DISTRIBUTIVE LATTICES ON GRAPH ORIENTATIONS

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ABSTRACT. Propp gave a construction method for distributive lattices on a class of orientations of a graph – called c-orientations. Given a distributive lattice we construct a set of graphs, realizing the distributive lattice as the lattice of their c-orientations. One distributive lattice may arise from different graphs. We describe the set of graphs which generate the same distributive lattice.

1. INTRODUCTION

This paper deals with finite directed graphs and finite distributive lattices. Given a directed graph D we follow Propp and define two operations flip and flop which modify the orientation locally. The set of all graphs D' which are reachable from D via a sequence of flips and flops comes with a natural order relation. In fact it is a distributive lattice, the flip flop poset (Theorem 1.1). Remarkably, the elements of the flip flop poset of D can also be characterized as the set of all orientations of the underlying graph G with a certain numerical invariant on the cycles of D (Theorem 1.2). The contributions of this paper are the following results:

- Every finite distributive lattice is the flip flop poset of a digraph D (Theorem 2.1).
- A description of the class of digraphs which generate the same flip flop poset (Theorem 3.10).

For a more general approach to flip flop posets induced by oriented matroids and general sign matrices, see [4]. This paper is based on the fourth chapter of [4].

1.1. **Preliminaries.** The main object of this paper are directed graphs $D = (V, A)$ with vertex set V and arc set $A \subseteq V \times V$. We will generally forbid loops, i.e. arc of the form (v, v) are not allowed. To avoid confusion when necessary we shall specify vertex and arc set of D as $V(D)$ and $A(D)$, respectively. Given an undirected graph $G = (V, E)$ we call a directed graph $D = (V, A)$ an **orientation** of G if there is a bijection $f : E \to A$ such that $f(\{u, v\}) \in \{(u, v), (v, u)\}$. If D is an orientation of G then G is unique up to isomorphism and we call the isomorphism class of G the **underlying graph** of D. Given D we denote its underlying graph as $\underline{D}$. For digraphs D, D' we say that D' is a **reorientation** of D if $\underline{D} = \underline{D'}$. Clearly, two digraphs D, D' are orientations of isomorphic graphs if and only if D and D' have identical underlying graphs which by definition is equivalent to D' being a reorientation of D. So instead of looking at a set of orientations of an undirected graph we will choose one reference orientation and a set of its reorientations. A feature of this point of view is that given a directed graph $D = (V, A)$ any reorientation D' can be viewed as a subset $R \subseteq A$, i.e. the arcs in R have to be reversed to obtain D' from D. The reason for still having the word *orientation* in the title and in the abstract is that previous research ([3, 6]) has been formulated using this term and that it sounds better.

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We define a **cut** of D as an arc set $S[X] \subseteq A$ induced by $X \subseteq V$. The cut consists of all the arcs that are incident to X and $V \backslash X$, i.e.

$$S[X] := \{(v, w) \in A \mid \#(\{v, w\} \cap X) = 1\},$$

where $\#$ maps a set to its cardinality.

A cut is **directed** if either all its arcs point from X to $V \backslash X$ **(positively directed)** or from $V \backslash X$ to X **(negatively directed)**. A cut of the form $S[\{v\}]$ for $v \in V$ is called a **vertex cut** and denoted by $S[v]$. Vertices that induce positively and negatively directed vertex cuts are also called **sources** and **sinks**, respectively.

We introduce two operations on directed graphs to obtain a partial order on some of the reorientations of a given connected digraph D. Reversing the orientation on all the arcs of a positively directed vertex cut is called a **flip**. The inverse operation, i.e. reversing the orientation on a negatively directed vertex cut, is called a **flop**. Fix an arbitrary vertex $\top$ of D – **the forbidden vertex** – and allow flipping and flopping all the other vertex cuts of D. For reorientations of D' and D'' of D, we define $D' \leq D''$ if D'' can be obtained from D' by a sequence of flips at vertices different from $\top$. The binary relation $\leq$ is a partial order on the set of reorientations of D that can be reached from D via a sequence of flips and flops. This set of reorientations of D partially ordered by $\leq$ is called the **flip flop poset** $P_{\text{ff}}(D, \top)$. See Figure 1 for an example.

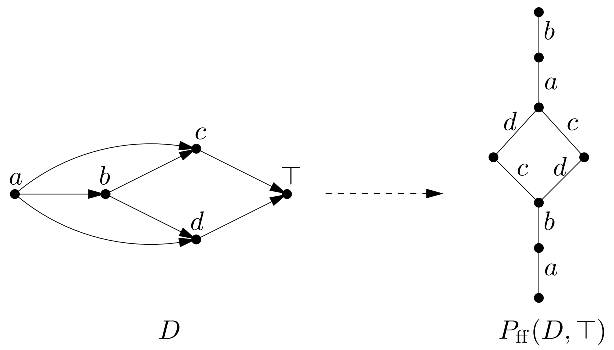

FIGURE 1. A digraph D and its flip flop poset with additional edge labels. An edge is labelled with the vertex of D whose flip it corresponds to. The digraph D is the bottom-element of $P_{\text{ff}}(D, \top)$. Every other reorientation in $P_{\text{ff}}(D, \top)$ can be reached by a sequence of flips.

It turns out that $P_{\text{ff}}(D, \top)$ is highly structured.

Theorem 1.1 ([6]). *Let $D = (V, A)$ be a connected digraph and $\top \in V$. Then the flip flop poset $P_{\text{ff}}(D, \top)$ with forbidden vertex $\top$ is a distributive lattice.*

To explain a second feature of the set $P_{\text{ff}}(D, \top)$ we need to introduce some more terminology. We call a sequence $W = (v_1, a_1, v_2, \ldots, a_{k-1}, v_k)$ of vertices $v_i \in V(D)$ and arcs $a_i \in A(D)$ a **walk** if $a_i \in \{(v_i, v_{i+1}), (v_{i+1}, v_i)\}$ for $1 \leq i \leq k-1$ and every arc appears at most once in W. An arc a_i in W is a **forward** arc if $a_i = (v_i, v_{i+1})$ and **backward** arc if $a_i = (v_{i+1}, v_i)$. For a walk W denote by $F(W)$ set of its forward arcs. If a walk contains every vertex at most once it is called **path** or (v_1, v_k)-path. If $v_1 = v_k$ and every other vertex appears at most once in the walk then we call it a **cycle**. A walk is called **directed** if it contains no forward or no backward arcs. A digraph is called **acyclic** if it has no directed cycles.

Now we define c-reorientations: Given a directed graph D denote its set of cycles by $\mathcal{C}(D)$. For every reorientation D' of D there is a unique arc set $R_{D'} \subseteq A(D)$ such that changing the orientation of the arcs in $R_{D'}$ yields D'. The set $R_{D'}$ yields a bijection of $\mathcal{C}(D)$ and $\mathcal{C}(D')$. For a $C \in \mathcal{C}(D)$ change the orientation of all its arcs that are in $R_{D'}$. This gives the corresponding cycle $R_{D'}(C)$ of D'.

