c^+$ not incident with any edge of E_c^+. For every $e \in \mathscr{B}^+$, E_c^+ is peripheral in $\pi_e(G')$, thus there is an edge in E_e connecting v and an edge in E_c. Thus for every $e \in \mathscr{B}^+$, there is a path from v to E_c^- first crossing an edge in E_e and then an edge in E_c. Vertex v together with the end-vertices of these paths form a collection of vertices pairwise at distance 2. They have no common neighbor, since this neighbor would have to be in E_c^-, but v has no neighbor in E_c^-. Since all pc-minors of G' are in AG, Lemma 5.2 implies that the convex hull C of these vertices is isomorphic to Q_n^- or Q_n^{--} for some $n > 0$. This convex subset is crossed exactly by all the Θ-classes of $\mathscr{B}^+$ and E_c.

Contracting to $\pi_{e_1}(G')$, C is is contracted into a hypercube crossed by all $\mathscr{B}^+ \setminus \{e_1\}$ and E_c. Then the lemma holds. □

The graph G'' and all its pc-minors are in AG by Claim 6.4 and all its contractions are antipodal. By Lemma 5.3, we have $G'' \cong Q_n^-$. Now, we will consider the sequence of expansions of G'' leading back the first contraction $\pi_{e_1}(G')$. Note that G'' is crossed by precisely those Θ-classes that are not in $\mathscr{C}$. These are the Θ-classes $\mathscr{A} \cup \mathscr{B}^-$, and the Θ-classes $\mathscr{B}^+ \setminus \mathscr{C}$.

By Claims 6.6 and 6.7, every Θ-class in $\mathscr{A} \cup \mathscr{B}^-$ is peripheral in $\pi_{e_1}(G')$ thus also in every contraction of it. For each $e \in \mathscr{A} \cup \mathscr{B}^- \setminus \{c\}$, without loss of generality, say that E_e^+ is the peripheral halfspace of it. For E_c the halfspace E_c^+ is peripheral, since $e_1 \in \mathscr{B}^+$, so we can assume that E_e^+ is peripheral for each $e \in \mathscr{A} \cup \mathscr{B}^-$. Since $G'' \cong Q_n^-$ is crossed by each E_e and E_e^- is non-peripheral in it, E_e^- is also non-peripheral in every expansion in the sequence.

Let $|\mathscr{A} \cup \mathscr{B}^-| = k$ and $|\mathscr{B}^+\backslash\mathscr{C}| = \ell$. Then $G'' \cong Q_n^-$ with $n = k + \ell$, can be seen as a collection of 2^k disjoint subgraphs spanned on the edges of $\mathscr{B}^+\backslash\mathscr{C}$ each isomorphic to Q_ℓ, except for one isomorphic to Q_ℓ^-, and all connected in a hypercube manner by edges of $\mathscr{A} \cup \mathscr{B}^-$. The subgraph isomorphic to Q_ℓ^- is precisely the subgraph $\bigcap_{e\in\mathscr{A}\cup\mathscr{B}^-} E_e^+$.

In other words $\pi_{\mathscr{B}^+\backslash\mathscr{C}}(G'') \cong Q_k$ and for $X \in \{+,-\}^{\mathscr{A}\cup\mathscr{B}^-}$ we have $\rho_X(G'') \cong Q_\ell$ unless if $X = (+,\dots,+)$ in which case $\rho_X(G'') \cong Q_\ell^-$. Next we prove that this structure is preserved when expanding back towards $\pi_{e_1}(G')$.

Claim 6.9. *Let $\bar{\ell} = \ell + p - 1$, where $p = |\mathscr{C}|$. We have $\pi_{\mathscr{B}^+\backslash\mathscr{C}}(\pi_{e_1}(G')) \cong Q_k$ and for $X \in \{+,-\}^{\mathscr{A}\cup\mathscr{B}^-}$ we have $\rho_X(\pi_{e_1}(G')) \cong Q_{\bar{\ell}}$ if and only if $X \neq (+,\dots,+)$.*

Proof. We will prove the claim by induction on the number p, where G_j is the jth expansion, starting from $G'' = G_0$ and ending in $\pi_{e_1}(G') = G_{p-1}$. By the paragraph before the claim, the claim holds for G''.

By induction assumption let the claim hold for G_j, $j \geq 0$, and let G_{j+1} be its expansion, in the sequence of the expansions leading to $\pi_{e_1}(G')$. Let $f \in \mathscr{C}$ be such that $\pi_f(G_{j+1}) = G_j$ and H_1, H_2 be the subgraphs of G_j we expand along.

First, we will prove that each copy of $Q_{\ell+j}$ in G_j not crossed by the Θ-classes $\mathscr{A} \cup \mathscr{B}^-$ is contained in $H_1 \cap H_2$. Denote $\mathscr{A} \cup \mathscr{B}^- = \{f_1,\dots,f_s\}$. Consider an edge $uv \in E_{f_i}$ in G_j such that $u \in H_1 \cap H_2$. If $u \in E_{f_i}^+$ and $v \in E_{f_i}^-$, then $E_{f_i}^+$ is peripheral in the expanded graph G_{j+1}. Thus, $v \in H_1 \cap H_2$.

Now assume that $u \in E_{f_i}^-$ and $v \in E_{f_i}^+$. We will prove that if additionally $v \notin \bigcap_{e\in\mathscr{A}\cup\mathscr{B}^-} E_e^+$, then we can also conclude $v \in H_1 \cap H_2$. So let $u, v \in E_{f_{i'}}^-$ for some other $f_{i'} \in \mathscr{A} \cup \mathscr{B}^- \setminus \{f_i\}$. For the sake of contradiction assume that $v \notin H_1 \cap H_2$. Without loss of generality $v \in H_1$. If v has a neighbor in $E_{f_{i'}}^+$ – say v', then by the above arguments, if $v' \in H_1 \cap H_2$ then also $v \in H_1 \cap H_2$. Thus, if v' exists, then it is in $H_1\backslash H_2$.

Consider the contractions $\pi_{f_{i'}}(G_{j+1})$ and $\pi_{f_{i'}}(G_j)$. Since $f_{i'} \in \mathscr{A} \cup \mathscr{B}^-$ both graphs are antipodal. By Lemma 2.6, the expansion from $\pi_{f_{i'}}(G_j)$ to $\pi_{f_{i'}}(G_{j+1})$ corresponds to sets $\pi_{f_{i'}}(H_1)$ and $\pi_{f_{i'}}(H_2)$. Since $v, v' \in H_1\backslash H_2$, their image $\pi_{f_{i'}}(v) = \pi_{f_{i'}}(v')$ is in $\pi_{f_{i'}}(H_1)\backslash\pi_{f_{i'}}(H_2)$. Since $u \in H_1 \cap H_2$, its image is in $\pi_{f_{i'}}(H_1) \cap \pi_{f_{i'}}(H_2)$. Let $-\pi_{f_{i'}}(v), -\pi_{f_{i'}}(u)$ be the antipodes of $\pi_{f_{i'}}(v), \pi_{f_{i'}}(u)$ in $\pi_{f_{i'}}(G_j)$, respectively. Since $\pi_{f_{i'}}(G_{j+1})$ is an antipodal expansion of $\pi_{f_{i'}}(G_j)$ by Lemma 2.14 we have, $-\pi_{f_{i'}}(v) \in \pi_{f_{i'}}(H_2)\backslash\pi_{f_{i'}}(H_1)$ and $-\pi_{f_{i'}}(u) \in \pi_{f_{i'}}(H_1) \cap \pi_{f_{i'}}(H_2)$.

Since $v \in E_{f_i}^+$ also $\pi_{f_{i'}}(v) \in E_{f_i}^+$, thus $-\pi_{f_{i'}}(v) \in E_{f_i}^-$. Similarly, since $u \in E_{f_i}^-$ also $\pi_{f_{i'}}(u) \in E_{f_i}^-$, thus $-\pi_{f_{i'}}(u) \in E_{f_i}^+$.

Consider the contraction from G_j to $\pi_{f_{i'}}(G_j)$. By induction hypothesis, the structure of G_j is such that $E_{f_i}^-$ is isomorphic to a hypercube crossed by $E_{f_{i'}}$ thus two vertices x, z of G_j are contracted to $-\pi_{f_{i'}}(v)$. Since $-\pi_{f_{i'}}(v) \in \pi_{f_{i'}}(H_2)\backslash\pi_{f_{i'}}(H_1)$, we have $x, z \in H_2\backslash H_1$.

On the other hand, at least one vertex $w \in H_1 \cap H_2$ of G_j is contracted to $-\pi_{f_{i'}}(u)$. It also holds that the vertices $x, z \in E_{f_i}^-$ and $w \in E_{f_i}^+$. Since uv is an edge, $-\pi_{f_{i'}}(u) - \pi_{f_{i'}}(v)$ is an edge. Hence, xw or zw has to be an edge - say without loss of generality zw is an edge. Then we have a pair of adjacent vertices w, z such that $w \in H_1 \cap H_2 \cap E_{f_i}^+$ and $z \in H_1\backslash H_2 \cap E_{f_i}^-$. We have already shown that this is impossible. Thus, $u \in H_1 \cap H_2$ implies $v \in H_1 \cap H_2$.

Now consider the copies of $Q_{\ell+j}$ in G_j. By Claim 6.8, $\pi_{e_1}(G')$ contains a hypercube crossed by exactly all the Θ-classes of $\mathscr{B}^+\backslash\{e_1\}$. The latter implies that G_{j+1} contains a hypercube crossed by precisely the Θ-classes crossing copies $Q_{\ell+j}$ in G_j plus the Θ-class obtained while expanding to G_{j+1}. In particular this implies that one of the copies of $Q_{\ell+j}$ in G_j gets completely expanded, i.e. that copy is completely in $H_1 \cap H_2$. Since none of the copies of $Q_{\ell+j}$ in G_j is in $\bigcap_{e\in\mathscr{A}\cup\mathscr{B}^-} E_e^+$, by the above paragraph this property propagates to all the copies of $Q_{\ell+j}$, i.e. all of them are completely in $H_1 \cap H_2$. Since the non-cube $\bigcap_{e\in\mathscr{A}\cup\mathscr{B}^-} E_e^+$ cannot expand to a cube, G_{j+1} is as stated in the claim. □

The following claim will suffice to finish the proof of the lemma.

Claim 6.10. *If the expansion G' of $\pi_{e_1}(G')$ is such that $\pi_c(G')$ is antipodal and E_c is not peripheral in G', then G' is not in* AG.

Proof. Consider the expansion along sets G_1 and G_2. Since $e_1 \in \mathscr{B}^+$, the halfspace E_c^+ is peripheral in $\pi_{e_1}(G')$. Moreover, by Claim 6.9 we have $E_c^- \cong Q_{k+\bar{\ell}-1}$ and $E_c^+ \ncong Q_{k+\bar{\ell}-1}$. On the other hand, since E_c is not peripheral in G', there is an edge $zz' \in \pi_{e_1}(G') \cap E_c$ such that $z \in (G_1 \setminus G_2) \cap E_c^-$ and $z' \in G_1 \cap G_2 \cap E_c^+$.

Note that $\pi_c(G')$ is an antipodal expansion of $\pi_{e_1}(\pi_c(G')) \cong Q_{k+\bar{\ell}-1}$. By Lemma 2.6, it is an expansion along sets $\pi_c(G_1)$ and $\pi_c(G_2)$. In the following we will use Lemma 2.6 to get properties of G_1, G_2 from the expansion $\pi_c(G_1), \pi_c(G_2)$.

We consider two cases, either $\pi_c(G')$ is a full expansion of $\pi_{e_1}(\pi_c(G')) \cong Q_{k+\bar{\ell}-1}$ or not. In the first case, every vertex of $\pi_{e_1}(\pi_c(G'))$ is in $\pi_c(G_1) \cap \pi_c(G_2)$, thus for any edge E_c in $\pi_{e_1}(G')$ we have that one of its endpoints is in $G_1 \cap G_2$ and any vertex not incident to E_c is in $G_1 \cap G_2$, as well. Since $E_c^+ \ncong Q_{k+\bar{\ell}-1}$ there is a missing vertex in it.

Let v be a vertex in $(G_1 \setminus G_2) \cap E_c^-$ that is as close as possible to a missing vertex. We have proved above that v exists (one candidate would be z). Since the expansion is full, by the above paragraph, v is not adjacent to the missing vertex. Then by the above paragraph its neighbor in E_c^+ and by the choice of v all its neighbors in E_c^- closer to the missing vertex are in $G_1 \cap G_2$. In the expansion G', the expansions of these vertices together with v give rise to vertices pairwise at distance 2. Since v is not expanded they have no common neighbor. By Claim 6.4 G' and all its pc-minors are in AG, thus by Lemma 5.2, their convex hull C is a cube minus a vertex or a cube minus two antipodes. Moreover, by the choice of v and since $E_c^- \cong Q_{k+\bar{\ell}-1}$ in $\pi_{e_1}(G_1)$, the hull C also has the missing vertex. However, now also its expansion is missing, thus two adjacent vertices are missing from C. Thus, C is not isomorphic to a cube minus a vertex or a cube minus two antipodes.

Now consider that $\pi_c(G')$ is not a full expansion of $\pi_{e_1}(\pi_c(G')) \cong Q_{k+\bar{\ell}-1}$. Then there exists a vertex v of $\pi_{e_1}(\pi_c(G'))$ with $v \in \pi_c(G_2) \setminus \pi_c(G_1)$. Then there is a vertex w of $\pi_{e_1}(G')$ in the preimage of v such that $w \in (G_2 \setminus G_1) \cap E_c^-$. Let again $zz' \in \pi_{e_1}(G')$ be an edge as in the first paragraph, i.e. $zz' \in E_c$ such that $z \in (G_1 \setminus G_2) \cap E_c^-$ and $z' \in G_1 \cap G_2 \cap E_c^+$. Then let $y \in E_c^-$ be the closest vertex to z such that $y \in G_2 \setminus G_1$ (one candidate would be w). Now, let x be a vertex in the interval from z to y such that $x \in G_1 \setminus G_2$ and x is as close as possible to y. Then all the vertices in the interval from x to y are in $G_1 \cap G_2$ except for x, y. Then since $E_c^- \cong Q_{k+\bar{\ell}-1}$ the interval from x to y is isomorphic to a hypercube, i.e. antipodal. Then, also its expansion is an antipodal graph, but one preimage of z' has no gate in it by Lemma 2.19. □

By the above claim the sequence of contractions cannot exist which gives a contradiction. Thus there is no G', G and the lemma holds. □

Proof of Theorem 6.1. Since Lemma 6.2 holds, by Lemma 6.3 we obtain the theorem. □

7 Further characterizations and recognition of tope graphs

In this section we will describe the important implications of Theorem 1.1. It can be read without having gone through the technicalities of the previous sections. In particular, we give polynomial time recognition algorithm for tope graphs of COMs, specialize our results to tope graphs of OMs, AOMs, and LOPs (of bounded rank), and prove a conjecture of [11]. But first of all, Theorem 1.1 can be used to obtain a characterization of $\mathscr{G}_{\mathrm{COM}}$ in terms of zone graphs that is a generalization of a result of Handa [26].

Corollary 7.1. *A graph G is the tope graph of a COM, i.e. $G \in \mathscr{G}_{\mathrm{COM}}$, if and only if G is a partial cube such that all iterated zone graphs are well-embedded partial cubes.*

Proof. If $G \in \mathscr{G}_{\mathrm{COM}}$, then by Lemma 4.12 all its zone graphs are well-embedded partial cubes and in $\mathscr{G}_{\mathrm{COM}}$. Hence, the argument can be iterated to prove that $\zeta_{e_1,\ldots,e_k}(G)$ is a partial cube for any sequence of hyperspaces.

If $G \notin \mathcal{G}_{\mathrm{COM}}$, then by Theorem 1.1 there is an $H \in \mathcal{Q}^-$ such that $H = \pi_A(\rho_X(G))$ for some Θ-classes of G. By Lemma 5.5 we can thus find a sequence $e_1, \ldots, e_k$ such that $\zeta_{e_1,\ldots,e_k}(\pi_A(\rho_X(G)))$ is not a partial cube. If $\zeta_{e_i,\ldots,e_k}(G)$ was a well-embedded partial cube for all $1 \leq i \leq k$, then by Lemma 2.9 we could find sets of Θ-classes A' and X' in it such that $\zeta_{e_1,\ldots,e_k}(\pi_A(\rho_X(G))) = \pi_{A'}(\rho_{X'}(\zeta_{e_1,\ldots,e_k}(G)))$. But then the latter would be a partial cube, contradicting the choice of $e_1, \ldots, e_k$. Therefore there is an iterated zone graph of G that is not a partial cube. □

Theorem 1.1 and Corollary 7.1 specialize to other systems of sign-vectors. Using Proposition 4.13 we immediately get the following corollary. Recall that for a set X of partial cubes we denote by $\mathcal{F}(X)$ the class of partial cubes that do not have any graph from X as a partial cube minor.

Corollary 7.2. *For a graph G the following conditions are equivalent:*

(i) G is the tope graph of an OM, i.e., $G \in \mathcal{G}_{\mathrm{OM}}$,

(ii) G is an antipodal partial cube and all its antipodal subgraphs are gated,

(iii) G is in $\mathcal{F}(\mathcal{Q}^-)$ and antipodal,

(iv) G is an antipodal partial cube and all its iterated zone graphs are well-embedded partial cubes.

Note that the equivalence (i)⟺(ii) corresponds to a characterization of tope sets of OMs due to da Silva [14] and (i)⟺(vi) corresponds to a characterization of tope sets of Handa [27].

Let us call an affine subgraph G' of an affine partial cube G ***conformal*** if for all $v \in G'$ we have $-_{G'}v \in G' \iff -_G v \in G$. We give an intrinsic characterization of $\mathcal{G}_{\mathrm{AOM}}$:

Corollary 7.3. *For a graph G the following conditions are equivalent:*

(i) G is the tope graph of an AOM, i.e., $G \in \mathcal{G}_{\mathrm{AOM}}$,

(ii) G is an affine partial cube and all its antipodal and conformal subgraphs are gated,

(iii) G is in $\mathcal{F}(\mathcal{Q}^-)$ (or $G \in \mathcal{G}_{\mathrm{COM}}$), affine, and all its conformal subgraphs are gated.

Proof. By Proposition 4.14, $G \in \mathcal{G}_{\mathrm{AOM}}$ if and only if G is an affine partial cube such that the antipodal $\widetilde{G}$ containing G as a halfspace is in $\mathcal{G}_{\mathrm{OM}}$. By Corollary 7.2 this is equivalent to all antipodal subgraphs of $\widetilde{G}$ being gated. It is easy to see, that the antipodal subgraphs of $\widetilde{G}$ correspond to the antipodal and the conformal subgraphs of G and they are gated if and only if the corresponding antipodal and conformal subgraphs of G are. This proves (i)⟺(ii).

□

For lopsided sets the (i)⟺(ii) part of the corollary corresponds to a characterization due to Lawrence [30]. For the complete statement denote $\mathcal{Q}^{--} := \{Q_n^{--} \mid n \geq 3\}$.

Corollary 7.4. *For a graph G the following conditions are equivalent:*

(i) G is the tope graph of a LOP, i.e., $G \in \mathcal{G}_{\mathrm{LOP}}$,

(ii) G is a partial cube and all its antipodal subgraphs are hypercubes,

(iii) G is in $\mathcal{F}(\mathcal{Q}^{--})$.

Proof. By Proposition 4.15 and Theorem 1.1 $G \in \mathcal{G}_{\mathrm{LOP}}$ is equivalent to the property that G has all antipodal subgraphs gated and isomorphic to hypercubes. But hypercubes are always gated. This gives (i)⟺(ii).

The implication (ii)⇒(iii) follows from the fact that $\mathcal{G}_{\mathrm{LOP}}$ is pc-minor closed while the graphs from $\mathcal{Q}^{--}$ are (pc-minor minimal) non-hypercube antipodal graphs.

To prove (iii)⇒(ii), let G be a pc-minor-minimal partial cube with an antipodal subgraph not isomorphic to a hypercube. Then G is antipodal and not isomorphic to a hypercube. Let G be embedded in a Q_n for some minimal $n \in \mathbb{N}$. Now, G can be seen as Q_n minus a set of antipodal pairs of vertices. Take a pair u, v of vertices in Q_n not in G at a minimal distance in Q_n. If $d(u, v) = 1$, contracting the Θ-class of uv contract G to an antipodal subgraph not isomorphic to a hypercube, contradicting that G is pc-minor-minimal. The interval $\mathrm{conv}(u, v)$ in Q_n intersected with G is a convex subgraph of G isomorphic to a hypercube minus a vertex. Note that $d(u, v) > 2$, since otherwise the subgraph is not connected. Since G is pc-minor-minimal, the subgraph is the whole G and thus G is in $\mathcal{Q}^{--}$. □

Recall that by Lemma 4.2 the rank of a COM is the dimension of a maximal hypercube to which its tope graph can be contracted. Considering COMs of bounded rank, we can reduce the set of excluded pc-minors to a finite list. For any $r \geq 3$ define the following finite sets

$$\mathcal{Q}_r^- := \{Q_n^{-*}, Q_n^{--}(m), Q_{r+2}^{--}(r+2), Q_{r+1} \mid 4 \leq n \leq r+1; 1 \leq m \leq n\} \subset \mathcal{Q}^- \cup \{Q_{r+1}\},$$

and

$$\mathcal{Q}_r^{--} := \{Q_n^{--}, Q_{r+1} \mid 3 \leq n \leq r+1\} \subset \mathcal{Q}^{--} \cup \{Q_{r+1}\}.$$

Corollary 7.5. *For a graph G and an integer $r \geq 3$ we have:*

- *$G \in \mathcal{G}_{\mathrm{COM}}$ of rank at most $r \iff G \in \mathcal{F}(\mathcal{Q}_r^-)$.*
- *$G \in \mathcal{G}_{\mathrm{OM}}$ of rank at most $r \iff G \in \mathcal{F}(\mathcal{Q}_r^-)$ and G is antipodal.*
- *$G \in \mathcal{G}_{\mathrm{AOM}}$ of rank at most $r \iff G \in \mathcal{F}(\mathcal{Q}_r^-)$, G is affine and all its conformal subgraphs are gated.*
- *$G \in \mathcal{G}_{\mathrm{LOP}}$ of rank at most $r \iff G \in \mathcal{F}(\mathcal{Q}_r^{--})$.*

Proof. The respective proofs follow immediately from Theorem 1.1 and Corollaries 7.2, 7.3, and 7.4 together with the observation that the largest hypercube that is a contraction minor of Q_n^{-*} and $Q_n^{--}(m)$ is of dimension $n-1$ for $0 \leq m \leq n-1$ and of $Q_n^{--}(n) \in \mathcal{Q}^-$ is of dimension $n-2$. Now, the graphical interpretation of the rank as given in Lemma 4.2 gives the result. □

Note that using Proposition 2.4 can easily be seen to yield a polynomial time recognition algorithm for the recognition of the bounded rank classes above. However, Theorem 1.1 also yields polynomial time recognition algorithms for the unrestricted classes.

Corollary 7.6. *The recognition of graphs from any of the classes $\mathcal{G}_{\mathrm{COM}}, \mathcal{G}_{\mathrm{AOM}}, \mathcal{G}_{\mathrm{OM}}, \mathcal{G}_{\mathrm{LOP}}$ can be done in polynomial time.*

Proof. By [19], partial cubes can be recognized and embedded in a hypercube in quadratic time. For a partial cube embedded in a hypercube checking if it is antipodal can be done in linear time by checking if every vertex has its antipode. Note that the convex hull of any subset can be computed in linear time in the number of edges (for instance by using Lemma 2.1 for a graph embedded in a hypercube) and checking if a convex subgraph is gated is linear by Lemma 4.6.

We proceed by designing recognition algorithms using the (i)⇔(ii) part of the respective characterizations. We start by computing $\mathrm{conv}(u, v)$ for all pairs of vertices u, v and storing them. We check if $\mathrm{conv}(u, v)$ is antipodal, and if so we check if it is gated and if it is isomorphic to a hypercube, i.e. $|\mathrm{conv}(u, v)| = 2^{d(u,v)}$. If all the antipodal graphs obtained in this way are gated, then $G \in \mathcal{G}_{\mathrm{COM}}$, otherwise we do not proceed. If G is among the antipodal subgraphs, then $\mathcal{G}_{\mathrm{OM}}$. Moreover, if all the antipodal subgraphs are isomorphic to hypercubes, then $G \in \mathcal{G}_{\mathrm{LOP}}$.

We continue by checking for each $\mathrm{conv}(u, v)$ if it is an affine subgraphs. For each pair $u', v' \in \mathrm{conv}(u, v)$ such that $|\mathrm{conv}(u', v')| < |\mathrm{conv}(u, v)|$ we store the pair in $NA(u, v)$, and we search for a pair $w, -_{\mathrm{conv}(u,v)}w \in \mathrm{conv}(u, v)$ such that the set of Θ-classes on a shortest (u', w)-path and on a shortest

$(v', -_{\mathrm{conv}(u,v)}w)$-path are disjoint. Note that the convex hulls are already computed. If this is the case for all such u', v', we store $\mathrm{conv}(u, v)$ as an affine subgraph. The correctness follows from Proposition 2.16.

Now we check if the whole graph is affine, in this case say $G = \mathrm{conv}(u, v)$. Then for every affine subgraph $\mathrm{conv}(u', v')$ and vertex $w \in \mathrm{conv}(u', v')$, we check if the pair $w, -_{\mathrm{conv}(u',v')}w$ is a pair in $NA(u', v')$ if and only if $w, -_G w$ is a pair in $NA(u, v)$. If this is the case, $\mathrm{conv}(u', v')$ is a conformal subgraph and we check if it is gated. Finally, if all conformal subgraphs are gated, then $G \in \mathscr{G}_{\mathrm{AOM}}$. □

A partial cube is called ***Pasch***, their class is denoted by $\mathscr{S}_4$, if any two disjoint convex subgraphs lie in two disjoint halfspaces. We confirm a conjecture from [11]:

Corollary 7.7. *The class $\mathscr{S}_4$ of Pasch graphs is a subclass of $\mathscr{G}_{\mathrm{COM}}$.*

Proof. Correcting the list given in [9, 10], in [11] the list of excluded pc-minors of Pasch graphs has been given, see Figure 9. With this it is easy to verify that $\mathscr{S}_4 \subset \mathscr{F}(\mathscr{Q}^-)$. Another way of seeing the inclusion is using [11, Theorem A], which confirms that the convex closure of any isometric cycle in a Pasch graph is gated. By Lemma 2.15 in particular we have that all antipodal subgraphs are gated and the claim follows from Theorem 1.1. □

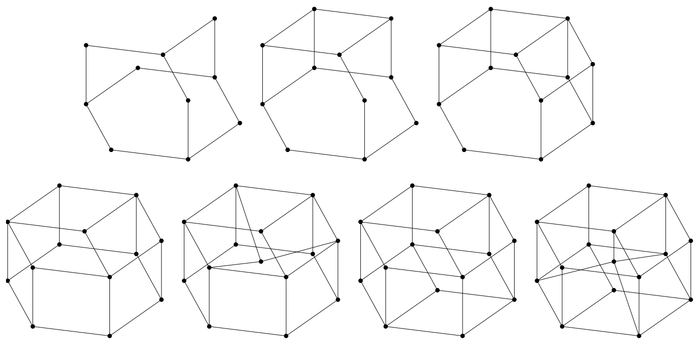


Figure 9: The set of minimal forbidden pc-minors of $\mathscr{S}_4$.

Note that together with a recent paper [34], Corollary 7.7 implies that ***netlike*** partial cubes are tope graphs of COMs. Moreover, it provides an alternative proof for the fact that hypercellular graphs are tope graphs of COMs, see [11], and therefore also median graphs, and bipartite cellular graphs.

8 Problems

This final section resumes the central open problems that either result from our paper or might be attacked from a new perspective using our results.

Let us start with the following:

Remark 8.1. *Any partial cube is a convex subgraph of an antipodal partial cube.*

To see this, consider an arbitrary partial cube G embedded in a hypercube Q_n. Construct an induced subgraph A_G of the hypercube Q_{n+3} where the convex subgraph with the last three coordinates equal to 1 is G, the convex subgraph with the last three coordinates equal to -1 is $-G$ and the other convex subgraphs given by fixing the last three coordinates are isomorphic to Q_n. It is easily checked that A_G

is an antipodal partial cube with G a convex subgraph of it. Nevertheless even if $G \in \mathcal{G}_{\text{COM}}$, A_G is not necessary in $\mathcal{G}_{\text{COM}}$, e.g., it is easy to check that A_{C_6} has a Q_4^{*-}-minor. Thus, let us restate a conjecture from [3, Conjecture 1] in terms of tope graphs.

Conjecture 1. *Every tope graph of a COM is a convex subgraph of a tope graph of an OM.*

Note that an affirmative answer to this question would immediately give a topological representation theorem for COMs.

A question arising from Corollaries 7.2 and 7.3 is based on the observation that $\mathcal{G}_{\text{AOM}}$ and $\mathcal{G}_{\text{OM}}$ are closed under contraction.

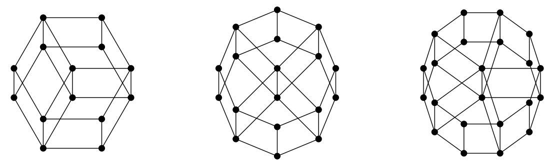
Problem 1. *Find the list of minimal excluded affine and antipodal contraction-minors for the classes $\mathcal{G}_{\text{AOM}}$ and $\mathcal{G}_{\text{OM}}$, respectively.*

Using Proposition 2.16 it is easy to see, that the affine partial cubes in $\mathcal{Q}^-$ are exactly the graphs of the form $Q_n^{--}(m)$. So this gives a family of excluded contraction minors for $\mathcal{G}_{\text{AOM}}$. However, it is not complete since some graphs of this family, e.g. the one arising from $Q_4^{--}(1)$, have halfspaces that are COMs which therefore have to be excluded as well. Moreover, the antipodal subgraphs obtained as in the proof of Proposition 2.16 from the affine graphs $Q_n^{--}(m)$ give a family of excluded contraction minors for $\mathcal{G}_{\text{OM}}$. In particular, this implies that the only non-matroidal antipodal graphs with at most five Θ-classes are the ones coming from the four graphs $Q_4^{--}(1), \dots, Q_4^{--}(4)$ – a result originally due to Handa [27]. However, also this family is incomplete, since the graph A_{C_6} constructed above has no halfspace in $\mathcal{Q}^-$.

Several characterizations of planar partial cubes are known [1, 16], where the latter corrects a characterization from [33]. A particular consequence of Corollary 7.5 is that tope graphs of OMs of rank at most three are characterized by the finite list of excluded pc-minors $\mathcal{Q}_3^-$. By a result of Fukuda and Handa [23] this graph class coincides with the class of antipodal planar partial cubes. Since planar partial cubes are closed under pc-minors, we wonder about an extension of this result to general planar partial cubes:

Problem 2. *Find the list of minimal excluded pc-minors for the class of planar partial cubes.*

It is easy to see, that any answer here will be an infinite list, since a (strict) subfamily is given by the set $\{G_n \square K_2 \mid n \geq 3\}$, where G_n denotes the ***gear graph*** (also known as ***cogwheel***) on $2n+1$ vertices, see Figure 10.



$G_3 \square K_2$ $G_4 \square K_2$ $G_5 \square K_2$

Figure 10: The first three members of an infinite family of minimal obstructions for planar partial cubes.

This in particular shows, that if a pc-minor closed class is contained in an (ordinary) minor closed graph class it can still have an infinite list of excluded minors. A tempting probably quite hard problem is the following:

Problem 3. *Give a criterion for when a pc-minor closed class has a finite list of excluded pc-minors.*

Acknowledgments

We wish to thank Emanuele Delucchi and Yida Zhu for discussions on AOMs, Ilda da Silva for insights on OMs, Hans-Jürgen Bandelt and Victor Chepoi for several fruitful discussions on COMs and their tope graphs, and Matjaž Kovše for being part of the very first sessions on tope graphs. The first author was supported by grants ANR-16-CE40-0009-01, ANR-17-CE40-0015, and ANR-17-CE40-0018, the second author by grants P1-0297 and J1-9109. Finally, we wish to thank the referees for comments that clearly improved the quality of this paper.

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Partial cubes without Q_3^- minors

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Abstract. We investigate the structure of isometric subgraphs of hypercubes (i.e., partial cubes) which do not contain finite convex subgraphs contractible to the 3-cube minus one vertex Q_3^- (here contraction means contracting the edges corresponding to the same coordinate of the hypercube). Extending similar results for median and cellular graphs, we show that the convex hull of an isometric cycle of such a graph is gated and isomorphic to the Cartesian product of edges and even cycles. Furthermore, we show that our graphs are exactly the class of partial cubes in which any finite convex subgraph can be obtained from the Cartesian products of edges and even cycles via successive gated amalgams. This decomposition result enables us to establish a variety of results. In particular, it yields that our class of graphs generalizes median and cellular graphs, which motivates naming our graphs hypercellular. Furthermore, we show that hypercellular graphs are tope graphs of zonotopal complexes of oriented matroids. Finally, we characterize hypercellular graphs as being median-cell – a property naturally generalizing the notion of median graphs.

1. Introduction

Partial cubes are the graphs which admit an isometric embedding into a hypercube. They comprise many important and complex graph classes occurring in metric graph theory and initially arising in completely different areas of research. Among them there are the graphs of regions of hyperplane arrangements in $\mathbb{R}^d$ [15], and, more generally, tope graphs of oriented matroids (OMs) [16], median graphs (alias 1-skeleta of CAT(0) cube complexes) [7, 31], netlike graphs [38–41], bipartite cellular graphs [5], bipartite graphs with S_4 convexity [23], graphs of lopsided sets [8, 37], 1-skeleta of CAT(0) Coxeter zonotopal complexes [34], and tope graphs of complexes of oriented matroids (COMs) [9]. COMs represent a general unifying structure for many of the above classes of partial cubes: from tope graphs of OMs to median graphs, lopsided sets, cellular graphs, and graphs of CAT(0) Coxeter zonotopal complexes. Median graphs are obtained by gluing in a specific way cubes of different dimensions. In particular, they give rise not only to contractible but also to CAT(0) cube complexes. Similarly, lopsided sets yield contractible cube complexes, while cellular graphs give contractible polygonal complexes whose cells are regular even polygons. Analogously to median graphs, graphs of CAT(0) Coxeter zonotopal complexes can be viewed as partial cubes obtained by gluing zonotopes. COMs can be viewed as a common generalization of all these notions: their tope graphs are the partial cubes obtained by gluing tope graphs of OMs in a lopsided (and thus contractible) fashion.

In this paper, we investigate the structure of a subclass of zonotopal COMs, in which all cells are gated subgraphs isomorphic to Cartesian products of edges and even cycles, see Figure 1(a) for such a cell. More precisely, we study the partial cubes in which all finite convex subgraphs can be obtained from Cartesian products of edges and even cycles by successive gated amalgamations. We show that our graphs share and extend many properties of bipartite cellular graphs of [5]; they can be viewed as high-dimensional analogs of cellular graphs. This is why we call them *hypercellular graphs*, see Figure 1(b) for an example. There is another way of describing hypercellular graphs, requiring a few definitions.

Djoković [27] characterized partial cubes in the following simple but pretty way: *a graph $G = (V, E)$ can be isometrically embedded in a hypercube if and only if G is bipartite and for any edge uv, the sets $W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$ and $W(v, u) = \{x \in V : d(x, v) < d(x, u)\}$ are convex.* In this case, $W(u, v) \cup W(v, u) = V$, whence $W(u, v)$ and $W(v, u)$ are complementary convex subsets of G, called

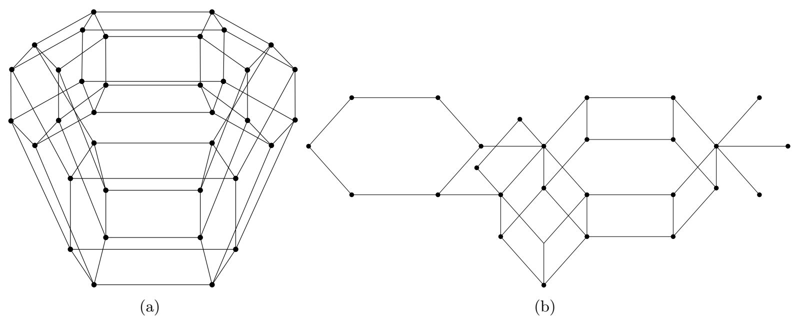

FIGURE 1. (a) a four-dimensional cell isomorphic to $C_6 \times C_6$. (b) a hypercellular graph with eight maximal cells: C_6, C_4, C_4, $K_2 \times K_2 \times K_2$, $C_6 \times K_2$, and three K_2.

halfspaces. The edges between $W(u,v)$ and $W(v,u)$ correspond to a coordinate in a hypercube embedding of G.

Moreover, partial cubes have the separation property S_3: any convex subgraph G' of a partial cube G can be represented as an intersection of halfspaces of G [1,4,20]. We will call such a representation (or simply the convex subgraph G') a *restriction* of G. A *contraction* of G is the partial cube G' obtained from G by contracting all edges corresponding to a given coordinate in a hypercube embedding. Now, a partial cube H is called a *partial cube-minor* (abbreviated, *pc-minor*) of G if H can be obtained by a sequence of contractions from a convex subgraph of G. If $T_1, \dots, T_m$ are finite partial cubes, then $\mathcal{F}(T_1, \dots, T_m)$ is the set of all partial cubes G such that *no* $T_i, i = 1, \dots, m$, can be obtained as a pc-minor of G. We will say that a class of partial cubes $\mathcal{C}$ is *pc-minor-closed* if we have that $G \in \mathcal{C}$ and G' is a minor of G imply that $G' \in \mathcal{C}$. As we will see in Section 2.2, for any set of partial cubes $T_1, \dots, T_m$, the class $\mathcal{F}(T_1, \dots, T_m)$ is pc-minor-closed.