We want to investigate the set of reorientations of D which agree with the same number of forward arcs among corresponding cycles, i.e. we call

$$\textit{reor}_c(D) := \{D' \mid \underline{D'} = \underline{D} \textit{ and } \#F(R_{D'}(C)) = \#F(C) \forall C \in \mathcal{C}(D)\}$$

the set of **c-reorientations** of D.

Theorem 1.2 ([6]). *For an acyclic, connected digraph $D = (V, A)$ we have*

$$P_{\text{ff}}(D, \top) = \textit{reor}_c(D).$$

The definition of a flip flop poset can be modified such that Theorem 1.2 holds for arbitrary digraphs. Just flip cuts induced by strong components instead of vertex cuts and choose one *forbidden strong component* per connected component. For the sake of simplicity we will restrict ourselves to connected acyclic digraphs.

2. Every Distributive Lattice is the Flip Flop Poset of a Digraph

We describe a method which from a distributive lattice L constructs a set $[D]$ of acyclic digraphs with unique source such that $P_{\text{ff}}(\widetilde{D}, \top) \cong L$ for every $\widetilde{D} \in [D]$.

An essential ingredient for this will be *Birkhoff's Representation Theorem for Finite Distributive Lattices*, see [1]. To state the theorem we need to make a few definitions.

Given a distributive lattice L denote by $\mathcal{J}(L)$ the subposet of join-irreducible elements of L, i.e. $\mathcal{J}(L)$ consists of those elements of L that cover exactly one element of L. As usual we say x **covers** y and denote it by $x \succ y$ whenever x is minimal with $x > y$ in L.

Given on the other hand a poset P, denote by $\mathcal{O}(P)$ the inclusion order on the ideals (downsets) of P.

Birkhoff's Theorem asserts that $\mathcal{O}(P)$ is a distributive lattice and $\mathcal{J}(\mathcal{O}(P)) \cong P$ and $\mathcal{O}(\mathcal{J}(L)) \cong L$ for distributive lattices L.

Now we turn back to the construction of a graph class $[D]$ whose elements have a given distributive lattice L as their flip flop poset. Define a directed graph D' with a vertex $v' \in V(D')$ for every element $v \in \mathcal{J}(L)$ and arcs

$$(u', v') \in A(D') :\Longleftrightarrow u \prec_{\mathcal{J}(L)} v.$$

Add a vertex $\top$ to D' and introduce arcs from the sinks and sources of D' to $\top$. Call this new graph D. Since $V(D) = V(D') \cup \{\top\}$ the construction yields a bijection

$$f : \mathcal{J}(L) \to V(D) \backslash \{\top\}$$

by mapping $v \in \mathcal{J}(L)$ to $v' \in V(D') = V(D) \backslash \{\top\}$.

Denote by $[D]$ the set of digraphs that can be obtained from D by adding transitive arcs, where an arc (u, v) is called **transitive** if there is a directed (u, v)-path in D. Since $V(\widetilde{D}) = V(D)$ for every $\widetilde{D} \in [D]$ the bijection f can be seen as a bijection from $\mathcal{J}$ to $V(\widetilde{D}) \backslash \{\top\}$. The construction of $[D]$ is exemplified by Figure 2.

Theorem 2.1. *Let L be a distributive lattice and $\widetilde{D} \in [D]$. Then $P_{\text{ff}}(\widetilde{D}, \top)$ is isomorphic to L.*

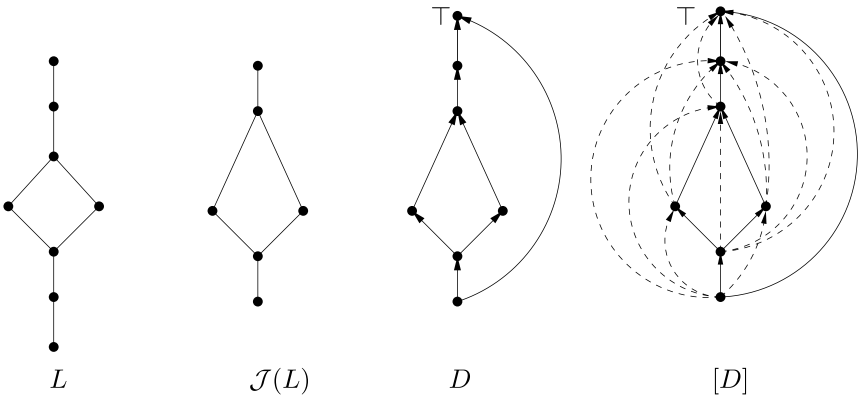

FIGURE 2. The construction of $[D]$ from L. The dashed arcs in the drawing of $[D]$ are the transitive arcs of D. Every element of $[D]$ results from D by adding a set of transitive arcs to D.

Proof. By construction $\widetilde{D}$ has no negatively directed vertex cuts apart from $\top$, i.e. no flops can be performed at $\widetilde{D}$, i.e. $\widetilde{D}$ is a minimum of $P_{\text{ff}}(\widetilde{D},\top)$ by definition of $P_{\text{ff}}(\widetilde{D},\top)$. According to Theorem 1.1 $P_{\text{ff}}(\widetilde{D},\top)$ is a distributive lattice. In particular $P_{\text{ff}}(\widetilde{D},\top)$ has a unique minimum, which consequently must be $\widetilde{D}$. Thus every element of $P_{\text{ff}}(\widetilde{D},\top)$ can be reached by a sequence of flips starting at $\widetilde{D}$.

By Birkhoff's Theorem, it is enough to show $\mathcal{O}(\mathcal{J}(L)) \cong P_{\text{ff}}(\widetilde{D},\top)$.

Let $g : P_{\text{ff}}(\widetilde{D},\top) \to \{\widetilde{I} \mid \widetilde{I} \subseteq V(\widetilde{D})\backslash\{\top\}\}$ map $\widetilde{D}' \in P_{\text{ff}}(\widetilde{D},\top)$ to a vertex set $\widetilde{I}$ whose vertex cuts $\{S[v] \mid v \in \widetilde{I}\}$ can be flipped in some order to reach $\widetilde{D}'$ from $\widetilde{D}$.

Since flipping different vertex sets $\widetilde{I},\widetilde{I}' \subseteq V(\widetilde{D})\backslash\{\top\}$ yields different reoriented arc sets $S[\widetilde{I}], S[\widetilde{I}']$, there is a unique $\widetilde{I} \subseteq V(\widetilde{D})\backslash\{\top\}$ whose vertex cuts have to be flipped to obtain $\widetilde{D}'$ from $\widetilde{D}$, for every $\widetilde{D}' \in P_{\text{ff}}(\widetilde{D},\top)$. This is, for every $\widetilde{I} \subseteq V(\widetilde{D})\backslash\{\top\}$ whose vertex cuts can be flipped there is $g^{-1}(\widetilde{I}) \in P_{\text{ff}}(\widetilde{D},\top)$.