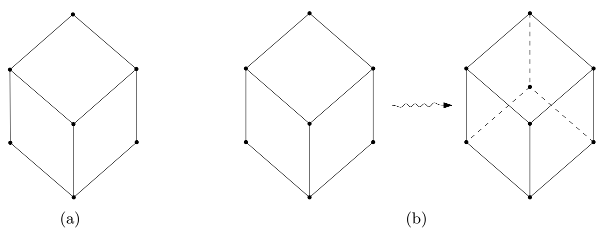

FIGURE 2. (a) Q_3^- – the 3-cube minus one vertex.(b) the 3-cube condition.

It turns out that the class of hypercellular graphs coincides with the minor-closed class $\mathcal{F}(Q_3^-)$, where Q_3^- denotes the 3-cube minus one vertex, see Figure 2(a). In a sense, this is the first nontrivial class $\mathcal{F}(T)$. Indeed, the class $\mathcal{F}(C)$, where C is a 4-cycle, is just the class of all trees. Also, the classes $\mathcal{F}(T)$ where T is the union of two 4-cycles sharing one vertex or one edge are quite special. Median graphs, graphs of lopsided sets, and tope graphs of COMs are pc-minor closed, whereas tope graphs of OMs are only closed under contractions but not under restrictions. Another class of pc-minor closed partial cubes is the class $\mathcal{S}_4$ also known as *Pasch graphs*. It consists of bipartite graphs in which the geodesic convexity satisfies the separation property S_4 [20,23], i.e., any two disjoint convex sets can be separated by disjoint half-spaces. It is shown in [20, 23] that $\mathcal{S}_4 = \mathcal{F}(T_1, \ldots, T_m)$, where all T_i are isometric subgraphs of Q_4; see Figure 3 for the complete list, from which T_5 and T_7 were missing in [20, 23]. In particular, Q_3^- is a pc-minor of all of the T_i. Thus, $\mathcal{F}(Q_3^-) \subseteq \mathcal{S}_4$.

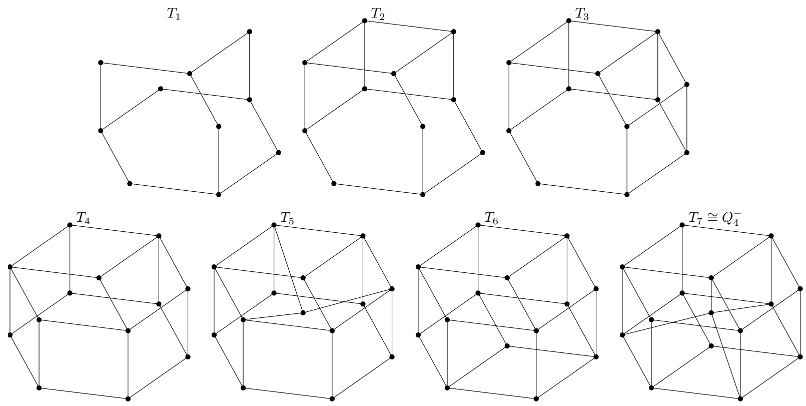

FIGURE 3. The set of minimal forbidden pc-minors of $\mathcal{S}_4$.

Our results mainly concern the cell-structure of graphs from $\mathcal{F}(Q_3^-)$. It is well-known [2] that median graphs are exactly the graphs in which the convex hulls of isometric cycles are hypercubes; these hypercubes are gated subgraphs. Moreover, any finite median graph can be obtained by gated amalgams from cubes [33, 43]. Analogously, it was shown in [5] that any isometric cycle of a bipartite cellular graph is a convex and gated subgraph; moreover, the bipartite cellular graphs are exactly the bipartite graphs which can be obtained by gated amalgams from even cycles. We extend these results in the following way:

Theorem A. *The convex closure of any isometric cycle of a graph $G \in \mathcal{F}(Q_3^-)$ is a gated subgraph isomorphic to a Cartesian product of edges and even cycles. Moreover, the convex closure of any isometric cycle of a graph $G \in \mathcal{S}_4$ is a gated subgraph, which is isomorphic to a Cartesian product of edges and even cycles if it is antipodal.*

In view of Theorem A we will call a subgraph X of a partial cube G a *cell* if X is a convex subgraph of G which is a Cartesian product of edges and even cycles. Note that since a Cartesian product of edges and even cycles is the convex hull of an isometric cycle, by Theorem A the cells of $\mathcal{F}(Q_3^-)$ can be equivalently defined as convex hulls of isometric cycles. Notice also that if we replace each cell X of G by a convex polyhedron $[X]$ which is the Cartesian product of segments and regular polygons (a segment for each edge-factor and a regular polygon for each cyclic factor), then we associate with G a cell complex $\mathbf{X}(G)$.

We will say that a partial cube G satisfies the *3-convex cycles condition* (abbreviated, *3CC-condition*) if for any three convex cycles C_1, C_2, C_3 that intersect in a vertex and pairwise intersect in three different edges the convex hull of $C_1 \cup C_2 \cup C_3$ is a cell; see Figure 4 for an example. Notice that the absence of cycles satisfying the preconditions of the 3CC-condition together with the gatedness of isometric cycles characterizes bipartite cellular graphs [5].

Defining the dimension of a cell X as the number of edge-factors plus two times the number cyclic factors (which corresponds to the topological dimension of $[X]$) one can give a natural generalization of the 3CC-condition. We say that a partial cube G (or its cell complex $\mathbf{X}(G)$) satisfies the *3-cell condition*

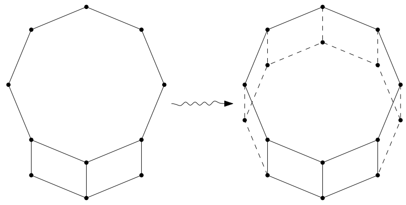

FIGURE 4. The 3-convex cycles condition.

(abbreviated, *3C-condition*) if for any three cells X_1, X_2, X_3 of dimension $k+2$ that intersect in a cell of dimension k and pairwise intersect in three different cells of dimension $k+1$ the convex hull of $X_1 \cup X_2 \cup X_3$ is a cell. In case of cubical complexes $\mathbf{X}$, the 3-cell condition coincides with Gromov's flag condition [30] (which can be also called cube condition, see Figure 2(b)), which together with simply connectivity of $\mathbf{X}$ characterize CAT(0) cube complexes. By [24, Theorem 6.1], median graphs are exactly the 1-skeleta of CAT(0) cube complexes (for other generalizations of these two results, see [14, 18]).

The following main characterization of graphs from $\mathcal{F}(Q_3^-)$ establishes those analogies with median and cellular graphs, that lead to the name hypercellular graphs.

Theorem B. *For a partial cube $G = (V, E)$, the following conditions are equivalent:*

- (i) $G \in \mathcal{F}(Q_3^-)$, *i.e., G is hypercellular;*
- (ii) *any cell of G is gated and G satisfies the 3CC-condition;*
- (iii) *any cell of G is gated and G satisfies the 3C-condition;*
- (iv) *each finite convex subgraph of G can be obtained by gated amalgams from cells.*

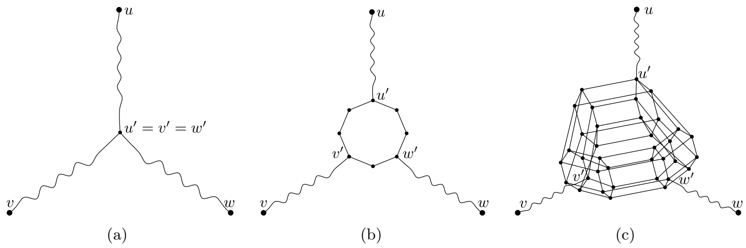

FIGURE 5. (a) a median-vertex. (b) a median-cycle. (c) a median-cell.

A further characterization of hypercellular graphs is analogous to median and cellular graphs, see the corresponding properties in Figure 5(a) and 5(b), respectively. We show that hypercellular graphs satisfy the so-called *median-cell property*, which is essentially defined as follows: for any three vertices u, v, w of G there exists a unique gated cell X of G such that if u', v', w' are the gates of u, v, w in X, respectively, then u', v' lie on a common (u, v)-geodesic, v', w' lie on a common (v, w)-geodesic, and w', u' lie on a common (w, u)-geodesic, see Figure 5(c) for an illustration. Namely, we prove:

Theorem C. *A partial cube G satisfies the median-cell property if and only if G is hypercellular.*

Theorem B has several immediate consequences, which we formulate next.

Theorem D. *Let G be a locally finite hypercellular graph. Then $\mathbf{X}(G)$ is a contractible zonotopal complex. Additionally, if G is finite, then G is a tope graph of a zonotopal COM.*

Theorem B also immediately implies that median graphs and bipartite cellular graphs are hypercellular. Furthermore, a subclass of netlike partial cubes, namely partial cubes which are gated amalgams of even cycles and cubes [40], are hypercellular. In particular, we obtain that these three classes coincide with $\mathcal{F}(Q_3^-, C_6)$, $\mathcal{F}(Q_3^-, Q_3)$, and $\mathcal{F}(Q_3^-, C_6 \times K_2)$, respectively. Other direct consequences of Theorem B concern convexity invariants (Helly, Caratheodory, Radon, and partition numbers) of hypercellular graphs which are shown to be either a constant or bounded by the topological dimension of $\mathbf{X}(G)$.

Let G be a hypercellular graph. For an equivalence class E_f of edges of G (i.e., all edges corresponding to a given coordinate f in a hypercube embedding of G), we denote by $N(E_f)$ the *carrier* of f, i.e., subgraph of G which is the union of all cells of G crossed by E_f. It was shown in [9, Proposition 5] that carriers of COMs are also COMs. A *star* $\mathrm{St}(v)$ of a vertex v (or a star $\mathrm{St}(X)$ of a cell X) is the union of all cells of G containing v (respectively, X). The *thickening* G^Δ of G is a graph having the same set of vertices as G and two vertices u, v are adjacent in G^Δ if and only if u and v belong to a common cell of G. Finally, a graph H is called a *Helly graph* if any collection of pairwise intersecting balls has a nonempty intersection. Helly graphs play an important role in metric graph theory as discrete analogs of injective spaces: any graph embeds isometrically into a smallest Helly graph (for this and other results, see the survey [7] and the recent paper [18]). It was shown in [12] that the thickening of median graphs are finitely Helly graphs (for a generalization of this result, see [18, Theorem 6.13]).

Theorem E. *Let G be a hypercellular graph. Then all carriers $N(E_f)$ and stars $\mathrm{St}(X)$ of G are gated. If additionally G is locally-finite, then the thickening G^Δ of G is a Helly graph.*

Finally, we generalize fixed box theorems for median graphs to hypercellular graphs and prove that in this case the fixed box is a cell. More precisely, we conclude the paper with the following:

Theorem F. *Let G be a hypercellular graph.*

(i) *if G does not contain infinite isometric rays, then G contains a cell X fixed by every automorphism of G;*
(ii) *any non-expansive map f from G to itself fixing a finite set of vertices (i.e., $f(S) = S$ for a finite set S) also fixes a finite cell X of G. In particular, if G is finite, then any non-expansive map f from G to itself fixes a cell of G;*
(iii) *if G is finite and regular, then G is a single cell, i.e., G is isomorphic to a Cartesian product of edges and cycles.*

Structure of the paper: In Section 2 we introduce preliminary definitions and results needed for this paper. In particular, we discuss convex and gated subgraphs in partial cubes, the notion of partial cube minors and their relation with convexity and gatedness. We also briefly discuss the properties of Cartesian products central to our work. Section 3 is devoted to the structure of cells in hypercellular graphs and graphs from $\mathcal{S}_4$; in particular, we prove Theorem A. Section 4 is devoted to amalgamation and decomposition of hypercellular graphs; we prove Theorem B. In Section 5 we discuss the median cell property of hypercellular graphs and prove Theorem C. Section 6 provides a rich set of properties of hypercellular graphs. In Subsection 6.1 we expose relations to other classes of partial cubes and in particular prove Theorem D. Subsection 6.2 gives several properties with respect to convexity parameters. Subsection 6.3 is devoted to the proof of Theorem E. In Subsection 6.4 we prove several fixed cell results for hypercellular graphs, in particular, we prove Theorem F. We conclude the paper with several problems and conjectures in Section 7.

2. Preliminaries

2.1. **Metric subgraphs and partial cubes.** All graphs $G = (V, E)$ occurring in this paper are simple, connected, without loops or multiple edges, but not necessarily finite. The *distance* $d(u, v) := d_G(u, v)$ between two vertices u and v is the length of a shortest (u, v)-path, and the *interval* $I(u, v)$ between u and v consists of all vertices on shortest (u, v)–paths, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) := \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

An induced subgraph of G (or the corresponding vertex set A) is called *convex* if it includes the interval of G between any two of its vertices. Since the intersection of convex subgraphs is convex, for every subset $S \subseteq V$ there exists the smallest convex set $\mathrm{conv}(S)$ containing S, referred to as the *convex hull* of S. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. In particular, convex subgraphs are isometric.

A subset W of V or the subgraph H of G induced by W is called *gated* (in G) [29] if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x'. It is easy to see that if x has a gate in H, then it is unique and that gated sets are convex. Gated sets enjoy the finite *Helly property* [44, Proposition 5.12 (2)], that is, every finite family of gated sets that pairwise intersect has a nonempty intersection. Since the intersection of gated subgraphs is gated, for every subset $S \subseteq V$ there exists the smallest gated set $\langle\langle S \rangle\rangle$ containing S, referred to as the *gated hull* of S. A graph G is a *gated amalgam* of two graphs G_1 and G_2 if G_1 and G_2 constitute two intersecting gated subgraphs of G whose union is all of G.

A graph $G = (V, E)$ is *isometrically embeddable* into a graph $H = (W, F)$ if there exists a mapping $\varphi : V \to W$ such that $d_H(\varphi(u), \varphi(v)) = d_G(u, v)$ for all vertices $u, v \in V$, i.e., $\varphi(G)$ is an isometric subgraph of H. A graph G is called a *partial cube* if it admits an isometric embedding into some hypercube $Q(\Lambda) = \{-1, +1\}^{\Lambda}$. From now on, we will always suppose that a partial cube $G = (V, E)$ is an isometric subgraph of the hypercube $Q(\Lambda) = \{-1, +1\}^{\Lambda}$ (i.e., we will identify G with its image under the isometric embedding). If this causes no confusion, we will denote the distance function of G by d and not d_G.

For an edge $e = uv$ of G, define the sets $W(u, v) = \{x \in V : d(x, u) < d(x, v)\}$ and $W(v, u) = \{x \in V : d(x, v) < d(x, u)\}$. By Djoković's theorem [27], a graph G is a partial cube if and only if G is bipartite and for any edge $e = uv$ the sets $W(u, v)$ and $W(v, u)$ are convex. The sets of the form $W(u, v)$ and $W(v, u)$ are called *complementary halfspaces* of G. To establish an isometric embedding of G into a hypercube, Djoković [27] introduces the following binary relation Θ – called *Djoković-Winkler relation* – on the edges of G: for two edges $e = uv$ and $e' = u'v'$ we set $e\Theta e'$ iff $u' \in W(u, v)$ and $v' \in W(v, u)$. Under the conditions of the theorem, it can be shown that $e\Theta e'$ iff $W(u, v) = W(u', v')$ and $W(v, u) = W(v', u')$, whence Θ is an equivalence relation. Let $\mathcal{E} = \{E_i : i \in \Lambda\}$ be the equivalence classes of Θ and let b be an arbitrary fixed vertex taken as the basepoint of G. For an equivalence class $E_i \in \mathcal{E}$, let $\{H_i^-, H_i^+\}$ be the pair of complementary convex halfspaces of G defined by setting $H_i^- := W(u, v)$ and $H_i^+ := W(v, u)$ for an arbitrary edge $uv \in E_i$ with $b \in W(u, v)$.

2.2. **Partial cube minors.** Let $G = (V, E)$ be an isometric subgraph of the hypercube $Q(\Lambda) = \{-1, +1\}^{\Lambda}$. Given $f \in \Lambda$, an *elementary restriction* consists in taking one of the subgraphs $G(H_f^-)$ or $G(H_f^+)$ induced by the complementary halfspaces H_f^- and H_f^+, which we will denote by $\rho_{f^-}(G)$ and $\rho_{f^+}(G)$, respectively. These graphs are isometric subgraphs of the hypercube $Q(\Lambda \setminus \{f\})$. Now applying twice the elementary restriction to two different coordinates f, g, independently of the order of f and g, we will obtain one of the four (possibly empty) subgraphs induced by the $H_f^- \cap H_g^-, H_f^- \cap H_g^+, H_f^+ \cap H_g^-$, and $H_f^+ \cap H_g^+$. Since the intersection of convex subsets is convex, each of these four sets is convex in G and consequently induces an isometric subgraph of the hypercube $Q(\Lambda \setminus \{f, g\})$. More generally, a *restriction* is a subgraph of G induced by the intersection of a set of (non-complementary) halfspaces of G. We denote a restriction by $\rho_A(G)$, where $A \in \Lambda^{\{+,-\}}$ is a signed set of halfspaces of G. For subset S of the vertices of G, we denote $\rho_A(S) := \rho_A(G) \cap S$. The following is well-known:

Lemma 1 ([1,4,20])**.** *The set of restrictions of a partial cube G coincides with its set of convex subgraphs. In particular, the class of partial cubes is closed under taking restrictions.*

For $f \in \Lambda$, we say that the graph G/E_f obtained from G by contracting the edges of the equivalence class E_f is an (f-)*contraction* of G. For a vertex v of G, we will denote by $\pi_f(v)$ the image of v under the f-contraction in G/E_f, i.e., if uv is an edge of E_f, then $\pi_f(u) = \pi_f(v)$, otherwise $\pi_f(u) \neq \pi_f(v)$. We will apply π_f to subsets $S \subset V$, by setting $\pi_f(S) := \{\pi_f(v) : v \in S\}$. In particular we denote the *f-contraction* of G by $\pi_f(G)$.

It is well-known and easy to prove and in particular follows from the proof of the first part of [21, Theorem 3] that $\pi_f(G)$ is an isometric subgraph of $Q(\Lambda \setminus \{f\})$. Since edge contractions in graphs commute, i.e., the resulting graph does not depend on the order in which a set of edges is contracted, we have:

Lemma 2. *Contractions commute in partial cubes, i.e., if $f, g \in \Lambda$ and $f \neq g$, then $\pi_g(\pi_f(G)) = \pi_f(\pi_g(G))$. Moreover, the class of partial cubes is closed under contractions.*

Consequently, for a set $A \subset \Lambda$, we can denote by $\pi_A(G)$ the isometric subgraph of $Q(\Lambda \setminus A)$ obtained from G by contracting the classes $A \subset \Lambda$ in G.

A partial cube G is an *expansion* of a partial cube G' if $G' = \pi_f(G)$ for some equivalence class f of $\Theta(G)$. More generally, let G' be a graph containing two isometric subgraphs G'_1 and G'_2 such that $G' = G'_1 \cup G'_2$, there are no edges from $G'_1 \setminus G'_2$ to $G'_2 \setminus G'_1$, and $G'_0 := G'_1 \cap G'_2$ is nonempty. A graph G is an *isometric expansion* of G' with respect to G_0 (notation $G = \psi(G')$) if G is obtained from G' by replacing each vertex v of G'_1 by a vertex v_1 and each vertex v of G'_2 by a vertex v_2 such that u_i and v_i, $i = 1, 2$ are adjacent in G if and only if u and v are adjacent vertices of G'_i and v_1v_2 is an edge of G if and only if v is a vertex of G'_0. The following is well-known:

Lemma 3 ([20, 21])**.** *A graph G is a partial cube if and only if G can be obtained by a sequence of isometric expansions from a single vertex.*

Lemma 4. *Contractions and restrictions commute in partial cubes, i.e., if $f, g \in \Lambda$ and $f \neq g$, then $\rho_{g^+}(\pi_f(G)) = \pi_f(\rho_{g^+}(G))$.*

Proof. Let $f, g \in \Lambda$ and $f \neq g$. The crucial property is that E_g is an edge-cut of G and $E_g \cap E_f = \emptyset$. If we see vertices as sign vectors in the hypercube, the vertex set of $\pi_f(\rho_{g^+}(G))$ can be described as $\{x \in V(G) : x_g = +\}/E_f = \{x \in V(G) \setminus V(E_f) : x_g = +\} \cup \{xy \in E_f : x_g = y_g = +\}$. The vertex set of $\rho_{g^+}(\pi_f(G))$ is $\{x \in V(G) \setminus V(E_f)\} \cup \{xy \in E_f\} \setminus H_g^-$ which again equals $\{x \in V(G) \setminus V(E_f) : x_g = +\} \cup \{xy \in E_f : x_g = y_g = +\}$. Furthermore, identifying a vertex of the form $\{x, y\} \in E_f$ with the vector z arising from x or y by omitting the f-coordinate, adjacency is defined the same way in both graphs, namely by taking the induced subgraph of the hypercube. This concludes the proof. □

The previous lemmas show that any set of restrictions and any set of contractions of a partial cube G provide the same result, independently of the order in which we perform the restrictions and contractions. The resulting graph G' is also a partial cube, and G' is called a *partial cube-minor* (or *pc-minor*) of G. In this paper we will study classes of partial cube excluding a given set of minors.

2.3. **Partial cube minors versus metric subgraphs.** In this section we present conditions under which contractions and restrictions preserve metric properties of subgraphs.

Let $G = (V, E)$ be an isometric subgraph of the hypercube $Q(\Lambda)$ and let S be a subgraph of G. Let f be any coordinate of Λ. We will say that E_f *crosses* S iff $S \cap H_f^- \neq \emptyset$ and $S \cap H_f^+ \neq \emptyset$. We will say that E_f *osculates* S iff E_f does not cross S and there exists an edge $e = uv \in E_f$ such that $\{u, v\} \cap S \neq \emptyset$. Otherwise, we will say that E_f is *disjoint* from S.

Lemma 5. *If S is a convex subgraph of G and $f \in \Lambda$, then $\rho_{f^+}(S)$ is a convex subgraph of $\rho_{f^+}(G)$. If E_f crosses S or is disjoint from S, then also $\pi_f(S)$ is a convex subgraph of $\pi_f(G)$.*

Proof. Let S be convex. Then by Lemma 1, S can be written as $\rho_A(G)$, where A is a signed set of those Θ-classes that osculate with S. Again by Lemma 1, $\rho_{f^+}(S) = \rho_{f^+}(\rho_A(G))$ is a convex subgraph of $\rho_{f^+}(G)$, proving the first assertion. Now, if $f \in \Lambda \setminus A$, then $\pi_f(S) = \pi_f(\rho_A(G))$, which by Lemma 4 equals $\rho_A(\pi_f(G))$, i.e., $\pi_f(S)$ is a convex subgraph of $\pi_f(G)$. □

Lemma 6. *If S' is a convex subgraph of G' and G is obtained from G' by an isometric expansion ψ, then $S := \psi(S')$ is a convex subgraph of G.*

Proof. Let $f \in \Lambda$ be such that $G' = \pi_f(G)$ and S' a convex subgraph of G'. By Lemma 1, there exists a signed set of Θ-classes $A \subset \Lambda \setminus \{f\}$, such that $S' = \rho_A(G') = \rho_A(\pi_f(G))$. By Lemma 4, $S' = \pi_f(\rho_A(G))$, thus $\rho_A(G) \subset S$. For every $g^+ \in A$, we have $\pi_f(\rho_{g^-}(G)) = \rho_{g^-}(G')$, thus it is disjoint with S. Then $S = \rho_A(G)$, which is convex by Lemma 1. □

Lemma 7. *If S is a subset of vertices of G and $f \in \Lambda$, then $\pi_f(\mathrm{conv}(S)) \subseteq \mathrm{conv}(\pi_f(S))$. If E_f crosses S, then $\pi_f(\mathrm{conv}(S)) = \mathrm{conv}(\pi_f(S))$.*

Proof. Let $y' \in \pi_f(\mathrm{conv}(S))$, i.e., there is a $y \in \pi_f^{-1}(y')$ on a shortest path P in G between vertices $x, z \in S$. Contracting f yields a shortest path P_f in G_f between two vertices on $\pi_f(S)$ containing y'. This proves $\pi_f(\mathrm{conv}(S)) \subseteq \mathrm{conv}(\pi_f(S))$.

For the second claim note that since $\mathrm{conv}(S) \supseteq S$, we have $\pi_f(\mathrm{conv}(S)) \supseteq \pi_f(S)$ and $\mathrm{conv}(\pi_f(\mathrm{conv}(S))) \supseteq \mathrm{conv}(\pi_f(S))$. Finally, since E_f crosses S it also crosses $\mathrm{conv}(S)$ and by Lemma 5 we have that $\mathrm{conv}(\pi_f(\mathrm{conv}(S))) = \pi_f(\mathrm{conv}(S))$, yielding the claim. □

We call a subgraph S of a graph $G = (V, E)$ *antipodal* if for every vertex x of S there is a vertex x^- of S such that $S = \mathrm{conv}(x, x^-)$ in G. Note that antipodal graphs are sometimes defined in a different but equivalent way (graphs satisfying our definition are also called symmetric-even, see [13]). By definition, antipodal subgraphs are convex.

Lemma 8. *Let S be an antipodal subgraph of G and $f \in \Lambda$. If E_f is disjoint from S, then $\rho_{f^+}(S)$ is an antipodal subgraph of $\rho_{f^+}(G)$. If E_f crosses S or is disjoint from S, then $\pi_f(S)$ is an antipodal subgraph of $\pi_f(G)$.*

Proof. If E_f is disjoint from S, then $S\rho_{f^+}(S) = S$ and by Lemma 5 is convex. This yields the first assertion. For the second assertion, again by Lemma 5, $\pi_f(S)$ is convex. Moreover, by Lemma 7 if $\mathrm{conv}(x, x^-) = S$, then $\pi_f(S) = \pi_f(\mathrm{conv}(x, x^-)) = \mathrm{conv}(\pi_f(\{x, x^-\}))$. Since every vertex in $\pi_f(S)$ is an image under the contraction, $\pi_f(S)$ is antipodal. □

Lemma 9. *If S is an antipodal subgraph of G, then S contains an isometric cycle C such that $\mathrm{conv}(C) = S$.*

Proof. Let $x \in S$ and let $P = (x = x_0, x_1, \ldots, x_k = x^-)$ be a shortest path in S to the antipodal vertex x^- of x. It is well-known that the mapping $x \mapsto x^-$ is a graph automorphism of S, thus $C = (x = x_0, x_1, \ldots, x_k = x_0^-, x_1^-, \ldots, x_k^- = x)$ is a cycle. Furthermore, by the properties of the map $x \mapsto x^-$ every subpath of C of length at most k is a shortest path. Thus, C is an isometric cycle of S. Since C contains antipodal vertices of S, we have $\mathrm{conv}(C) = S$. □

Lemma 10. *If S is a gated subgraph of G, then $\rho_{f^+}(S)$ and $\pi_f(S)$ are gated subgraphs of $\rho_{f^+}(G)$ and $\pi_f(G)$, respectively.*

Proof. Let $x \in G$ with gate $y \in S$, $z \in S$, and let P be a shortest path from x to z passing via y. To prove that $\rho_{f^+}(S)$ is gated, suppose that $x, z \in \rho_{f^+}(G)$. This implies $y \in \rho_{f^+}(G)$, thus y is also the gate of x in $\rho_{f^+}(S)$ in the graph $\rho_{f^+}(G)$.

To prove the second assertion, notice that the distance in $\pi_f(G)$ between x and z decreases by one if and only if P crosses E_f and remains unchanged otherwise, thus $\pi_f(P)$ is a shortest path in $\pi_f(G)$. This shows that $\pi_f(y)$ is the gate of $\pi_f(x)$ in $\pi_f(S)$ in the graph $\pi_f(G)$. □

2.4. **Cartesian products.** The *Cartesian product* $F_1 \times F_2$ of two graphs $F_1 = (V_1, E_1)$ and $F_2 = (V_2, E_2)$ is the graph defined on $V_1 \times V_2$ with an edge $(u, u')(v, v')$ if and only if $u = v$ and $u'v' \in E_2$ or $u' = v'$ and $uv \in E_1$. This definition generalizes in a straightforward way to products of sets of graphs. If $G = F_1 \times \ldots \times F_k$, then each F_i is called a *factor* of G. A *subproduct* of such G is a product $F'_1 \times \ldots \times F'_k$, where F'_i is a subgraph of F_i for all $1 \leq i \leq k$. A *layer* is a subproduct, where all but one of the F'_i consist of a single vertex and the remaining F'_i coincides with F_i.

It is well-known that products of partial cubes are partial cubes, and thus products of even cycles and edges are partial cubes, which we will be particularly interested in. It is easy to see that any contraction of a product of even cycles and edges is a product of even cycles and edges. Furthermore, any Cartesian product of even cycles and edges is antipodal, since taking the antipode with respect to all factors gives the antipode with respect to the product. By Lemma 9 any such product is the convex hull of an isometric cycle. We will use the following properties of these graphs frequently (and sometime without an explicit reference):

Lemma 11. *Let $G \cong F_1 \times \ldots \times F_k$ be a Cartesian product of edges and even cycles and let H be an induced subgraph of G. Then H is a convex subgraph if and only if H is a Cartesian product $F'_1 \times \ldots \times F'_m$, where each F'_i either coincides with F_i or is a convex subpath of F_i. Furthermore, H is a gated subgraph of G if and only if H is a Cartesian product $F'_1 \times \ldots \times F'_m$, where each F'_i either coincides with F_i or is a vertex or an edge of F_i.*

Proof. It is well known (see for example [44] that convex subsets (respectively, gated subsets) of Cartesian products of metric spaces are exactly the Cartesian products of convex (respectively, gated) subsets of factors. Now, the proper convex subsets of an even cycle C are exactly the convex paths, while the proper gated subsets of C are the vertices and the edges of C. □

Lemma 12. *Let $G \cong F_1 \times \ldots \times F_k$ be a Cartesian product of edges and even cycles and let G' be a connected induced subgraph of G. If for every 2-path P of G' its gated hull $\langle\langle P\rangle\rangle$ is included in G', then G' is a gated subgraph of G.*

Proof. Let H be a maximal gated subgraph of G'. By Lemma 11 H is a subproduct $F'_1 \times \ldots \times F'_k$ of G, such that for all $1 \leq i \leq k$ we either have $F'_i = F_i$ or F'_i is a vertex or an edge of F_i. Suppose by way of contradiction that $H \neq G'$. Since G' is connected, there exists an edge ab in G' such that $a \in H$ and $b \in G' \backslash H$. Without loss of generality, assume that ab is an edge arising from the factor F_1. Thus ab can be represented as $a_1b_1 \times v_2 \times \ldots \times v_k$, where a_1b_1 is an edge of F_1, $a_1 \in F'_1$ and $b_1 \notin F'_1$. Consider the subgraph $H' = (F'_1 \cup a_1b_1) \times F'_2 \times \ldots \times F'_k$. We assert that H' is a subgraph of G'. For any $i > 1$, consider the layer L'_i of H' passing via the vertex a. Let L''_i be the subgraph of G obtained by shifting L'_i along the edge ab (thus both L'_i and L''_i are isomorphic to F'_i). We assert that L''_i is also included in G'. This is trivial if L''_i is a vertex, because then $L''_i = b$. Otherwise, using that the gated hull of any 2-path of G' is included in G', L''_i is connected and L'_i is in G', one can easily conclude that L''_i is also included in G'. Propagating this argument through the graph, we obtain that H' is a subgraph of G'. However, either its factor $(F'_1 \cup a_1b_1)$ is an edge and H' is gated by Lemma 11 or it is a 2-path and thus the gated subgraph $\langle\langle F'_1 \cup a_1b_1\rangle\rangle \times F'_2 \times \ldots \times F'_k$ is contained in G. This contradicts the maximality of H and shows that $H = G'$. The proof is complete. □

3. Cells in hypercellular graphs and graphs from $\mathcal{S}_4$

Let $\mathcal{F}(Q_3^-)$ be the class of all partial cubes not containing the 3-cube minus one vertex Q_3^- as a *pc*-minor. Our subsequent goal will be to establish a cell-structure of such graphs in the following sense. We show that for $G \in \mathcal{F}(Q_3^-)$, the convex hull of any isometric cycle C of G is gated in G and furthermore isomorphic to a Cartesian product of edges and even cycles. Using these results we establish that a finite partial cube G belongs to $\mathcal{F}(Q_3^-)$ if and only if G can be obtained by gated amalgams from Cartesian products of edges and even cycles. Throughout this paper, we will call a subgraph H of a graph G a *cell*, if H is convex and isomorphic to a Cartesian product of edges and even cycles.

Some of the results of this section extend to bipartite graphs satisfying the *separation property* S_4. This is, any two disjoint convex sets A, B can be separated by complementary convex sets H', H'', i.e., $A \subseteq H', B \subseteq H''$. By [20] and Theorem 7 of [23], the S_4 separation property is equivalent to the Pasch axiom: for any triplet of vertices $u, v, w \in V$ and $x \in I(u, v), y \in I(u, w)$, we have $I(v, y) \cap I(w, x) \neq \emptyset$. The bipartite graphs with S_4 convexity have been characterized in [20] and Theorem 10 of [23]: these are the partial cubes without any pc-minor among six isometric subgraphs of Q_4 five of which where listed in [23] plus Q_4^- – the cube Q_4 minus one vertex. Note that we correct here the result in [20, 23], where Q_4^- was missing from the list. All these six forbidden graphs can be obtained from Q_3^- by an isometric expansion and thus if we denote by $\mathcal{S}_4$ the class of bipartite graphs with S_4, then hypercellular graphs are in $\mathcal{S}_4$.

The *full subdivision* H' of a graph H is the graph obtained by subdividing every edge of H once. The vertices of H in H' are called the *original* vertices of the full subdivision.

Proposition 1. *Let $G = (V, E)$ be a partial cube and $S \subseteq V$. If $\operatorname{conv}(S)$ is not gated, then either there exists $f \in \Lambda$ such that $\operatorname{conv}(\pi_f(S))$ is not gated in $\pi_f(G)$ or there is an $m \geq 2$ such that:*

(i) *G contains a full subdivision H of K_{m+1} as an isometric subgraph,*

(ii) *H contains a full subdivision H' of K_m, such that no vertex of G is adjacent to all original vertices of H'.*

Furthermore, if S is an isometric cycle of G, then $m \geq 3$.

Proof. Suppose that G contains a subset S such that $X := \text{conv}(S)$ is not gated. We can assume that G is selected in a such a way that for any element $f \in \Lambda$, the convex hull of $\pi_f(S)$ is gated in $\pi_f(G)$. Since any f-contraction of an isometric cycle C of size at least 6 is an isometric cycle $\pi_f(C)$ of $\pi_f(G)$, this assumption is also valid for proving the claim in the case that $S = C$, because 4-cycles are always gated.

Let v be a vertex of G that has no gate in X and is as close as possible to X, where $d_G(v, X) = \min\{d_G(v, z) : z \in X\}$ is the distance from v to X. Let $P_v := \{x \in X : d_G(v, x) = d_G(v, X)\}$ be the *metric projection* of v to X. Let also $Q_v := \{x \in X : I(v, x) \cap X = \{x\}\}$. Obviously, $P \subseteq Q$. Notice that $u \in V$ has a gate in X if and only if $Q_u = P_u$ and P_u consists of a single vertex. We will denote the vertices of P_v by $x_1, \ldots, x_m$. For any vertex $x_i \in P_v$, let γ_i be a shortest path from v to x_i. Let v_i be the neighbor of v in γ_i. From the choice of v we conclude that each vertex v_i has a gate in X. From the definition of P_v it follows that x_i is the gate of v_i in X. Notice that this implies that the vertices $v_1, \ldots, v_m$ are pairwise distinct. Since $x_i \in \gamma_i$ we have $x_i \in W(v_i, v)$. Furthermore, for any $y \in Q_v \setminus \{x_i\}$ we have $y \in W(v, v_i)$ since otherwise $x_i, y \in Q_{v_i}$, which contradicts that v_i has a gate in X. Denote the equivalence classes of Θ containing the edges $vv_1, \ldots, vv_m$ by $E_{g_1}, \ldots, E_{g_m}$, respectively. Then each $E_{g_1}, \ldots, E_{g_m}$ crosses X. For any edge zz' of γ_i comprised between v_i and x_i with z closer to v_i than z', we have $X \subseteq W(z', z)$ and $v \in W(z, z')$. Thus any such edge zz' belongs to an equivalence class E_f which separates X from v. Therefore such E_f crosses any shortest path between v and a vertex $y \in Q_v$. Denote the set of all such $f \in \Lambda$ by A. Notice that $d_G(v, X) = d_G(v, x_i) = |A| + 1$ for any $x_i \in P_v$.

We continue with a claim:

Claim 1. *Each equivalence class E_e crossing X coincides with one of the equivalence classes $E_{g_i}, i = 1, \ldots, m$.*

Proof. Assume that $vv_1 \notin E_e$. Denote $X' := \text{conv}(\pi_e(S))$ in $\pi_e(G)$. Let also $v' := \pi_e(v)$ and $x'_1 := \pi_e(x_1)$. By Lemma 7, X' coincides with $\pi_e(X)$. Let $x' \in X'$ be the gate of v' in X'. Since for each $f \in A$, E_f separates v and X, E_f also separates v' and x'. Thus $d_{G'}(v', X') \geq |A|$. On the other hand, since $vv_1 \notin E_e$, $d_{G'}(v', x'_1) = |A| + 1$. But x'_1 cannot be the gate in X', thus $d_{G'}(v', x') = d_{G'}(v', X') = |A|$. Let y be such that $x' = \pi_e(y)$. Since E_e crosses X and $X' = \pi_e(X)$, we have $y \in X$. Applying the expansion, the distance between two vertices can only increase by one, thus $d_G(v, y) \leq |A| + 1$. On the other hand, it holds that $d_G(v, y) \geq |A| + 1$ since $d_G(v, X) = |A| + 1$. Thus $y \in P$, say $y = x_i$, and every shortest (v, y)-path traverses an edge in E_e. Since E_e crosses X, we have $e \notin A$, whence $E_e = E_{g_i}$. □

First we prove that $P_v = Q_v$. Suppose by way of contradiction that $z \in Q_v \setminus P_v$. Then $d_G(v, z) > d_G(v, X) = |A| + 1$. For any $f \in A$, E_f separates v from z. On the other hand, since $z \in W(v, v_i)$, neither of the equivalence classes $E_{g_i}, i = 1, \ldots, m$, separates v from z. Hence there exists an equivalence class E_e with $e \notin A$ separating v from z. Since $e \notin A$, E_e does not separate v from any vertex x_i of P_v. Hence E_e crosses X, contrary to Claim 1. This shows that $P_v = Q_v$.

Now we will prove that $d_G(x_i, x_j) = 2$ for any two vertices $x_i, x_j \in P_v$, which will later yield the existence of H' as claimed. Indeed, since x_i is the gate of v_i in X and x_j is the gate of v_j in X,

$$d_G(v_i, x_j) = d_G(v_i, x_i) + d_G(x_i, x_j) \leq d_G(v_i, v_j) + d_G(v_j, x_j) = 2 + d_G(v_j, x_j).$$

Analogously, $d_G(v_j, x_i) = d_G(v_j, x_j) + d_G(x_i, x_j) \leq 2 + d_G(v_i, x_i)$. Summing up the two inequalities we deduce that $d_G(x_i, x_j) \leq 2$. Since G is bipartite, x_i and x_j cannot be adjacent, thus $d_G(x_i, x_j) = 2$.

Finally, we will show that $d_G(v, X) = 2$, yielding H as claimed. Suppose by way of contradiction that $d_G(v, X) \geq 3$. Pick any $f \in A$. Consider the graph $G' := \pi_f(G)$ and denote the convex hull of the set $S' := \pi_f(S)$ in G' by X'. By Lemma 7, $\pi_f(X) \subseteq X'$. Since any class $E_{f'}$ with $f' \in A$, separates v from X (and therefore from S) in G, any equivalence class $E_{f'}$ with $f' \in A \setminus \{f\}$ separates $v' := \pi_f(v)$ from S' in G'. Therefore, X' is contained in the intersection of the halfspaces defined by $f' \in A \setminus \{f\}$ that contain S'. This implies that $d_{G'}(v', X') \geq |A| - 1$. On the other hand, since $d_{G'}(v', \pi_f(x_i)) = d_G(v, x_i) - 1$ for any $i = 1, \ldots, m$ and $\pi_f(x_i) \in X'$, we conclude that $d_{G'}(v', X') \leq d_G(v, X) - 1 = |A| + 1 - 1 = |A|$.

From the choice of the graph G, in the graph G' the vertex v' must have a gate x' in the set X'. Since $d_{G'}(v', \pi_f(x_i)) = d_G(v, x_i) - 1$, $i = 1, \ldots, m$, the vertex x' cannot be one of the vertices of $\pi_f(P_v)$. Thus $d_{G'}(v', x') = d_{G'}(v', X') = |A| - 1 = d_G(v, X) - 2$.

Let $x'_i := \pi_f(x_i)$, $i = 1, \ldots, m$. Since x' is the gate of v' in X' and $x'_i \in X'$, we have $d_{G'}(v', x'_i) = d_{G'}(v', x') + d_{G'}(x', x'_i)$. On the other hand, since $d_{G'}(v', x'_i) = d_G(v, X) - 1$ and $d_{G'}(v', x') = d_G(v, X) - 2$, this implies that $d_{G'}(x', x'_i) = 1$ for any $i = 1, \ldots, m$. Since G' is obtained by f-contraction of a partial cube G, we conclude that G contains a vertex x such that $\pi_f(x) = x'$ and for any $i = 1, \ldots, m$ either $d_G(x, x_i) = 1$ or $d_G(x, x_i) = 2$. Since $d_G(x_i, x_j) = 2$ and G is bipartite, either x is adjacent to all x_i, $i = 1, \ldots, m$ or $d_G(x, x_i) = 2$ for all $i = 1, \ldots, m$. First assume $m \geq 2$. In the second case, the vertices $x_1, \ldots, x_m$ and x together with their common neighbors define the required full subdivision H of K_{m+1}. In the first case, we conclude that $x \in I(x_i, x_j) \subset X$ and since $d_G(v, x) = d_G(v, x_i) - 1$, we obtain a contradiction with the choice of x_i from the metric projection P_v of v on X. So, assume that $m = 1$. Since $P_v = Q_v$, this implies that X is gated, contrary to our assumption.

Finally suppose that S is an isometric cycle C of G whose convex hull X is not gated. Then the length of C is at least 6 (if C is a 4-cycle, then $X = C$ is gated). Hence there exist at least three different equivalence classes $E_{e_1}, E_{e_2}, E_{e_3}$ crossing C and X. By Claim 1, each of these classes coincides with a class $E_{g_i}, i = 1, \ldots, m$. Hence $m \geq 3$. □

Proposition 2. *Let C be an isometric cycle of $G \in \mathcal{S}_4$. Then the convex hull* $\operatorname{conv}(C)$ *of C in G is gated.*

Proof. The class $\mathcal{S}_4$ is closed by taking pc-minors [23, Theorem 10]. Therefore we can suppose that G is maximally contracted graph from $\mathcal{S}_4$ containing an isometric cycle C with $X := \operatorname{conv}(C)$ not gated. By Proposition 1, X contains 3 vertices x_1, x_2, x_3 at pairwise distance 2 and a vertex v at distance 2 from each of the vertices x_1, x_2, x_3. Let v_1, v_2, v_3 be the common neighbors of v and x_1, x_2, x_3, respectively. Let also z_i be a common neighbor of x_j and x_k for all $\{i, j, k\} = \{1, 2, 3\}$. By Proposition 1, the set $T = \{v, x_1, x_2, x_3, v_1, v_2, v_3, z_1, z_2, z_3\}$ defines four 6-cycles which are isometric cycles of G. The convex hull in G of each of these 6-cycles is a subgraph of a 3-cube. On the other hand, T is contained in each of the three intervals $I(v_i, z_i)$. Since $d_G(v_i, z_i) = 4$, the convex hull of T is a subgraph of a 4-cube. The convex hull of the 6-cycle $C_1 = (x_1, z_3, x_2, z_1, x_3, z_2)$ cannot be a 3-cube. We conclude that one of the 2-paths $(x_1, z_3, x_2), (x_2, z_1, x_3), (x_3, z_2, x_1)$, say (x_1, z_3, x_2), is a convex path of G. Consider the convex sets $I(x_1, x_2)$ and $I(v, x_3)$. They are disjoint, otherwise z_3 must be adjacent to v and x_3, which is impossible. Let H, H' be two complementary halfspaces separating $I(x_1, x_2)$ and $I(v, x_3)$, say $I(x_1, x_2) \subset H$ and $I(v, x_3) \subset H'$. Then necessarily $z_1, z_2 \in H'$, otherwise, if say $z_1 \in H$, then $x_3 \in C_1 \subset I(x_1, z_1) \subset H$, a contradiction. But then $x_1 \in I(z_2, v) \subset H'$ and $x_2 \in I(z_1, v) \subset H'$, which is impossible. This final contradiction shows that $I(x_1, x_2)$ and $I(v, x_3)$ cannot be separated, i.e., $G \notin \mathcal{S}_4$. Thus, the convex hull of any isometric cycle C of a partial cube G from $\mathcal{S}_4$ is gated. □

Analogously to [5], we will compare the Djoković-Winkler relation Θ to the following relation Ψ^*. First say that two edges xy and $x'y'$ of a bipartite graph G are in relation Ψ if they are either equal or are opposite edges of some convex cycle C of G. Then let Ψ^* be the transitive closure of Ψ. Let $\mathcal{C}(G)$ denote the set of all convex cycles of G and let $\mathbf{C}(G)$ be the 2-dimensional cell complex whose 2-cells are obtained by replacing each convex cycle C of length $2j$ of G by a regular Euclidean polygon $[C]$ with $2j$ sides.

Recall that a cell complex $\mathbf{X}$ is *simply connected* if it is connected and if every continuous map of the 1-dimensional sphere S^1 into $\mathbf{X}$ can be extended to a continuous mapping of the disk D^2 with boundary S^1 into $\mathbf{X}$. Note that a connected complex $\mathbf{X}$ is simply connected iff every continuous map from S^1 to the 1-skeleton of $\mathbf{X}$ is null-homotopic.

Lemma 13. *If G is a partial cube, then the relations Θ and Ψ^* coincide. In particular, $\mathbf{C}(G)$ is simply connected.*

Proof. The proof of the first assertion is the content of Proposition 5.1 of [36] (it also follows by adapting the proof of Lemma 1 of [5]). To prove that $\mathbf{C}(G)$ is simply connected it suffices to show that any cycle C of G is contractible in $\mathbf{C}(G)$. Let $k(C)$ denote the number of equivalence classes of Θ crossing C. By induction on $k(C)$, we will prove that any cycle C of G is contractible to any of its vertices

$w \in C$. Let E_f be an equivalence class of Θ crossing C and let uv and $u'v'$ be two edges of C from Θ. By the first assertion, there exists a collection $C_1, C_2, \ldots, C_m$ of convex cycles and a collection of edges $e_0 = uv, e_1, \ldots, e_{m-1}, e_m = u'v' \in E_f$ such that $e_i \in C_i \cap C_{i+1}$ for any $i = 1, \ldots, m-1$. Suppose that $u, u' \in H_f^+$ and $v, v' \in H_f^-$. Let $P_i^+ := C_i \cap H_f^+, P_i^- := C_i \cap H_f^-$ for $i = 1, \ldots, m$. Let P' be the path between u and u' which is the union of the paths $P_1^+, \ldots, P_m^+$. Analogously, let P'' be the path between v and v' which is the union of the paths $P_1^-, \ldots, P_m^-$. Finally, let $P^+ := C \cap H_f^+$ and $P^- := C \cap H_f^-$, and suppose without loss of generality that the vertex w belongs to the path P^+. Let C' be the cycle which is the union of the paths P' and P^+ and let C'' be the cycle which is the union of the paths P^- and P''. Since G is a partial cube, any equivalence class of Θ crossing P' or P'' also crosses the paths P^+ and P^-. On the other hand, E_f does not cross the cycles C' and C''. This implies that $k(C') < k(C)$ and $k(C'') < k(C)$. By induction assumption, C'' can be contracted in $\mathbf{C}(G)$ to any of its vertices, in particular to the vertex v. On the other hand, the union $\bigcup_{i=1}[C_i]$ can be contracted to the path P' in a such a way that each edge e_i is contracted to its end from P''. In particular, v is mapped to u. Finally, by induction assumption, C' can be contracted to the vertex w. Composing the three contractions (C'' to v, $\bigcup_{i=1}[C_i]$ to P', and C' to w), we obtain a contraction of C to w. □

Let C be an even cycle of length $2n$. Let G_1 be a subgraph of C isomorphic to a path of length ℓ at least 2 and at most n. Let $\mathrm{Ex}_\ell(C)$ be an expansion of C with respect to G_1 and $G_2 = C$. We will call the graphs $\mathrm{Ex}_\ell(C)$ *half-expanded cycles.*

Proposition 3. *Let G' be a Cartesian product of edges and even cycles and let G be an isometric expansion with respect to the subgraphs G'_1 and G'_2 of G', such that G contains no convex subgraph isomorphic to a half-expanded cycle. Then either G is a Cartesian product of edges and even cycles or one of G'_1, G'_2 coincides with G' while the other is isomorphic to a subproduct of edges and cycles.*

Proof. Let $G' = F_1 \times F_2 \times \ldots \times F_m$, where each $F_i, i = 1, \ldots, m$ is either a K_2 or an even cycle C. Then G' is a partial cube from $\mathcal{F}(Q_3^-)$. The graph G is obtained from G' by an isometric expansion with respect to G'_1 and G'_2, i.e., G'_1 and G'_2 are two isometric subgraphs of G' such that $G' = G'_1 \cup G'_2$, $G'_0 := G'_1 \cap G'_2 \neq \emptyset$, there is no edge between $G'_1 \setminus G'_2$ and $G'_2 \setminus G'_1$, and G is obtained from G' by expansion along G'_0. If $G'_1 = G'_2 = G'$, then the expansion of G' with respect to G'_1 and G'_2 is the product $G' \times K_2$ and we are done. Thus we can assume that G'_0 is a proper subgraph of G'.

Claim 2. *Let L be a layer of G', i.e., $L = \{v_1\} \times \ldots \times \{v_{i-1}\} \times F_i \times \{v_{i+1}\} \times \ldots \times \{v_m\}$ with $v_j \in F_j$ for all $j \neq i$. If F_i is a cycle and $G'_1 \cap L$ or $G'_2 \cap L$ is different from L and contains a path of length at least 2, then G is a Cartesian product of edges and cycles. More precisely, $G = F_1 \times \ldots \times F_{i-1} \times F'_i \times F_{i+1} \times \ldots \times F_m$, where F'_i is an isometric expansion of F_i along two opposite vertices of F_i.*

Proof. Since we can reorder the factors, suppose without loss of generality that $i = 1$ and denote $G'' = F_2 \times \ldots \times F_m$. We have $L = F_1 \times \{v\} = C \times \{v\}$ with $v \in V(G'')$, such that $G'_1 \cap L$ includes a path P_1 of length at least 2 but differs from L. Since L is a convex $2j$-cycle of G', P_1 is a shortest path of G'. If L is included in G'_2, then the expansion of L along P_1 is isomorphic to a half-extended cycle and is a convex subgraph of G by Lemma 6, which is impossible. Thus L is not included in G'_2, yielding that $L \cap G'_2$ is a shortest path P_2 of G'. Since P_1 and P_2 cover the cycle L, the only possibility is that P_1 and P_2 intersect in two antipodal vertices of the cycle L. Thus the image of L in G is a cycle of length $2j + 2$. Consider any layer $L' = C \times \{v'\}$ of G' adjacent to L, i.e., $v'v \in E(G'')$. Then $L \cup L'$ is a convex subgraph of G', thus by Lemma 6 the expansion of $L \cup L'$ is a convex subgraph H of G. If L' is contained in G'_1, then the intersection of H with the half-space of G corresponding to G'_1 is a convex subgraph isomorphic to a half-extended cycle. Thus L' cannot be entirely in G'_1, and for the same reason it cannot be entirely in G'_2. Again the only possibility is that $G'_1 \cap L'$ and $G'_2 \cap L'$ are shortest paths of L' that intersect in two opposite vertices of L'.

Let $v_1, v_2 \in L \cap G'_0$ and $u_1, u_2 \in L' \cap G'_0$. We assert that after a possible relabeling, v_1 is adjacent to u_1 and v_2 is adjacent to u_2. Suppose that this is not true. Then the neighbors v'_1 and v'_2 of v_1 and v_2, respectively, in L' are both different from u_1 and u_2. Analogously, the neighbors u'_1 and u'_2 of u_1 and u_2, respectively, in L are both different from v_1 and v_2. We can assume without loss of generality that v_1' and u_1' are not in G_1', otherwise we can exchange v_1 and v_2 or u_1 and u_2. We assert that G_1' is not an isometric subgraph of G'. Indeed, the distance in G' between u_1 and v_1 is at most j and the interval $I(v_1, u_1)$ is contained in the union $Q \cup Q'$, where Q is the subpath between v_1 and u_1' of the path between v_1 and v_2 passing via u_1' and Q' is the subpath between v_1' and u_1 of the path between v_1' and v_2' passing via u_1. Since all vertices of Q except v_1 belong only to G_2' and v_1' does not belong to G_1', we conclude that any shortest path in G' between v_1 and u_1 contains at least one vertex from $G_2' \setminus G_1'$, showing that G_1' is not an isometric subgraph of G'. Hence v_1 is adjacent to u_1 and v_2 is adjacent to u_2. Notice that then the both layers have the same side of the cycles in G_1' and G_2' since there is no edge between $G_1' \setminus G_2'$ and $G_2' \setminus G_1'$. Propagating this argument through the graph, we conclude that all layers L'' parallel to L have the same vertices in G_1' and G_2'. Hence the traces on $F_i = C$ with respect to G_1' and G_2' of L and L'' coincide: they are two paths P_1 and P_2 of C covering the cycle and intersecting in two opposite vertices x', x'' of C. Therefore, the graph G_0' with respect to which we perform the isometric expansion is the subgraph of G' induced by $(\{x'\} \times V(G'')) \cup (\{x''\} \times V(G''))$, G_1' is the subgraph induced by $V(P_1) \times V(G'')$, and G_2' is the subgraph induced by $V(P_2) \times V(G'')$. Consequently, the expansion of G' with respect to G_1' and G_2' produces a graph G isomorphic to $C' \times G''$, where the length of the cycle C' is two more than the length of C. This establishes the claim. □

By Claim 2, we can further assume that every layer L of G' coming from a cyclic factor satisfies one of the following two conditions: either both G_1' and G_2' include L, or one of G_1', G_2' includes an edge, a vertex, or nothing while the other includes the whole layer L. Consequently, for each cyclic factor $F_i \cong C$ of G' and each layer $L := \{v_1\} \times \ldots \times \{v_{i-1}\} \times C \times \{v_{i+1}\} \times \ldots \times \{v_m\}$, the intersection $L \cap G_0'$ is the whole layer L, an edge, a vertex, or empty.

We will now analyze the structure of the subgraph G_0' of G' along which we perform the isometric expansion. Suppose that G' is obtained from G by contracting the equivalence class E_f.

Claim 3. *If $R' = (u_1, v_1, v_2, u_2)$ is a 4-cycle in G' such that the edges u_1v_1 and v_1v_2 do not lie in the same layer and $u_1, v_1, v_2 \in V(G_0')$, then u_2 also belongs to $V(G_0')$.*

Proof. If this is not the case, then assume without loss of generality that $u_2 \in G_1' \setminus G_2'$. Since the 4-cycle R' is a convex subgraph of G', by Lemma 6 the expansion of R' along G_0' is a convex subgraph of G isomorphic to Q_3^-, thus is a half-extended cycle, a contradiction. This contradiction shows that $u_2 \in V(G_0')$. □

We continue with an auxiliary assertion:

Claim 4. *Any convex cycle Z of G crossed by E_f is a 4-cycle.*

Proof. Assume by way of contradiction that Z has length $\ell(Z) \geq 6$. Therefore Z is contracted to a convex cycle Z' of length $\ell(Z) - 2$ of G'. The convex sets in a Cartesian product are products of convex sets of the factors. Thus either Z' is a layer of G' or Z' is a 4-cycle which is a product of two edges from two different factors. In the first case Z' is a layer which has two antipodal vertices in G_0', one path between these vertices in $G_1' \setminus G_2'$, and the other path in $G_2' \setminus G_1'$, and this case was covered by Claim 2. Thus assume that Z' is a 4-cycle (v_1, v_2, u_2, u_1) that has edges v_1v_2, u_1u_2 projected to factor F_1 and edges v_1u_1, v_2u_2 projected to factor F_2. Moreover, let $v_1, u_2 \in V(G_1') \cap V(G_2')$, $v_2 \in V(G_1') \setminus V(G_2')$, and $u_1 \in V(G_2') \setminus V(G_1')$. If both factors F_1, F_2 are isomorphic to K_2, then they can be treated as a single cyclic factor because $K_2 \times K_2$ is a 4-cycle and Z' is a layer. Then the result follows from Claim 2. Thus assume that F_1 is a cycle of length at least 6 – otherwise we are in the above case. Let $L_1 = (v_1, v_2, \ldots, v_{2i}, v_1)$ and $L_2 = (u_1, u_2, \ldots, u_{2i}, u_1)$ be the two layers of F_1 that include Z'. They include vertices u_1, v_2 which are not in G_0'. Since $i \geq 3$ by isometry of G_1' and G_2', we have $v_3 \in G_1' \setminus G_2'$ and $u_3 \in G_2' \setminus G_1'$. But v_3 and u_3 are adjacent, which is impossible. This establishes that any convex cycle Z of G crossed by E_f has length 4. □

Claim 5. *G_0' is a subgraph of G' of the form $H_1 \times H_2 \times \ldots \times H_m$, where each factor satisfies $H_i \subseteq F_i$ and is either a vertex, an edge, or the entire F_i. In particular, G_0' is convex in G'.*

Proof. First we prove that G_0' is connected. Let a_1a_2 and b_1b_2 be any two edges in the equivalence class E_f. Edges a_1a_2 and b_1b_2 get contracted to vertices a', b' of G'. By Lemma 13, a_1a_2 and b_1b_2 can be connected by a sequence $\mathcal{C} = C_1, C_2, \ldots, C_k$ of convex cycles of G such that $a_1a_2 \in C_1, b_1b_2 \in C_k$, and any two consecutive cycles C_i and C_{i+1} intersect in an edge of E_f. Hence the cycles of $\mathcal{C}$ are contracted in G' to a path Q' between a' and b'. Since all cycles C_i of $\mathcal{C}$ are crossed by E_f, by Claim 4 each $C_i, i = 1, \ldots, k$, is a 4-cycle. Thus, additionally to a', b' also all other vertices of the path Q' belong to G'_0. Consequently, a' and b' belong to a common connected component of G'_0. Since a_1a_2 and b_1b_2 are arbitrary edges from E_f, the graph G'_0 is connected.

To prove the second assertion, let I be a maximal subgraph of G'_0 of the form $I_1 \times I_2 \times \ldots \times I_m$, where each I_i is a connected nonempty subgraph of F_i. We claim that I coincides with G'_0. If not, since I and G'_0 are connected, there exists an edge vw of G'_0 such that $v \in V(I)$ and $w \in V(G'_0) \setminus V(I)$. Let $L := \{v_1\} \times \ldots \times \{v_{i-1}\} \times F_i \times \{v_{i+1}\} \times \ldots \times \{v_m\}$ be the layer of G' that includes the edge vw. Suppose that the ith coordinates of v and w are the adjacent vertices v'_i and v''_i of F_i, respectively. Set $I' := I_1 \times \ldots \times I_{i-1} \times \{v'_i\} \times I_{i+1} \times \ldots \times I_m$ and $I'' := I_1 \times \ldots \times I_{i-1} \times \{v''_i\} \times I_{i+1} \times \ldots \times I_m$. Then $v \in V(I') \subseteq V(I)$ and $w \in V(I'')$. Since $w \notin V(I)$, by the definition of I, the subgraph I'' contains a vertex not belonging to G'_0. Let x be a closest to w vertex of $V(I'') \setminus V(G'_0)$. Let y be a neighbor of x in $I(x, w)$. Since I'' is convex, $y \in I(x, w) \subset V(I'')$. By the choice of x, we deduce that y is a vertex of G'_0. Let x' and y' be the neighbors of respectively x and y in I' (such vertices exist by the definitions of I' and I'' and the fact that v'_i and v''_i are adjacent in F_i). Since $V(I') \subset V(I)$, the vertices x' and y' belong to G'_0. Since the 4-cycle (x, y, y', x') does not belong to a single layer and y, y', x' are vertices of G'_0, by Claim 3 also x is a vertex of G'_0, a contradiction with its choice. This establishes that I coincides with G'_0.

Finally, we assert that each $I_i, i = 1, \ldots, m$, is a vertex, an edge, or the whole factor F_i. The assertion obviously holds if F_i is an edge. Now, let $F_i = C$ be an even cycle. Since $I_i \neq \emptyset$ and $G'_0 = I_1 \times I_2 \times \ldots \times I_m$, the assertion follows from the conclusion after Claim 2, that the intersection of G'_0 with any layer is the whole layer, an edge, a vertex, or empty. □

Let H' be the subgraph of G' induced by all vertices of G' not belonging to G'_0.

Claim 6. *H' is either empty or is a connected subgraph of G'.*

Proof. By Claim 5, G'_0 is a connected subgraph of G' of the form $H_1 \times H_2 \times \ldots \times H_m$, where each H_i is a vertex, an edge of F_i, or the whole factor F_i. Suppose that G'_0 is a proper subgraph of G'. By renumbering the factors in the product $F_1 \times F_2 \times \ldots \times F_m$ we can suppose that there exists an index $m' \leq m$, such that for each $i \leq m'$, H_i is a proper subgraph of F_i and that for each $m' < i \leq m$, we have $H_i = F_i$. For each $i \leq m'$, let F'_i be the (nonempty) connected subgraph of F_i induced by $V(F_i) \setminus V(H_i)$. For any $i \leq m$, let H'_i be the subgraph $F_1 \times \ldots \times F_{i-1} \times F'_i \times F_{i+1} \times \ldots \times F_m$ of $G' = F_1 \times \ldots \times F_m$. Obviously, each such H'_i is a connected subgraph of H' (and of G'). Moreover, $V(H') = \bigcup_{i=1}^{m} V(H'_i)$ and any two H'_i and H'_j with $i, j \leq m$ share a vertex. This shows that H' is a connected subgraph of G'. □

Now, we are ready to conclude the proof of the proposition. If both G'_1 and G'_2 are proper subgraphs of G', then G'_0 is also a proper subgraph of G'. By Claim 6, the subgraph H' of G' induced by all vertices not in G'_0 is connected. This implies that G' contains edges running between the vertices of $G'_1 \setminus G'_2$ and $G'_2 \setminus G'_1$, which is impossible. Consequently, we can suppose that G'_1 coincides with G' and G'_2 coincides with G'_0. By Claim 5, $G'_2 = G'_0$ has the form $H_1 \times H_2 \times \ldots \times H_m$, where each H_i is a vertex or an edge of F_i, or the whole factor F_i. □

Since each half-extended cycle can be contracted to a Q_3^-, we immediately have the following lemma.

Lemma 14. *If $G \in \mathcal{F}(Q_3^-)$, then G has no convex subgraph isomorphic to a half-extended cycle.*

Now we are ready to prove the first part of Theorem A.

Theorem 1. *The convex closure of any isometric cycle of a graph G in $\mathcal{F}(Q_3^-)$ is a gated subgraph isomorphic to a Cartesian product of edges and even cycles.*

Proof. Let G be a minimal graph in $\mathcal{F}(Q_3^-)$ for which we have to prove that the convex closure of an isometric cycle C of G is a product of cycles and edges. Since G is minimal and convex subgraphs of graphs in $\mathcal{F}(Q_3^-)$ are also in $\mathcal{F}(Q_3^-)$, we conclude that G coincides with the convex closure of C. If C

is a 4-cycle, then C is a convex subgraph of G and we are done. Analogously, if C is a 6-cycle, then since $G \in \mathcal{Q}_3^-$, either C is convex or the convex hull of C is the 3-cube Q_3. So, assume that the length of C is at least 8. By minimality of G, any equivalence class of G crosses C. Any contraction of G is a graph G' in $\mathcal{F}(Q_3^-)$ and it maps C to an isometric cycle C' of G'. By Lemma 7, G' is the convex hull of C', thus by minimality choice of G, G' is a Cartesian product of cycles and edges, say G' is isomorphic to $F_1 \times F_2 \times \ldots \times F_m$, where each $F_i, i = 1, \ldots, m$, is either a K_2 or an even cycle C. The graph G is obtained from G' by an isometric expansion, i.e., there exist isometric subgraphs G'_1 and G'_2 of G' such that $G' = G'_1 \cup G'_2$, $G'_0 := G'_1 \cap G'_2 \neq \emptyset$, there is no edge between $G'_1 \setminus G'_2$ and $G'_2 \setminus G'_1$, and G is obtained from G' by expansion along G'_0.

By Proposition 3 and Lemma 14, either G is a Cartesian product of edges and even cycles or G'_1 coincides with G' and $G'_2 = G'_0$ is a proper convex subgraph of G' of the form $H_1 \times H_2 \times \ldots \times H_m$, where each H_i is a vertex, an edge of F_i, or the whole factor F_i. In the first case we are done, so suppose that the second case holds. Let G_j be the image of G'_j after the expansion, for $j = 0, 1, 2$. Since $G'_2 = G'_0$ is convex, G_0 is a convex subgraph of G isomorphic to $G'_0 \times K_2$. If G' is the f-contraction of G, then G_1 and G_2 are the subgraphs induced by the halfspaces H_f^+ and H_f^- of G. Let a_1a_2 and b_1b_2 be two opposite edges of C belonging to E_f. Since G is the convex hull of C, the cycle C intersects every equivalence class of the relation Θ in G. In particular, this implies that contracting E_f, the edges a_1a_2 and b_1b_2 are contracted to vertices a' and b' of G'_0. Since G'_0 is a convex subgraph of G', the image of C under this contraction is an isometric cycle C' of G'. Since $a', b' \in C'$, C' is contained in G'_0. Since by Lemma 7 G' is the convex hull of C', we conclude that $G'_1 = G'_0 = G'_2$, contrary to the assumption that G'_0 is a proper subgraph of G'. □

Together with Proposition 2, the following gives the second part of Theorem A.

Proposition 4. *The antipodal subgraphs of graphs from $\mathcal{S}_4$ are gated and are products of edges and cycles.*

Proof. Let G be a antipodal graph not in $\mathcal{F}(Q_3^-)$. Then G can be contracted to a graph G' that contains a convex subgraph X isomorphic to Q_3^-. By Lemma 8 any contraction of an antipodal graph is an antipodal graph, thus we can assume that G' is maximally contracted, i.e. every contraction of G' is in $\mathcal{F}(Q_3^-)$. Denote the central vertex of X with x, the isometric cycle around it with $(v_0, v_1, \ldots, v_5)$, and assume that x is adjacent to exactly v_0, v_2 and v_4. Let $x', v'_0, \ldots, v'_5 \in V(X')$ be the antipodes of vertices in X.

First assume that G' has exactly three Θ-classes, namely $E_{xv_0}, E_{xv_2}, E_{xv_4}$. Then either $G' = X$ or $G' \cong Q_3$. In the first case G' is not antipodal, while in the second case X is not a convex subgraph of G'. Thus assume that there exists another Θ-class, say E_{wz}. Contracting this class we obtain a graph G'' that has no Q_3^- convex subgraphs, thus the convex closure X' of X in G'' must be isomorphic to Q_3. Let y be a vertex in G' that gets mapped to the vertex in $X' \setminus X$ in G''. Vertex y and v_1 are adjacent in G'' with edge in E_{xv_4}, but since X is convex in G' any path from v_1 to y in G' must be of length 2 and first cross an edge in E_{wz} and then an edge in E_{xv_4}. Thus there is only one such path, say P, and it is a convex subgraph. On the other hand, the path (v_3, v_4, v_5) is convex in X, thus it is convex in G'. But then the paths (v_3, v_4, v_5) and P are convex subgraphs that cannot be separated by two complementary halfspaces. The latter holds since there is a path between them consisting of edges in $E_{xv_4}, E_{xv_0}, E_{xv_2}$, but each of these Θ-classes intersects either one convex set or another. Thus G' is not in $\mathcal{S}_4$. By [23, Theorem 10], contracting a graph in $\mathcal{S}_4$ gives a graph in $\mathcal{S}_4$. Thus also G is not in $\mathcal{S}_4$. □

The example in Figure 6 shows that the second condition of Theorem A does not characterize bipartite graphs with S_4 convexity.

4. Gated amalgamation in hypercellular graphs

This section is devoted to the proof of Theorem B. First, we present the 3CC-condition for partial cubes G in a seemingly stronger but equivalent form:

3-convex cycles condition (*3CC-condition*): for three convex cycles C_1, C_2, C_3 of G such that any two cycles C_i, C_j, $1 \leq i < j \leq 3$, intersect in an edge e_{ij} with $e_{12} \neq e_{23} \neq e_{31}$ and the three cycles intersect in a vertex, the convex hull of $C_1 \cup C_2 \cup C_3$ is a cell of G isomorphic to $C_i \times K_2$ and C_j, C_k are 4-cycles.

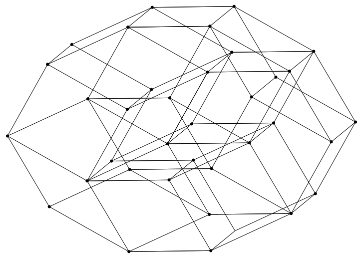

FIGURE 6. An expansion of Q_4^- that is apiculate and lopsided. In particular, the graph is not in S_4, the convex hull of any isometric cyle is gated, and its antipodal subgraphs are cubes and thus products of edges and cycles.

Any cell X' which is contained in a cell X of a partial cube G is called a *face* of X. By Lemma 11 equivalently, the faces of X are the gated subgraphs of G included in X. We denote by $X(G)$ the set of all cells of G and call $X(G)$ the *combinatorial complex* of G. The *dimension* $\dim(X)$ of a cell X of G is the number of edge-factors plus two times the number of cyclic factors. Let us now recall the stronger 3C-condition for partial cubes G:

3-cell condition (*3C-condition*): for all $k \geq 0$ and three $(k+2)$-dimensional cells X_1, X_2, X_3 of G such that each of the pairwise intersections X_{12}, X_{23}, X_{13} is a cell of dimension $k+1$ and the intersection X_{123} of all three cells is a cell of dimension k, the convex hull of $X_1 \cup X_2 \cup X_3$ is a $(k+3)$-dimensional cell.

The proof of Theorem B is organized in the following way. We start by showing that any hypercellular graph satisfies the 3CC-condition. Together with Theorem A, this shows that (i)⇒(ii). We then obtain (ii)⇒(iii), while (iii)⇒(ii) holds trivially. To prove (ii)⇒(i), we show that the class of partial cubes satisfying (ii) is closed by taking minors. Since Q_3^- does not satisfies the 3CC-condition, we conclude that all such graphs are hypercellular. The last and longest part of the section is devoted to the proof of the equivalence (i)⇔(iv).

Since by Theorem A, hypercellular graphs have gated cells, the following lemma completes the proof of (i)⇒(ii).

Lemma 15. *Any hypercellular graph G satisfies the 3CC-condition.*

Proof. Let C_1, C_2, C_3 be three convex cycles of a partial cube $G \in \mathcal{F}(Q_3^-)$ such that any two cycles C_i, C_j, $1 \leq i < j \leq 3$, intersect in an edge e_{ij} and the three cycles intersect in a vertex v. We proceed by induction on the number of vertices of G. By induction assumption we can suppose that G is the convex hull of the union $C_1 \cup C_2 \cup C_3$. If each of the cycles C_1, C_2, C_3 is a 4-cycle, then their union is an isometric subgraph H of G isomorphic to Q_3^-. Since $G \in \mathcal{F}(Q_3^-)$, H is not convex. Therefore the convex hull of $H = C_1 \cup C_2 \cup C_3$ is the 3-cube Q_3, and we are done. Thus suppose that one of the cycles, say C_1, has length ≥ 6. Let the edge e_{12} be of the form v_1v. Let u be the neighbor of v_1 in C_1 different from v. Let E_f be the equivalence class of Θ defined by the edge uv_1. We claim that E_f does not cross C_2 and C_3, or, equivalently, that $C_2 \cup C_3 \subset W(v_1, u)$. Since $G \in \mathcal{F}(Q_3^-)$, by Proposition 2, each of the cycles C_1, C_2, C_3 is a gated subgraph of G. Since u is adjacent to $v_1 \in C_2$, the vertex v_1 is the gate of u in C_2, whence $C_2 \subset W(v_1, u)$. Analogously, since C_1 is gated and $v \in C_1 \cap C_3$, the gate of u in C_3 must belong to $I(u, v) \subset C_1$. Since the length of C_1 is at least 6, this gate cannot be adjacent to u and v, thus v is the gate of u in C_3. Since $v_1 \in I(u, v)$, again we conclude that $C_3 \subset W(v_1, u)$. Hence E_f does not cross the cycles C_2 and C_3. Let $G' := \pi_f(G)$ and $C_i' := \pi_f(C_i)$, for $i = 1, 2, 3$. Since each $C_i, i = 1, 2, 3$ is gated, by Lemma 10 each $C_i', i = 1, 2, 3$ is a gated subgraph of G' and by Lemma 7 G' is the convex hull of $C_1' \cup C_2' \cup C_3'$. Notice that the three cycles C_1', C_2', C_3' pairwise intersect in the same edges as the cycles C_1, C_2, C_3 and all three in the vertex v.

Since $G' \in \mathcal{F}(Q_3^-)$, by induction assumption G' is isomorphic to the Cartesian product $C \times K_2$, where C is isomorphic to one of C_1', C_2', C_3'. The graph G is obtained from the graph G' by an isometric expansion with respect to the subgraphs G_1' and G_2' of G'. By Proposition 3 and Lemma 14,

(i) G is a Cartesian product of edges and even cycles or

(ii) G_1' coincides with G' and G_2' is isomorphic to a subproduct of edges and cycles.