Every $\widetilde{I} \subseteq V(\widetilde{D})$ corresponds to an $I \subseteq \mathcal{J}(L)$ via the bijection f given in the construction of $[D]$.

First we show that f maps every set $\widetilde{I}$ of vertices whose cuts can be flipped to reach an element of $P_{\text{ff}}(\widetilde{D},\top)$ to an order ideal $I \subseteq \mathcal{J}(L)$, i.e. $f(g(P_{\text{ff}}(\widetilde{D},\top))) \subseteq \mathcal{O}(\mathcal{J}(L))$.

The vertex cuts at the sources of $\widetilde{D}$ are positively directed. But the sources are connected to $\top$, which is not allowed to be flipped, so the cuts of the sources can be flipped exactly once.

Furthermore every vertex cut $S[v]$ can be flipped only after the vertex cuts $S[u]$ for $(u,v) \in A(\widetilde{D})$ have been flipped. Iteratively every vertex cut can be flipped at most as often as the sources. Together we have that every vertex cut can be flipped at most once in a flip sequence and f maps the set of vertices $\widetilde{I}$, whose cuts have been flipped in any flip sequence to an ideal I of $\mathcal{J}(L)$.

It remains to be shown that for every ideal I of $\mathcal{J}(L)$ the vertex cuts of the set $f^{-1}(I) \subseteq V(\widetilde{D})$ can be flipped. So take an ideal I. The corresponding vertex set $\widetilde{I} := f^{-1}(I)$ induces a directed cut $S[\widetilde{I}]$ in $\widetilde{D}$. It is an elementary fact from graph theory that a cycle in a digraph contributes as many forward arcs as backward arcs to its intersection with a directed cut. So reversing the orientation on $S[\widetilde{I}]$ leaves the number of forward arcs among the cycles of $\widetilde{D}$ invariant. Thus reversing $S[\widetilde{I}]$ yields a c-reorientation of $\widetilde{D}$. By Theorem 1.2 the cut $S[\widetilde{I}]$ can be reversed by a flip sequence of vertex cuts. Every vertex cut in $\widetilde{D}$ can be flipped at most once. Therefore for every arc $(u,v) \in S[\widetilde{I}]$ exactly one of the vertex cuts $S[u]$ and $S[v]$ has to be flipped. But to flip $S[v]$ one must have flipped $S[u]$ before. This is, $S[u]$ and not $S[v]$ must have been flipped. We have $u \in \widetilde{I}$ and by construction f^{-1} maps the vertices of $\widetilde{I}$ incident to $S[\widetilde{I}]$ to the maxima of I. So to reverse $S[\widetilde{I}]$ the vertex cuts in $f^{-1}(I)$ must be flipped, i.e. $g^{-1}(f^{-1}(\mathcal{O}(\mathcal{J}(L)))) \subseteq P_{\text{ff}}(\widetilde{D}, \top)$. □

This theorem is a slight generalization of a result in [5] which has been obtained in the disguise of edge firing games. In [5] instead of the set $[D]$ only the digraph D is considered. Note that the elements of $[D]$ form a Boolean lattice under arc set inclusion.

3. Digraphs with the Same Flip Flop Poset

The construction for $[D]$ used in Theorem 2.1 does not generally yield all the graphs which flip flop generate L. Figure 3 where three quite different digraphs generate the same distributive lattice is one example. On the left we have the digraph from Figure 1. The digraph in the middle results from the lattice by the construction of Theorem 2.1 and the right one will remain the running example for the rest of the paper. The aim of the present section is to understand the set of different digraphs generating the same distributive lattice.

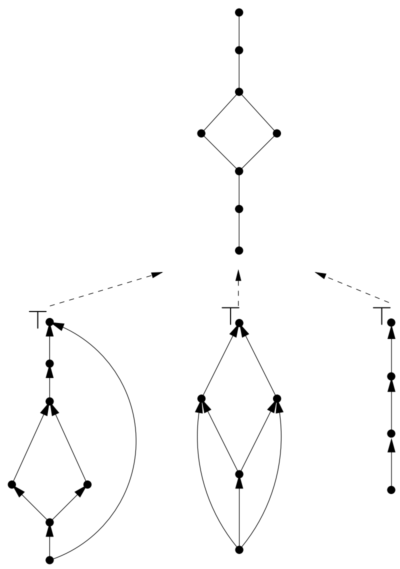

Figure 3. Three digraphs generating the same distributive lattice.

3.1. **The Embedded Flip Flop Poset.**

We analyze the digraphs which generate a given distributive lattice. In the following we describe how a digraph not only generates a distributive lattice but also leads to an embedding of the flip flop poset into some $\mathbb{Z}^d$. We take $\mathbb{Z}^d$ together with the **dominance order**, i.e.

$$x \leq y :\Longleftrightarrow x(i) \leq y(i) \ \ \forall_{1 \leq i \leq d},$$

where $x(i)$ refers to the ith entry of a vector x. Denote by e_i the d-dimensional ith **unit vector**, i.e $e_i(i) = 1$ and $e_i(j) = 0$ if $i \neq j$. By $[d]$ we refer to the set $\{1, \dots d\}$. Given a distributive lattice L an injection $\phi : L \hookrightarrow \mathbb{Z}^d$ is called an **embedding** if for every $p, q \in L$ we have

- $p \succ q \Rightarrow \phi(p) - \phi(q) \in \{e_i \mid i \in [d]\}$
- $\phi(p) \geq \phi(q) \Rightarrow p \geq q$

This is a slightly stronger version of the usual term *order embedding*, since by the first condition ϕ has to be even cover preserving.

In the following we will construct a canonical embedding ϕ_{ff} of $P_{\text{ff}}(D, \top)$ into $\mathbb{Z}^{\#V-1}$. For convenience we assume the digraph D to have $\top$ as unique sink. This means no flops can be performed in D, i.e. D is the minimum of $P_{\text{ff}}(D, \top)$. Thus every c-reorientation can be reached by some sequence $s = (S[v_1], \dots, S[v_k])$ of flips. For any such sequence s of length k define the vector $z_s \in \mathbb{Z}^{\#V-1}$ with entries

$$z_s(v) := \#\{i \in [k] \mid S[v_i] = S[v]\}$$

for every $v \in V \backslash \top$. Now given $D' \in P_{\text{ff}}(D, \top)$ which can be reached via a flip sequence s from D define $\phi_{\text{ff}}(D') := z_s$.

We will denote the **embedded flip flop poset** $\phi_{\text{ff}}(P_{\text{ff}}(D, \top))$ by $\mathcal{P}_{\text{ff}}(D, \top)$. For an example see Figure 4.