The only convex cycles of length at least 6 in a product of edges and cycles are layers. Therefore in the case (i) C_1 must be a layer L in the product. Each of cycles C_2 and C_3 shares an edge with C_1. The only such cycles are 4-cycles between layer L and any other layer adjacent to L. Since also C_2 and C_3 share an edge, they must both be between L and some layer L'. Then L and L' form a cell isomorphic to $L \times K_2 \cong C_1 \times K_2$ that is the convex hull of $C_1 \cup C_2 \cup C_3$.

Finally, assume (ii) holds. Since $G' \cong C \times K_2$, G_2' is either a vertex, an edge, a 4-cycle, a layer isomorphic to C, or the whole G'. Thus, G_2' intersects no cyclic layer in just two antipodal vertices, i.e., no convex cycle of G' gets extended. A contradiction, since C_1' should be extended. □

We will now establish the implication (ii)⇒(iii), while (iii)⇒(ii) trivially holds.

Proposition 5. *If G is a partial cube in which cells are gated and which satisfies the 3CC-condition, then G satisfies the 3C-condition.*

Proof. Since the properties of G are closed under restriction, without loss of generality we consider $G = \mathrm{conv}(X_1 \cup X_2 \cup X_3)$. Since cells are gated in G, by Lemma 11 X_{123} is a subproduct of X_{12}, X_{23}, X_{13} and X_{ij} is a subproduct of X_i and X_j for all $i, j \in \{1, 2, 3\}$, where in all cases the factors of the subproducts are vertices, edges, or factors of the superproducts. Indeed by the conditions on the dimensions, the subproducts all have the same factors than their superproducts except that either one edge-factor from the superproduct is a vertex in the subproduct or one cyclic factor from the superproduct is an edge in the subproduct. This gives that any $v \in X_{123}$ has exactly one neighbor $v_{ij} \in X_{ij} \setminus X_{123}$ for all $i, j \in \{1, 2, 3\}$. Now, since the cells X_1, X_2, X_3 are products, a path of the form $(v_{ij}, v, v_{ik}) \subset X_i$ is contained in the unique convex cycle C_i of X_i accounting for the two supplementary dimensions of X_i compared to X_{123}, for all $\{i, j, k\} = \{1, 2, 3\}$. Since G satisfies the 3CC-condition, $\mathrm{conv}(C_1 \cup C_2 \cup C_3)$ is a cell X_v of G isomorphic to $C_i \times K_2$ and $C_j \cong C_k$ are 4-cycles, for some $\{i, j, k\} = \{1, 2, 3\}$. Moreover, by the product structure of X_i the Θ-classes of C_i and their order on C_i do not depend on the choice of $v \in X_{123}$, for all $i \in \{1, 2, 3\}$. Therefore, for all $v, w \in X_{123}$ and some $i \in \{1, 2, 3\}$ we have $X_v \cong X_w \cong C_i \times K_2$, where corresponding edges are in the same Θ-class of G. Since they are separated by Θ-classes crossing X_{123}, for different $v, w \in X_{123}$, the cells X_v and X_w are disjoint and by construction the union of all of them covers $X_1 \cup X_2 \cup X_3$. We obtain that $X_1 \cup X_2 \cup X_3 \subsetneq X_{123} \times C_i \times K_2$, which is a $(k+3)$-dimensional cell of G, thus gated and thus convex. Since $G = \mathrm{conv}(X_1 \cup X_2 \cup X_3)$, we get $\mathrm{conv}(X_1 \cup X_2 \cup X_3) \cong X_{123} \times C_i \times K_2$, which establishes the claim. □

To show (ii)⇒(i), in Proposition 6 we prove that the class of partial cubes satisfying (ii) is minor-closed. Since Q_3^- does not satisfy the 3CC-condition, the graphs satisfying (ii) cannot be contracted to Q_3^-, thus they are hypercellular.

Proposition 6. *The family of partial cubes having gated cells and satisfying the 3CC-condition is a pc-minor-closed family.*

Proof. If a condition of the proposition is violated for a convex subgraph of a partial cube G, then it is also violated for G. Therefore the family in question is closed under restrictions.

Let now G be a partial cube satisfying the conditions of the proposition and let G' be a contraction of G along some equivalence class E_f. Pick a cell X' in G'. By Lemma 6, the expansion X of X' is a convex subgraph of G, thus X also satisfies the conditions of the proposition. By the 3CC-condition, X has no convex subgraph isomorphic to a half-expanded cycle. Thus Proposition 3 provides us with the structure of X; in particular, X includes a cell Y such that $\pi_f(Y) = X'$. Since the cells of G are gated, by Lemma 10, X' is gated. Therefore the cells of G' are gated.

Now, let C_1', C_2', C_3' be three convex cycles of G' such that any two cycles C_i', C_j', $1 \le i < j \le 3$, intersect in an edge e_{ij}' and the three cycles intersect in a vertex v'. Let G_1' and G_2' be the subgraphs of G' with respect to which we perform the expansion of G' into G. By Proposition 3 (or directly using the fact that C'_1, C'_2, C'_3 are convex cycles), for each C'_i, $i \in \{1,2,3\}$, we have one of the following three options: (a) either both $G'_1 \cap C'_i$ and $G'_2 \cap C'_i$ coincide with C'_i, or (b) one of $G'_1 \cap C'_i$ and $G'_2 \cap C'_i$ is the whole cycle C'_i and other is an edge, a vertex, or empty, or (c) both $G'_1 \cap C'_i$ and $G'_2 \cap C'_i$ are paths corresponding to halves of C'_i with intersection in two antipodal vertices of C'_i. Using this trichotomy, we divide the analysis in the following cases.

Case 1. For G'_1 or G'_2, say for G'_1, and for at least two of the three cycles C'_1, C'_2, C'_3, say for C'_1, C'_2, we have $G'_1 \cap C'_i = C'_i$ and $G'_1 \cap C'_j = C'_j$.

Then the edges e'_{13} and e'_{23} are in G'_1. Thus either $G'_1 \cap C'_3 = C'_3$ or $G'_1 \cap C'_3$ is a half of C'_3 that includes e'_{13} and e'_{23}. Then in the expansion we have 3 convex cycles C_1, C_2, C_3 pairwise sharing an edge and a vertex in the intersection of all three, such that $\pi_f(C_i) = C'_i$ for $i \in \{1,2,3\}$. Since the 3CC-condition holds in G, two of the cycles C_1, C_2, C_3 are 4-cycles and all three are included in a cell X isomorphic to $C_k \times K_2$ where C_k is the third cycle. Since a contraction can only shorten the cycles, at least two of the cycles C'_1, C'_2, C'_3 are 4-cycles and the convex hull of all three must be included in a cell $\pi_f(X)$. The only contraction of $C_k \times K_2$ that has at least three convex cycles is isomorphic to $C'_k \times K_2$.

Case 2. For G'_1 or G'_2, say for G'_1, among C'_1, C'_2, C'_3 there exists a unique cycle, say C'_1, such that $G'_1 \cap C'_1 = C'_1$.

By symmetry and in view of Case 1, for two other cycles C'_2 and C'_3 we have only one of the following options: (1) either $G'_2 \cap C'_j = C'_j$ for exactly one $j \in \{2,3\}$ or (2) for all $j \in \{2,3\}$ and $k \in \{1,2\}$ we have that $G'_k \cap C'_j$ is a half of the cycle C'_j.

First consider the option (2). By properties of C'_1, in the half-space G_1 of G corresponding to G'_1 in the expansion there exists a convex cycle C_1 of the same length as C'_1 such that $\pi_f(C_1) = C'_1$. Moreover C'_2 and C'_3 get expanded to convex cycles C_2, C_3 each sharing exactly one edge with C_1 and having one vertex in the intersection of all three. Since the cells of G are gated, the cycles C_2, C_3 are gated. The cycles C_2, C_3 share at least one edge. If a vertex of e'_{23} is in $G'_1 \cap G'_2$, then C_2 and C_3 share 2 edges, which impossible because C_2 and C_3 are gated. By the 3CC-condition, two of C_1, C_2, C_3 are 4-cycles, which is impossible because in G' one of those 4-cycles C_i will get contracted to an edge and not to the cycle C'_i.

Now consider the option (1) that $G'_2 \cap C'_2 = C'_2$ and both $G'_1 \cap C'_3, G'_2 \cap C'_3$ are halves of C'_3. Since $v \in G'_1 \cap G'_2$, the antipode u of v in C'_3 also belongs to $G'_1 \cap G'_3$. Hence C'_3 gets expanded to v, its antipode u, and $e'_{12} \in G'_1 \cap G'_2$. Therefore to ensure that we do not have $G'_k \cap C'_j = C'_j$ for some $k \in \{1,2\}$ and $j \in \{1,2,3\}$, we must have $G'_1 \cap C'_2 = e'_{12}$ and $G'_2 \cap C'_1 = e'_{12}$. Now the expansion of $C'_1 \cup C'_2 \cup C'_3$ has 4 convex cycles: the expansion C_3 of C'_3, the convex cycles C_1 and C_2 that get mapped to C'_1 and C'_2 by the contraction and a 4-cycle C_4 between C_1 and C_2. Cycles C_1, C_3, C_4 pairwise intersect in three different edges and all have a common vertex, thus their convex closure X is isomorphic to $C_3 \times K_2$ (since the cycle C_3 must have length at least 6). This proves that C_1 is a 4-cycle. Let C_5 be the fourth cycle, sharing edges with C_3 and C_4 different from C_1. Then C_5 shares two edges with C_2 which is possible only if $C_2 = C_5$. Thus we see that again $\pi_f(X)$ is a gated cell including C'_1, C'_2, C'_3.

Case 3. For every $i \in \{1,2,3\}$, $G'_1 \cap C'_i$ and $G'_2 \cap C'_i$ are halves of C'_i intersecting in two antipodal vertices of C'_i.

If $G'_1 \cap G'_2$ contains any vertex of e'_{ij} for $i,j \in \{1,2,3\}$, then there exist convex cycles C_i and C_j in G that share two edges, which is impossible. Thus C'_1, C'_2, C'_3 get extended to cycles C_1, C_2, C_3 pairwise sharing an edge and a vertex in common. Then two of them must be 4-cycles, which is not the case because then two of them get contracted to edges. We have proved that the 3CC-condition also holds for G', thus the class we consider is closed under contractions. This finishes the proof. □

The remaining part of this section is devoted to the proof of the equivalence (i)⇔(iv). For an equivalence class E_f of Θ, we denote by $N(E_f)$ the *carrier* of f, i.e., the subgraph of G which is the union of all cells of G which are crossed by E_f. The carrier $N(E_f)$ splits into its positive and negative parts $N^+(E_f) := N(E_f) \cap H_f^+$ and $N^-(E_f) := N(E_f) \cap H_f^-$.

Lemma 16. *Let G be a hypercellular graph and $e, f \in \Lambda, e \neq f$. Then $\pi_e(N(E_f))$ is the carrier of E_f in $\pi_e(G)$.*

Proof. Let $Y \in N(E_f)$ be a cell of G. Since contractions of products are products, $\pi_e(Y)$ is a product of edges and even cycles in $\pi_e(G)$ and clearly crosses E_f. Furthermore, since $Y = \text{conv}(C)$ for a cycle C in G, we have by Lemma 7, that $\pi_e(Y) \subseteq \text{conv}(\pi_e(C))$. Since $\pi_e(G)$ is hypercellular, $\text{conv}(\pi_e(C))$ is a cell by Theorem A. Thus, $\pi_e(Y)$ is convex in $\pi_e(G)$ by Lemma 11. Therefore $\pi_e(Y)$ is a cell of $\pi_e(N(E_f))$.

Conversely, let Y be a cell in the carrier of f in $\pi_e(G)$ and Y' be its expansion with respect to e. By Lemma 6, Y' is convex and by Proposition 3 Y' is either a product of cycles and thus a cell of $N(E_f)$, or Y' consists of two cells Y'', Y''' separated by E_e, where say Y'' is isomorphic to Y. Since f crosses Y'', Y'' is in $N(E_f)$ and Y arises as its contraction, so we are done. □

Lemma 17. *Let G be a hypercellular graph. Then any two cells Y', Y'' of $X(G)$ either are disjoint or intersect in a cell of $X(G)$.*

Proof. Let Y', Y'' be two arbitrary intersecting cells of G. Let $Y_0 := Y' \cap Y''$. Since Y' and Y'' are gated subgraphs of G, Y_0 is also gated. In particular, Y_0 is a gated subgraph of Y'. Since Y' is a product of edges and cycles $F_1 \times \ldots \times F_m$, by Lemma 11, Y_0 is a product $F'_1 \times \ldots \times F'_m$, where each F'_i is a vertex, an edge, or the whole factor F_i. Hence Y_0 is a cell and we are done. Now suppose that some $F'_i \cong P = (x, \ldots, y, \ldots, z)$ is a path of length ≥ 2 within the cyclic factor F_i. Since P is convex, the length of P must be less than half of the length of F_i. Thus the antipodal vertex of y in the F_i, say y', is not in P. Now, y' cannot have a gate w in P, since if w is between x and y there is no shortest path from y' through w to z. Symmetrically, if w is between y and z there is no shortest path from y' through w to x. □

Lemma 18. *Let G be a hypercellular graph and $f \in \Lambda$. If two cells Y', Y'' of $N(E_f)$ intersect, then they share an edge of E_f.*

Proof. Let $y \in Y' \cap Y''$ and suppose without loss of generality that $y \in H^-_f$. Since $Y' \in N(E_f)$, there exists an edge $u'v' \in E_f$ with $u', v' \in Y'$. Suppose $u' \in H^-_f$ and $v' \in H^+_f$. If $v' \in Y''$, then $u' \in I(v', y) \subset Y''$ by convexity of Y'', thus the edge $u'v'$ belongs to $Y' \cap Y''$ and we are done. So, suppose $v' \notin Y''$. Let v be the gate of v' in Y'' and let x be a vertex of $Y'' \cap H^+_f$ (such a vertex exists because $Y'' \in N(E_f)$). Since $v \in I(v', x)$ and H^+_f is convex, we conclude that $v \in H^+_f$. Since $y \in H^-_f$, on any shortest path P from v to y we will meet an edge $v''u''$ of E_f. Since $v'', u'' \in I(v, y)$, $v, y \in Y''$, and Y'' is convex, the edge $v''u''$ belongs to Y''. On the other hand, since $v \in I(v', y)$ and Y' is convex, we conclude that the edge $v''u''$ also belongs to Y'. □

Proposition 7. *For any equivalence class E_f of a hypercellular graph G, the carrier $N(E_f)$ is a gated subgraph of G. Therefore, $N^+(E_f)$ is gated in the halfspace H^+_f, $N^-(E_f)$ is gated in H^-_f, and the extended halfspaces $H^+_f \cup N(E_f)$ and $H^-_f \cup N(E_f)$ are gated in G.*

Proof. First, since by Lemma 13 the relations Θ and Ψ^* coincide, $N^+(E_f)$, $N^-(E_f)$ and consequently $N(E_f)$ are connected subgraphs of G.

Through Claims 7, 8, and 9 we will prove that $N^+(E_f)$ is convex. Suppose that $N^+(E_f)$ is not convex. Choose two vertices $y, z \in N^+(E_f)$ with minimal distance $k(y, z) := d_{N^+(E_f)}(y, z)$ that can be connected by a shortest path R of G outside $N^+(E_f)$. Let P be a shortest y, z-path in $N^+(E_f)$. Let us prove that P is a shortest path of G. If this was not the case, we could replace y by its neighbor y' in P. But from the minimality in the choice of y, z, we conclude that $I(y', z) \subseteq N^+(E_f)$. Thus, the subpath of P between y' and z is a shortest path of G. Now, since $y \notin I(y', z)$, we have $z \in W(y', y)$, yielding that P is a shortest path of G. Again by the choice of y, z, we conclude that P and R intersect only in their common endvertices y, z.

Claim 7. *Any shortest path between a vertex of P and a vertex of R passes via y or z. In particular, $C := P \cup R$ is an isometric cycle of G.*

Proof. We claim that if Q is a shortest path connecting two interior vertices p of P and r of R, then Q passes via y or z. Suppose that this is not the case. Then we can find a shortest path $Q = (p := q_0, q_1, \ldots, q_{k-1}, q_k := r)$ between two interior vertices p of P and r of R such that $Q \cap C = \{p, r\}$. Since $p, r \in I(y, z)$ and $Q \subset I(p, r)$, we conclude that $Q \subset I(y, z)$, because intervals of partial cubes are convex. This yields $q_1 \in I(p, y) \cup I(p, z)$, since otherwise $y, z \in W(p, q_1)$ and $q_1 \in I(y, z)$ contradict the convexity of $W(p, q_1)$. Since $k(p, y) < k(z, y)$ and $k(p, z) < k(z, y)$, we conclude that $I(p, y) \cup I(p, z) \subset N^+(E_f)$, giving $q_1 \in N^+(E_f)$. We can iterate this argument by first replacing p by q_1 and q_1 by q_2, etc., and obtain that all vertices $q_1, q_2, \ldots, q_{k-1}, q_k = r$ belong to $N^+(E_f)$ and $k(q_i, y) < k(y, z), k(q_i, z) < k(y, z)$. In particular, $r \in N^+(E_f)$ and $k(r, y) < k(y, z), k(r, z) < k(y, z)$, thus by our assumption $R \subset I(r, y) \cup I(r, z) \subset N^+(E_f)$. This contradiction shows that the path Q does not exist, i.e., any shortest path between a vertex of P and a vertex of R passes via y or z. In particular, this implies that $C = P \cup R$ is an isometric cycle of G. □

Claim 8. *$C = P \cup R$ is a convex cycle of G.*

Proof. We proceed as in the proof of Lemma 13. If C is not convex, then by Claim 7 there exist two vertices $p, p' \in P$ connected by a shortest path P' which intersects P only in p, p' or there exist two vertices $r, r' \in R$ connected by a shortest path R' which intersects R only in r, r'. Let P'' be the subpath of P between p and p' in the first case and let R'' be the subpath of R between r and r' in the second case. Let C' be the cycle obtained from C by replacing the path P'' by P' in the first case and let C'' be the cycle obtained from C by replacing the path R'' by R' in the second case. If the first case occurs and $\{p, p'\} \neq \{y, z\}$, then $k(p, p') < k(y, z)$, whence $P' \subset N^+(E_f)$. Therefore applying Claim 7 to the cycle C' instead of C, we conclude that C' is an isometric cycle of G. Analogously, if $\{r, r'\} \neq \{y, z\}$, then no vertex of R' belongs to $N^+(E_f)$. Indeed, if say $w \in R' \cap N(E_f)$, then $k(w, y) < k(y, z), k(w, z) < k(y, z)$ and by minimality the vertices r and r' belong to $N(E_f)$, contrary to the assumption that $R \cap N(E_f) = \{y, z\}$. Again applying Claim 7 to the cycle C'' instead of C, we conclude that C'' is an isometric cycle of G. Finally, if $\{p, p'\} = \{y, z\}$, then $P'' = P$ and one can see that either $P' \subseteq N^+(E_f)$ and C' is an isometric cycle of G or $P' \cap N^+(E_f) = \{y, z\}$ and we redefine C' as the cycle formed by P and P', which is isometric by Claim 7. Similarly, if $\{r, r'\} = \{y, z\}$, then $R'' = R$ and we can suppose that C'' is an isometric cycle of G sharing with C either the path P or the path R. Consequently, in all cases we derive a new isometric cycle of G (C' or C'') obtained by replacing either a subpath P'' of P by P' or replacing a subpath R'' of R by R'. Suppose without loss of generality that we are in the first case, i.e., the new isometric cycle is C'.

Let x'' be a vertex of P'' different from p, p'. Let x be the opposite of x'' in the cycle C and let x' be the opposite of x in the cycle C'. Since C and C' are isometric cycles of the same length of G, x' is a vertex of P' different from p, p'. Then $p, p' \in I(x, x')$, thus by convexity of $I(x, x')$ we obtain that $x'' \in I(x, x')$. But this is impossible because x' and x'' have the same distance to x because they are opposite to x in C and C', respectively, and C and C' have the same length. This proves that C is convex. □

Since the cells of $N(E_f)$ are convex, the path $P \subset N^+(E_f)$ of C cannot be contained in a single cell. Thus there exist two consecutive edges $e' = uv$ and $e'' = vw$ of P and two cells Y', Y'' of $N(E_f)$ such that $u \in Y' \setminus Y'', w \in Y'' \setminus Y'$, and $v \in Y' \cap Y''$. By the following claim this is impossible.

Claim 9. *If there exist two cells Y', Y'' of $N(E_f)$ and a convex cycle C with edges $e' = uv, e'' = vw$ on it, such that $u \in Y' \setminus Y'', w \in Y'' \setminus Y'$, and $v \in Y' \cap Y''$, then there exists a cell in $N(E_f)$ that includes C.*

Proof. By Lemma 17, the intersection $Y' \cap Y''$ is a product of edges and cycles. Since $Y', Y'' \in N(E_f)$, Lemma 18 yields that $Y' \cap Y''$ contains at least one edge from E_f. Let F be a factor of $Y' \cap Y''$ and L be a corresponding layer that is crossed by E_f. We will establish the claim by proving that e', e'' lie in a cell of $N(E_f)$, that is isomorphic to $L \times C$. Pick any edge $e = vz$ in $Y' \cap Y''$ that lies in the layer L.

First assume that e and e' lie in the same layer of Y'. The factor F is an edge or an even cycle, but it must be a strict subset of a factor of Y' since $e' \notin Y' \cap Y''$. Thus L is isomorphic to an edge and this edge must be in E_f. In particular, $e \in E_f$. Since Y' and Y'' are products of edges and cycles, there exists a convex cycle C' of Y' passing via the edges e' and e and there exists a convex cycle C'' of Y'' passing via the edges e and e''. By Lemma 15, C, C', and C'' are contained in a cell Y of G. Since $e \in E_f$, this cell is in $N(E_f)$.

By symmetry, we are left with the case that e and e' as well as e and e'' lie in different layers of Y' and Y'', respectively. Consequently, there exists a 4-cycle $C' = (u, v, z, u')$ of Y' passing via the edges e' and e and a 4-cycle $C'' = (w, v, z, w')$ of Y'' passing via the edges e and e''. By Lemma 15, C, C', and C'' are contained in a cell $Y \cong C \times K_2$ of G, where e lies in a layer corresponding to the factor K_2. If $e \in E_f$, then Y is a cell in $N(E_f)$ and we are done. If $e \notin E_f$, then L is isomorphic to an even cycle. Consider z, and let z' be its neighbor, different from v, that lies in $Y' \cap Y''$ in L. Since z is in Y, z is incident to a cycle C' isomorphic to C and lies on the path (u', z, w') of C'. Considering C' and edges $u'z$, zw', and zz', we can as before with C, e', e'', and e, obtain a cell Z isomorphic to $C \times K_2$. The union of Y and Z is isomorphic to $C \times P_3$, where P_3 is the path on 3 vertices. Inductively picking neighbors in the layer L we obtain a graph isomorphic to $C \times F$, that contains C. □

We have sown that $N^+(E_f)$ and symmetrically $N^-(E_f)$ are convex. To see that $N(E_f)$ is convex, pick two vertices $x \in N^+(E_f), y \in N^-(E_f)$, a shortest (x, y)-path R and a vertex z of R. Since R connects a vertex of H_f^+ with a vertex of H_f^-, necessarily R contains an edge $x'y'$ in E_f, say $x' \in H_f^+$ and $y' \in H_f^-$. The vertex z belongs to one of the two subpaths of R between x and x' or y' and y, say the first. Then $z \in I(x, x')$. Since $x, x' \in N^+(E_f)$ and $N^+(E_f)$ is convex, we conclude that $z \in N^+(E_f) \subset N(E_f)$, showing that the carrier $N(E_f)$ is convex.

Now, suppose that G is a minimal graph in $\mathcal{F}(Q_3^-)$ containing a non-gated carrier $N(E_f)$. Since $N(E_f)$ is convex and by Lemma 16 any contraction $\pi_e(N(E_f))$ for $e \in \Lambda, e \neq f$, is the carrier of E_f in $G' = \pi_e(G)$ and thus is gated in G', by Proposition 1 there exist two vertices $x_1, x_2 \in N(E_f)$ with $d_G(x_1, x_2) = 2$ and a vertex $v \notin N(E_f)$ at distance 2 from x_1, x_2, such that the vertices v, x_1, x_2 do not contain a common neighbor. Since $G \in \mathcal{F}(Q_3^-)$, the last condition implies that the convex hull of v, x_1, x_2 is a 6-cycle C_1. Let u be the unique common neighbor of x_1 and x_2 in C. Since $N(E_f)$ is convex, u also belongs to $N(E_f)$. Namely, if say $v \in H_f^+$, then $x_1, u, x_2 \in N^+(E_f)$. Since by Proposition 2 each cell of G is gated, the vertices x_1 and x_2 cannot belong to a common cell. Thus there exist two cells Y', Y'' of $N(E_f)$ such that the edge x_1u belongs to Y' and ux_2 belongs to Y''. By Claim 9, there exists a cell Y of the carrier $N(E_f)$ that includes C, contrary to the assumption that the vertex v of C does not belong to $N(E_f)$. This establishes that $N(E_f)$ is gated. By Lemma 10 also $N^+(E_f)$ is gated in H_f^+ and $N^-(E_f)$ is gated in H_f^-. Consequently, the extended halfspaces $H_f^+ \cup N(E_f)$ and $H_f^- \cup N(E_f)$ are gated in G. □

Now, we are ready to prove the following result (the equivalence (i)⇔(iv) of Theorem B):

Theorem 2. *A partial cube G is hypercellular if and only if each finite convex subgraph of G can be obtained by gated amalgams from Cartesian products of edges and even cycles.*

Proof. First suppose that a finite graph G is obtained by gated amalgam from two graphs $G_1, G_2 \in \mathcal{F}(Q_3^-)$. Suppose by way of contradiction that $G \notin \mathcal{F}(Q_3^-)$ and suppose that G is a minimal such graph. Then any proper convex subgraph H of G is either contained in one of the graphs G_1, G_2 or is the gated amalgam of $H \cap G_1$ and $H \cap G_2$, thus $H \in \mathcal{F}(Q_3^-)$ by minimality of G. Thus there exists a sequence of contractions of G to the graph Q_3^-. Let E_f be the first such contraction, i.e., the graph $G' := \pi_f(G)$ does not belong to $\mathcal{F}(Q_3^-)$. On the other hand, by Lemma 10, $G_1' := \pi_f(G_1)$ and $G_2' := \pi_f(G_2)$ are gated subgraphs of G'. Moreover, G_1' and G_2' belong to $\mathcal{F}(Q_3^-)$ because G_1 and G_2 belong to $\mathcal{F}(Q_3^-)$ and $\mathcal{F}(Q_3^-)$ is closed by contractions. As a result we obtain that the graph $G' \notin \mathcal{F}(Q_3^-)$ is the gated amalgam of the graphs $G_1', G_2' \in \mathcal{F}(Q_3^-)$, contrary to the minimality of G. This establishes that the subclass of $\mathcal{F}(Q_3^-)$ consisting of finite graphs from $\mathcal{F}(Q_3^-)$ is closed by gated amalgams.

Conversely, suppose that G is an arbitrary finite convex subgraph of a graph from $\mathcal{F}(Q_3^-)$. We follow the schema of proof of implication (3)⇒(4) of [5, Theorem 1]. If G is a single cell, then we are done. Otherwise, we claim that G is a gated amalgamation of two proper gated subgraphs G_1 and G_2.

First suppose that there exist two disjoint maximal cells Y' and Y''. Let $y' \in Y'$ and $y'' \in Y''$ be two vertices realizing the distance $d(Y', Y'') = \min\{d(x, z) : x \in Y', z \in Y''\}$. Since $Y' \cap Y'' = \emptyset$, necessarily $y' \neq y''$. Since Y' and Y'' are gated, from the choice of y', y'' it follows that y' is the gate of y'' in Y'

and y'' is the gate of y' in Y''. Let y be a neighbor of y' on a shortest path between y' and y''. Suppose that the edge $y'y$ belongs to the equivalence class E_f. Notice that y'' is also the gate of y in Y'' and y' is the gate of y in Y'. Therefore $Y' \subseteq W(y', y) = H_f^+$ and $Y'' \subseteq W(y, y') = H_f^-$. Consequently, Y' and Y'' are not contained in the carrier $N(E_f)$, thus $H_f^+ \setminus N(E_f)$ and $H_f^- \setminus N(E_f)$ are nonempty. By Proposition 7, $N(E_f)$, $H_f^+ \cup N(E_f)$, and $H_f^- \cup N(E_f)$ are gated subgraphs of G, thus G is the gated amalgam of $H_f^+ \cup N(E_f)$ and $H_f^- \cup N(E_f)$ along the common gated subgraph $N(E_f)$.

Thus further we may suppose that all maximal cells of G pairwise intersect. Since they are gated and G is finite, by the Helly theorem for gated sets [44, Proposition 5.12 (2)], the maximal cells of G intersect in a non-empty cell X_0.

Claim 10. *There exists an equivalence class E_f of G such that the carrier $N(E_f)$ of E_f does not contain all maximal cells of G and E_f contains an edge uv with $v \in X_0$ and $u \notin X_0$. Moreover, all maximal cells of the carrier $N(E_f)$ contain the edge uv.*

Proof. By definition, X_0 is a proper face of each maximal cell X of G. Therefore, there exists an edge uv with $v \in X_0$ and $u \in X \setminus X_0$. Suppose that uv belongs to the equivalence class E_f of G. Then $X_0 \subseteq W(v, u)$. Notice that X belongs to the carrier $N(E_f)$ of E_f. Since $u \notin X_0$, there exists a maximal cell X' such that $u \notin X'$. Since $v \in X'$, we assert that X' does not belong to $N(E_f)$. Indeed, suppose E_f contains an edge $u''v''$ with both ends in X'. Assume without loss of generality that $W(u, v) = W(u'', v'')$ and $W(v, u) = W(v'', u'')$. Since $u \in I(v, u'')$ and $v, u'' \in X'$, by the convexity of X' we conclude that $u \in X'$, a contradiction. This shows that $N(E_f)$ consists of all maximal cells containing the edge uv. □

Let E_f be an equivalence class of G as in Claim 10, in particular, uv is an edge of E_f with $v \in X_0$ and $u \notin X_0$. Let $X_1, \ldots, X_k$ be the maximal cells of G containing the edge uv. By the second assertion of Claim 10, $N(E_f)$ coincides with the union $\bigcup_{j=1}^k X_j$. Let $X_{k+1}, \ldots, X_m$ be the remaining maximal cells of G, i.e., the maximal cells not containing the vertex u (such cells exist by the choice of E_f). Set $Y := \bigcup_{i=k+1}^m X_i$ and notice that by the choice of X_0 we have $Y \subseteq W(v, u)$.

Let Z be the subgraph of G induced by the intersection of $N(E_f)$ with Y, i.e., $Z = \bigcup_{i=k+1}^m Z_i$, where $Z_i := N(E_f) \cap X_i$, $i = k+1, \ldots, m$. By Proposition 7, $N(E_f)$ is gated. Since by Proposition 2 each cell X_i of Y is also gated, each Z_i is gated, and thus is a face of X_i, $i = k+1, \ldots, m$, by Lemma 11.

Now we define a gated subgraph Z^* of G, which extends Z and *separates* $N(E_f)$ from Y, i.e., it contains their intersection and there is no edge from $N(E_f) \setminus Z^* \neq \emptyset$ to $Y \setminus Z^* \neq \emptyset$. Each maximal cell X_j in $N(E_f)$ is a Cartesian product of edges and even cycles, say $X_j = F_1 \times \ldots \times F_p$. Let L_j be the layer of X_j containing the edge uv. Suppose that $L_j = \{v_1\} \times \ldots \{v_{l-1}\} \times F_l \times \{v_{l+1}\} \times \ldots \times \{v_p\}$, where F_l is the lth factor of X_j and v_s is a vertex of the factor $F_s, s \neq l$. If $L_j = uv$, i.e. L_j comes from an edge-factor $F_l = u'_j v'_j$, then set $Z_j^* := F_1 \times \ldots \times F_{l-1} \times \{v'_j\} \times F_{l+1} \times \ldots \times F_p$. Since $u \notin Z_j^*$, Z_j^* is a proper gated subgraph of X_j. Now, suppose that L_j comes from a cyclic factor F_l of X_j. Let vw_j be the edge of L_j incident to v and different from uv. Suppose that the edges uv and vw_j of L come from the edges $u'_j v'_j$ and $v'_j w'_j$ of F_l, respectively. Set $Z_j^* := F_1 \times \ldots \times F_{l-1} \times \{v'_j, w'_j\} \times F_{l+1} \times \ldots \times F_p$. Again, since $u \notin Z_j^*$, Z_j^* is a proper gated subgraph of X_j. Equivalently, Z_j^* is the subgraph of X_j induced by all vertices of X_j whose gates in the gated cycle L_j is either v or w_j. Notice also that in both cases Z_j^* is a proper face of X_j included in $W(v, u)$. Finally, set $Z^* := \bigcup_{j=1}^k Z_j^*$.

Claim 11. *For each $j = 1, \ldots, k$, we have $Z^* \cap X_j = Z_j^*$.*

Proof. By definition, $Z_j^* \subseteq Z^* \cap X_j$. To prove the converse inclusion, it suffices to show that for any $j' \in \{1, \ldots, k\}$, $j' \neq j$, we have $Z_{j'}^* \cap X_j \subseteq Z_j^*$. Consider the layers L_j of X_j and $L_{j'}$ of $X_{j'}$ containing the edge uv. Each of them consists either of the edge uv or is a gated cycle of G. If L_j is uv, then Z_j^* coincides with $X_j \cap W(v, u)$. Since $Z_{j'}^* \subseteq W(v, u)$, necessarily $Z_{j'}^* \subseteq X_j \subseteq Z_j^*$. Now suppose that L_j is an even cycle. Suppose by way of contradiction that $Z_{j'}^* \cap X_j$ contains a vertex x not included in Z_j^*. Since $x \in W(v, u)$, the gate of x in L_j is a vertex x' of $L_j \cap W(v, u)$ different from v. Since $X_{j'}$ is convex, $x', v \in I(x, u) \subset X_{j'}$. Since $X_{j'}$ is gated and contains three different vertices u, v, x' of the gated cycle L_j, necessarily $X_{j'}$ contains the entire cycle L_j. This implies $L_{j'} = L_j$ and $w_j = w_{j'}$. By definition of

$Z^*_{j'}$, we also conclude that $x' = w_{j'}$. Since $x \in X_j$, by definition of Z^*_j we must have $x \in Z^*_j$, contrary to the choice of x. □

Claim 12. *For each $i = k+1, \ldots, m$, we have $Z^* \cap X_i = Z_i$. In particular, $Z^* \cap Y = Z$.*

Proof. For each maximal cell $X_j, i = 1, \ldots, k$, of $N(E_f)$, consider the intersection Z_{ji} of X_j with each cell $X_i, i = k+1, \ldots, m$, of Y. From the definition of Z it follows that each $Z_i, i = k+1, \ldots, m$, can be viewed as the union of all Z_{ji}, $j = 1, \ldots, k$, thus Z can be viewed as the union of all Z_{ji}, $j = 1, \ldots, k, i = k+1, \ldots, m$.

Now, let $k+1 \le i \le m$. First we prove that for any $1 \le j \le k$ the set Z_{ji} is included in $X_j \cap Z^*$ (which coincides with Z^*_j by Claim 11). This is obviously so if the layer L_j is the edge uv: in this case, since $X_i \subset W(v,u)$, $Z_{ji} = X_j \cap X_i$ is a subset of $W(v,u) \cap X_j = Z^*_j$. Now, suppose that L_j is an even cycle. Suppose by way of contradiction that $Z_{ji} = X_j \cap X_i$ contains a vertex x whose gate x' in L_j is different from v and w_j. Since $x \in W(v,u)$, necessarily w_j and x' belong to the interval $I(x,v)$. Since $x, v \in Z_{ji}$ and Z_{ji} is convex, $w_j, x' \in Z_{ji}$. Since Z_{ji} and L_j are gated and $Z_{ji} \cap L_j$ contains the vertices u, v, x', necessarily L_j must be included in Z_{ji}. Since $u \in L_j \setminus Z_{ji}$, we obtained a contradiction. This establishes the inclusion $Z_{ji} \subseteq Z^* \cap X_i \subseteq Z^*$.

We have $Z^* \cap X_i = (\bigcup_{j=1}^k Z^*_j) \cap X_i = \bigcup_{j=1}^k (Z^*_j \cap X_i)$. By Claim 11, the latter equals to $\bigcup_{j=1}^k (Z^* \cap X_j \cap X_i) = \bigcup_{j=1}^k (Z^* \cap Z_{ji}) = \bigcup_{j=1}^k Z_{ji}$, where the last equation holds by the inclusion established above. Finally, by the definition, $\bigcup_{j=1}^k Z_{ji} = Z_i$. □

Claim 13. *Let S be a subgraph of G such that the intersection of S with any maximal cell of G is non-empty and gated (i.e., a face by Lemma 11). Then S is a gated subgraph of G.*

Proof. Let X be a maximal cell of G, $x \in X$ a vertex, and $S^* := S \cap X$. By our assumptions, S^* is a nonempty face of X, thus a gated subgraph of G. Let x^* be the gate of x in S^*. We assert that x^* is also the gate of x in the set S, i.e., for any vertex $y \in S$, we have $x^* \in I(x,y)$. Suppose that y belongs to a maximal by inclusion cell R in S. Let $R_0 := X \cap R$ and let x_0 be the gate of x in R_0. Since $R \subseteq S$, necessarily $R_0 \subseteq S^*$, whence $x^* \in I(x, x_0)$. Therefore, to prove that $x^* \in I(x,y)$ it suffices to show that $x_0 \in I(x,y)$. For this it is enough to prove that x_0 is the gate of x in R. Suppose by way of contradiction that the gate of x in R is a vertex x' different from x_0. Then $x' \in I(x, x_0) \subset X$ because X is convex. Since $x' \in R$, we conclude that $x' \in X \cap R = R_0$. This contradicts the assumption that x_0 is the gate of x in R_0. Hence x^* is the gate of x in S, establishing that S is gated. □

By Claims 11 and 12, the intersection of Z^* with each cell $X_i, i = 1, \ldots, m$, of G is a proper face of X_i (and thus a gated subgraph of G). Hence Z^* satisfies the conditions of Claim 13, thus Z^* is a gated subgraph of G. Since $Z^* \subseteq N(E_f) \cap W(v,u)$ and $u \in N(E_f) \setminus Z^*$, Z^* is a proper subgraph of $N(E_f)$. Since by Claim 12 $Z^* \cap Y = Z$ and Z is a proper subgraph of Y, the gated subgraph Z^* separates any vertex of $N(E_f) \setminus Z^* \ne \emptyset$ from any vertex of $Y \setminus Z^* = Y \setminus Z \ne \emptyset$. Consequently, G is the gated amalgam of $N(E_f)$ and $Y \cup Z^*$ along Z^*, concluding the proof of the theorem. □

5. The median cell property

Three (not necessarily distinct) vertices x, y, z of a graph G are said to form a *metric triangle* xyz if the intervals $I(x,y), I(y,z)$, and $I(z,x)$ pairwise intersect only in the common end vertices. A (degenerate) equilateral metric triangle of size 0 is simply a single vertex. We say that a metric triangle xyz is a *quasi-median* of the triplet u, v, w if

$$d(u,v) = d(u,x) + d(x,y) + d(y,v),$$
$$d(v,w) = d(v,y) + d(y,z) + d(z,w),$$
$$d(w,u) = d(w,z) + d(z,x) + d(x,u).$$

Observe that, for every triplet u, v, w, a quasi-median xyz can be constructed in the following way: first select any vertex x from $I(u,v) \cap I(u,w)$ at maximal distance to u, then select a vertex y from $I(v,x) \cap I(v,w)$ at maximal distance to v, and finally select any vertex z from $I(w,x) \cap I(w,y)$ at maximal distance to w. In the case that the quasi-median is degenerate ($x = y = z$), it is a median of the triplet u, v, w.

We continue with the following characterization of metric triangles in hypercellular graphs:

Proposition 8. *If G is a hypercellular graph and xyz is a metric triangle of G, then x, y, z belong to a common cell of G. In particular, the gated hull $\langle\langle x, y, z\rangle\rangle$ coincides with the convex hull conv(x, y, z) and is a cell of G.*

Proof. First we prove the result for an arbitrary finite hypercellular graph G. By Theorem B either G is a single cell and we are done, or G is a gated amalgam of two proper gated subgraphs G_1 and G_2. Suppose without loss of generality that $y, z \in V(G_1)$. If $x \in V(G_1)$, then we can apply induction hypothesis to G_1 and conclude that x, y, z belong to a common cell of G_1, and thus to a common cell of G. Now suppose that $x \in V(G_2) \setminus V(G_1)$. Let x' be the gate of x in G_1. Since x' belongs to G_1 and x not, $x' \neq x$. Since $x' \in I(x, y) \cap I(x, z)$, we obtain a contradiction with the assumption that xyz is a metric triangle of G. Thus x, y, z belong to a common cell of G. Since each cell of G is gated and xyz is a metric triangle, the gated hull of x, y, z coincides with the convex hull conv(x, y, z) and is a cell.

Now, suppose that G is an arbitrary hypercellular graph. Let G' be the subgraph induced by the convex hull of x, y, and z. Then G' is a finite hypercellular graph. By the above result for finite graphs, we have that G' is a convex Cartesian product of edges and even cycles. Therefore, G' is the convex hull of an isometric cycle of G. By Theorem A, G' is a gated cell of G. □

For a triple of vertices u, v, w of a graph G, a *u-apex* relative to v and w is a vertex $x := (uvw) \in I(u, v) \cap I(u, w)$ such that $I(u, x)$ is maximal with respect to inclusion. A graph G is *apiculate* [6] if and only if for any vertex u the vertex set of G is a meet-semilattice with respect to the base-point order $\preceq_u$ defined by $v \preceq_u v' \Leftrightarrow v \in I(u, v')$, that is, $I(u, v) \cap I(u, w) = I(u, (uvw))$ for any vertices v, w. Note that many partial cubes are not apiculate, see [15] for this discussion with respect to tope graphs of oriented matroids. For any triplet u, v, w of vertices of an apiculate graph G, the vertices u, v, w admit unique apices $x := (uvw), y := (vuw)$, and $z := (wuv)$ and admit a unique quasi-median defined by the metric triangle xyz.

Lemma 19. *[6, Proposition 2] Every Pasch graph G is apiculate. Consequently, every hypercellular graph is apiculate.*

We say that a triplet u, v, w of vertices in an apiculate graph G admits a *median cell* (respectively, a *median cycle*) if the gated hull $\langle\langle x, y, z\rangle\rangle$ of the unique quasi-median xyz of u, v, w is a Cartesian product of vertices, edges, and cycles (respectively, a cycle or a single vertex). Notice that any median-cell is either a vertex or is a Cartesian product of even cycles of length ≥ 6. A graph G is called *cell-median* (respectively, *cycle-median*) if G is apiculate and any triplet u, v, w of G admits a unique median cell (respectively, unique median cycle or vertex). By Proposition 3 of [5], bipartite cellular graphs are cycle-median. This result has been extended in [40] by showing that all graphs which are gated amalgams of even cycles and hypercubes are cycle-median, and those are exactly the netlike cycle-median partial cubes. Now, we are ready to prove Theorem E.

Theorem 3. *A partial cube $G = (V, E)$ is cell-median if and only if G is hypercellular.*

Proof. First we prove that hypercellular graphs are cell-median. By Corollary 3 and Lemma 19 it follows that any graph G from $\mathcal{F}(Q_3^-)$ is apiculate. Therefore, to show that G is cell-median it suffices to show that if xyz is a metric triangle of G, then the gated hull $\langle\langle x, y, z\rangle\rangle$ of x, y, z is a cell; this is Proposition 8.

Conversely, to prove that cell-median partial cubes are hypercellular graphs we will use Theorem B(ii). Namely, we have to prove that a cell-median partial cube G satisfies the 3CC-condition and that any cell X of G is gated. Suppose by way of contradiction, that G contains a cell X and a vertex not having a gate in X. Let v be such a vertex closest to X. Since v does not have a gate, we can find two vertices $x, y \in X$ such that $I(x, v) \cap X = \{x\}, I(y, v) \cap X = \{y\}$, and x is closest to v in X. From the choice of v, we conclude that $I(v, x) \cap I(v, y) = \{v\}$. Hence, the vertices v, x, and y define a metric triangle of G. By the median-cell property, the convex hull of v, x, y is a gated cell Y of G. Let $Z := X \cap Y$. Notice that $x, y \in Z$ and $v \notin Z$. Notice also that Z is convex but not gated, otherwise we will get a contradiction with the choice of v. Since Z is convex, Z is a subproduct of X and Y and is a Cartesian product of convex paths and cycles. Let $Z = Z_1 \times Z_2 \times \ldots \times Z_m$. Suppose also that $X = X_1 \times X_2 \times \ldots \times X_m$ and $Y = Y_1 \times Y_2 \times \ldots \times Y_m$, where each $X_i, i = 1, \ldots, m$, and each $Y_j, j = 1, \ldots, m$, is an even cycle, an edge, or a vertex, and each Z_i is a convex subgraph of each X_i and Y_i, $i = 1, \ldots, m$. Since Z is not gated, at least one factor, say Z_1, is a convex path of length at least 2, and X_1 and Y_1 are even cycles.

Let z be a vertex of $Z = Z_1 \times Z_2 \times \ldots \times Z_m$ of the form $z = z_1 \times z_2 \times \ldots \times z_m$. Then the layers $X_1 \times z_2 \times \ldots \times z_m$ of X and $Y_1 \times z_2 \times \ldots \times z_m$ of Y are respectively a convex and a gated cycle of G. These two cycles intersect in a path of length at least two, namely in $Z_1 \times z_2 \times \ldots \times z_m$. By the following Claim 14, this is impossible. This contradiction establishes that the cell X is gated.

Claim 14. *Let C_1, C_2 be two distinct convex cycles of a partial cube G. If C_2 is gated, then $C_1 \cap C_2$ is empty, a vertex, or an edge of G.*

Proof. Suppose by way of contradiction that $C_1 \cap C_2$ contains a path (v_1, v, v_2) of length 2. Let u be the antipodal to v vertex of C_1. If $u \in C_2$, then $u, v \in C_2$ and by convexity of C_2 we deduce that $C_1 = I(u, v) \subseteq C_2$, thus $C_1 = C_2$, a contradiction. Consequently, $u \notin C_2$. Let x be the gate of u in C_2. Since $v_1, v_2 \in C_2$, $x \in I(u, v_1) \cap I(u, v_2)$. From these inclusions we conclude that either $x = v$ or x is the antipodal to v vertex of C_2. Since $v_1, v_2 \in I(u, v)$, necessarily $x \neq v$. But if x is the antipode of v in C_2, then $C_2 \subset I(v_1, u) \cup I(v_2, u)$, which is only possible if $C_1 = C_2$. □

To establish the 3CC-condition, let C_1, C_2, C_3 be three convex cycles of G such that any two cycles C_i, C_j, $1 \leq i < j \leq 3$, intersect in an edge e_{ij} and the three cycles intersect in a vertex x. Since the cells of G are gated, C_1, C_2, C_3 are gated cycles of G. Let $e_{12} = xx_2, e_{23} = xx_0$, and $e_{13} = xx_3$. Let v_1, v_2, and v_3 be the vertices of respectively C_1, C_2, and C_3 antipodal to x. If v_1, v_2, and v_3 define a metric triangle, then the gated hull of v_1, v_2, v_3 is a Cartesian product of vertices, edges, and even cycles containing C_1, C_2, and C_3, and we are done. So suppose without loss of generality that there exists a vertex $u_1 \in I(v_1, v_2) \cap I(v_1, v_3)$ adjacent to v_1. Notice that x_2 and x_3 are the gates of v_1 in the cycles C_2 and C_3, respectively. In fact this is true since the gate of v_1 in C_2 must be in $I(v_1, x_2) \cap C_2 = \{x_2\}$ and the gate of v_1 in C_3 must be in $I(v_1, x_3) \cap C_3 = \{x_3\}$.

Since u_1 is adjacent to v_1, one can easily show that the gates y_2 and y_3 of u_1 in C_2 and C_3 are two vertices adjacent to x_2 and x_3, respectively. If y_2 or y_3 coincides with x, then $u_1 \in I(v_1, x)$, contrary to the assumption that the cycle C_1 is convex. Thus y_2 is the second neighbor of x_2 in C_2 and y_3 is the second neighbor of x_3 in C_3. Since $y_2, y_3 \in W(u_1, v_1), x_2, x, x_3 \in W(v_1, u_1)$, and $W(u_1, v_1)$ is convex, we deduce that $d(y_2, y_3) = 2$. Consequently, y_2 and y_3 have a common neighbor z_0. First suppose that $z_0 \neq x_0$, i.e., x_0 is not adjacent to one of the vertices y_2, y_3, say x_0 and y_2 are not adjacent. Since C_2 and C_3 are convex, z_0 cannot be adjacent to x. Thus $d(z_0, x) = 3$, whence the 6-cycle $C_0 := (z_0, y_2, x_2, x, x_3, y_3)$ is isometric. Since C_0 intersects C_2 and C_3 along paths of length 2, by Claim 14, this cycle cannot be gated and thus cannot be convex. Since G is cell-median, the convex hull of C_0 cannot be a Q_3^-, thus its convex hull is a 3-cube Q_3. Therefore the intervals $I(y_2, x)$ and $I(x, y_3)$ are squares of G which necessarily must coincide with C_2 and C_3. Consequently, x_0 is adjacent to y_2 and y_3, contrary to the assumption that x_0 and y_2 are not adjacent. Now, suppose that $z_0 = x_0$, i.e., $C_2 = (x, x_2, y_2, x_0)$ and $C_3 = (x, x_0, y_3, x_3)$. In this case, $y_2 = v_2$ and $y_3 = v_3$. If C_1 is also a 4-cycle, then we get an isometric Q_3^-, which must be completed to a 3-cube, otherwise v_1, y_2, and y_3 define a metric triangle whose gated hull is not a cell.

So, C_1 is a cycle of length at least 6. We assert that the gated hull of v_1, y_2, and y_3 is a cell isomorphic to $C_1 \times K_2$. For the sake of contradiction, assume that this is not the case and assume that C_1 has minimal length among all convex cycles with two 4-cycles attached to them such that they pairwise intersect in three different edges, all three in a vertex, and their convex hull is not a cell. If the vertices y_2, y_3 have a second common neighbor p, then we get an isometric Q_3^- which must be completed to a Q_3. Consequently, x_2 and x_3 have a common neighbor different from x, which is impossible because C_1 is convex. Thus x_0 is the unique common neighbor of y_2 and y_3. Let u_1^* be the apex of u_1 with respect to the pair y_2, y_3. We assert that $u_1 = u_1^*$. Suppose not and let u_1' be a neighbor of u_1 in $I(u_1, u_1^*)$. Consider the gate of u_1' in C_1. If this gate is not the vertex v_1, then it must be one of the neighbors of v_1 in C_1 and u_1' must be adjacent to this vertex. But if this is say the neighbor v_1' of v_1 in the path $I(v_1, x_2)$, then v_1' cannot belong to a shortest path between u_1' and x_3, whence v_1' cannot serve as a gate of u'_1. Thus v_1 must be the gate of u'_1 in C_1. In this case, $d(u'_1, x_2) = 2 + d(v_1, x_2) = d(u'_2, y_2) + 1$. Since $d(u'_1, y_2) = d(u_1, y_2) - 1$ and $d(u_1, y_2) + 1 = d(u_1, x_2) = 1 + d(v_1, x_2)$, we will obtain a contradiction. This shows that $u_1^* = u_1$, i.e., $I(u_1, y_2) \cap I(u_1, y_3) = \{u_1\}$. Since y_2 and y_3 are closer to u_1 than x_0 and x_0 is the unique common neighbor of y_2 and y_3, we conclude that the triplet u_1, y_2, y_3 defines a metric triangle. Hence $\langle\langle u_1, y_2, y_3\rangle\rangle$ is a gated cell U of G.

Since (y_2, x_0, y_3) is a convex path of length 2 of the cell $U = U_1 \times \ldots \times U_m$, necessarily (y_2, x_0, y_3) is contained in a layer of U which is a gated cycle C'_1 of G, say $C'_1 = U_1 \times u_2 \times \ldots \times u_m$ for a cyclic factor U_1 of length ≥ 6. First suppose that $u_1 \notin C'_1$. Then the length of C'_1 is smaller than the length of C_1. From the choice of C_1 and since C'_1 pairwise intersects the cycles C_2 and C_3, we conclude that the gated hull of $C'_1 \cup C_2 \cup C_3$ is a cell U' isomorphic to $C'_1 \times K_2$. But then in U' we can find a gated cycle C''_1 isomorphic to C'_1 and containing the convex path (x_2, x, x_3). Since C''_1 is shorter than C_1 and $x_1, x, x_3 \in C''_1 \cap C_1$, we obtain a contradiction with Claim 14. Now, let $u_1 \in C'_1$. Then obviously the cell U coincides with C'_1. Since C'_1 and C_1 are gated cycles of the same length and we have the edges v_1u_1, x_2y_2, xx_0, and x_3y_3, one can easily show that any vertex z' of C'_1 is adjacent to a unique vertex z of C_1 such that the subgraph H of G induced by $C_1 \cup C'_1$ is isomorphic to $C_1 \times K_2$. To conclude the proof of the 3CC-condition, it remains to show that H is a convex subgraph of G. For this it suffices to show that for any vertex $q \notin V(H)$ adjacent to a vertex p of H, q does not belong to a shortest path between p and some vertex q' of H. Suppose without loss of generality that $p \in C_1$ and let p' be the unique neighbor of p in C'_1. Then obviously p is the gate of q in C_1, thus $p \in I(q, r_1)$ for every $r_1 \in C_1$. Analogously, p' must be the gate of q in C'_1, otherwise since $d(q, p') = 2$, the gate of q must be one of the neighbors of p' in C'_1 and we obtain a $K_{2,3}$, which is forbidden in partial cubes. Therefore $p' \in I(q, r_2)$ for any vertex $r_2 \in C'_1$. Since $p \in I(q, p')$, we conclude that $p \in I(q, r_2)$. This implies that $p \in I(q, q')$, thus q cannot lie in $I(p, q')$. This establishes the 3CC-condition and concludes the proof of the theorem. □

6. Properties of hypercellular graphs

We continue with several properties of hypercellular graphs, in particular we prove Theorems D, E, and F. First, we show how hypercellular graphs are related with other known classes of partial cubes. We also establish some basic properties of geodesic convexity in hypercellular graphs and establish a fixed-cell property. Some of these results directly follow from Theorem B.

6.1. **Relations with other classes of partial cubes.** By one of their characterizations provided in [5], *bipartite cellular graphs* are the bipartite graphs in which all isometric cycles are gated. It is shown in [5] that bipartite cellular graphs are partial cubes and that any finite bipartite graph is a bipartite cellular graph if it can be can be obtained by successive gated amalgamations from its isometric cycles. In [40], Polat investigated a class of netlike partial cubes in which each finite convex subgraph is a gated amalgam of even cycles - let us call them *Polat graphs* for now. They are exactly the netlike partial cubes satisfying the median cycle property and generalize bipartite cellular graphs as well as median graphs. Theorem B and Theorem C have the following corollary:

Corollary 1. *Bipartite cellular graphs are precisely the graphs in $\mathcal{F}(Q_3^-, Q_3)$, while median graphs are precisely the graphs in $\mathcal{F}(Q_3^-, C_6)$ and Polat graphs are $\mathcal{F}(Q_3^-, C_6 \times K_2)$. In particular, the latter class contains the first two and all three classes are contained in the class of hypercellular graphs.*

Proof. Since the hypercellular graphs are exactly the graphs from $\mathcal{F}(Q_3^-)$, the last assertion follows from the first ones. Median graphs, bipartite cellular graphs, and Polat graphs are pc-minor closed families. Since Q_3^- and Q_3 are not cellular, Q_3^- and C_6 are not median, and Q_3^- and $C_6 \times K_2$ are not Polat graphs, this settles the inclusion of all three families in $\mathcal{F}(Q_3^-, Q_3)$, $\mathcal{F}(Q_3^-, C_6)$, and $\mathcal{F}(Q_3^-, C_6 \times K_2)$, respectively.

Conversely, let G be a graph from $\mathcal{F}(Q_3^-, Q_3)$. Since G is hypercellular, by Theorem B any finite convex subgraph of G can be obtained by successive gated amalgamations from cells. Since Q_3 is a forbidden pc-minor, all cells of G are edges or even cycles. Thus G is a bipartite cellular graph.

Analogously, let G be a graph from $\mathcal{F}(Q_3^-, C_6)$. Then G does not contain convex cycles of length ≥ 6. Hence any cell of G is a cube. Consequently, any finite convex subgraph of G can be obtained by successive gated amalgamations from cubes, i.e., G is median. Alternatively, by Theorem C, G satisfies the median cell property. Since, any cell of G is a cube, all median cells of G are vertices and therefore G is a median graph.

Finally, let G be a graph from $\mathcal{F}(Q_3^-, C_6 \times K_2)$. Since G is hypercellular, by Theorem B any finite convex subgraph of G can be obtained by successive gated amalgamations from cells. Since $C_6 \times K_2$ is a forbidden pc-minor, all cells of G are even cycles or cubes. Thus, G is a Polat graph. □

With a cell $X = F_1 \times \ldots \times F_m$ of G we associate a convex polyhedron $[X]$ obtained as a Cartesian product of segments and regular polygons, where each face F_i which is a K_2 is replaced by a unit segment and any face F_i which is an even cycle C of length $2n$ is replaced by a regular polygon with $2n$ sides. Hence $\dim(X)$ can be viewed as the (topological) dimension of $[X]$. Since by Lemma 17, in a hypercellular graph G the intersection of any two cells is also a cell, the union of all convex polyhedra $[X], X \in \mathbf{X}(G)$, can be viewed as a polyhedral cell complex, which we denote by $\mathbf{X}(G)$. The *dimension* $\dim(G)$ of a graph G from $\mathcal{F}(Q_3^-)$ is the dimension of this cell complex, i.e., the maximum dimension of a cell of G. Notice that the 1-skeleton of $\mathbf{X}(G)$ coincides with G and the 2-skeleton of $\mathbf{X}(G)$ coincides with $\mathbf{C}(G)$.

The following was announced as Theorem D in the introduction:

Corollary 2. *Any finite hypercellular graph G is the tope graph of a COM, more precisely, G is a tope graph of a zonotopal COM. Consequently, the zonotopal cell complex $\mathbf{X}(G)$ of any locally-finite hypercellular graph G is contractible.*

Proof. By [9, Proposition 3], each COM can be obtained from its maximal faces (which are all oriented matroids) using COM amalgamations. Since a gated amalgamation is a stronger version of a COM amalgamation and each Cartesian product of edges and even cycles is the tope graph of a realizable oriented matroid, Theorem B implies that each finite graph G from $\mathcal{F}(Q_3^-)$ is the tope graph of a zonotopal COM. From the contractibility of the cell complexes of all COMs established in [9, Proposition 14], it follows that for any finite hypercellular graph G its zonotopal complex $\mathbf{X}(G)$ is contractible.

Now, we will prove the contractibility of $\mathbf{X}(G)$ for any locally-finite hypercellular graph G. For this, we will represent G as a directed union of finite convex subgraphs G_i of G. Let v_0 be an arbitrary fixed vertex and let $B_i(v_0)$ be the ball of radius i centered at v_0. Since G is locally-finite, each such ball $B_i(v_0)$ is finite. Moreover, since G is a partial cube, the convex hull $\text{conv}(A)$ of any finite set A of G is finite (because $\text{conv}(A)$ coincides with the intersection of $V(G)$ with the smallest hypercube H of $H(\Lambda)$ hosting A and H is finite-dimensional). Hence the subgraph G_i of G induced by $\text{conv}(B_i(v_0))$ is a finite convex subgraph of G, and thus hypercellular. Therefore, by the first part, each of the zonotopal complexes $\mathbf{X}(G_i)$, $i \geq 0$, is contractible. Consequently, $\mathbf{X}(G)$ is the direct union $\bigcup_{i\geq 0} \mathbf{X}(G_i)$ of contractible complexes, thus $\mathbf{X}(G)$ is contractible by Whitehead's theorem. □

6.2. **Convexity properties.** The geodesic convexity of a graph $G = (V, E)$ satisfies the *join-hull commutativity property* (JHC) if for any convex set A and any vertex $x \notin A$, $\text{conv}(x \cup A) = \bigcup\{I(x, v) : v \in A\}$ [44] holds. It is well-know and easy to prove that JHC property is equivalent to the Peano axiom: if u, v, w is an arbitrary triplet of vertices, $x \in I(u, w)$ and $y \in I(v, x)$, then there exists a vertex $z \in I(v, w)$ such that $y \in I(u, z)$. A graph G is called a *Pasch-Peano graph* [10, 44] if the geodesic convexity of G satisfies the Pasch and Peano axioms. In particular, such a graph is in $\mathcal{S}_4$.

Corollary 3. *Any hypercellular graph G is a Pasch-Peano graph.*

Proof. Both the Pasch and the Peano axioms concern triplets of vertices u, v, w and vertices included in the convex hull of u, v, w. Since the convex hull of any finite set of vertices in a partial cube is finite, to prove that a hypercellular graph is Pasch-Peano, it suffices to prove that each finite hypercellular graph is Pasch-Peano. Since each of the Pasch and Peano axioms are preserved by gated amalgams and Cartesian products [10, 44], now the result directly follows Theorem B and the fact that cycles and edges are Pasch-Peano graphs. □

The *Helly number* $h(G)$ of a graph G is the smallest number $h \geq 2$ such that every finite family of (geodesically) convex sets meeting h by h has a nonempty intersection. The *Caratheodory number* $c(G)$ is the smallest number $c \geq 2$ such that for any set $A \subset V$ the convex hull of A is equal to the union of the convex hulls of all subsets of A of size c. The *Radon number* $r(G)$ of a graph G is the smallest number

$r \geq 2$ such that any set of vertices A of G containing at least $r+1$ vertices can be partitioned into two sets A_1 and A_2 such that $\text{conv}(A_1) \cap \text{conv}(A_2) \neq \emptyset$. More generally, the *mth partition number (Tverberg number)* is the smallest integer $p_m \geq 2$ such that any set of vertices A of G containing at least $p_m + 1$ vertices can be partitioned into m sets $A_1, \ldots, A_m$ such that $\cap_{i=1}^{m} \text{conv}(A_i) \neq \emptyset$. For a detailed treatment of all these fundamental parameters of abstract and graph convexities, see [44].

The following result is straightforward:

Lemma 20. *For $G \cong K_2$, $h(G) = r(G) = 2$ and $c(G) = 1$. If $G \cong C$, then $h(G) = r(G) \leq 3$ and $c(G) = 2$ ($h(G) = r(G) = 3$ if C is of length at least 6).*

Corollary 4. *Let G be a hypercellular graph. Then $h(G) \leq 3$, $c(G) \leq 2dim(G)$, and $r(G) \leq 10dim(G)+1$. More generally, $p_m \leq (6m-2)dim(G) + 1$.*

Proof. We will use the results of [44, Chapter II, §2] for Cartesian products and of [10] or [44, Chapter II, §3] for gated amalgams of convexity structures. Notice also that since in partial cubes convex hulls of finite sets are finite, it suffices to establish our results for finite hypercellular graphs G. By these results, $h(G_1 \times G_2) = \max\{h(G_1), h(G_2)\}$ and if G is the gated amalgam of G_1 and G_2, then $h(G) = \max\{h(G_1), h(G_2)\}$. By these formulas, Lemma 20, and Theorem B , we conclude that $h(G) \leq 3$ for any hypercellular graph G. In case of the Caratheodory number, we have $c(G_1 \times G_2) \leq c(G_1) + c(G_2)$ and $c(G) = \max\{c(G_1), c(G_2)\}$ if G is a gated amalgam of G_1 and G_2. By the first formula and Lemma 20, we conclude that if X is a Cartesian product of k' cyclic factors and k'' edges, then $c(X) \leq 3k' + 2k''$ and $\dim(X) = 2k' + k''$, yielding $c(X) \leq 2\dim(X)$. To deduce the upper bounds for Radon and partitions numbers, we will use the following inequality of [28] (see also [44, 5.15.1]) for all convexities: $p_m(G) \leq c(G)(m \cdot h(G) - 1) + 1$. Replacing in this formula $h(G) = 3$ and $c(G) \leq 2\dim(G)$, we obtain the required inequalities. □

For cellular graphs, an exact bound $p_m \leq 3m$ for partition number was obtained in [26].

We conclude this subsection with a local-to-global characterization of convex and gated sets in hypercellular graphs. A similar characterization of gated sets was obtained for bipartite cellular graphs [5, Proposition 1] and netlike graphs [38, Theorem 6.2]. Notice also that other local characterizations of convex and gated sets are known for weakly modular graphs [22]. The following result is similar to the content of Claim 13.

Proposition 9. *A connected subgraph H of a hypercellular graph G is convex (respectively, gated) if and only if the intersection of H with each cell of G is convex (respectively, gated).*

Proof. We closely follow the proof of Proposition 1 of [5]. Necessity is evident: any cell X of G is convex and gated, therefore X intersect each convex (respectively, gated) subgraph in a convex (respectively, gated) subgraph.

As to the converse, in both cases we will first show that H is convex. For two vertices y, z we denote by $k(y, z) := d_H(y, z)$ the distance between y and z in H. Suppose the contrary and let v, x be two vertices of H minimizing $k(y, z)$ such that $I(v, x)$ is not included in H. Then there exists a shortest (v, x)-path Q whose inner vertices do not belong to H. Let P be any path of minimal length joining v and x inside H. We assert that P is a shortest path of G. By the choice of v, x, the paths P and Q intersect only in v and x. Let u be a neighbor of v in P. If P were longer than Q, then $u \notin I(v, x)$, whence $v \in I(u, x)$. Since $k(u, x) < k(v, x)$, by the minimality in the choice of the pair v, x we conclude that $Q \subset I(u, x) \subset H$, a contradiction. Thus P is a shortest path of G. Let w be the neighbor of v in Q. If the vertices w and u have a common neighbor y different from v, since $u, w \in I(v, x)$ and $y \in I(w, u)$, from the convexity of the interval $I(v, x)$ we conclude that $y \in I(v, x)$. This implies that $y \in I(w, x) \cap I(u, x)$. Since $I(u, x) \subset H$, we conclude that $y \in H$. Since $w \notin H$, the intersection of H with the square (v, u, y, w) (which is a cell of G) is not convex. This contradiction shows that $I(u, w) = \{u, v, w\}$. Hence $I(u, w) \cap I(u, x) = \{u\}$ and $I(w, u) \cap I(w, x) = \{w\}$. On the other hand, the minimality in the choice of the pair v, x implies that $I(x, u) \cap I(x, w) = \{x\}$. Consequently, the triplet x, u, w defines a metric triangle of G. By Proposition 8, this metric triangle xuw is included in a cell X of G. Since X is convex and $v \in I(u, w)$, we obtain $v \in X$. But then $X \cap H$ is not convex because $v, x \in X \cap H$ and $w \in I(v, x) \setminus H$. This contradiction shows that H is convex.

Now suppose that the intersection of H with each cell of G is gated. Suppose that H is not gated. Choose a vertex z at minimum distance to H having no gate in H. Let x be a vertex of H closest to z, and let y be a vertex of H such that the interval $I(z,y)$ does not contain x, where $d(x,y)$ is as small as possible. Then the intervals $I(x,y), I(y,z)$, and $I(z,x)$ intersect each other only in the common end vertices. Hence x, y, and z define a metric triangle xyz. By Proposition 8, xyz is contained in a cell X. Since $z \notin H$ and $x, y \in H$, the choice of the vertices z and x, y implies that $X \cap H$ is not gated. This contradiction establishes that H is gated and concludes the proof. □

6.3. **Stars and thickening.** A *star* $\mathrm{St}(v)$ of a vertex v (or a star $\mathrm{St}(X)$ of a cell X) is the union of all cells of G containing v (respectively, X).

Proposition 10. *For any cell X of a hypercellular graph G in which all cells are of finite dimension, the star $\mathrm{St}(X)$ is gated.*

Proof. Since $\mathrm{St}(X)$ is a connected subgraph of G, by Proposition 9 it is enough to prove that the intersection of $\mathrm{St}(X)$ with any cell Y of G is gated. We apply Lemma 12 to the cell Y to show that $\mathrm{St}(X) \cap Y$ is gated. First, notice that $\mathrm{St}(X) \cap Y$ is connected. Indeed, let $X_1, X_2 \dots$ be the maximal cells of $\mathrm{St}(X)$ intersecting Y. Since $X_1, X_2 \dots$ intersect in X and each of these cells intersects Y, by the Helly property for gated sets, $Y \cap X_1 \cap X_2 \dots$ is non-empty and gated. Thus, any two vertices of $\mathrm{St}(X) \cap Y$ can be connected with a path in $\mathrm{St}(X) \cap Y$ passing via this intersection, whence $\mathrm{St}(X) \cap Y$ is connected.

Let $P = (y', x, y'')$ be any 2-path in $\mathrm{St}(X) \cap Y$ and let $C = \langle\langle P \rangle\rangle$ be its gated hull in Y. We will prove that C is included in $\mathrm{St}(X)$. This is obviously so if the path P is included in a single cell X_i of $\mathrm{St}(X)$. Indeed, in this case C is included in the gated subgraph $X_i \cap Y$. So, assume that the edges $y'x$ and xy'' do not belong to a common cell of $\mathrm{St}(X)$. Notice that each of these edges belong to a cell of $\mathrm{St}(X)$: for example, the edge xy' belongs to all cells of $\mathrm{St}(X)$ that contain a furthest from X vertex of the pair $\{x, y'\}$. Let X_i be a cell of $\mathrm{St}(X)$ including the edge xy'. Analogously, let X_j be a cell of $\mathrm{St}(X)$ including the edge xy''. By what was assumed above, X_i and X_j are not included in each other, in particular $X_i \neq X_j$. Let $X_{ij} := X_i \cap X_j$. Then X_{ij} is a cell of $\mathrm{St}(X)$ containing x but not containing y' and y''.

Let Z be a maximal cell of the form $Z = \langle\langle C \cup Z' \rangle\rangle$ for some subcell Z' of X_{ij} containing x. Note that such a cell exists since Z' can be chosen to be x and C is a cell of Y containing x. Since the intersection of cells is a cell, we can further assume that $Z' := X_{ij} \cap Z$. We assert that $Z = \langle\langle C \cup X_{ij} \rangle\rangle$. Suppose that this is not the case. Then there exists an edge $zw \in X_{ij}$ with $z \in Z' = X_{ij} \cap Z$ and $w \notin Z$. Let k be the dimension of Z'. We will use the following property of cells of hypercellular graphs, which is a direct consequence of Lemma 11:

Claim 15. *If D' is a subcell of dimension ℓ of a cell D of G and $v'v$ is an edge with $v' \in D'$ and $v \in D \setminus D'$, then* $\dim(\langle\langle D' \cup v'v \rangle\rangle) = \ell + 1$.

Since $Z' \cup xy' \subset X_i$ with $x \in Z', y' \notin Z' \subset X_{ij}$, by the previous claim the dimension of $\langle\langle Z' \cup xy' \rangle\rangle$ is $k+1$. Similarly, the dimension of $\langle\langle Z' \cup xy'' \rangle\rangle$ is $k+1$. Moreover, $Z' \cup zw \subset X_{ij}$ with $z \in Z', w \notin Z'$, thus $\langle\langle Z' \cup zw \rangle\rangle$ has also dimension $k+1$. Now we have $xy', \langle\langle Z' \cup zw \rangle\rangle \subset X_i$ with $x \in \langle\langle Z' \cup zw \rangle\rangle$ since $x \in Z'$ and $y' \notin \langle\langle Z' \cup zw \rangle\rangle$ since y' is not in X_{ij}, thus $\langle\langle Z' \cup zw \cup xy' \rangle\rangle$ has dimension $k+2$. Analogously, $\langle\langle Z' \cup zw \cup xy'' \rangle\rangle$ has dimension $k+2$. Finally, since $xy'', \langle\langle Z' \cup xy' \rangle\rangle \subset Z$ with $x \in \langle\langle Z' \cup xy' \rangle\rangle$ and $y'' \notin \langle\langle Z' \cup xy' \rangle\rangle$, the dimension of $\langle\langle Z' \cup xy' \cup xy'' \rangle\rangle$ is also $k+2$.

Consequently, we have proved that $\langle\langle Z' \cup zw \cup xy' \rangle\rangle$, $\langle\langle Z' \cup zw \cup xy'' \rangle\rangle$, and $\langle\langle Z' \cup xy' \cup xy'' \rangle\rangle$ are three cells of dimension $k+2$ that pairwise intersect in the cells $\langle\langle Z' \cup zw \rangle\rangle$, $\langle\langle Z' \cup xy' \rangle\rangle$, and $\langle\langle Z' \cup xy'' \rangle\rangle$ of dimension $k+1$ and the intersection of all three cells is the cell Z' of dimension k. By Theorem B(iii), there is a $(k+3)$-dimensional cell W that includes all of them. In particular, $W = \langle\langle C \cup Z' \cup zw \rangle\rangle$. Since the gated hull of $Z' \cup wz$ is a subcell Z'' of X_{ij} properly containing Z' and since $W = \langle\langle C \cup Z'' \rangle\rangle$, we obtain a contradiction to the maximality of Z.

Thus, $Z = \langle\langle C \cup X_{ij} \rangle\rangle$ is a cell of $\mathrm{St}(X) \cap Y$ including C and $X \subset X_{ij}$, whence $C \subset \mathrm{St}(X)$. Lemma 12 implies that $\mathrm{St}(X) \cap Y$ is gated. □

The *thickening* G^{Δ} of a hypercellular graph G is a graph having the same set of vertices as G and two vertices u, v are adjacent in G^{Δ} if and only if u and v belong to a common cell of G. A graph H is called a *Helly graph* if any collection of pairwise intersecting balls of G has a nonempty intersection [7]. Analogously, H is called a *1-Helly graph* (respectively, *clique-Helly graph*) if any collection of pairwise intersecting 1-balls (balls of radius 1) of G (respectively, of maximal cliques) has a nonempty intersection.