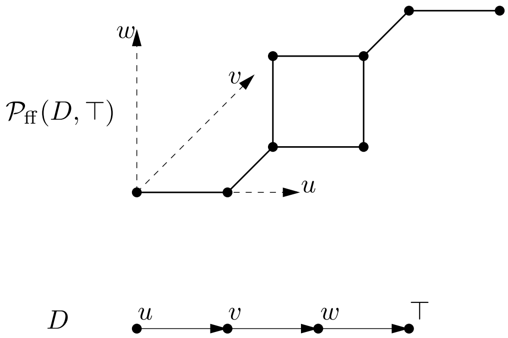

FIGURE 4. A digraph D with associated embedded flip flop poset $\mathcal{P}_{\text{ff}}(D, \top)$ in $\mathbb{Z}^3$.

It is not even clear that ϕ_{ff} is a mapping, but we will prove this and more in the following:

Proposition 3.1. *The above defined ϕ_{ff} is an embedding of $P_{\text{ff}}(D, \top)$ into $\mathbb{Z}^{\#V-1}$.*

Proof. Before proving that ϕ_{ff} is a mapping, we prove that in this case it would be injective as well. Let s, t be two flip sequences of vertex cuts, which can be applied to $D' \in P_{\text{ff}}(D, \top)$. We show, if $z_s = z_t$ then s and t lead to the same reorientation D''. The equation $z_s = z_t$ encodes that s and t flip the same vertex cuts the same number of times – only in a different order. This means that every arc in D' is flipped the same number of times along s and along t. Since s and t start from the same reorientation D' the resulting reorientations must coincide.

Now we show that ϕ_{ff} is indeed a mapping.

Suppose two flip sequences s, t connect reorientations D' and D'', but $z_s \neq z_t$. Since in a distributive lattice all the maximal chains connecting a pair of elements have the same length s and t have the same length – say k. Take s, t of minimal length among all the counterexamples. Among these shortest length counterexamples take a pair s, t with maximal starting orientation D' with respect to $P_{\text{ff}}(D, \top)$. Let $S[v_1]$ and $S[w_1]$ be the first elements of s and t, respectively. Let $s' := (S[v_1], \ldots S[v_i])$ the initial subsequence of s such that $S[v_{i+1}]$ is the first occurrence of $S[w_1]$ in s or $i = k$. Since $S[w_1]$ is positively directed in D' and all s' can be flipped in D' every vertex cut in s' is disjoint from $S[w_1]$. Now flip $S[w_1]$ call the resulting reorientation D'''. We distinguish two cases:

Case 1: $S[w_1] \in s$.

Since all the elements of s' are disjoint from $S[w_1]$ we can apply s' in D'''. Since $z_{S[w_1],s'} = z_{s',S[w_1]}$ the reorientation resulting from flipping first $S[w_1]$ and then s' and the one resulting from first flipping s' and then $S[w_1]$ coincide. So $\widetilde{s} = (S[w_1], s', S[v_{i+2}], \ldots, S[v_k])$ is another flip sequence leading from D' to D''.

Since t and $\widetilde{s}$ coincide in the first entry, deleting this entry results in two sequences t' and $\widetilde{s}'$, which start in D''' and end in D''. Moreover $z_t \neq z_s = z_{\widetilde{s}}$ together with $z_{\widetilde{s}} - z_{S[w_1]} = z_{\widetilde{s}'}$ and $z_{t'} = z_t - z_{S[w_1]}$ implies $z_{t'} \neq z_{\widetilde{s}'}$. Thus $\widetilde{s}'$ and t' are counterexamples of smaller length and contradict our minimality assumption in the choice of s and t.

Case 2: $S[w_1] \notin s$.

Again since all the elements of s' are disjoint from $S[w_1]$ we can apply s' in D'''. Since $S[w_1] \notin s$ we have $s' = s$. Let D'''' be reorientation, which results from flipping s in D'''. Define t' by deleting the first entry $S[w_1]$ of t and attaching it to the end. Recall that $S[w_1]$ is positively directed in D'', because it is positively directed in D' and all vertex cuts in s are disjoint to $S[w_1]$. So applying all but the last flip of t' to D''', yields D'' we can flip all t' in D'''. Now $z_{s'} = z_s \neq z_t = z_{t'}$ but s' and t' start at $D'' \succ D'$. This contradicts the maximality assumption in the choice of D'.

Now that we have shown that ϕ_{ff} is an injective mapping, we prove the order preserving properties.

If $D' \succ D''$ then there is $v \in V$ such that flipping $S[v]$ in D'' leads to D'. This is, $\phi_{\text{ff}}(D') = \phi_{\text{ff}}(D'') + e_v$.

To prove $\phi(D') \geq \phi(D'') \Rightarrow D' \geq D''$ suppose the contrary. So take D', D'' incomparable in $P_{\text{ff}}(D, \top)$ with $\phi(D') > \phi(D'')$, which minimize $|\phi(D') - \phi(D'')|$. Here we denote by $|\cdot|$ the sum over the entries of a vector.

Let D''' be the meet of D' and D'', i.e. D''' is the maximal element smaller than both D' and D''. The unique existence of D''' is a property of the distributive lattices. Now D' and D'' can be reached by flip sequences s and t from D''', respectively. Let $S[w_1]$ be the first element of t and $S[v_1]$ the first element of s. Since both vertex cuts are positively directed in D''' they must be disjoint. Thus, after flipping $S[w_1]$ the cut $S[v_1]$ must still be positively directed and we can flip the initial subsequence $s' = (S[v_1], \ldots S[v_i])$ of s with $S[v_{i+1}] = S[w_1]$, which necessarily occurs because $\phi(D') > \phi(D'')$, i.e. $z_s > z_t$. The same orientation can be obtained by flipping s in D''' until (inclusively) the first occurrence of $S[w_1]$. Thus the orientation which results from D''' by flipping $S[w_1]$ is smaller than D' and D'', but bigger than D'''. This contradicts that D''' was the meet of D' and D''. □

Note that all the digraphs in the set $[D]$ constructed in the previous section lead to the same embedding.

Now that we have defined a construction for the embedding of the flip flop poset we derive a description for the embedded point set. To do so, we find criteria when a vertex can be flipped for the kth time in terms of how many times other vertices have been flipped.

For every $v \in V(D)$ denote by $v^{\downarrow}$ the set of vertices w such that there is a directed (w, v)-path and by $v^{\uparrow}$ the vertices w such that a directed (v, w)-path exists. Furthermore denote by $\text{dist}(v, w)$ the minimum number of arcs on a directed (v, w)-path in D.

Lemma 3.2. *To flip $S[v]$ for the kth time every $S[w]$ with $w \in v^{\uparrow}$ has to be flipped exactly $k - \mathit{dist}(v, w)$ times before.*

Proof. Let $w \in v^{\uparrow}$. We proceed by induction on $\text{dist}(v, w)$.