Proposition 11. *The thickening G^{Δ} of a locally-finite hypercellular graph G is a Helly graph.*

Proof. Pick any vertex v of G and let $B_1(v)$ denote the ball of radius 1 of G^{Δ} centered at v. From the definition of G^{Δ} it immediately follows that $B_1(v)$ is isomorphic to the star $\mathrm{St}(v)$ of v in G. Since G is locally-finite, $\mathrm{St}(v)$ is finite. By Proposition 10, $\mathrm{St}(v)$ is a gated subgraph of G. By the Helly property of finite gated sets, we conclude that G^{Δ} is a 1-Helly graph. Any maximal clique K of G^{Δ} is the intersection of all 1-balls centered at the vertices of K, therefore the family of maximal cliques of G^{Δ} can be obtained as the intersections of 1-balls of G^{Δ}. By [18, Remark 3.6], G^{Δ} is a clique-Helly graph. By Theorem D the zonotopal cell complex $\mathbf{X}(G)$ of G is contractible and therefore simply connected. This easily implies that the clique complex of G^{Δ} is simply connected. Consequently, G^{Δ} is a clique-Helly graph with a simply connected clique complex. By [18, Theorem 3.7], G^{Δ} is a Helly graph. □

Propositions 10 and 11 together with Proposition 7 conclude the proof of Theorem E.

6.4. **Fixed cells.** In this subsection we prove Theorem F. First, we follow ideas of Tardif [42] to generalize fixed box theorems for median graphs to hypercellular graphs. We will prove that the fixed box in the case of hypercellular graphs is a cell. We obtain this cell verbatim as in the case of median graphs. Set

$$F(G) := \{W : W \text{ is an inclusion maximal proper halfspace of } G \text{ and } V(G) \setminus W \text{ is not}\}.$$

Let $Z(G) = \bigcap_{W \in F(G)} W$ (if $F(G) = \emptyset$, then set $Z(G) := G$). Now we recursively define $Z^{\infty}(G)$. Set $Z^0(G) := G$ and for every ordinal α, let $Z^{\alpha+1}(G) := Z(Z^{\alpha}(G))$ if $Z^{\alpha}(G)$ has been defined and let $Z^{\alpha}(G) := \cap_{\beta<\alpha} Z^{\beta}(G)$ if α is a limit ordinal. For every graph G there exists a minimal ordinal γ such that $Z^{\gamma}(G) = Z^{\gamma+1}(G)$. Finally, define $Z^{\infty}(G) := \cap_{\gamma} Z^{\gamma}(G)$.

Lemma 21. *Let G be a hypercellular graph not containing infinite isometric rays. Then $Z^{\infty}(G)$ is a finite cell of G.*

Proof. First notice that since G does not contain infinite isometric rays and all cells of G are Cartesian products of cycles and edges, all cells of G are finite. Since $Z^{\infty}(G)$ is an intersection of convex subgraphs, $Z^{\infty}(G)$ is convex. Since every cell X of G is a Cartesian product of edges and even cycles, any proper halfspace of X is maximal by inclusion, hence $F(X) = \emptyset$. Therefore, $Z(X) = X$. Suppose by way of contradiction that there exists a convex subgraph H of G which is not a cell and such that $Z(H) = H$. Since H is convex, H is hypercellular. Since there are no infinite isometric rays, $Z(H) = H$ if and only if for every edge ab of H the halfspaces $W(a, b)$ and $W(b, a)$ are maximal, which is equivalent to the condition that for every edge ab the carrier $N(E_{ab})$ is the whole graph H. Let X be the intersection of all maximal cells of H; consequently, X is a cell of H. As in the proof of Theorem 2, if there exist two disjoint maximal cells of H, then for every edge f on a shortest path between them, the carrier $N(E_f)$ is not the whole graph H. Hence the maximal cells of H pairwise intersect. Since they are finite and gated, by the Helly property for gated sets, X is nonempty. If $X \neq H$, by Claim 10, there exists an edge of H whose carrier does not include all maximal cells, a contradiction. Hence $H = X$, i.e., H is a finite cell. Consequently, $Z^{\infty}(G)$ is a finite cell of G. □

We continue with the proof of assertion (i) of Theorem F.

Proposition 12. *If G is a hypercellular graph not containing infinite isometric rays, then there exists a finite cell X in G fixed by every automorphism of G.*

Proof. Every automorphism φ of G maps maximal halfspaces to maximal halfspaces, thus $\varphi(Z^{\infty}(G)) = Z^{\infty}(G)$. By Lemma 21, $Z^{\infty}(G)$ is a finite cell, thus every automorphism of G fixes the cell $Z^{\infty}(G)$. □

A *non-expansive map* from a graph G to a graph H is a map $f : V(G) \to V(H)$ such that for any $x, y \in V(G)$ it holds $d_H(f(x), f(y)) \leq d_G(x, y)$.

Lemma 22. *Let G be a hypercellular graph and f be a non-expansive map from G to itself. Let u, v, w be any three vertices of G and let $X = \langle\langle u', v', w' \rangle\rangle$ be their median-cell. If $f(u) = u, f(v) = v, f(w) = w$, then f fixes each of the apices $u' = (uvw), v' = (vwu), w' = (wuv)$ and $f(X) = X$.*

Proof. Denote by $u'v'w'$ the unique quasi-median of the triplet u, v, w. The map f fixes each of the vertices u, v, w and maps shortest paths between them to shortest paths. This implies that f fixes the vertices of the metric triangle $u'v'w'$. Let $X = \langle\langle u', v', w' \rangle\rangle$ be the gated cell induced by this triplet. Since $u'v'w'$ is a metric triangle, we conclude that $X \cong C_1 \times \ldots \times C_k$, where each C_i is an even cycle of length n_i at least 6. Moreover, we can suppose without loss of generality that u', v', w' are embedded in this product as $u' = (0, 0, \ldots, 0)$, $v' = (i_1, i_2, \ldots, i_k)$, and $w' = (j_1, j_2, \ldots, j_k)$, where $i_m, j_m - i_m, n_m - j_m < n_m/2$ for all $1 \le m \le k$. Since $f(u') = u', f(v') = v', f(w') = w'$ and any vertex of X lies on a shortest path between one of the pairs of u', v', w', we conclude that $f(X) \subset X$. It remains to prove that $f(X) = X$.

Without loss of generality assume that among all $j_m - i_m$ with $1 \le m \le k$, the difference $j_1 - i_1$ is minimal. The vertex $y = (i_1, 0, 0, \ldots, 0)$ belongs to $I(u', v')$ and is located at distance $j_1 - i_1$ from $z = (j_1, 0, 0, \ldots, 0) \in I(w', u')$. The vertices of $I(u', v')$ at distance i_1 from u' have the form $(i_1 - y_1, y_2, \ldots, y_k)$, for $0 \le y_i \le i_m, 1 \le m \le k$, with $y_2 + \ldots + y_k = y_1$. On the other hand, the vertices of $I(w', u')$ at distance $n_1 - j_1$ from u' have the form $(j_1 + z_1, n_2 - z_2, \ldots, n_k - z_k)$ for $0 \le z_m \le n_m - j_m, 1 \le m \le k$ with $z_2 + \ldots + z_k = z_1$, where the m-th coordinate is computed in $\mathbb{Z}_{n_m}$. We will now find all pairs (y', z') where $y' \in I(u', v'), z' \in I(w', v')$ and y' and z' are at distance $j_1 - i_1$.

We distinguish two cases. On one hand assume that for a chosen pair y', z' there exists a coordinate $m, 1 \le m \le k$ such that the projections of (y', z')-shortest paths to the m-th coordinate belong to the interval between i_m and j_m. Then the distance between y' and z' is at least $j_m - i_m$, and since $j_m - i_m \ge j_1 - i_1$, we have $y' = (0, \ldots, 0, i_m, 0, \ldots, 0)$ and $z' = (0, \ldots, 0, j_m, 0, \ldots, 0)$ with $j_m - i_m = j_1 - i_1$ and $i_m = i_1, n_m - j_m = n_1 - j_1$. This implies $j_m = j_1$ and $n_m = n_1$. Assume now that f maps y, z to y', z'. There exists an automorphism φ of X that swaps coordinates 1 and m of X and fixes u', v', w'. Since proving that $f(X) = X$ is the same as proving that $\varphi(f(X)) = X$, we can, in this case, assume that f fixes y, z.

On the other hand, if for a pair y', z' and every coordinate $m, 1 \le m \le k$, the projection of (y', z')-shortest paths to the m-th coordinate does not belong to the interval between i_m and j_m, then the distance between $y' = (i_1 - y_1, y_2, \ldots, y_k)$ and $z' = (j_1 + z_1, n_2 - z_2, \ldots, n_k - z_k)$ is $((i_1 - y_1) + (n_1 - j_1 - z_1)) + (y_2 + z_2) + \ldots + (y_k + z_k) = n_1 - (j_1 - i_1) > n/2 > j_1 - i_1$. Since this is impossible, we can by the previous paragraph assume that f fixes y and z. Then f fixes $(0, 0, \ldots, 0), (i_1, 0, \ldots, 0), (j_1, 0, \ldots, 0)$, thus it must fix every $(x, 0, \ldots, 0)$, $0 \le x < n_1$.

Now we will prove that every cyclic layer of the form $(C_1, x_2, \ldots, x_k)$ is mapped by f to a cyclic layer of the form $(C_1, x_2', x_3', \ldots, x_k')$. We proceed by induction on $x_2 + x_3 + \ldots + x_k$. It holds for $(C_1, 0, 0, \ldots, 0)$. Without loss of generality consider only $f(C_1, x_2 + 1, x_3, \ldots, x_k)$, assuming that $f(C_1, x_2, \ldots, x_k) = (f_1(C_1), y_2, y_3, \ldots, y_k)$ for some automorphism f_1 of C_1.

For every $x_1 \in C_1$, the vertex $y = f(x_1, x_2 + 1, x_3, \ldots, x_k)$ must be equal or adjacent to $(f_1(x_1), y_2, y_3, \ldots, y_k)$, thus it must be of the form $(f_1(x) + s, y_2, y_3, \ldots, y_k)$ or $(f_1(x), y_2, \ldots, y_m + s, \ldots, y_k)$ for some $2 \le m \le k$ and $s \in \{-1, 0, 1\}$. Now we analyze the options for $a = f(x + n_1/2 - 1, x_2 + 1, x_3, \ldots, x_k)$ and $b = f(x + n_1/2 + 1, x_2 + 1, x_3, \ldots, x_k)$. Both must be at distance at most two from each other, at distance at most $n_1/2 - 1$ from y, and a must be adjacent or equal to $(f_1(x + n_1 - 1), y_2, y_3, \ldots, y_k)$ while b must be adjacent or equal to $(f_1(x + n_1 + 1), y_2, y_3, \ldots, y_k)$. If $y = (f_1(x) + s, y_2, y_3, \ldots, y_k)$, this implies that $a = (f_1(x + n_1/2 - 1) + s, y_2, y_3, \ldots, y_k)$ and $b = (f_1(x + n_1/2 + 1) + s, y_2, y_3, \ldots, y_k)$. If $y = (f_1(x), y_2, \ldots, y_m + s, \ldots, y_k)$, then $a = (f_1(x + n_1/2 - 1), y_2, \ldots, y_m + s, \ldots, y_k)$ and $b = (f_1(x + n_1/2 + 1), y_2, \ldots, y_m + s, \ldots, y_k)$. In each case y, a, b spans a cycle, since the length of C_1 is at least six and f is a non-expansive map. Thus $f(C_1, x_2 + 1, x_3, \ldots, x_k)$ is a cycle of the form $(f_1(C) + s, y_2, y_3, \ldots, y_k)$ or $(f_1(C), y_2, \ldots, y_m + s, \ldots, y_k)$ for some $2 \le m \le k$ and $s \in \{-1, 0, 1\}$, proving the assertion.

We have proved that f acting on X has blocks of imprimitivity of the form $(C_1, x_2, \ldots, x_k)$ and it holds $f(C_1, 0, \ldots, 0) = (C_1, 0, \ldots, 0)$, $f(C_1, i_2, \ldots, i_k) = (C_1, i_2, \ldots, i_k)$ and $f(C_1, j_2, \ldots, j_k) = (C_1, j_2, \ldots, j_k)$. By the induction on the number of factors of X, f acts as an automorphism on the quotient graph, thus f acts as an automorphism on X. □

An endomorphism r of G with $r(G) = H$ and $r(v) = v$ for all vertices v in H is called a *retraction* of G and H is called a *retract* of G.

Corollary 5. *A retract H of a hypercellular graph G is a hypercellular graph.*

Proof. Let r be a retraction of G to H. For arbitrary vertices u, v, w of H it holds $r(u) = u, r(v) = v, r(w) = w$, thus by Lemma 22 $X = r(X) \subseteq r(G) = H$, where X is the median cell of u, v, w. This proves that H satisfies the median-cell property and by Theorem C, H is hypercellular. □

We continue with the proof of assertion (ii) of Theorem F.

Proposition 13. *Let G be a hypercellular graph and let f be a non-expansive map from G to itself such that $f(S) = S$ for some finite set S of vertices of G. Then there exists a finite cell X of G that is fixed by f. In particular, if G is a finite hypercellular graph, then it has a fixed cell.*

Proof. Let H be the subgraph of G induced by the set of all vertices v in $\text{conv}(S)$ for which there exists an integer $n_v > 0$ such that $f^{n_v}(v) = v$. Since S is finite, also $\text{conv}(S)$ is finite, therefore H is finite and nonempty. Notice that $f(\text{conv}(S)) \subseteq \text{conv}(S)$, thus $H \supseteq f(H) \supseteq f(f(H)) \supseteq \ldots$, but since for every $v \in V(H)$ there exists n_v such that $f^{n_v}(v) = v$, the inclusions cannot be strict. Thus $f(H) = H$ and f acts as an automorphism on H.

Let $u, v, w \in V(H)$ be arbitrary vertices of H. Let n be the least common multiple of n_u, n_v, n_w. Then f^n fixes each of the vertices u, v, w. By Lemma 22, $f^n(X) = X$ where X is the median cell of u, v, w. Since X is finite, this proves that also $X \subset H$. Therefore H satisfies the median-cell property and, by Theorem C, H is hypercellular. Applying Proposition 12 to H, we deduce that there exists a finite fixed cell. □

The above proposition follows the ideas from [42], but the main difficulty is to prove Lemma 22. In the case of median graphs this lemma is not needed since X is a single vertex. The next proposition uses ideas from [32] to generalize yet another classical result on median graphs and to prove assertion (iii) of Theorem F.

Proposition 14. *If G is a finite regular hypercellular graph, then G is a single cell, i.e., G is isomorphic to a Cartesian product of edges and even cycles.*

Proof. Pick an arbitrary edge ab in G such that $W(a, b)$ is an inclusion minimal halfspace. We will prove that also $W(b, a)$ is minimal. The carrier $N(E_{ab})$ is the union of maximal cells of G crossed by E_{ab}. For each such maximal cell X of $N(E_{ab})$ there exists a unique automorphism of X that fixes edges of $E_{ab} \cap X$ and maps $X \cap W(a, b)$ to $X \cap W(b, a)$ and vice versa. This automorphisms extends to an automorphism φ of $N(E_{ab})$ that maps $N(E_{ab}) \cap W(a, b)$ to $N(E_{ab}) \cap W(b, a)$ and vice versa. For the sake of contradiction assume now that $W(b, a)$ is not minimal and that there exists an edge cd with $c \in N(E_{ab}) \cap W(b, a)$ and $d \notin N(E_{ab})$. The vertex $c' = \varphi(c) \in W(a, b)$ has the same degree in $N(E_{ab})$ as c. Since G is regular, there must exist an edge $c'd'$ with $d' \notin N(E_{ab})$. Since $N(E_{ab})$ is gated, $E_{c'd'}$ does not cross the carrier $N(E_{ab})$, thus $W(d', c') \subsetneq W(a, b)$. This contradicts the choice of ab.

Since G is finite, we have proved that for every edge ab, $W(a, b)$ and $W(b, a)$ are minimal halfspaces. Thus for every edge ab, we have $N(E_{ab}) = G$. This implies that $Z(G) = G$, thus $Z^{\infty}(G) = G$. By Lemma 21, G is a single cell. □

7. Conclusions and open questions

In the present paper we have established a rich cell-structure for hypercellular graphs. In particular, we have obtained that they generalize bipartite cellular and median graphs in a natural way. On the other hand, we expect that other properties and characterizations of median graphs or cellular graphs extend naturally to hypercellular graphs. Some of those questions concern the metric structure of hypercellular graphs, while other questions ask for replacing metric conditions by topological or algebraic conditions. Namely, in all our results we characterized hypercellular graphs among partial cubes. On can ask to what extent we can characterize hypercellular graphs and their cell complexes among all graphs and complexes.

For example, as we noticed already, by a result of Gromov [30], CAT(0) cube complexes are exactly the simply connected cube complexes satisfying the cube condition, i.e., the cubical version of the 3C-condition. As proved in [24], median graphs are exactly the graphs whose associated cube complexes are CAT(0). In fact, it is shown in [24] (see also [14] for other similar results) that median graphs are exactly the graphs of square complexes which are simply connected and satisfy the square version of the 3CC-condition: any three squares pairwise intersecting in three edges and all three intersecting in a vertex belong to a 3-cube. We believe that a similar result holds for hypercellular graphs. Namely, let $\mathbf{X}$ be a hyperprism complex, i.e., a polyhedral cell complex whose cells are Cartesian products of segments and regular polygons with an even number of sides and glued in a such a way that the intersection of any two cells is a cell. The 1-skeleton of $\mathbf{X}$ is the graph $G(\mathbf{X})$ having the 0-cells of $\mathbf{X}$ as vertices and 1-cells as edges. Finally, we call a cell complex of a hypercellular graph a *hypercellular complex*. We conjecture that hypercellular graphs can be characterized in the following way:

Conjecture 1. For a graph G, the following conditions are equivalent:

(i) G is hypercellular;
(ii) G is the 1-skeleton of a simply connected hyperprism complex $\mathbf{X}$ satisfying the 3C-condition;
(iii) G is the 1-skeleton of a simply connected polygonal complex $\mathbf{X}$ (whose 2-cells are regular polygons with an even number of sides) satisfying the 3CC-condition.

Moreover, all hypercellular cell complexes are CAT(0) spaces.

Since CAT(0) spaces obey the fixed point property [17], the fact that hypercellular cell complexes are CAT(0) spaces would immediately imply Proposition 13.

Median graphs are exactly the discrete median algebras (for this and other related results see the paper [11] and the survey [7]). In [6], the apex algebras of weakly median graphs have been characterized. The *apex algebra* of an apiculate graph G associates to each triplet of vertices u, v, w, the apex (uvw) of u with respect to v and w.

Problem 1. Characterize the apex ternary algebras of hypercellular graphs.

In view of the fact that median graphs are precisely retracts of hypercubes [3], we believe that the following is true:

Conjecture 2. A partial cube G is hypercellular if and only if G is a retract of a Cartesian product of bipartite cellular graphs.

A group F *acts by automorphisms* on a cell complex $\mathbf{X}$ if there is an injective homomorphism $F \to \mathrm{Aut}(\mathbf{X})$ called an *action of* F. The action is *geometric* (or F *acts geometrically*) if it is proper (i.e., cells stabilizers are finite) and cocompact (i.e., the quotient $\mathbf{X}/F$ is compact). A group F is called a *Helly group* [19] if F acts geometrically on the clique complex of a Helly graph. Analogously, we will say that a group F is *hypercellular* if F acts geometrically on a cell complex $\mathbf{X}(G)$ of a hypercellular graph G (in this case, G is locally-finite). Analogously to [18, Proposition 6.32] one can show that F acts geometrically on the clique complex of the thickening G^{Δ} of G. By Proposition 11, G^{Δ} is a Helly graph, thus any hypercellular group is a Helly group. Since all Helly groups are biautomatic [19], we obtain the following corollary:

Corollary 6. *Any hypercellular group is a Helly group and thus is biautomatic.*

In Theorem D we have shown that finite hypercellular graphs are tope graphs of zonotopal COMs. In [9] the question is raised whether zonotopal COMs are fibers of realizable COMs. In our case this specializes to solving the following:

Problem 2. Is every finite hypercellular graph a convex subgraph of the tope graph of a realizable oriented matroid?

In the first part of the paper we have obtained a few results for $\mathcal{S}_4$ similar to those for hypercellular graphs. We believe, that it is possible to use analogous amalgamation techniques as we did for Theorem B and Theorem D in order to prove:

Conjecture 3. Every finite graph in $\mathcal{S}_4$ is the tope graph of a COM.

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Corners and simpliciality in oriented matroids and partial cubes

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April 21, 2021

Abstract

Building on a recent characterization of tope graphs of Complexes of Oriented Matroids (COMs), we tackle and generalize several classical problems in Oriented Matroids, Lopsided Sets (aka ample set systems), and partial cubes via Metric Graph Theory. These questions are related to the notion of simpliciality of topes in Oriented Matroids and the concept of corners in Lopsided Sets arising from computational learning theory.

Our first main result is that every element of an Oriented Matroid from a class introduced by Mandel is incident to a simplicial tope, i.e, such Oriented Matroids contain no mutation-free elements. This allows us to refute a conjecture of Mandel from 1983, that would have implied the famous Las Vergnas simplex conjecture.

The second main contribution is the introduction of corners of COMs as a natural generalization of corners in Lopsided Sets. Generalizing results of Bandelt and Chepoi, Tracy Hall, and Chalopin et al. we prove that realizable COMs, rank 2 COMs, as well as hypercellular graphs admit corner peelings. On the way we introduce the notion of cocircuit graphs for pure COMs and disprove a conjecture about realizability in COMs of Bandelt et al.

Finally, we study extensions of Las Vergnas' simplex conjecture in low rank and order. We first consider antipodal partial cubes – a vast generalization of oriented matroids also known as acycloids. We prove Las Vergnas' conjecture for acycloids of rank 3 and for acycloids of order at most 7. Moreover, we confirm a conjecture of Cordovil-Las Vergnas about the connectivity of the mutation graph of Uniform Oriented Matroids for ground sets of order at most 9. The latter two results are based on the exhaustive generation of acycloids and uniform oriented matroids of given order, respectively.

1 Introduction

The *hypercube* Q_n of dimension n is the graph whose vertex set is $\{+,-\}^n$ and two vertices are adjacent if they differ in exactly one coordinate. A graph G is called a *partial cube* if G is an isometric subgraph of a hypercube Q_n, i.e., $d_G(u,v) = d_{Q_n}(u,v)$ for all $u, v \in G$. This class is central to Metric Graph Theory, has applications from Chemistry [16] to Media Theory [17], and contains many graph classes appearing naturally in many places, e.g. diagrams of distributive lattices, antimatroids, median graphs, skeleta of CAT(0) cube complexes, linear extensions graphs of posets, region graphs of pseudoline arrangements and hyperplane arrangements, and Pasch graphs. The classes of interest in this paper contain all the above, see [4, 29]. Namely, we are concerned with tope graphs of Complexes of Oriented Matroids (COMs) [4], Affine Oriented Matroids (AOMs) [6], Lopsided Sets (LOPs) [33], and most of all Oriented Matroids (OMs) and Uniform Oriented Matroids (UOM) [6]. COMs are a recent common generalization of the other more established classes listed above. However, this generalization has been acknowledged already several times in its short existence, see [5, 15, 23, 25, 31, 38, 41, 44].

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While all the above families can be seen as subfamilies of partial cubes, they have been subject to much more research than partial cubes in general. However, the point of view of partial cubes allows an intuitive approach to these structures. In this paper we aim at promoting an analysis of Oriented Matroids and related structures through their graph structure. The tope graph determines its OM uniquely up to isomorphism [6], but only recently a good graph theoretic characterization has been found [29] from the point of view of COMs.

Based on these results, we start by presenting OMs, COMs, AOMs, UOMs, and LOPs solely as graphs, in particular as a subfamily of partial cubes. We begin with necessary standard definitions from Metric Graph Theory: For a vertex v of the hypercube we call the vertex with all its coordinates flipped the *antipode* of v. As stated above, a partial cube is an isometric subgraph of a hypercube, where the minimal dimension it embeds into is called its *isometric dimension*. In a COM or OM the isometric dimension of its tope graph corresponds to the size of its ground set. We call a partial cube G of isometric dimension n *antipodal* if when embedded in Q_n for every vertex v of G also its antipode with respect to Q_n is in G. Antipodal partial cubes are the tope graphs of *acycloids* introduced in [20] as generalizations of OMs. A subgraph H of G is *convex*, if all the shortest paths in G connecting two vertices of H are also in H. Convex subgraphs of partial cubes are partial cubes. A subgraph of G is *antipodal* if it is a convex subgraph of G and an antipodal partial cube on its own. Finally, we call a subgraph H of a graph G *gated* if for every vertex x of G there exists a vertex v_x of H such that for every $u \in H$ there is a shortest path from x to u passing through v_x. Gated subgraphs are convex. Embedding a partial cube G in Q_n yields a partition of the edges of G into so-called Θ-*classes* corresponding to the dimensions of Q_n. Each Θ-class consists of the edges corresponding to a flip of a fixed coordinate. We denote by E_f a Θ-class of G where f corresponds to a coordinate in $\{1,\dots,n\}$. We write E_f^+ for the vertices having the coordinate f equal to $+$ and analogously define E_f^-. The sets E_f^+ and E_f^- are called *halfspaces* of G. Note that in the standard language of oriented matroids, Θ-classes correspond to *elements* of the OM, hence also isometric dimension is sometimes just called the number of elements.

We are ready to state a characterisation – that serves as a definition in this paper – of (simple) COMs, OMs, AOMs, LOPs, and UOMs:

Theorem 1.1 ([29]). *There is a one to one correspondence between the classes of (simple):*

(i) COMs and partial cubes whose antipodal subgraphs are gated,

(ii) OMs and antipodal partial cubes whose antipodal subgraphs are gated,

(iii) AOMs and halfspaces of antipodal partial cubes whose antipodal subgraphs are gated,

(iv) LOPs and partial cubes whose antipodal subgraphs are hypercubes,

(v) UOMs and antipodal partial cubes whose proper antipodal subgraphs are hypercubes.

While the proof of the above theorem is rather complicated, the correspondence is simple: in all of the above structures seen as sets of covectors, one obtains a graph by considering its *tope graph*, i.e., considering the subgraph induced in the hypercube by all covectors without 0-entries. Conversely, one can get the covectors from the tope graph by associating to its antipodal subgraphs sign-vectors that encode their relative position to the halfspaces. This is explained in more detail in Section 2.

It is well-known that the isometric embedding of a partial cube into a hypercube of minimum dimension is unique up to automorphisms of the hypercube, see e.g. [40, Chapter 5]. Indeed, partial cubes $G_1, G_2 \subseteq Q_n$ are isomorphic as graphs if and only if there is $f \in \mathrm{Aut}(Q_n)$ such that $f(G_2) = G_1$. This leads to the fact that isomorphisms of simple COMs and their tope graphs correspond to each other. Since (unlabeled, non-embedded) isomorphic tope graphs are sometimes considered as equal, this allow to speak about isomorphism classes of COMs in a natural way.

A more refined notion of isomorphisms for COMs and partial cubes are reorientations. For partial cubes $G_1, G_2 \subseteq Q_n$ one says that G_1 is a *reorientation* of G_2 if there is $f \in \mathbb{Z}_2^n \subseteq \mathrm{Aut}(Q_n)$ such that

$f(G_2) = G_1$, i.e., f only switches signs. This yields an equivalence relation whose classes are called *reorientation classes*. Since we represent COMs, OMs, AOMs, and LOPs as graphs we say that G is an OM, if G is in fact the tope graph of an OM, etc.

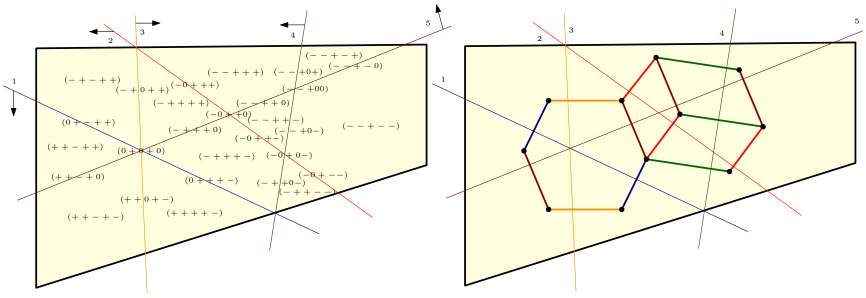

Figure 1: A realizable COM and its tope graph. Its bounded faces, edges and vertices are the antipodal subgraphs.

Some COMs, OMs, AOMs, and LOPs can be particularly nicely represented by a geometric construction. Let $\{H_1, \dots, H_n\}$ be a set of hyperplanes in an Euclidean space $\mathbb{R}^d$ and C a full-dimensional open convex set in $\mathbb{R}^d$. The hyperplanes cut C in d-dimensional chambers and to every point in C one can associate a vector in $\{+, -, 0\}^n$ that denotes its relative position to the hyperplanes (if they are given with positive and negative side). These are the covectors. One can form a graph G whose vertices are the chambers and two chambers are adjacent if and only if they are separated by exactly one hyperplane. Such graphs are always COMs. If $C = \mathbb{R}^d$ then the graph is an AOM, and if moreover all $\{H_1, \dots, H_n\}$ cross the origin point, then the graph is an OM. If $\{H_1, \dots, H_n\}$ are the coordinate hyperplanes of $\mathbb{R}^d$ and C is arbitrary, then the graph is a LOP. If a COM, OM, AOM, or LOP G is isomorphic to a graph obtained in this way, we call it *realizable*. See Figure 1 for a realizable COM and its tope graph. Realizable COMs embody many nice classes such as linear extension graphs of posets, see [4] and are equivalent to special convex neural codes, namely so-called stable hyperplane codes, see [25, 31].

An operation that is well known in the study of partial cubes is a *contraction* π_f that for a coordinate f contracts all the edges in a Θ-class E_f. The family of partial cubes is closed under the operation of contraction as well as are the families of COMs, OMs, AOMs, and LOPs (in their graph representation) as well as the class of antipodal partial cubes, see [29]. In fact the contraction operation defined directly on the latter structures defined as covector systems is known as (one element) *deletion*.

The *rank* $r(G)$ of a partial cube G is the largest r such that G can be transformed to Q_r by a sequence of contractions. The definition of rank in oriented matroid theory is equivalent, see [14]. Furthermore, notice that viewing the vertices of $G \subseteq Q_n$ as a set $\mathscr{S}$ of subsets of $\{1, \dots, n\}$, $r(G)$ coincides with the VC-dimension of $\mathscr{S}$, see [52]. The latter correspondence has led to some recent interest in partial cubes of bounded rank, see [12]. See Figure 2 for an antipodal partial cube of rank 3.

We call a vertex v of an antipodal partial cube G *simplicial* if $\deg(v) = r(G)$. In an OM G simplicial vertices correspond to simplicial topes and it is a well known fact that the degree of each vertex must be at least $r(G)$. We are ready to formulate the well-known simplex conjecture of Las Vergnas [32] in terms of tope graphs of OMs.

Conjecture 1 (Las Vergnas). *Every OM has a simplicial vertex.*

The conjecture is motivated by the fact that it holds for all realizable OMs [50]. Moreover, in [20] it is shown that OMs of rank at most 3 are exactly the planar antipodal partial cubes. Hence, for rank 3 OMs Conjecture 1 can be easily deduced using Euler's Formula. We extend this result by showing that all antipodal partial cubes of rank at most 3 have simplicial vertices in Section 6.

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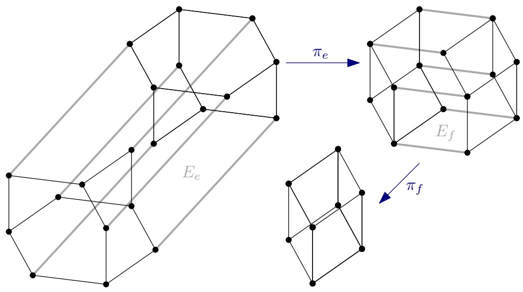

Figure 2: Contracting an antipodal partial cube to Q_3.

Conjecture 1 has furthermore been verified for UOMs of rank 4 up to 12 elements [8]. In Section 6 we prove that even general antipodal partial cubes of isometric dimension up to 7 have simplicial vertices.

The largest class known to satisfy Conjecture 1 was found in [35, Theorem 7]. We call that class *Mandel* here and consider it in depth in Section 4. Realizable OMs and OMs of rank at most 3 are *Euclidean* and the latter are Mandel, but the class is larger. Indeed, Mandel [35, Conjecture 8] even conjectured the following as a "wishful thinking statement", since by the above it would imply the conjecture of Las Vergnas:

Conjecture 2 (Mandel). *Every OM is Mandel.*

Let us now consider some strengthenings of Las Vergnas' conjecture. First consider the property that every Θ-class of G contains an edge incident to a simplicial vertex. We say that such G is *Θ-Las Vergnas*. In the language of OMs this means that G has no mutation-free elements. It is known that rank 3 OMs are Θ-Las Vergnas [34]. In Proposition 6.5 we extend this result to all antipodal partial cubes of rank 3. In Theorem 4.5 we extend the class of Θ-Las Vergnas OMs significantly, by showing that Mandel OMs are Θ-Las Vergnas.

On the other hand, UOMs of rank 4 violating this property of isometric dimension 21 [45], 17 [8], and 13 [51] have been discovered. See Figure 3 for an illustration of the latter. Thus, together with Theorem 4.5 this disproves Mandel's conjecture (Corollary 4.6).

We generalize the notion of simpliciality from OMs to COMs as follows: a vertex $v \in G$ is *simplicial* if it is contained in a unique maximal antipodal subgraph $A \subseteq G$ and $\deg(v) = r(A)$. In LOPs simplicial vertices are usually called *corners*, see [9]. If G is a LOP and also an AOM, i.e, a halfspace E_e^+ of an OM G', then G has a corner if and only if G' has a simplicial vertex incident to E_e. By the examples from [3,9,51] this proves that there are LOPs without corners. This was first observed in [9], where it is translated to an important counter example in computational learning theory. In Section 5.2 we consider the concept of a corner in COMs and show that realizable COMs (Proposition 5.5), COMs of rank 2 (Theorem 5.11), and hypercellular graphs (Theorem 5.14) admit corner peelings. This generalizes results of [3, 9, 51]. Furthermore, together with the examples from [8,45,51] this yields locally realizable COMs that are not realizable and refutes a conjecture of [4, Conjecture 2] (Remark 5.6).

Let us present another strengthening of Las Vergnas' conjecture for UOMs. If v is a simplicial vertex in a UOM G of rank r, then v is contained in a unique convex hypercube minus a vertex, lets denote it by Q_r^-. This is a well known assertion and it directly follows from Lemma 4.3. If one fills in the missing vertex of Q_r^- and instead removes v and does the same to the antipodes of the vertices, one obtains a new UOM G of rank r. This operation is called a *mutation*. Hence, a mutation is an operation that transforms a UOM into another UOM. A simple analysis shows that the operation is reversible, i.e. the inverse operation is also a mutation, and that the rank of both UOMs is equal. Thus, one can now consider a *mutation graph* whose vertices are UOMs embedded into Q_n, for some $n \in \mathbb{N}$ of fixed rank r and edges are corresponding

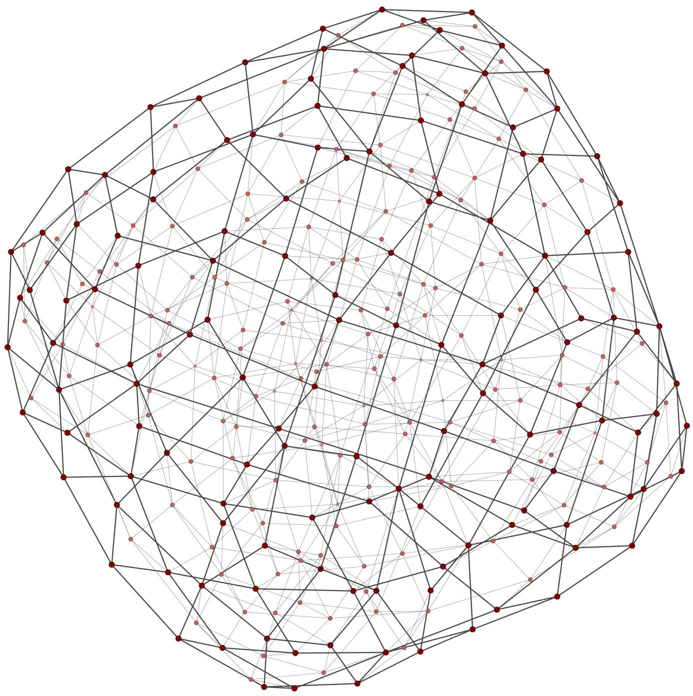

Figure 3: A halfspace E_e^+ of a non-Mandel OM G, where E_e is the Θ-class witnessing that G is not Θ-Las Vergnas. The bold subgraph is an OM of rank 3 on the vertices incident with E_e in G.

to mutations. In fact, one can consider three mutation graphs corresponding to the different notions of equivalence of OMs introduced above:

- $\overline{\mathscr{G}}^{n,r}$ is the graph whose vertices are UOMs of rank r and isometric dimension n, embedded into Q_n. Two graphs are connected if and only if there exists a mutation between them.
- $\mathscr{G}^{n,r}$ is the graph whose vertices are reorientation classes of UOMs of rank r and isometric dimension n embedded into Q_n. Two reorientation classes are connected if and only if there exists a mutation between them.
- $\underline{\mathscr{G}}^{n,r}$ is the graph whose vertices are graph isomorphism classes of UOMs of rank r and isometric dimension n. Two classes are connected if and only if there exists a mutation between them.

While mutation graphs $\overline{\mathscr{G}}^{n,r}$ and $\underline{\mathscr{G}}^{n,r}$ seem natural in the graph theoretic language of OMs, the situation is the same if one consider OMs in the standard definition since an isomorphism of a OM directly translates to an isomorphism of its tope graph, see e.g. [7] or [6]. The graphs $\mathscr{G}^{n,r}$ are motivated by the topological representation of OMs and are the most studied ones. In particular, by Ringel's Homotopy Theorem [46, 47] it follows that $\mathscr{G}^{n,3}$ is connected. Moreover, the induced subgraph of $\mathscr{G}^{n,r}$ on all the realizable UOMs is connected by [48].

Las Vergnas' conjecture implies the above graphs have minimum degree at least 1 (where loops can occur). A much stronger affirmation for $\mathscr{G}^{n,r}$ appears in [48]:

Conjecture 3 (Cordovil-Las Vergnas). *For all r, n the graph $\mathscr{G}^{n,r}$ is connected.*

Naturally one can conjecture the same assertion for all three graphs. Note that there is a hierarchical structure of these three open questions. Indeed, it is easy to see that connectivity of $\overline{\mathscr{G}}^{n,r}$ implies connectivity of $\mathscr{G}^{n,r}$ which implies that $\underline{\mathscr{G}}^{n,r}$ is connected (Observation 3.1).

Our results with respect to Conjecture 3 include that $\overline{\mathscr{G}}^{n,3}$ is connected, which is a consequence of Ringel's Homotopy Theorem [46, 47] and strengthens the fact that $\mathscr{G}^{n,3}$ is connected (Proposition 3.2). Moreover, we show that connectivity of $\underline{\mathscr{G}}^{n,r}$ implies connectivity of $\mathscr{G}^{n,r}$ (Proposition 3.3). Together with the fact that Conjecture 3 (as well as Conjecture 1) is closed under duality, see [7, Exercise 7.9], this allows us to verify Conjecture 3 for all $n \leq 9$, computationally. See Table 1 for orders of the graphs $\underline{\mathscr{G}}^{n,r}$ and Figure 5 for a depiction of $\underline{\mathscr{G}}^{8,4}$.

2 Preliminaries

We have already introduced contractions in partial cubes above. The inverse of a contraction is an expansion. In fact one can look at expansions in the following way: if H is a contraction of G, i.e. $H = \pi_e(G)$, then one can consider in H sets $H_1 = \pi_e(E_e^+)$ and $H_2 = \pi_e(E_e^-)$. Notice that they completely determine the expansion, since G can be seen as a graph on the disjoint union of H_1 and H_2 where edges between them correspond to $H_1 \cap H_2$. By *expansion* of H we refer to the subgraphs H_1, H_2 and sometimes to G. In case that $H_1 = H$ or $H_2 = H$ we say that the expansion is *peripheral*. A peripheral expansion is *proper* if $H_1 \neq H_2$. If G and H are OMs, then we say that H_1, H_2 is an *OM-expansion*, also called *single-element extension* in OM theory. More generally, if H is an antipodal partial cube, then an expansion H_1, H_2 of H is *antipodal* if the expanded graph G is again antipodal. It is well-known, see e.g [29, 43], that G is antipodal if and only if $H_1 = -H_2$. Clearly, a OM-expansion is antipodal. An OM-expansion H_1, H_2 is *in general position* if the expansion is properly peripheral on all maximal proper antipodal subgraphs, i.e. each maximal proper antipodal subgraph is either completely in H_1 or completely in H_2 but not in $H_1 \cap H_2$. Expansions in general position are central to the notion of Mandel OMs as well as to the definition of corners in COMs, i.e., in both Section 4 and Section 5.2. It is well-known, that every OM admits an expansion and indeed even an expansion in general position, see [49, Lemma 1.7] or [7, Proposition 7.2.2].

Besides contractions and restrictions, there is another operation of particular interest in partial cubes to obtain a smaller graph from a partial cube. This is the zone graph $\zeta_f(G)$ of G with respect to a Θ-class f, see [28]. In general the zone graph is not a partial cube, but indeed a characterization of COMs from [29] (generalizing a result of Handa for OMs [22]) allows the following definition in COMs. Let G be a COM, and F a subset of its Θ-classes. The *zone graph* $\zeta_F(G)$ is the graph obtained from G, whose vertices are the minimal antipodal subgraphs of G that are crossed by all the classes in F. It turns out that all such antipodal subgraphs have the same rank, say r. Two such antipodal subgraphs are connected in $\zeta_F(G)$ if they lie in a common antipodal subgraph of G of rank $r + 1$. The above mentioned characterizing property of COMs is that $\zeta_F(G)$ is always a COM. In the standard language of OMs, zone graphs are known as contractions of OMs. The zone graph operation will be used frequently in Section 5.2.

In the paper we will talk about COMs as graphs, hence we introduced them this way. Nevertheless, for certain results we need to define covectors and cocircuits of COMs which are one of the standard ways to introduce OMs. Usually the covectors are represented as a subset $\mathscr{L} \subset \{+, -, 0\}^n$ and have to satisfy certain axioms in order to encode a COM, OM, AOM, LOP or UOM. If $X \in \mathscr{L}$ and $e \in [n]$ is a coordinate of X, we shall write $X_e \in \{+, -, 0\}$ for the value of X in coordinate e. When considering tope graph, one restricts usually to simple systems. Here, a system of sign-vectors $\mathscr{L}$ is *simple* if it has no "redundant" elements, i.e., for each $e \in [n]$, $\{X_e : X \in \mathscr{L}\} = \{+, -, 0\}$ and for each pair $e \neq f$ in $[n]$, there exist $X, Y \in \mathscr{L}$ with $\{X_e X_f, Y_e Y_f\} = \{+, -\}$. We will assume simplicity without explicit mention. By the graph-theoretical representation of COMs given in [29], the covectors correspond to the antipodal subgraphs of a COM G. Indeed, in a partial cube every convex subgraph is an intersection of halfspaces, see e.g. [1], and one can assign to any convex subgraph H a unique sign-vector $X(H) \in \{0,+,-\}^n$ by setting for any coordinate $e \in \{1,\dots,n\}$:

$$X(H)_e = \begin{cases} + & \text{if } X \subseteq E_e^+, \\ - & \text{if } X \subseteq E_e^-, \\ 0 & \text{otherwise.} \end{cases}$$

This correspondence yields a dictionary in which important concepts on both sides, graphs and sign-vectors translate to each other. An easy example of this is that if the $v \in G$ is a vertex of an antipodal partial cube, then for its antipode u we have $X(u) = -X(v)$. Thus we will often denotes the antipodes of a set of vertices H just by $-H$. Another noteworthy instance is the relation of gates and composition. The *composition* of two sign vectors $X, Y \in \{0,+,-\}^n$ is defined as the sign-vector obtained by setting for any coordinate $e \in \{1,\dots,n\}$:

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise.} \end{cases}$$

Proposition 2.1. *[29] If H is a gated subgraph of a partial cube G, then for a vertex v with gate u in H, we have $X(u) = X(H) \circ X(v)$.*

A set of covectors forms a COM if it satisfies the following two axioms:

(FS) $X \circ -Y \in \mathscr{L}$ for all $X, Y \in \mathscr{L}$.

(SE) for each pair $X, Y \in \mathscr{L}$ and for each $e \in [n]$ such that $X_e Y_e = -1$, there exists $Z \in \mathscr{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in [n]$ with $X_f Y_f \neq -1$.

For similar axiomatizations for LOPs, AOMs, UOMs, and OMs we refer to [4, 5]. We will content ourselves with the setting of COMs and otherwise use Theorem 1.1 as a definition. In order to formulate (SE) in terms of antipodal subgraphs, we give some more terminology about how subgraphs and θ-classes can relate. We will say that a Θ-class E_f *crosses* a subgraph H of G if at least one of the edges in H is in E_f. Moreover, E_f *separates* subgraphs H, H' if $H \subseteq E_f^+$ and $H' \subseteq E_f^-$ or the other way around. We collect the coordinates separating H and H' in the set $S(H, H')$ – usually called the *separator*. Let us now state the axiom of *strong elimination for graphs*:

(SE) Let X, Y be antipodal subgraphs of G and E_e a Θ-class such that $X \subseteq E_e^+$ and $Y \subseteq E_e^-$, i.e., $e \in S(X, Y)$. There is an antipodal subgraph Z of G that is crossed by E_e and for all $f \neq e$ we have:
- if $X, Y \subseteq E_f^+$ then $Z \subseteq E_f^+$, and if $X, Y \subseteq E_f^-$ then $Z \subseteq E_f^-$
- if E_f crosses one of X, Y and the other is a subset of E_f^+ then $Z \subseteq E_f^+$, and if E_f crosses one of X, Y the other is a subset of E_f^- then $Z \subseteq E_f^-$
- if E_f crosses both X and Y, then it crosses Z.

In some parts of the paper we will abuse a bit the distinction between covectors and antipodal subgraphs in the way that we have suggested in the above definitions.

If G is an OM, then the set of covectors or antipodal subgraphs is usually ordered by reverse inclusion to yield the *big face lattice* $\mathscr{F}(G)$ whose minimum is G itself. Indeed, if G is of rank r, then, it follows from basic OM theory that the $\mathscr{F}(G)$ is atomistic and graded, where $r+1$ is the length of any maximal chain. Moreover, an intersection of any two antipodal subgraphs is a (possibly empty) antipodal subgraph. The antipodal subgraphs of rank $r-1$ are themselves OMs and correspond to what is called *cocircuits* in the standard theory of OMs. In other words the *cocircuits* of G are the atoms of $\mathscr{F}(G)$, i.e., the maximal proper antipodal subgraphs of G.

Another graph associated to an OM G of rank r is its *cocircuit graph* of G, i.e., the graph G^* whose vertices are the antipodal subgraphs of G of rank $r-1$ and two vertices are adjacent if their intersection in G is an antipodal subgraph of rank $r-2$. We denote the cocircuit graph of G by G^*, since it generalizes planar duality in rank 3, see [20], however in higher rank $(G^*)^*$ is not well-defined, because the cocircuit graph does not uniquely determine the tope graph, see [13]. There has been extensive research on cocircuit graphs [2, 18, 30, 39]. However their characterization and recognition remains open. Cocircuit graphs play a crucial role for the notion of Euclideaness and Mandel in Section 4.

In a general COM G, the poset $\mathscr{F}(G)$ remains an upper semilattice, since antipodal subgraphs are closed under intersection but there is no minimal element. There are different possible notions of cocircuits that allow to axiomatize COMs, see [4]. We consider cocircuits in the setting of *pure* COMs G, i.e., all maximal antipodal subgraphs of G are of the same rank and G^* is connected. If G is a non-antipodal pure COM, then its cocircuits are just the maximal antipodal subgraphs. We will introduce the cocircuit graph of pure COMs in Section 5.2, where it will serve for proving the existence of corners in COMs of rank 2.

3 Mutation graphs of uniform oriented matroids

In this section we present results on the mutation graphs. The three different mutation graphs as defined in the introduction are related as follows:

Observation 3.1. *Let $0 \leq r \leq n$. We have $\overline{\mathscr{G}}^{n,r}$ connected $\implies \mathscr{G}^{n,r}$ connected $\implies \underline{\mathscr{G}}^{n,r}$ connected.*

Proof. If $\overline{\mathscr{G}}^{n,r}$ is connected, then also $\mathscr{G}^{n,r}$ is, since there is a weak homomorphism from the first to the second, mapping an OM to its equivalence class. Similarly, if $\mathscr{G}^{n,r}$ is connected then also $\underline{\mathscr{G}}^{n,r}$ is, since a reorientation can be seen as an isomorphism. □

We start by analyzing the connectivity of mutation graphs for small rank. Since OMs of rank 1 or 2 are simply isomorphic to an edge or an even cycle, respectively, the first interesting case is when the rank of OMs is 3. It is a well known fact that OMs of rank 3 can be represented as pseudo-line arrangements on a sphere which can be further represented by wiring diagrams (see Figure 4). By Ringel's Homotopy Theorem [46, 47] any two simple pseudo-line arrangements on a sphere can be transformed one into another by performing mutations. Since it does not deal with orientations, Ringel's Homotopy Theorem implies that $\mathscr{G}^{n,3}$ is connected. As stated in Observation 3.1, this implies that also $\underline{\mathscr{G}}^{n,3}$ is connected. Now we present new results on the topic:

Proposition 3.2. *For every n the graph $\overline{\mathscr{G}}^{n,3}$ is connected.*

Proof. We use the proof of Ringel's Theorem as shown in [7], which in fact first uses that any pseudo-line arrangement can be represented as a wiring-diagram and second uses representation in Coxeter groups. In fact, a stronger statement is shown that any two labeled simple wiring diagrams can be transformed one into another by performing mutations of bounded cells. This will suffice to deduce the claim.

Suppose that we have two UOMs of rank three $\mathscr{M}$ and $\mathscr{M}'$ and want to transform one into the other by mutations. Both graphs can be represented as pseudo-line arrangements on a sphere which can be further represented by wiring diagrams, as stated above. Then by Ringel's Theorem the transformation using mutation can be done modulo reorientations, i.e Ringel's Theorem does not deal with orientations in the wiring. So we only need to consider the case where $\mathscr{M}$ and $\mathscr{M}'$ differ in the reorientation of one element e, but also this can be done performing mutations on bounded cells. Represent $\mathscr{M}$ as a wiring-diagram $\mathscr{A}$, where e is the top-element on the left and construct a wiring diagram $\mathscr{A}'$ as shown in Figure 4, with e being the bottom element on the left. In fact, when representing a pseudo-line arrangement with wiring diagrams the top (or bottom) element can be chosen. Note that $\mathscr{A}'$ represents

8

$\mathscr{M}'$, and the mutations transforming $\mathscr{A}$ into $\mathscr{A}'$ do only mutations on bounded faces, i.e., push the line e without changing its orientation. Thus, we have obtained $\mathscr{M}'$ from $\mathscr{M}$ by mutations. □

By Observation 3.1, this implies that also $\mathscr{G}^{n,3}$ and $\underline{\mathscr{G}}^{n,3}$ are connected.

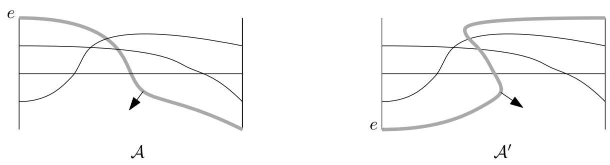

Figure 4: How to construct the new pseudoline arrangement in the proof of Proposition 3.2. The arrow points to the positive side of e.

We show that two of the open questions about the connectivity of mutation graphs are equivalent.

$n\backslash r$	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3		1	1	1	4	11	135	482	312356
4			1	1	1	11	2628	9276595	?
5				1	1	1	135	9276595	?
6					1	1	1	4382	?
7						1	1	1	312356
8							1	1	1
9								1	1
10									1

Table 1: Known orders of $\underline{\mathscr{G}}^{n,r}$, retrieved from `http://www.om.math.ethz.ch/`.

Proposition 3.3. *The graph $\mathscr{G}^{n,r}$ is connected if and only if $\underline{\mathscr{G}}^{n,r}$ is connected.*

Proof. As stated in Observation 3.1, the mapping from $\mathscr{G}^{n,r}$ to $\underline{\mathscr{G}}^{n,r}$ defined by mapping the class of OMs up to the reorientation into their isomorphism classes is a weak homomorphism of graphs. Hence if $\mathscr{G}^{n,r}$ is connected, so is $\underline{\mathscr{G}}^{n,r}$.

Conversely, notice that the property of being a realizable OM is independent of reorientation or permuting the elements. If $\underline{\mathscr{G}}^{n,r}$ is connected, then there exists a sequence of mutations from any $[A] \in \underline{\mathscr{G}}^{n,r}$ to a realizable class $[B] \in \underline{\mathscr{G}}^{n,r}$. This sequence can then be lifted to a sequence of mutations from any $A' \in \mathscr{G}^{n,r}$ to a realizable $B' \in \mathscr{G}^{n,r}$. This proves that there exists a path from every $A' \in \mathscr{G}^{n,r}$ to a reorientation classes of realizable OMs. Since by [48] the induced subgraph of all realizable classes in $\mathscr{G}^{n,r}$ is connected, this proves that $\mathscr{G}^{n,r}$ is connected. □

Proposition 3.3 allows to approach Conjecture 3 for small values of n and r from the computational perspective, since it allows computations on the smaller graph $\underline{\mathscr{G}}^{n,r}$. To provide an idea of the computational weight of this task, Table 1 shows the known orders of such graphs $\underline{\mathscr{G}}^{n,r}$ for small n and r. Moreover, Figure 5 displays the graph $\underline{\mathscr{G}}^{8,4}$.

We verified computationally that for all the parameters from Table 1 where the isomorphism classes of OMs are known, their mutation graph $\underline{\mathscr{G}}^{n,r}$ is connected. By Proposition 3.3 also the corresponding $\mathscr{G}^{n,r}$ are connected. This was possible by considering UOMs as (tope)graphs in which finding possible mutations is easy, since only degrees of vertices need to be checked. We calculated the isomorphism

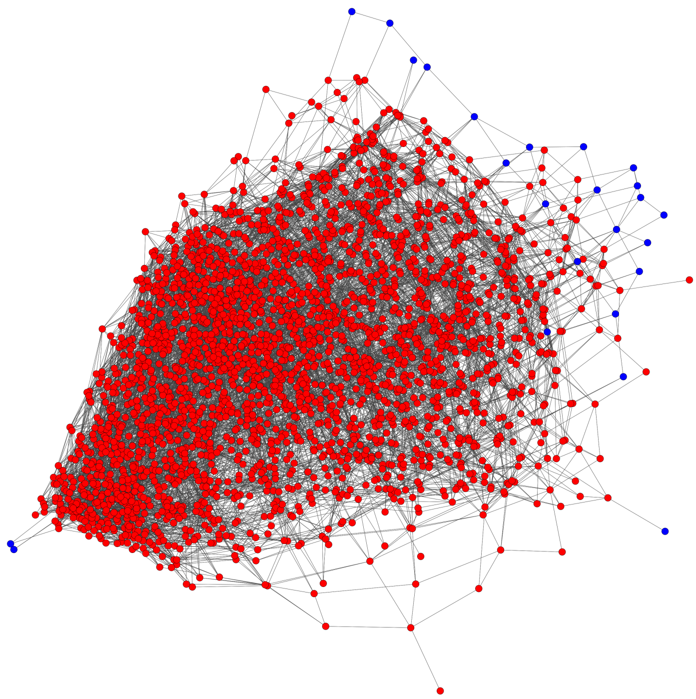

Figure 5: The graph $\underline{\mathscr{G}}^{8,4}$ where red vertices are isomorphism classes of realizable UOMs and blue ones are non-realizable

class of the mutated graphs using the software Bliss [26] that is designed for calculations of isomorphisms of graphs. The computationally most demanding task was the graph $\underline{\mathscr{G}}^{9,4}$ where efficient graph representation was needed. Checking connectivity of $\overline{\mathscr{G}}^{n,r}$ is far more demanding.

4 Mandel's OMs and Euclideanness

In this section we focus on Las Vergnas' and Mandel's Conjectures. The main result is that Mandel OMs are Θ-Las Vergnas and therefore not all OMs are Mandel. The concept of Euclideanness is based on the structure of antipodal subgraphs of an OM, as discussed in Section 2. Furthermore, the property of being Mandel relies on the definition of extensions in general postilion and the cocircuit graph, also introduced in Section 2.

Let G be an OM of rank r. Recall the definition of cocircuit graph from Section 2, i.e., the graph G^* whose vertices are the antipodal subgraphs of G of rank $r-1$ and two vertices are adjacent if their intersection in G is an antipodal subgraph of rank $r-2$.

Consider now a maximal path $A_1,\dots,A_n$ in G^* such that for all $1 < i < n$ the set $A_{i-1} \cap A_i$ is the set of antipodes of $A_i \cap A_{i+1}$ with respect to A_i. It follows from the topological representation of OMs [35], that $A_1,\dots,A_n$ induce a cycle in G^*. Moreover, every $A_{n/2+i}$ is the set of antipodes of A_i with respect to G, for $1 \le i \le n/2$ and all intersections $A_i \cap A_{i+1}$ are crossed by the same set F of Θ-classes. Indeed, this cycle can be seen as the line graph of the the zone-graph $\zeta_F(G)$. Furthermore, each Θ-class $E_f \notin F$ crosses exactly two pairwise antipodal A_i and $A_{n/2+i}$. The cocircuit graph G^* is the (edge-disjoint) union of such cycles.

Considering now a halfspace H of G, i.e. H is an AOM. The induced subsequence $A_k, \ldots, A_\ell$ of the above cycle is called a *line* L in H. This name comes from the fact that in the topological representation the sequence corresponds to a pseudo-line, see [35]. Let now $E_e \in \mathscr{E}$ be a Θ-class of H. We say that E_e *crosses* a line L of G, if there exists A_i on L that is crossed by E_e but $A_{i-1} \cap A_i$ or $A_i \cap A_{i+1}$ is not crossed by it. Note that in a line L of an AOM the crossed A_i is unique if it exists. This allows to define *the orientation of L with respect to E_e*: If L is not crossed by E_e we leave its edges undirected. Otherwise, let A_i be the element of L that is crossed by E_e and assume that $A_j \subset E_e^-$ for $j < i$ and $A_j \subset E_e^+$ for $j > i$. We orient all edges of the form A_j, A_{j+1} from A_j to A_{j+1}. This is, the path L is directed from E_i^- to E_i^+.

The edges of the cocircuit graph G^* of an AOM G are partitioned into lines and for every Θ-class E_e we obtain a partial orientation of G^* by orienting every line with respect to E_e. Let us call this mixed graph the orientation of G^* with respect to E_e. Following Mandel [35, Theorem 6], an AOM is *Euclidean* if for every Θ-class E_e the orientation of the cocircuit graph G^* with respect to E_e is *strictly acyclic*, i.e., any directed cycle (following undirected edges or directed edges in the respective orientation) consists of only undirected edges. In other words, any cycle that contains a directed edge contains one into each direction. Euclidean AOMs are important since they allow a generalization of linear programming from realizable AOMs.

Following Fukuda [19], an OM is called *Euclidean* if all of its halfspaces are Euclidean AOMs. Since non-Euclidean AOMs exist, see [19, 35], also non-Euclidean OMs exist. However, there is a larger class of OMs that inherits useful properties of Euclidean AOMs and that was introduced by Mandel [35].

We call an OM *Mandel* if it has an expansion in general position such that G_1 and G_2 are Euclidean AOMs. Mandel [35, Theorem 7] proved (and it is up to today the largest class known to have this property) that these OMs satisfy the conjecture of Las Vergnas:

Theorem 4.1 ([35]). *If an OM G is Mandel, then it has a simplicial vertex.*

As stated in the introduction, Mandel [35, Conjecture 8] conjectured that every OM is Mandel as a "wishful thinking statement", since with Theorem 4.1 it would imply the conjecture of Las Vergnas (Conjecture 1). In the following we use the ideas from [35] to improve Theorem 4.1 to such an extent, that we can disprove Conjecture 2.

We shall repeatedly use the following fact, that can be easily seen through the topological representation of OMs but is a bit less trivial in the graph view. Nevertheless, it was also proved with graphs in [29, Lemma 6.2].

Lemma 4.2. *Let G, G' be COMs such that G is an expansion of G'. Then for every antipodal subgraph A' of G' the expansion restricted to A' is either an OM expansion or a peripheral expansion. In particular, there exists an antipodal subgraph A of G that contracts to A'.*

The following is a characterization of simplicial vertices in an OM, that can be found in [35, Proposition 5] and [32, Proposition 1.4]. We provide a formulation in terms of the tope graph. Recall that in an OM G a vertex v is simplicial if the degree of v coincides with the rank of G.

Lemma 4.3. *Let G be an OM of rank r with Θ-classes $\mathscr{E}$. A vertex $v \in G$ is simplicial if and only if there is a set $F \subseteq \mathscr{E}$ of size r and r maximal proper antipodal subgraphs incident with v such that for each $E_e \in F$ there is a corresponding antipodal subgraph which is crossed by $F \setminus \{E_e\}$ but not by E_e. Moreover, in this case v is incident with exactly the Θ-classes of F and exactly the aforementioned r antipodal subgraphs.*

As a last basic ingredient for the Theorem 4.5 we need the following. Let G be the tope graph of a simple OM and E_e its Θ-class. Let $v \in E_e^+$. Since the lattice of antipodal subgraphs is atomistic and graded, there are is a set of maximal proper antipodal subgraphs such that their composition is exactly v. In particular, at least one of the latter maximal proper antipodal subgraphs must be in E_e^+. This is:

11

Observation 4.4. *Let E_e be a Θ-class in OM G. There exists a maximal proper antipodal subgraph A such that neither A nor $-A$ cross E_e.*

We are prepared to give the main theorem of the section. In the proof we will make use of the *Cartesian product* of graphs G_1, G_2, being defined as the graph $G_1 \square G_2$ whose vertices are $V(G_1) \times V(G_2)$ and two vertices $(x_1, y_1), (x_2, y_2)$ adjacent if and only if $x_1 = x_2$ and y_1 is adjacent to y_2 in G_2, or $y_1 = y_2$ and x_1 is adjacent to x_2 in G_1.

Theorem 4.5. *Let G be a simple, Mandel OM of rank r and E_e a Θ-class of G. Then there is a vertex of degree $r \in G$ incident with E_e, i.e., G is Θ-Las Vergnas.*

Proof. Let G_1, G_2 define an expansion of G in general position such that G_1 is a Euclidean AOM and let $E_e \in \mathscr{E}$ be a Θ-class of G. We prove the slightly stronger assertion that there exists a vertex v of degree r in $G_1 \setminus G_2$ incident with E_e. We will proceed by induction on the size of G and distinguish two cases:

Case 1. *G is not the Cartesian product with factor K_2 corresponding to E_e.*

By Observation 4.4, G has at least two antipodal subgraphs of rank $r-1$ not crossed by E_e. Then half of them lie in G_1 implying that at least one is in one of E_e^- or E_e^+. Without loss of generality assume that it is in E_e^-, otherwise reorient E_e. This is, there is a line in G_1 that has at least one antipodal subgraph A of rank $r-1$ completely in E_e^-.

Orient the lines of G_1 with respect to E_e. Note that every line of G_1 is a subline of a cycle in G^* that either has all its vertices (maximal proper antipodal subgraphs) crossed by E_e or is crossed by E_e in exactly two maximal proper antipodal subgraphs. Since the expansion according to G_1 and G_2 is in general position this implies that every line of G_1 is either crossed by E_e in exactly one maximal proper antipodal subgraph or in all its maximal proper antipodal subgraphs are crossed by E_e. Thus all the lines are oriented, except the ones with all maximal proper antipodal subgraphs on E_e.

By the definition of Euclideaness, the orientation is strictly acyclic, hence we can find in $G_1 \cap E_e^-$ an antipodal subgraph A of rank $r-1$ such that all its out-neighbors in the graph of cocircuits are intersected by E_e. Let A^0 be the set of all Θ-classes that cross A. Let H be the contraction of G along all its Θ-classes besides E_e and the ones in A^0. Let H_1, H_2 be the respective images of G_1, G_2 in H. Then H_1 and H_2 are isometric subgraphs, and every antipodal subgraph in H is an image of an antipodal subgraph in G, by Lemma 4.2, hence it lies completely in H_1 or in H_2. Moreover, every line in H is an image of a line in G and its orientation with respect to Θ-class E_e is inherited from an orientation of G, since the orientation is still pointing from E_e^- to E_e^+. Hence the orientation of H with respect to E_e is strictly acyclic as well. This proves that H is Mandel by the expansion in general position according to H_1 and H_2.

By definition of H is obtained by contracting the Θ-classes not crossing A. It is clear from OM theory and directly follows from gatedness of antipodal subgraphs in an OM, that antipodal subgraphs contract to antipodal subgraphs. Hence antipodal subgraph A is not affected by any of the contractions, hence with a slight abuse of notation we can say that H also contains A as antipodal subgraph. Thus, the rank of H is r since it properly contains A of rank $r-1$.

Let A' be the set of antipodes in H of vertices in A. Then A' is also an antipodal subgraph of H disjoint from A and only edges in E_e are connecting them. Since A is antipodal, all the vertices in A have their neighbor in A'. Thus $H \cong A \square K_2$. Since we are in Case 1, this gives that H is strictly smaller than G.

By the induction assumption, H has a vertex v of degree r in $H_1 - H_2$. Since $H \cong A \square K_2$ with the K_2 factor corresponding to E_e, all the vertices in H are incident with E_e. In particular, v is incident with E_e. By Lemma 4.3, there is a set $\mathscr{B}$ of r maximal proper antipodal subgraphs incident with v such that v has degree $r-1$ in each member of $\mathscr{B}$. Since a vertex of degree r cannot be incident with more than r maximal proper antipodal subgraphs and v is incident with A, we have $A \in \mathscr{B}$. Since $v \in H_1 - H_2$ and H_1, H_2 is an expansion in general position, all members of $\mathscr{B}$ are in H_1. By Lemma 4.2, there is a set $\mathscr{C}$ of r maximal proper antipodal subgraphs of G, such that each member of $\mathscr{C}$ contracts to a member of $\mathscr{B}$ in H. Moreover, since the members of $\mathscr{B}$ are in H_1, the graphs in $\mathscr{C}$ are in G_1. Clearly, $A \in \mathscr{C}$. Consider the vertex $v \in A$ in G. To prove that v has degree r and is incident with E_e in G, by Lemma 4.3, it suffices

12

to prove that the graphs in $\mathscr{C}$ are incident with v. Let $D \in \mathscr{C} \setminus \{A\}$, and D' the corresponding graph in $\mathscr{B}$. Since D' and A intersect in a rank $r-2$ antipodal subgraph and are both in H_1, then A and D lie on a line in G_1. Moreover, since D' is crossed by E_e, so is D. Thus, this line is oriented from A towards D, thus by the choice of A they are adjacent in G^*, and in particular intersect in a rank $r-2$ antipodal subgraph. Moreover this subgraph must be the subgraph that contracts to the intersection of D' and A. Hence, v is in the intersection, thus in D. This proves that the vertex v has degree r and is incident with E_e also in G. By construction it lies in $G_1 - G_2$.

Case 2. *G is the Cartesian product $G' \square K_2$ with factor K_2 corresponding to E_e.*

If G has a Θ-class E_f such that G is not a Cartesian product with factor K_2 corresponding to E_f, then by Case 1, G has a vertex v of degree r in $G_1 - G_2$. Moreover, in this case all the vertices of G are incident with E_e, in particular also v is.

If all of the Θ-classes of G correspond to factors K_2, then G is a hypercube and all its vertices are simplicial and incident to E_e. □

OMs with a Θ-class not incident to a simplicial vertex have been found of different sizes [8, 45, 51]. We conclude:

Corollary 4.6. *There exist OMs that are not Mandel. The smallest known on has* 598 *vertices (topes), isometric dimension* 13 *(elements) and is uniform of rank* 4*. See Figure 3.*

5 Corners and corner peelings

In the present section we introduce corners and corner peelings for general COMs. The first subsection is concerned with the first definitions and results, and in particular contains a proof for existence of corner peelings of realizable COMs. The second subsection contains corner peelings for COMs of rank 2 and hypercellular graphs.

5.1 First definitions and basic results

We will approach our general definition of corner of a COM, that generalizes corners on LOPs and has strong connections with simplicial vertices in OMs. The intuitive idea of a corner in a COM, is a set of vertices whose removal gives a new (maximal) COM. As a matter of fact it is convenient for us to first define this remaining object and moreover within an OM.

Recall the definition of an expansion in general position from Section 2. We will say that the subgraph T of an OM H is a *chunk* of H, if H admits an expansion in general position H_1, H_2, such that $T = H_1$. We call the complement $C = H \setminus H_1$ a *corner* of H. In the case that H has rank 1, i.e. H is isomorphic to an edge K_2, then a corner is simply a vertex of H.

This definition extends to COMs by setting C to be a *corner* of a COM G if C is contained in a unique maximal antipodal subgraph H and C is a corner of H. We need two more helpful observation:

Lemma 5.1. *If G' is an isometric subgraph of a COM G such that the antipodal subgraphs of G' are antipodal subgraphs of G, then G' is a COM.*

Proof. By Theorem 1.1 all antipodal subgraphs of G are gated, but since G' is an isometric subgraph and it has no new antipodal subgraph also the antipodal subgraphs of G' are gated. Thus, by Theorem 1.1 G' is a COM. □

We are now ready to prove that chunks and corners as we defined them achieve what we wanted. This proof uses the correspondence between sign-vectors and convex subgraphs as introduced in Section 2.

Lemma 5.2. *If C is a corner of a COM G, then the chunk $G \setminus C$ is an inclusion maximal proper isometric subgraph of G that is a COM.*

Proof. Let us first consider the case where $G = H$ is an OM. Let $T, -T$ be an expansion of H in general position, i.e., $-T$ is the set of antipodes of T. Since expansions in general position are OM-expansions, T is a halfspace of an OM. Thus, T is a COM — even an AOM. This proves that T is a sub-COM. Assume that it is not maximal and let $R \supsetneq T$ be a COM contained in H. Let $X \subseteq R$ be an antipodal subgraph of H that is not completely in T, and is maximal with this property. Since the expansion is in general position, it holds $X \subseteq -T$ and $-X \in T \subseteq R$. Let $E_e \in S(X, -X)$, i.e., E_e separates X from $-X$. Such E_e clearly exists, since R is a proper sub-COM of H, thus $X \neq H$.

Considering R as a COM we apply (SE) to $X, -X$ with respect to E_e in order to obtain $Z \subset R$ that is crossed by E_e. Note that X and $-X$ are crossed by the same set of Θ-classes X^0. By (SE) the set Z^0 of Θ-classes crossing Z strictly contains X^0. Thus, if $S(Z, X) = \emptyset$, then Z is an antipodal subgraph containing X, i.e., X was not a maximal antipodal subgraph of R. Let otherwise $E_f \in S(Z, X)$. Apply (SE) to X, Z with respect to E_f in order to obtain $Z' \subset R$ which is crossed by E_f. Since $Z^0 \supsetneq X^0$, we have $Z'^0 \supsetneq X^0$ and furthermore $S(Z', X) \subsetneq S(Z, X)$. Proceeding this way, we will eventually obtain an $\widetilde{Z} \subseteq R$ with $\widetilde{Z}^0 \supseteq X^0$ and $S(\widetilde{Z}, X) = \emptyset$. Thus, $\widetilde{Z} \in R$ is an antipodal subgraph containing X. By the choice of X, $\widetilde{Z}$ is not completely in T. This violates the assumption that X was maximal. Thus, $R = T$.

Now, let G be a COM that is not an OM and let H be the unique maximal antipodal subgraph of G containing C. By the above $T = H \setminus C$ is an isometric subgraph of H and a COM. Now, it follows that $G \setminus C$ is an isometric subgraph of G. Namely, since no vertex of C is adjacent to a vertex of $G \setminus H$ and T being an isometric subgraph of H, all shortest paths in G through C, can be replaced by shortest paths through T. Finally, Lemma 5.1 implies that $G \setminus C$ is a COM. Maximality follows from the first paragraph. □

Recall that simplicial vertices in LOPs are called corners. Before providing further central properties of corners in COMs, let us see that we indeed generalize corners of LOPs.

Proposition 5.3. *A subset C of the vertices of a LOP G is a corner if and only if $C = \{v\}$ is contained in a unique maximal cube of G.*

Proof. OMs that are LOPs are cubes, so let v be a vertex of Q_n. The expansion with $H_1 = Q_n \setminus \{v\}$ and $H_2 = Q_n \setminus \{-v\}$ clearly is antipodal. Moreover, every proper antipodal subgraph of Q_n is contained in either H_1 or H_2. Thus, this expansion is in general position. Consequently v is a corner or Q_n. Since chunks are maximal sub-COMs, by Lemma 5.2, single vertices are precisely the corners of Q_n.

If now v is a corner of a LOP G, then by the definition of corners of COMs and the fact that in a LOP all proper antipodal subgraphs are cubes, v is contained in a unique maximal cube of G.

Conversely if v is a vertex of a LOP contained in a unique cube, then this cube is also the unique maximal antipodal subgraph of the LOP, since in a LOP all proper antipodal subgraphs are cubes. Thus v is a corner of the LOP. □

Note that, as mentioned earlier, every OM admits an expansion in general position, see [49, Lemma 1.7] or [7, Proposition 7.2.2]. This yields directly from the definition:

Observation 5.4. *Every OM has a corner.*

Note however that COMs do not always have corners, e.g., with Proposition 5.3 one sees that the AOMs obtained from the UOMs with a mutation-free element have no corner.

Lemma 5.2 yields the following natural definition. A *corner peeling* in a COM G is an ordered partition $C_1, \dots, C_k$ of its vertices, such that C_i is a corner in $G - \{C_1, \dots, C_{i-1}\}$. In the following we generalize a results from [51] for realizable LOPs.

Proposition 5.5. *Every realizable COM has a corner peeling.*

Proof. We show that a realizable COM G has a realizable chunk T. Represent G as a central hyperplane arrangement $\mathscr{H}$ in an Euclidean space intersected with an open polyhedron P given by open halfspaces $\mathscr{O}$. Without loss of generality we can assume that the supporting hyperplanes of the halfspaces in $\mathscr{O}$ are in general position with respect to the hyperplanes in $\mathscr{H}$. We shall call points in the Euclidean space, that can be obtained as intersection of subset of hyperplanes $\mathscr{H}$ minimal dimensional cells. It follows from the correspondence between antipodal subgraphs and covectors of a COM [29, Theorem 4.9], that topes (chambers) surrounding minimal dimensional cells correspond to antipodal subgraphs of G.