If $\text{dist}(v, w) = 1$, we have $a = (v, w) \in A(D)$. Vertex cuts can only be flipped when they are positively directed, so $S[v]$ and $S[w]$ can only be flipped in an alternating fashion starting with $S[v]$. Thus before the kth flip of $S[v]$ can be performed, $S[w]$ has been flipped exactly $k - 1$ times.

If $\text{dist}(v, w) > 1$ choose w' as the first vertex on a shortest (v, w)-path after leaving v. By the induction base $S[w']$ must be flipped $k-1$ times to flip $S[v]$ the desired k times. As $\text{dist}(w', w) < \text{dist}(v, w)$ the induction hypothesis yields that w has to be flipped exactly $k - 1 - \text{dist}(w', w) = k - \text{dist}(v, w)$ times. □

Lemma 3.3. *To flip $S[v]$ for the kth time every $S[w]$ with $w \in v^{\downarrow}$ has to be flipped exactly k times before.*

Proof. The proof works similarly to the one of Lemma 3.2. For details, see Lemma 4.2.1. on page 70 in [4]. □

Having proved necessary conditions for a vertex cut to be flipped, we will now provide a sufficient one. Denote by $N^-(v)$ the set $\{w \in V(D) \mid (w, v) \in A(D)\}$ and by $N^+(v) := \{u \in V(D) \mid (v, u) \in A(D)\}$. These terms always refer to the starting orientation D.

Lemma 3.4. *If $S[w]$ can be flipped k times for every $w \in N^-(v)$ and $S[u]$ can be flipped $k - 1$ times for every $u \in N^+(v)$ then $S[v]$ can be flipped k times.*

Proof. For every $u \in N^+(v)$ there is an reorientation such that $S[u]$ has been flipped $k-1$ times and for every $w \in N^-(v)$ there is a reorientations such that $S[w]$ has been flipped k times. Let $\widetilde{D}$ be the join of all these reorientations. The unique existence of $\widetilde{D}$ is asserted by Theorem 1.2. By Lemma 3.2 and Lemma 3.3 the cut $S[v]$ has been flipped at least $k-1$ times to obtain $\widetilde{D}$.

Suppose that v has been flipped exactly $k - 1$ times. So by Lemma 3.2 and Lemma 3.3 the set $N^+(v)$ has been flipped exactly $k - 1$ times and $N^-(v)$ has been flipped exactly k times. The arcs that formerly pointed into v have been reversed an odd number of times. Those which pointed away from v have been reversed an even number of times. This means that $S[v]$ is positively directed in $\widetilde{D}$, thus can be flipped again. □

It is a natural idea to apply the above lemmas along paths in D. This gives rise to a new definition. We call the function $\pi : V \to \mathbb{Z}_{\geq 0}$ the **potential function** of D which assigns to every v the minimum number of forward arcs in a $(v, \top)$-path.

Lemma 3.5. *For every $v \in V$ the value $\pi(v)$ gives the maximal number of $S[v]$-flips occurring in a flip sequence.*

Proof. To see that $S[v]$ can be flipped at most $\pi(v)$ times, assume the contrary.

So take v that minimizes $\pi(v)$ with the property that $S[v]$ can be flipped $k > \pi(v)$ times.

If $\pi(v) = 0$ then since $\top$ is a source we have $v = \top$ but $S[\top]$ cannot be flipped by definition – a contradiction. So let $\pi(v) > 0$ and let P be a $(v, \top)$-path which has $\pi(v)$

forward arcs. Let $a_i = (v_i, v_{i+1})$ be the first forward arc in P. Since $v_i \in v^{\downarrow}$ Lemma 3.3 yields that v_i can be flipped k times. Applying Lemma 3.2 $v_{i+1} \in v_i^{\uparrow}$ yields that v_{i+1} can be flipped $k-1$ times. On the other hand the continuation on P from v_{i+1} to $\top$ has $\pi(v)-1$ forward arcs, so $\pi(v_{i+1}) \leq \pi(v)-1 < k-1$ which contradicts the minimality assumption in the choice of v.

Now we prove that for every $n \in \mathbb{Z}_{\geq 0}$ and every $v \in V(D)$ the vertex cut $S[v]$ can be flipped at least $\min(n, \pi(v))$ times. Since π is bounded this clearly implies that every vertex cut $S[v]$ can be flipped at least $\min(n, \pi(v)) = \pi(v)$ times, by choosing some big enough n.

We proceed by induction on n.

For the induction base assume $n = 0$. We have to show that $S[v]$ can be flipped $\min(0, \pi(v))$ times for every $v \in V(D)$. Since for every $v \in V(D)$ we have $\min(0, \pi(v)) = 0$, we must show that every vertex can be flipped at least 0 times. This is trivially fulfilled.

Now let $n > 0$. The induction hypothesis is that for every $v \in V(D)$ the vertex cut $S[v]$ can be flipped at least $\min(n-1, \pi(v))$ times. We have to show that for every $v \in V(D)$ the vertex cut $S[v]$ can be flipped at least $\min(n, \pi(v))$ times.

So let $v \in V(D)$ be an arbitrary vertex. We can assume that $\pi(v) = n$. That is because $\pi(v) < n$ implies $\min(n, \pi(v)) = \min(n-1, \pi(v))$ which by induction hypothesis gives that $S[v]$ can be flipped at least $\min(n, \pi(v))$ times. If on the other hand $\pi(v) > n$ we can walk on a directed $(v, \top)$-path from v to a vertex v' with $\pi(v') = n$. If we can prove that $S[v']$ can be flipped n times then Lemma 3.3 yields that $S[v]$ can be flipped $n = \min(n, \pi(v))$ times as well.

By the definition of π we have $\min(\pi(w), n-1) = n-1$ for every $w \in N^+(v)$. Induction hypothesis yields that the vertex cuts induced by $N^+(v)$ can be flipped $n-1$ times. Then Lemma 3.3 gives that the vertex cuts induced by $v^{\downarrow} \cup \{v\}$ can be flipped at least $n-1$ times.

For every $w \in (v^{\downarrow} \cup \{v\})$ we have $\pi(w) \geq \pi(v) = n$. So every $u \in N^+(w)$ has $\pi(u) \geq n-1$, so $S[u]$ can be flipped at least $\min(\pi(u), n-1)) = n-1$ times by induction hypothesis. For every $w \in (v^{\downarrow} \cup \{v\})$ flip all the $S[u]$ with $u \in N^+(w)$ these $n-1$ times.

Now let $w \in (v^{\downarrow} \cup \{v\})$ be such that $S[w]$ has been flipped $n-1$ times.

Case 1: All the vertices in $N^-(w)$ have been flipped n times or $N^-(w) = \emptyset$. We can flip $S[w]$ the nth time by Lemma 3.4.

Case 2: There is a vertex $w' \in N^-(w)$, which has been flipped only $n-1$ times. Take w' and check which case applies to w'.