Now, take some halfspace $O \in \mathscr{O}$ and push it into P until it contains the first minimal dimensional cell C of $\mathscr{H}$. The obtained realizable COM T is a chunk of G, because restricting the antipodal subgraph (an OM) corresponding to the cell C with respect to O is taking a chunk of C, while no other cells of G are affected and the resulting graph T is a COM. □

In [4, Conjecture 2] it was conjectured that all *locally realizable* COMs, i.e., those whose antipodal subgraphs are realizable OMs, are realizable. Proposition 5.5 yields a disproof of this conjecture, since all antipodal subgraphs of a LOP are hypercubes, i.e., LOPs are locally realizable, but by the example in Figure 3 and others there are LOPs that do not have corner peelings. Thus, they cannot be realizable.

Remark 5.6. *There are locally realizable COMs, that are not realizable.*

Indeed , LOPs are even *zonotopally realizable*, i.e., one can choose realizations for all cells such that common intersections are isometric. It remains open whether every locally realizable COM is zonotopally realizable, see [4, Question 1].

5.2 Corners and corner peelings in further classes

In this section we consider the question of the existence of corners and corner peelings in various classes of graphs. By Proposition 5.3 simplicial vertices in LOPs are corners. Thus, Theorem 4.5 yields:

Corollary 5.7. *Every halfspace of a Mandel UOM has a corner.*

In the following we focus on COMs of rank 2 and hypercellular graphs. In both these proofs we use the zone graph of a partial cube, see Section 2. We start with some necessary observations on cocircuit graphs of COMs.

Cocircuit graphs of COMs

In the following we generalize the concept of orientation of the cocircuit graph introduced in Section 4 from AOMs to general COMs.

Lemma 5.8. *If G is a COM and a hypercube Q_r a minor of G, then there is an antipodal subgraph H of G that has Q_r a minor. In particular, the rank of a maximal antipodal subgraph of a COM G is the rank of G.*

Proof. Since Q_r is antipodal, by Lemma 4.2, there exist an antipodal subgraph H of G that contracts to it. Then H is the desired subgraph. □

We define the *cocircuit graph* of a non-antipodal rank r COM as the graph whose vertices are the rank r antipodal subgraphs and two vertices are adjacent if they intersect in a rank $r-1$ antipodal subgraph. By Lemma 5.8 the vertices of the cocircuit graph of a non-antipodal COM G correspond to the maximal antipodal subgraphs of G. The cocircuit graph of a COM can be fully disconnected hence we limit ourselves to COMs having all its maximal antipodal subgraphs of the same rank with G^* connected. We call them *pure* COMs. Note that AOMs are pure COMs.

Let G be a pure COM, $\{A_1, A_2\}$ be an edge in G^* and F be the set of Θ-classes crossing $A_1 \cap A_2$. We have seen in Section 4 that if G is an AOM, then the maximal proper antipodal subgraphs of G crossed by Θ-classes in F induce a path of G^* which we called a line. The following lemma is a generalization of the latter and of general interest with respect to cocircuit graphs of COMs, even if we will use it only in the case of rank 2.

Lemma 5.9. *Let G be a COM that is not an OM, $\{A_1, A_2\}$ be an edge in G^*, and F be the set of Θ-classes crossing $A_1 \cap A_2$. Then the maximal proper antipodal subgraphs of G crossed by Θ-classes in F induce a subgraph of G^* isomorphic to the line graph of a tree.*

Proof. Let G, A_1, A_2, F be as stated and r the rank of G. Consider the zone-graph $\zeta_F(G)$. Recall that its vertices are antipodal subgraphs of rank $r-1$ crossed by Θ-classes in F and two subgraphs are adjacent if they lie in a common rank r antipodal subgraph. Since $\zeta_F(G)$ is a COM, see e.g. [29] and has rank 1, we have that $\zeta_F(G)$ is a tree. By definition, the maximal proper antipodal subgraphs of G crossed by Θ-classes in F correspond to edges of $\zeta_F(G)$, with two such edges connected if they share a vertex. Hence they form a subgraph of G^* that is isomorphic to the line graph of $\zeta_F(G)$. □

Lemma 5.9 implies that G^* can be seen as the edge disjoint union of line graphs of trees. We can use this to orient edges of G^*. Similarly as in the settings of AOMs, we will call a *line* in G a maximal path $L = A_1, \ldots, A_n$ in the cocircuit graph G^* such that $A_{i-1} \cap A_i$ is the set of antipodes of $A_i \cap A_{i+1}$ with respect to A_i. Let now $E_e \in \mathscr{E}$ be a Θ-class of G. Similarly as before we say that E_e crosses a line L of G^* if there exists A_i on ℓ that is crossed by E_e but $A_{i-1} \cap A_i$ or $A_i \cap A_{i+1}$ is not crossed by it. If A_i exists, it is unique. *The orientation of L with respect to E_e* is the orientation of the path L in G^* from E_e^- to E_e^+ if E_e crosses L and not orienting the edges of L otherwise. Notice that in this way we can orient the edges of G^* with respect to E_e by orienting all the lines simultaneously. The orientation of each edge (if it is oriented) is well defined: If $\{A_j, A_{j+1}\} \in E_e^-$ is an edge in a line graph of a tree that is crossed by E_e in A_i and A_{j+1} is closer to A_i than A_j is, then $\{A_j, A_{j+1}\}$ is oriented from A_j to A_{j+1}. Similarly if $\{A_j, A_{j+1}\} \in E_e^+$ and A_j is closer to A_i than A_{j+1} is, then $\{A_j, A_{j+1}\}$ is oriented from A_j to A_{j+1}. Furthermore, $\{A_j, A_k\}$ is not oriented if A_j, A_k are at the same distance to A_i. See Figure 6 for an illustration.

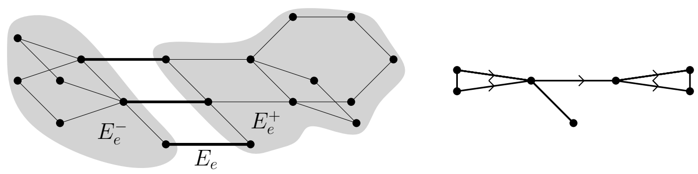

Figure 6: A pure rank 2 COM and its cocircuits graph oriented with respect to E_e.

COMs of rank 2

Mandel proved that every AOM of rank 2 is Euclidean, which by Corollary 5.7 implies that every rank 2 halfspace of a UOM has a corner. We generalize this result.

Let us first consider what corners in rank 2 COMs are. Up to isomorphism the only rank 2 OMs are even cycles. An expansion in general position of an even cycle $G = C_{2n}$ is given by $G_1, G_2 = -G_1$, where G_1 consists of an induced path on $n+1$ vertices. Hence a corner in a rank 2 COM consist of $n-1$ vertices inducing a path, included in a unique antipodal C_{2n}. For example, the COM in Figure 6 has 11 corners. Those contained in a square are single vertices and the ones contained in the C_6 are paths with two vertices.

Proposition 5.10. *Let G be a pure COM of rank 2 and E_e a Θ-class of G. Then G has a corner in E_e^+ and in E_e^-.*

Proof. Let $E_e \in \mathscr{E}$ be a Θ-class of a pure rank 2 COM G. We shall prove that G has a corner in E_e^+, by symmetry it follows that it has one in E_e^- as well. Without loss of generality assume that there is no Θ-class E_f completely contained in E_e^+, otherwise switch E_e^+ with E_f^+ or E_f^- depending on which is entirely in E_e^+. Orient the edges of G^* with respect to E_e.

16

Since the rank of G is 2, the maximal antipodal subgraphs are even cycles and each line consist of sequence of cycles pairwise crossing in edges from E_f, for some E_f. We will say that the line *follows* E_f. The induced subgraph of G^* of all the maximal antipodal subgraphs crossed by E_f is the line graph of a tree, by Lemma 5.9. We will denote it by G^*_f. Let G^*_f be crossed by some E_g in A_i. Then this splits the vertices of $G^*_f - \{A_i\}$ into the ones lying in E^+_g and the ones lying in E^-_g. We denote these by $E^+_g(G^*_f)$ and $E^-_g(G^*_f)$, respectively.

We shall prove that there is no directed cycle in $G^* \cap E^+_e$ consisting of only directed edges. For the sake of contradiction assume that such a cycle exists and take one that is the union of as few parts of lines as possible. Let the cycle be a union of a part of L_1, a part of L_2,..., and a part of L_n. Also denote with $E_{e_1}, \ldots, E_{e_n}$ the respective Θ-classes followed by $L_1, \ldots, L_n$.

Since L_i and L_{i+1} intersect, L_i must be crossed by $E_{e_{i+1}}$ and L_{i+1} must be crossed by E_{e_i}. Without loss of generality assume that L_2 passes E_{e_1} from $E^-_{e_1}$ to $E^+_{e_1}$, L_3 passes E_{e_2} from $E^-_{e_2}$ to $E^+_{e_2}$, ..., and L_1 passes E_{e_n} from $E^-_{e_n}$ to $E^+_{e_n}$, otherwise reorient $E_{e_1}, \ldots, E_{e_n}$.

First we show that each L_i passes $E_{e_{i+1}}$ from $E^+_{e_{i+1}}$ to $E^-_{e_{i+1}}$. Assuming otherwise the intersection of L_{i-1} and L_i lies in $E^-_{e_{i+1}}$, while the intersection of L_{i+2} and L_{i+3} lies in $E^+_{e_{i+1}}$, by the assumption in the previous paragraph. Then one of the lines $L_{i+4}, L_{i+5}, \ldots L_{i-1}$ must pass $E_{e_{i+1}}$ from $E^+_{e_{i+1}}$ to $E^-_{e_{i+1}}$, say L_j passes it. If this passing is in $E^+_{e_i}$, then the cycle is not minimal, since one could just replace the lines $L_{i+1}, \ldots, L_j$ by the line following $E_{e_{i+1}}$ starting from the intersection of L_i and L_{i+1} to the crossing of $E_{e_{i+1}}$ and L_j. In fact such a directed line exists since by assumption the lines following $E_{e_{i+1}}$ pass E_i from E^-_i to E^+_i thus the orientation of the shortest path from the intersection of L_i and L_{i+1} to the crossing of L_j and $E_{e_{i+1}}$ is correct.

On the other hand, assume the passing is in $E^-_{e_i}$. By assumption, the intersection of L_{i+1} and L_{i+2} is in $E^+_{e_i}$. Hence one of the lines $L_{i+3}, L_{i+4}, \ldots L_j$ must pass E_{e_i}, say L_l. In particular it must pass it in $E^+_{e_{i+1}}$, by the choice of L_j. But then again the cycle is not minimal, since one could just continue on the line following E_{e_i} starting from the intersection of L_{i-1} and L_i to the crossing of E_{e_i} and L_l. This cannot be.

Now we show that for each e_i, e_{i+1} it holds that $E^-_{e_{i+1}}(G^*_e) \subset E^-_{e_i}(G^*_e)$. Since L_{i+1} passes E_{e_i} from $E^-_{e_i}$ to $E^+_{e_i}$ it follows that L_{i+1} passes E_e in $E^-_{e_i}$. Since G^*_e is the line graph of a tree, this implies that either $E^+_{e_{i+1}}(G^*_e) \subset E^-_{e_i}(G^*_e)$ or $E^-_{e_{i+1}}(G^*_e) \subset E^-_{e_i}(G^*_e)$. On the other hand, L_i passes $E_{e_{i+1}}$ from $E^+_{e_{i+1}}$ to $E^-_{e_{i+1}}$ hence L_i passes E_e in $E^+_{e_{i+1}}$. Again since G^*_e is the line graph of a tree, this implies that $E^+_{e_i}(G^*_e) \subset E^+_{e_{i+1}}(G^*_e)$ or $E^-_{e_i}(G^*_e) \subset E^+_{e_{i+1}}(G^*_e)$. Hence, $E^-_{e_{i+1}}(G^*_e) \subset E^-_{e_i}(G^*_e)$ and $E^+_{e_i}(G^*_e) \subset E^+_{e_{i+1}}(G^*_e)$.

Inductively $E^-_{e_n}(G^*_e) \subset E^-_{e_{n-1}}(G^*_e) \subset \cdots \subset E^-_{e_1}(G^*_e) \subset E^-_{e_n}(G^*_e)$ – contradiction. This proves that there is no directed cycle in $G^* \cap E^+_e$ consisting of only directed edges.

We can now prove that G has a corner in E^+_e. First, assume that $G^* \cap E^+_e$ is non-empty and let $A \in G^* \cap E^+_e$ be a maximal antipodal subgraph, i.e., an even cycle, that has no out-edges in G^*. By the choice of E_e, each line L that passes A is crossed by E_e. We now analyze how lines pass A. Let L_1, L_2 be lines passing A, following E_{f_1}, E_{f_2}, respectively. Since E_{f_2} crosses at most one antipodal subgraph of $G^*_{f_1}$, this implies that L_1 and L_2 simultaneously pass only A. In particular each antipodal subgraph of G^*_e is passed by at most one line passing A. Since G^*_e is the line graph of a tree, its every vertex is a cut vertex. Then each line L, passing A and $A_f \in G^*_e$, and following some E_f, splits $G^*_e - \{A_f\}$ into two connected components, $E^+_f(G^*_e)$ and $E^-_f(G^*_e)$. Thus we can inductively find L such that any other line passing A passes an antipodal subgraph in G^*_e in $E^+_f(G^*_e)$, reorienting E_f if necessary.

We now show that A includes a corner. Let A' be an antipodal subgraph on L that is a neighbor of A. Then $A \cap A'$ corresponds to an edge in E_f. Define the set C to include all the vertices of A in E^-_f besides the one vertex lying in $A \cap A'$. Then C is a corner of A, we will show that C is a corner of G. For the sake of contradiction assume that a vertex v of C lies in a maximal antipodal subgraph A'' different from A.

We prove that we can choose A'' such that it shares an edge with A. Assuming otherwise, since G is a pure COM, there is a path in G^* between A and A''. This implies that there is a cycle C_k in G with subpath $v'vv''$, where $v' \in A - A''$ and $v'' \in A'' - A$. By [11, Lemma 13], the convex cycles span the cycle

17

space in a partial cube. If A is one of the convex cycles spanning C_k, then there is a convex cycle incident with A in v and sharing an edge with A. If A is not used to span C_k, then a convex cycle incident with edge $v'v$ must be used, thus again we have a convex cycle sharing v and an edge with A.

We hence assume that A and A'' share an edge g. By definition of C, ether $g \in E_f^-$, or $g \in E_f$ but not in $A \cap A'$. The latter case implies that L can be extended with A'', which cannot be since A in G^* has no out-edges. Moreover, by the choice of L, all the other lines passing A pass E_f form E_f^+ to E_f^-. Then in the former case, some other line passing A can be extended, leading to a contradiction. This implies that G has a corner.

Finally, consider the option that $G^* \cap E_e^+$ is empty. Since G_e^* is the line graph of a tree, we can pick $A \in G_e^*$ that corresponds to a pendant edge in a tree, i.e. an edge with one endpoint being a leaf. Then it is easily seen that A has a corner in E_e^+. This finishes the proof. □

The following is a common generalization of corresponding results for cellular bipartite graphs [3] (being exactly rank 2 hypercellular graphs, which in turn are COMS [11]) and LOPs of rank 2 [9].

Theorem 5.11. *Every rank 2 COM has a corner peeling.*

Proof. Notice that a rank 2 COM is pure if and only if it is 2 connected. Consider the blocks of 2-connectedness of a rank 2 COM G. Then a block corresponding to a leaf in the tree structure of the block graph has 2 corners by Proposition 5.10. This implies that G has a corner. Proposition 5.2 together with the observation that G minus the corner has rank at most 2 yield a corner peeling. □

Hypercellular graphs

Hypercellular graphs were introduced as a natural generalization of median graphs, i.e., skeleta of CAT(0) cube complexes in [11]. They are COMs with many nice properties one of them being that all their antipodal subgraphs are Cartesian products of even cycles and edges, called *cells*. See Figure 7 for an example.

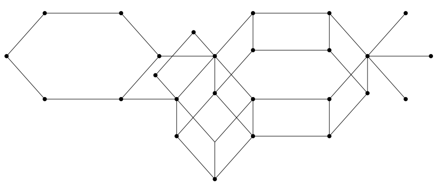

Figure 7: A hypercellular graph.

More precisely, a partial cube G is *hypercellular* if all its antipodal subgraphs are cells and if three cells of rank k pairwise intersect in a cell of rank $k-1$ and altogether share a cell of rank $k-2$, then all three lie in a common cell, for all $2 \leq k \leq r(G)$. See Figure 8 for three rank 2 cells (cycles) pairwise intersecting in rank 1 cells (edges) and sharing a rank 0 cell (vertex) lying in a common rank 3 cell (prism). Since median graphs are realizable COMs, see [37], which is also conjectured for hypercellular graphs [11], they have corner peelings by Proposition 5.5. Here, we prove that hypercellular graphs have a corner peeling, which can be seen as a support for their realizability.

The following lemma determines the structure of corners in hypercellular graphs, since the corners of an edge K_2 and an even cycle C_{2n} are simply a vertex and a path P_{n-1}, respectively.

Lemma 5.12. *Let $G = \square_i A_i$ be the Cartesian product of even cycles and edges. Then the corners of G are precisely sets of the form $\square_i D_i \subset G$, where D_i is a corner of A_i for every i.*

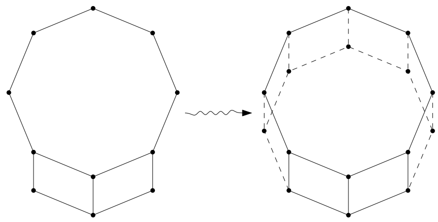

Figure 8: Any three cycles of a hypercellular graph that pairwise intersect in an edge and share a vertex lie in a common cell.

Proof. Let $D = G_2 - G_1$ be a corner of G, i.e. G_1, G_2 define an expansion in general position. Every subset of the form $A_1\Box \ldots \Box A_{i-1}\Box\{v\}\Box A_{i+1}\Box \ldots \Box A_n$ is an antipodal subgraph, thus it is either completely in G_1 or in G_2. We can use the latter to define an expansion in general position of A_i according to H_1, H_2, by $v \in H_j$ if $A_1\Box \ldots \Box A_{i-1}\Box\{v\}\Box A_{i+1}\Box \ldots \Box A_n \in G_j$, for $j \in \{1,2\}$. In fact each maximal antipodal subgraph A of A_i is either completely in H_1 or in H_2 (but not both) since $A_1\Box \ldots \Box A_{i-1}\Box A\Box A_{i+1}\Box \ldots \Box A_n$ is a maximal antipodal of G either completely in G_1 or in G_2 (but not both). Moreover, since $G_1 = -G_2$ also $H_1 = -H_2$.

It remains to prove that H_1, H_2 are isometric. Let $v_1, v_2 \in H_1$. If there exists a unique shortest v_1, v_2-path P in A_i, then pairs of vertices of $A_1\Box \ldots \Box A_{i-1}\Box\{v_1\}\Box A_{i+1}\Box \ldots \Box A_n$ and $A_1\Box \ldots \Box A_{i-1}\Box\{v_2\}\Box A_{i+1}\Box \ldots \Box A_n$ also have unique shortest paths. This implies that also $A_1\Box \ldots \Box A_{i-1}\Box P\Box A_{i+1}\Box \ldots \Box A_n \in G_1$, since G_1 is isometric. Thus $P \in H_1$. The only option that v_1, v_2 do not have a unique v_1, v_2-path is that A_i is a cycle of length $2k$, for $k > 2$, and v_1, v_2 antipodal vertices in A_i. Then for each vertex u of A_i different form v_1, v_2 holds that $A_1\Box \ldots \Box A_{i-1}\Box\{u\}\Box A_{i+1}\Box \ldots \Box A_n \in G_2$, thus $u \in H_2$. But then the neighbors of v_1 in A_i are in H_2 which by the previous case implies that $v_1 \in H_2$. Similarly $v_2 \in H_2$, which cannot be, since then $G_2 = G$. This proves that H_1, H_2 are isometric. In particular, if A_i is a cycle C_{2n}, then H_2, H_2 are paths P_{n+1}, and if A_i is an edge K_2, then H_1, H_2 are vertices. Thus each $D_i = H_2 - H_1$ is a corner.

By definition of the corners D_j, it holds $D = (G_2 - G_1) \subset \Box_i D_i$, i.e. $G - \Box_i D_i \subset G_1$. We prove that the equality holds. Let F be a set of Θ-classes that cross $\Box_i D_i$. Contracting all the Θ-classes in F gives a COM $\pi_F(G) = \Box_i A'_i$ where $A'_i = \pi_F(A_i)$ is a 4-cycle if A_i is a cycle and $A'_i = A_i$ if A_i is an edge. Thus the rank of $\pi_F(G)$ is the same as the rank of G. Now if $(G_2 - G_1) \subsetneq \Box_i D_i$, it holds $\pi_F(G_1) = \pi_F(G)$, since $\pi_F(\Box_i D_i)$ is a vertex. Defining the expansion of $\pi_F(G_1)$ according to $H_1 = \pi_F(G_1) = \pi_F(G), H_2 = \pi_F(G_2) = \pi_F(G)$, gives a graph H that has a higher rank than $\pi_F(G)$ and G. But H can be obtain as a contraction of the graph H' obtained by expanding G with respect to G_1 and G_2. Since G_1, G_2 define an expansion in general position H' has the same rank as G. This is impossible.

We have proved that if G has a corner, then it is of the form $\Box_i D_i$. As mentioned, every OM has a corner. By symmetry, every set of vertices of the form $\Box_i D_i$ is a corner of G. □

We shall need the following property about hypercellular graphs.

Lemma 5.13. *Every zone graph $\zeta_f(G)$ of a hypercellular graph G is hypercellular.*

Proof. Every zone graph of the Cartesian product of even cycles and edges is the Cartesian product of even cycles and edges, as it can easily be checked. Let $\zeta_f(G)$ be a zone graph of a hypercellular graph G. Then every cell of rank r in $\zeta_f(G)$ is an image of a cell of rank $r+1$ in G. Hence for every three rank r cells pairwise intersecting in rank $r-1$ cells and sharing a rank $r-2$ cell from $\zeta_f(G)$, there exist three rank $r+1$ cells pairwise intersecting in rank r cells and sharing a rank $r-1$ cell in G. Additionally the latter three cells lie in a common cell H in G. Then the image of H in $\zeta_f(G)$ is a common cell of the three cells from $\zeta_f(G)$. □

Let E_e be a Θ-class of a COM G. As usual, see e.g. [4, 11], we call the union of antipodal subgraphs crossed by E_e the *carrier* of E_e. The following is another generalization of the corresponding result for cellular graphs [3] and as mentioned above for median graphs.

Theorem 5.14. *Every hypercellular graph G has a corner peeling.*

Proof. We prove the assertion by induction on the size of G. The technical difficulty of the proof is that removing a corner in a hypercellular graph possibly produces a non-hypercellular graph. Hence we shall prove the above statement for the larger family $\mathscr{F}$ of COMs defined by the following properties:

(1) Every antipodal subgraph of $G \in \mathscr{F}$ is a cell.

(2) Every carrier of $G \in \mathscr{F}$ is convex.

(3) Every zone-graph of $G \in \mathscr{F}$ is in $\mathscr{F}$.

We first prove that hypercellular graph are a part of $\mathscr{F}$. By Lemma 5.13 only the first two properties must be checked. Now, (1) holds by definition of hypercellular graphs. Moreover, (2) follows from the fact that for any Θ-class E_e in a hypercellular graph the carrier of E_e is gated [11, Proposition 7], thus also convex.

We now prove that the graphs in $\mathscr{F}$ have a corner peeling. Let $G \in \mathscr{F}$ and E_e an arbitrary Θ-class in G. Since the carrier of E_e is convex the so-called Convexity Lemma [24] implies that for any edge $g \in E_e^+$ with exactly one endpoint in the carrier its Θ-class E_g does not cross the carrier. Now if the union of cells crossed by E_e does not cover the whole E_e^+, then for any edge g in E_e^+ with exactly one endpoint in the union, one of E_g^+ or E_g^- is completely in E_e^+. Repeating this argument with E_g one can inductively find a Θ-class E_f with the property that the carrier of E_f completely covers E_f^+, without loss of generality.

Let $\zeta_f(G)$ be the zone graph of G with respect to E_f, i.e., the edges of E_f are the vertices of $\zeta_f(G)$ and two such edges are connected if they lie in a common convex cycle. By (3) $\zeta_f(G)$ is in $\mathscr{F}$, thus by induction $\zeta_f(G)$ has a corner D_f. By definition there is a maximal antipodal subgraph A_f in $\zeta_f(G)$ such that the corner D_f is completely in A_f. Moreover, there exists a unique maximal antipodal subgraph A in G whose zone graph is A_f.

We lift the corner D_f from A_f to a corner D of A in the following way. If E_f in A corresponds to an edge factor K_2, then A is simply $K_2 \square A_f$. In particular we can define $D = \{v\} \square C_f$ where v is a vertex of K_2 in E_f^+. By Lemma 5.12, this is a corner of A. Since D_f lies only in the maximal antipodal graph A_f, D lies only in A.

Otherwise, assume E_f in A corresponds to a Θ-class of a factor C_{2k} (an even cycle). We can write $A = C_{2k} \square A'$. Then $A_f = K_2 \square A'$ with a corner $D_f = \{v\} \square A'$, by Lemma 5.12. We lift D_f to $D = P_{k-1} \square A'$. Here P_{k-1} is the path in C_{2k} consisting of the vertices in E_f^+ apart from the one lying on the edge not corresponding to v in the zone graph. As above since D_f lies only in the maximal antipodal graph A_f, D lies only in A.

We have proved that G has a corner D. To prove that it has a corner peeling it suffice to show that $G \backslash D$ is a graph in $\mathscr{F}$. Since removing a corner does not produce any new antipodal subgraph, all the antipodal subgraphs of $G \backslash D$ are cells, showing (1). The latter holds also for all the zone graphs of $G \backslash D$. To prove that (2) holds for $G \backslash D$ consider a Θ-class E_e of $G \backslash D$. By Lemma 5.2, $G \backslash D$ is an isometric subgraph of G, i.e. all the distances between vertices are the same in both graphs. Since the carrier of E_e in G is convex and removing a corner does not produce any new shortest path, the carrier of E_e is convex in $G \backslash D$. The same argument can be repeated in any zone graph of $G \backslash D$. This finishes the proof. □

We have shown corner peelings for COMs of rank 2 and hypercellular graphs. A common generalization are Pasch graphs [10, 11], which form a class of COMs [29] that exclude the examples from [3, 9, 51]:

Question 1. *Does every Pasch graph have a corner peeling?*

6 The minimum degree in antipodal partial cubes

Las Vergnas' conjecture can be seen as a statement about the minimum degree of an OM of given rank. Here we examine the relation of rank and minimum degree in general antipodal partial cubes.

6.1 Lower bounds

As stated in Section 1, if G is the tope graph of an OM, then $r(G) \leq \delta(G)$, see [7, Exercise 4.4]. In general rank r antipodal partial cubes the minimum degree is not bounded from below by r. More precisely:

Proposition 6.1. *For every $r \geq 4$ there is an antipodal partial cube of rank r and minimum degree 4. Moreover, there is an antipodal partial cube of rank 4 and minimum degree 3.*

Proof. In [29] it is been shown that every partial cube G with n Θ-classes – thus embeddable in Q_n – is a convex subgraph of an antipodal partial cube A_G. Here, A_G is obtained by replacing in a Q_{n+3} one Q_n by G and its antipodal Q_n by $-G$. It is straight-forward to see that the minimum degree of A_G is $\delta(G)+3$ and that the rank of A_G is at least $n+2$. Indeed, for instance taking G as a path of length $k>1$ we get $\delta(A_G)=4$ and $r(A_G)=k+2$.

Another construction is as follows. Take $Q_n^{--}(i)$, with $1 \leq i < n$ and $n \geq 4$, to be the graph obtained from Q_n by removing a vertex v, its antipode $-v$ and i neighbors of $-v$. Such a graph is affine and each antipode (in Q_n) of the removed neighbors of $-v$ is without the antipode in $Q_n^{--}(i)$, is of degree $n-1$ and of rank $n-1$. Then construct the antipodal graph taking two antipodal copies of it. Such graph will have minimum degree $n-1$ and rank n. For $n=4$ this gives the second part of the result. □

On the other hand, it is shown in [42] that if an antipodal partial cube G has $\delta(G) \leq 2$, then $r(G)=\delta(G)$. This implies that if an antipodal partial cube G has $r(G) \leq 3$, then $r(G) \leq \delta(G)$.

In relation to a question about cubic non-planar partial cubes we ask the following:

Question 2. *Are there antipodal partial cubes with minimum degree 3 and arbitrary rank?*

Indeed, since planar antipodal partial cubes are tope graphs of OMs of rank 3, see [20], any example for the above question has to be a non-planar antipodal partial cube of minimum degree 3. It has been wondered whether the only non-planar cubic partial cube is the (antipodal) Desargues graph [27], see the left of Figure 2. To our knowledge even the restriction to antipodal partial cubes remains open. For transitive cubic partial cubes it is known that the Desargues graphs is the only non-planar one, see [36]. On the other hand, it is open whether there are infinitely many non-planar partial cubes of minimum degree 3.

6.2 Upper bounds

Bounding the minimum degree in a partial cubes G from above by its rank is a generalization of Las Vergnas conjecture. As discussed in previous sections Las Vergnas conjecture is proved for OMs of rank at most 3. In fact tope graphs of OMs of rank 3 are even Θ-Las Vergnas, by Theorem 4.5 and the fact that they are Euclidean. We show that this property extends to general antipodal partial cubes of rank 3.

For this approach we introduce a couple of natural notions from [29]. A partial cube G is called *affine* if it is a halfspace E_e^+ of an antipodal partial cube. The *antipodes* $A(G)$ of an affine partial cube are those $u \in G$ such that there is $-u \in G$ such that the *interval*

$$[u,-u]=\{v \in G \mid \text{there is a shortest path from } u \text{ to } -u \text{ through } v\}$$

coincides with G. The antipodes of G are exactly the vertices of E_e^+ incident to E_e when G is viewed as subgraph of G'. We need a auxiliary statement about the rank of affine partial cubes.

Lemma 6.2. *If an affine partial cube G is a halfspace of an antipodal partial cube G' of rank r, then G has rank at most $r-1$.*

Proof. Suppose there is a sequence of contractions from G to Q_k. Then the same sequence of contraction in G' yields a minor H with Q_k as a halfspace. Since H is antipodal, $H = Q_{k+1}$. □

It was shown in [29] that affine partial cubes are closed under contractions. The following analyses the behavior of the set of antipodes under contraction.

Lemma 6.3. *Let G be affine with antipodes $A(G)$ and E_e a Θ-class. Then $A(\pi_e(G)) = \pi_e(A(G))$.*

Proof. Let $u, v \in V(G)$ such that $\pi_e(u) = -\pi_e(v)$ in $\pi_e(G)$. Since G is affine, by [29, Proposition 2.16] there is an $x \in A(G)$ such that $[x,u]$ and $[v,-x]$ cross disjoint sets of Θ-classes. Since $\pi_e(u) = -\pi_e(v)$ in $\pi_e(G)$, we have that $[u,v]$ crosses all classes of G except possibly E_e. Thus, without loss of generality either $u = x$ or u is incident with E_e its neighbors with respect to E_e is x and $v = -x$. But then $\pi_e(x) = u$. □

Lemma 6.4. *Every affine partial cube of rank at most 2 has a vertex of degree at most 2 among its antipodes.*

Proof. So, let the affine partial cube G be a minimal counterexample, i.e., all antipodes have degree at least 3, but (since affine partial cubes are closed under contraction) every contraction destroys this property.

Thus, let E_e be a Θ-class. By minimality, in $\pi_e(G)$ there are two antipodes of degree 2. Then by Lemma 6.3 there are two antipodal vertices $x, -x \in G$ such that in $\pi_e(G)$ they have degree 2. Then, $x, -x$ are incident with E_e, call their neighbor with respect to E_e, x' and $-x'$, respectively. Moreover, $x, -x$ have degree 3 and their other two neighbors are also incident with E_e. Thus, also x' and $-x'$ have at least two neighbors incident to E_e and incident with the neighbors of x and $-x$. Thus, contracting all other Θ-classes yields a Q_3-minor – contradiction. □

Proposition 6.5. *Let G be an antipodal partial cube of rank 3 and E_e a Θ-class. There is a degree 3 vertex incident to E_e.*

Proof. Suppose that the claim is false. Let G be a counterexample and E_e a Θ-class, such that all vertices incident to E_e have degree at least 4. Consider the contraction $G' = \pi_e(G)$ of G and let G'_2, G'_1 be the antipodal expansion of G' leading back to G. Since their preimage under π_e has degree at least 4, all vertices in $G'_1 \cap G'_2$ have degree at least 3. But $G'_1 \cap G'_2$ are the antipodes of the affine partial cube G'_1. Moreover, G'_1 is of rank 2 by Lemma 6.2. Thus, we have a contradiction with Lemma 6.4. □

While we have already used several times, that even OMs of rank 4 are not Θ-Las Vergnas, surprisingly enough Las Vergnas' conjecture could still hold for general antipodal partial cubes. We have verified it computationally up to isometric dimension 7. See Table 2 for the numbers.

n	2	3	4	5	6	7	8
antipodal	1	2	4	13	115	42257	?
OM	1	2	4	9	35	381	192449

Table 2: Numbers of antipodal partial cubes and OMs of low isometric dimension. The latter can also be retrieved from `http://www.om.math.ethz.ch/`.

Since already on isometric dimension 6 there are 13488837 partial cubes, instead of filtering those of isometric dimension 7 by antipodality, we filtered those of isometric dimension 6 by affinity. There are 268615 of them. We thus could create all antipodal partial cubes of dimension 7 and count them and verify Las Vergnas' conjecture also for this set. We extend the prolific Las Vergnas' conjecture to a much wider class.

Question 3. *Does every antipodal partial cube of rank r have minimum degree at most r?*

7 Conclusions and future work

We have shown that Mandel OMs have the Θ-Las Vergnas property, therefore disproving Mandel's conjecture. Finally, Las Vergnas' conjecture remains open and one of the most challenging open problems in OM theory. After computer experiments and a proof for rank 3, we dared to extend this question to general antipodal partial cubes, see Question 3. Another strengthening of Las Vergnas' conjecture is the conjecture of Cordovil-Las Vergnas. We have verified it by computer for small examples and it holds for low rank in general. However, here we suspect the existence of a counter example at least in the setting of $\overline{\mathscr{G}}^{n,r}$.

Our second main contribution is the introduction of corner peelings for COMs and the proof of their existence in the realizable, rank 2, and hypercellular cases. A class that is a common generalization of the latter two is the class $\mathscr{S}_4$ of Pasch graphs. Do these graphs admit corner peelings? See Question 1.

Let us close with two future directions of research that appear natural in the context of the objects discussed in this paper.

7.1 Shellability

There is a well-known notion of shellability of posets. In the context of an OM or AOM, a shelling is a special linear ordering of the vertices of its tope graph. See [7] for the definitions. It thus, is natural to compare corner peelings and shellings.

It is known that AOMs and OMs are shellable, see [7]. Shellability is defined for pure regular complexes thus the question of existence of shellings can be asked for all such COMs. It is not hard to see that a COM consisting of two 4-cycles joined in a vertex is not shellable.

Question 4. *Which COMs are shellable?*

A necessary condition might be r-connectedness, if a COM has rank r. Another useful hint might be the fact that an amalgamation procedure for COMs described in [4] is similar to the notion of constructibility, which is a weakening of shellability, see [21].

Corner peelings of LOPs are related to extendable shellability of the octahedron, see [9, 51]. While OMs have corners, AOMs do not always, as the example of Figure 3 shows. Hence, shellability does not imply the existence of a corner or a corner peeling. However, there still might be a connection:

Question 5. *If a shellable COM G has a corner peeling, can one find a shelling sequence that is a refinement of a corner peeling sequence of G?*

7.2 Murty's conjecture

An important open problem in OMs is a generalization of the *Sylvester-Gallai Theorem*, i.e., for every set of points in the plane that does not lie on a single line there is a line, that contains only two points.

The corresponding conjecture in OMs can be found in Mandel's thesis [35], where it is attributed to Murty. In terms of OMs it reads:

Conjecture 4 (Murty)**.** *Every OM of rank r contains a convex subgraph that is the Cartesian product of an edge and an antipodal graph of rank $r-2$.*

The realizable case of Murty's conjecture is shown by [50] and more generally holds for *Mandel* OMs [35]. Indeed, we suspect that along our strengthening of Mandel's theorem (Theorem 4.5) a Θ-version of Mandel's results can be proved:

Conjecture 5. *Every Θ-class in a Mandel OM of rank r is incident to a vertex of an antipodal graph that is the Cartesian product of an edge and an antipodal graph of rank $r-2$.*

On the other hand it would be interesting to find OMs, that do not have this strengthened property. Still Murty's conjecture in general seems out of reach. We propose a reasonable weaker statement to attack:

Question 6. *Does every OM of rank r contain a convex $Q_{\lceil \frac{r}{2} \rceil}$.*

8 Acknowledgments

We thank Arnaldo Mandel for providing us with a copy of his PhD thesis. This work was supported by the Slovenian Research Agency (research core funding No. P1-0297 and projects J1-9109, N1-0095, J1-1693). Tilen Marc wishes to express his gratitude to Institut français de Slovénie for supporting a visit in Marseille that started the present research. Kolja Knauer was furthermore supported by the ANR project DISTANCIA: ANR-17-CE40-0015 and by the Spanish *Ministerio de Economía, Industria y Competitividad* through grant RYC-2017-22701.

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