Because of acyclicity of D we always arrive at a vertex w' which falls into Case 1 and we can reduce the portion of vertices in $v^{\downarrow} \cup \{v\}$ which have been flipped less than n times. Iterating this process leads to the point where every $w \in (v^{\downarrow} \cup \{v\})$ has been flipped n times – particularly v. □

Combining the above lemmas we obtain the promised description of the embedded flip flop poset.

Theorem 3.6. *Let D be acyclic, connected with unique sink $\top$ then the canonically embedded flip flop poset*

$$\mathcal{P}_{\text{ff}}(D, \top) = \{\mathbf{0} \leq z \leq \pi \mid (v, w) \in A(D) \Rightarrow 0 \leq z(v) - z(w) \leq 1\}$$

as point sets in $\mathbb{Z}^{V(D) \setminus \top}$.

Proof. Before proving the theorem recall that $\mathcal{P}_{\mathrm{ff}}(D,\top) := \phi_{\mathrm{ff}}(P_{\mathrm{ff}}(D,\top))$ was defined previous to Proposition 3.1.

"$\subseteq$":

Lemma 3.5 implies $\mathcal{P}_{\mathrm{ff}}(D,\top) \subseteq \{\mathbf{0} \leq z \leq \pi\}$. Recall that any $z \in \mathcal{P}_{\mathrm{ff}}(D,\top)$ encodes in its v-entry how many times $S[v]$ has been flipped. So Lemma 3.2 implies that for $(v,w) \in A$ (particularly $w \in v^{\uparrow}$) we have $z(w) \geq z(v) - 1$. Lemma 3.3 implies that for $(v,w) \in A$ (particularly $v \in w^{\downarrow}$) we have $z(w) \leq z(v)$. This is

$$\mathcal{P}_{\mathrm{ff}}(D,\top) \subseteq \{\mathbf{0} \leq z \leq \pi \mid (v,w) \in A(D) \Rightarrow 0 \leq z(v) - z(w) \leq 1\}.$$

"$\supseteq$":

Let $\mathbf{0} \leq z \leq \pi$ and $(v,w) \in A(D) \Rightarrow 0 \leq z(v) - z(w) \leq 1$. We proceed by induction on $|z| := \sum_{i=1}^{n} z(i)$. If $|z| = 0$ then $z = \mathbf{0} \in \mathcal{P}_{\mathrm{ff}}(D,\top)$.

Let $|z| > 0$. First suppose that for every $v \in V\backslash\top$ there is $u \in N^{-}(v)$ with $z(u) > z(v)$ or a $w \in N^{+}(v)$ with $z(v) = z(w)$. So take v such that $z(v) \geq z(w)\ \forall_{w \in V\backslash\top}$. By the maximality of $z(v)$ there cannot be a $u \in N^{-}(v)$ with $z(u) > z(v)$ so there is an arc $(v,w) \in A$ such that $z(v) = z(w)$. So $z(w)$ is maximal as well and the same argument, which we applied to v to obtain w can be applied to w to obtain a $x \in N^{+}(w)$ with $z(x) = z(w)$ and so on. Since D is finite the iteration of this argument leads to a directed cycle in D – a contradiction.

We conclude that there exists $v \in V\backslash\top$ such that $z(N^{-}(v)) \equiv z(v)$ and $z(N^{+}(v)) \equiv z(v) - 1$. Where "$\equiv$" stands for z being constantly the claimed value on the claimed set. Since $|z| > 0 \Rightarrow z(v) > 0$ we have that $z' := z - e_v$ satisfies $\mathbf{0} \leq z' \leq \pi$ and $(v,w) \in A(D) \Rightarrow 0 \leq z'(v) - z'(w) \leq 1$ and $|z'| < |z|$. By induction hypothesis $z' \in \mathcal{P}_{\mathrm{ff}}(D,\top)$. So there is a sequence of flips leading from D to D' in which every vertex cut $S[v]$ has been flipped exactly $z'(v)$ times. We have $z'(N^{-}(v)) \equiv z(v)$ and $z'(v) \equiv z'(N^{+}(v)) \equiv z(v) - 1$ but after applying these flips $S[v]$ is positively directed in D'. So $S[v]$ can be flipped again and we obtain that $z \in \mathcal{P}_{\mathrm{ff}}(D,\top)$. $\square$

3.2. **Plissée Partitioned Posets.** Two different digraphs D and D' possibly generate the same flip flop poset L but the embeddings of L induced by them (cf. Theorem 3.6) may well differ. To understand the relation of D and D' in this case, we relate the embeddings of distributive lattices to chain partitions of the poset of join-irreducibles of L.

This will be done using a classical result of Dilworth [2], which naturally extends *Birkhoff's Representation Theorem for Finite Distributive Lattices* to a bijection between finite *embedded* distributive lattices and finite *chain-partitioned* posets. First we shall define the mappings, which generalize $\mathcal{J}$ and $\mathcal{O}$ from Birkhoff's Theorem.

Let $\phi : L \to \mathcal{L} \subset \mathbb{Z}^d$ be an embedding of a distributive lattice L into the dominance order on $\mathbb{Z}^d$. Define $\overline{\mathcal{J}}(\mathcal{L})$ as the pair $(\mathcal{J}(L), \{C_i\}_{i\in[d]})$, where $\{C_i\}_{i\in[d]}$ is the chain partition of $\mathcal{J}(L)$ defined by $C_i := \{x \in \mathcal{J}(\mathcal{L}) \mid \exists_{y\in L} : \phi(x) - \phi(y) = e_i\}$.

Given on the other hand a chain partitioned poset $(P, \{C_i\}_{i\in[d]})$ the corresponding distributive lattice of ideals $\mathcal{O}(P)$ can be embedded into $\mathbb{Z}^d$ as follows: associate a vector $z_I \in \mathbb{Z}^d$ to every $I \in \mathcal{O}(P)$, where $z_I(i) := |I \cap C_i|\ \forall_{i\in[d]}$. We denote this map by $\overline{\mathcal{O}}$.

Dilworth's Theorem asserts that for any chain-partitioned poset $(P, \{C_i\}_{i\in[d]})$ the mapping $\overline{\mathcal{O}}$ is an embedding of the distributive lattice $\mathcal{O}(P)$ into $\mathbb{Z}^d$. Moreover $\overline{\mathcal{J}}(\overline{\mathcal{O}}(P, \{C_i\}_{i\in[d]})) \cong (P, \{C_i\}_{i\in[d]})$ and $\overline{\mathcal{O}}(\overline{\mathcal{J}}(\mathcal{L})) \cong \mathcal{L}$ for an embedded distributive lattices $\mathcal{L}$.

In the following we give a characterization of those chain partitions which are the image under $\overline{\mathcal{J}}$ of the embedded flip flop posets of an acyclic digraph.

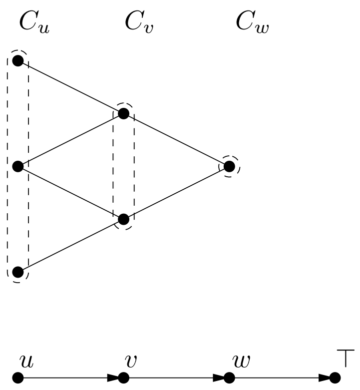

FIGURE 5. Good plissée partitioned poset, which comes from the embedded distributive lattice of Figure 4 via $\overline{\mathcal{J}}$ and its projection. Note that $\pi(i) = |C_i|$ for $i \in \{u, v, w\}$. Conversely the picture can be interpreted as a digraph with its potential poset carrying the canonical chain partition.

We call a chain partition $\{C_i\}_{i\in[m]}$ of a poset P **plissée** if $\forall_{i,j\in[m]}\forall_{p,q\in P} : C_i \ni p \succ q \in C_j$ implies that $C_i \cup C_j = \{r_1 \leq \ldots \leq r_{i+j}\}$ is a chain with

$$\{\{r_{2k+1} \mid 1 \leq 2k+1 \leq i+j\}, \{r_{2k} \mid 1 \leq 2k \leq i+j\}\} = \{C_i, C_j\}.$$

In this case we say that $C_i \cup C_j$ **alternates** between C_i and C_j.

Having P together with a plissée partition $\{C_i\}_{i\in[m]}$, we define the **projection** of $(P, \{C_i\}_{i\in[m]})$ as the directed graph $\Delta(P, \{C_i\}_{i\in[m]}) = (V, A)$, where the vertices are given by

$$V := \{Min(C_i) \mid i \in [m]\} \cup \{\top\}$$

and $(v, w) \in A$ if *either*

$v = Min(C_i)$ and $w = Min(C_j)$ and $Min(C_i) < Min(C_j)$ and $C_i \cup C_j$ alternates

or $v = Min(C_i)$ and $|C_i| = 1$ and $w = \top$.

Since $\Delta(P, \{C_i\}_{i\in[m]})$ is an acyclic digraph with unique sink $\top$ it induces a potential function π. A plissée partition $(P, \{C_i\}_{i\in[m]})$ is called **good** if for the potential function π of $\Delta(P, \{C_i\}_{i\in[m]})$ we have $v = Min(C_i) \Rightarrow \pi(v) = |C_i|$ for every $v \in V(\Delta(P, \{C_i\}_{i\in[m]}))$. See Figure 5 for an example.

Conversely, given an acyclic digraph D with unique sink $\top$ and potential function π, we define its **potential poset**, as the set $\Pi_D := \{v_i \mid i \in [\pi(v)], v \in V(D)\}$ together with the order relation transitively induced by

$$v_i \leq w_j :\Longleftrightarrow \begin{cases} i = j & \text{and} \quad (v, w) \in A(D) \\ & \textit{or} \\ i = j - 1 & \text{and} \quad (w, v) \in A(D) \end{cases}$$

The potential poset carries the **canonical chain partition**

$$\{C_v\}_{v\in V} := \{v_i \mid i \in [\pi(v)]\}_{v\in V(D)}.$$

Theorem 3.7. *The canonical chain partition of the potential poset is a good plissée partition.*

Proof. Let D be an acyclic connected digraph with unique sink $\top$. We have to prove that $(\Pi_D, \{C_v\}_{v\in V(D)})$ is a plissée partitioned poset. So let $C_w \ni w_j \succ v_i \in C_v$. Then *either*

$(i = j$ and $(v, w) \in A(D))$ *or* $(i = j - 1$ and $(w, v) \in A(D))$. In both cases it is easy to see that $w_j \leq v_{i+1}$ and $w_{j-1} \leq v_i$ by the respective other case – given that the claimed elements v_{i+1} and w_{j-1} exist.

Consider the case that $(i = j$ and $(v, w) \in A(D))$. Since $(v, w) \in A(D)$ we have $\pi(v) - \pi(w) \in \{0, 1\}$. If $\pi(v) = \pi(w)$ then $C_w \cup C_v = (v_1 \leq w_2 \leq v_2 \leq \ldots \leq v_{\pi(v)} \leq w_{\pi(w)})$ otherwise $C_w \cup C_v = (v_1 \leq w_2 \leq v_2 \leq \ldots \leq w_{\pi(w)} \leq v_{\pi(v)})$. So in both cases $C_w \cup C_v$ is an alternating chain.

If $(i = j - 1$ and $(w, v) \in A(D))$ the same argument works.

To see that this plissée partition is good, let $D' := \Delta(\Pi_D, \{C_v\}_{v \in V(D)})$ be the projection of the potential poset of D. Obviously $V(D') = V(D)$ and $A(D') \supseteq A(D)$.

We must show that the potential function π' of D' coincides with the potential function π of D. By the definition of potential function $A(D') \supseteq A(D)$ implies $\pi' \leq \pi$. So suppose $\pi' < \pi$. When we extend D to D' arc by arc the potential function decreases in some intermediate step D''.

So let $a = (u, v)$ be an arc in $A(D') \backslash A(D'')$. Suppose adding a to D'' decreases π. There are two possibilities how this can happen.

On the one hand a can decrease $\pi(u)$, i.e. $\pi(v) + 1 < \pi(u)$. By definition, a comes from an alternating $(C_u \cup C_v)$-chain in Π_D. Since $|C_u| = \pi(u)$ and $|C_v| = \pi(v)$ it is impossible for $C_u \cup C_v$ to be an alternating chain.

On the other hand introducing a could decrease $\pi(v)$, i.e. $\pi(u) < \pi(v)$. But this again contradicts the fact that $C_u \cup C_v$ is an alternating chain with minimal element in C_u by reasons of cardinality. □

Note that the arc set of the projection of $(\Pi_D, \{C_v\}_{v \in V(D)})$ possibly strictly contains $A(D)$.

Now we are ready to provide another way of describing the embedded flip flop poset of a digraph.

Theorem 3.8. *Let D be acyclic with unique sink $\top$ then*

$$\mathcal{P}_{\text{ff}}(D, \top) = \overline{\mathcal{O}}(\Pi_D, \{C_v\}_{v \in V}).$$

Proof. To every $z \in \mathcal{P}_{\text{ff}}(D, \top)$ we associate the set $Z := \{v_i \in \Pi_D \mid i \in [z(v)]\}$. It is clear that $z(v) = |Z \cap C_v|$ so if we show that the set of the so-obtained Z's coincides with $\mathcal{O}(\Pi_D)$ the definition of $\overline{\mathcal{O}}$ yields the theorem.

For $\mathcal{P}_{\text{ff}}(D, \top) \subseteq \overline{\mathcal{O}}(\Pi_D, \{C_v\}_{v \in V})$ we show $Z \in \mathcal{O}(\Pi_D)$:

Suppose Z is no ideal so there is $Z \ni v_k \succ w_\ell \notin Z$. By the definition of Z we have $v \neq w$. Since $\{C_v\}_{v \in V}$ is a plissée partition $C_v \cup C_w$ is an alternating chain. We distinguish two cases:

If $Min(C_v \cup C_w) = v_1$ then $(v, w) \in A$ and $\ell = k - 1$. We obtain $w_\ell \notin Z \Rightarrow z(w) < \ell = k - 1 = z(v) - 1$, which contradicts Theorem 3.6.

Similarly if $Min(C_v \cup C_w) = w_1$ then $(w, v) \in A$ and $\ell = k$. We obtain $w_\ell \notin Z \Rightarrow z(w) < \ell = k = z(v)$, which contradicts Theorem 3.6.

Now for $\mathcal{P}_{\text{ff}}(D, \top) \supseteq \overline{\mathcal{O}}(\Pi_D, \{C_v\}_{v \in V})$ we show that $\overline{\mathcal{O}}$ maps every $I \in \mathcal{O}(\Pi_D)$ to a $z \in \mathcal{P}_{\text{ff}}(D, \top)$:

Let $I \in \mathcal{O}(\Pi_D)$. Then vector $\overline{\mathcal{O}}(I)$ is mapped to the vector z with entries $z(v) := |C_v \cap I|$. By definition of the canonical chain partition we have $\mathbf{0} \leq z \leq \pi$. Let $(v, w) \in A$. Then $C_v \cup C_w$ is an alternating chain. Since I is an ideal we either have $z(v) = z(w) + 1$ or $z(v) = z(w)$. The description of $\mathcal{P}_{\text{ff}}(D, \top)$ by Theorem 3.6 yields that $z \in \mathcal{P}_{\text{ff}}(D, \top)$. □

Theorem 3.9. *Let $(P, \{C_i\}_{i\in[m]})$ be a chain partitioned poset. There is a digraph D such that $(P, \{C_i\}_{i\in[m]}) \cong \overline{\mathcal{J}}(\mathcal{P}_{\mathrm{ff}}(D, \top))$ if and only if $\{C_i\}_{i\in[m]}$ is a good plissée partition.*

Particularly if $\{C_i\}_{i\in[m]}$ is a good plissée partition then $(P, \{C_i\}_{i\in[m]}) \cong \overline{\mathcal{J}}(\mathcal{P}_{\mathrm{ff}}(\Delta(P, \{C_i\}_{i\in[m]}), \top))$.

Proof. Necessity. Let $(P, \{C_i\}_{i\in[m]}) \cong \overline{\mathcal{J}}(\mathcal{P}_{\mathrm{ff}}(D, \top))$. By Dilworth's Theorem $\overline{\mathcal{O}}(P, \{C_i\}_{i\in[m]}) \cong \mathcal{P}_{\mathrm{ff}}(D, \top)$, which by Theorem 3.8 equals $\overline{\mathcal{O}}(\Pi_D, \{C_v\}_{v\in V})$. So again by Dilworth's Theorem $(P, \{C_i\}_{i\in[m]}) \cong (\Pi_D, \{C_v\}_{v\in V})$ and by Theorem 3.7 this is a good plissée partition.

Sufficiency. Let $(P, \{C_i\}_{i\in[m]})$ be a poset with a good plissée partition. By definition we have $(P, \{C_i\}_{i\in[m]}) \cong (\Pi_{\Delta(P,\{C_i\}_{i\in[m]})}, \{C_v\}_{v\in V})$. Now Dilworth's Theorem together with Theorem 3.8 gives that the latter is isomorphic to $\overline{\mathcal{J}}(\mathcal{P}_{\mathrm{ff}}(\Delta(P, \{C_i\}_{i\in[m]}), \top))$. □

Let D be an acyclic directed graph with unique sink $\top$ and potential function π. Denote by $[D]$ the set of directed graphs which can be obtained from D by adding or deleting transitive arcs without changing π. We state the following:

Theorem 3.10. *Let L be a distributive lattice. The set D of acyclic directed graphs with unique sink $\top$, which flip flop generate L, i.e. $\{D \mid L \cong P_{\mathrm{ff}}(D, \top)\}$ equals*

$$\{D \in [\Delta(\mathcal{J}(L), \{C_i\}_{i\in[m]})] \mid \{C_i\}_{i\in[m]} \text{ is a good plissée partition of } \mathcal{J}(L)\}.$$

Proof. Let $L \cong P_{\mathrm{ff}}(D, \top)$. This is equivalent to $\mathcal{L} \cong \mathcal{P}_{\mathrm{ff}}(D, \top)$ for some embedding $\mathcal{L}$ of L. After applying Dilworth's Theorem to $\mathcal{L} \cong \mathcal{P}_{\mathrm{ff}}(D, \top)$, Theorem 3.9 says that this is equivalent to: There is a good plissée partition $\{C_i\}_{i\in[m]}$ of $\mathcal{J}(L)$ isomorphic to $\overline{\mathcal{J}}(\mathcal{P}_{\mathrm{ff}}(D, \top))$. Also by Theorem 3.9 we have that D can be taken to be $\Delta(P, \{C_i\}_{i\in[m]})$, i.e. $\mathcal{P}_{\mathrm{ff}}(\Delta(P, \{C_i\}_{i\in[m]}), \top) \cong \mathcal{P}_{\mathrm{ff}}(D, \top)$. It is easily seen that digraphs D and D' have $D \in [D']$ if and only if $\Pi_D \cong \Pi_{D'}$, which by Theorem 3.8 is equivalent to $\mathcal{P}_{\mathrm{ff}}(D, \top) \cong \mathcal{P}_{\mathrm{ff}}(D', \top)$. This yields equivalence to $D \in [\Delta(\mathcal{J}(L), \{C_i\}_{i\in[m]})]$. □

The digraphs in a class $[D]$ are naturally ordered by arc set inclusion. It can be verified that the projection Δ of the chain partitoned potential poset of D is the maximum of this order. On the other hand $[D]$ does not have a unique minimum in general, so Δ is somehow a natural representant for $[D]$.

The above theorem shows that constructing graphs that generate the same flip flop poset basically boils down to determining the good plissée partitions of the poset of join-irreducibles. For further discussion of this topic, see [4].

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REFERENCES

[1] B.A. Davey, H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, 1991.

[2] R.P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. **2** (1950), 161–166.

[3] S. Felsner, *Lattice structures from planar graphs*, Electron. J. Combin. **11** (2004).

[4] K.B. Knauer, *Partial orders on orientations via cycle flips*, `www.math.tu-berlin.de/~knauer/diplom.pdf`, 2007.

[5] M. Latapy, C. Magnien, *Coding distributive lattices with Edge Firing Games*, Inf. Process. Lett. **83** (2002), 125–128.

[6] J. Propp, *Lattice structure for orientations of graphs*, `arXiv:math/0209005v1[math.CO]`, 1993.

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