SIAM J. Applied Dynamical Systems
Vol. xx, pp. x

x–x

Classification of coupled dynamical systems with multiple delays: Finding the minimal number of delays

L. Lücken †, J. P. Pade †, and K. Knauer ‡

Abstract. In this article we study networks of coupled dynamical systems with time-delayed connections. If two such networks hold different delays on the connections it is in general possible that they exhibit different dynamical behavior as well. We prove that for particular sets of delays this is not the case. To this aim we introduce a componentwise timeshift transformation (CTT) which allows to classify systems which possess equivalent dynamics, though possibly different sets of connection delays. In particular, we show for a large class of semiflows (including the case of delay differential equations) that the stability of attractors is invariant under this transformation. Moreover we show that each equivalence class which is mediated by the CTT possesses a representative system in which the number of different delays is not larger than the cycle space dimension of the underlying graph. We conclude that the 'true' dimension of the corresponding parameter space of delays is in general smaller than it appears at first glance.

Key words. multiple delays, networks with delay, coupled dynamical systems, delay differential equations, dynamical equivalence, semidynamical systems, cycle space.

AMS subject classifications. 34K17, 34K20, 37L15, 37L05

1. Introduction.

Differential equations with time delays have been subject of intensive research in the last decades. Recently, major impulses for theoretical investigations came from the challenge of understanding and modeling the human brain [19, 15, 13]. Here, time delays appear due to the finite speed of action potentials propagating through axons [28, 38, 5]. It was observed that the qualitative behaviour of brain networks is closely related to their heterogeneous delay distributions. These were shown to play important roles in phenomena such as coherence in the resting state activity [13, 15], nonstationary bifurcations of equilibria [1] and enhanced synchronizability [12, 9]. Further examples of delay-coupled systems are interacting lasers where delays appear due to the finite speed of light travelling through optical fibers [39, 11, 36]. In population dynamics, delays correspond to maturing and gesturing times [22], and in gene regulatory networks, delays represent the time the system needs to produce a protein [23, 8]. Thus, delays are often included in order to account for the time a signal needs to propagate from one node of the network to another or for the processing time that it takes to emit a response to some input. An inevitable property of real networks of interacting systems with delays is that, generically, each delay time is different from any other.

Nevertheless, there exist comparatively few attempts to study systems with several different delays analytically. An important reason for this is that their analysis is usually much

†Humboldt University of Berlin, Institute of Mathematics, Unter den Linden 6, 10099 Berlin, Germany. luecken@math.hu-berlin.de, pade@math.hu-berlin.de. L. L. was supported by the German Research Foundation (DFG) in the framework of the Collaborative Research Center (SFB) 910. J. P. P was supported by the DFG International Research Training Group (IRTG) 1740.

‡Université Montpellier 2, Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier (LIRMM), Case Courrier 477, 161 rue Ada, 34095 Montpellier Cedex 5, France. K. K. was supported by the ANR Project EGOS: ANR-12-JS02-002-01

harder than for identical delays and techniques available for systems with one delay are in general not applicable in the case of multiple delays. In particular, the analysis of spectra of solutions becomes more complicated since the characteristic quasi-polynomials involve several exponential terms [4, 18]. At least for coupled systems with two different delays some analytical results are available. For instance, Nussbaum [29] proved the existence of periodic solutions in a system with two commensurable delays. Moreover some detailed studies on bifurcations in coupled systems with two different delay times were conducted [3, 7, 35, 6, 17]. In particular, we mention Shayer et al. [35], where the investigation initially assumes three different delays. In the course of the investigation, the authors discover that the stability of a steady state depends only on two values combined from these delays. This finding is a special case of the more general result that we present in this paper. Higher order scalar systems with two delays are considered, e.g., by Gu et al. [16], who study stability crossing curves for this system. For the same system, Ruan and Wei [34] refine techniques for the determination of the roots of characteristic quasi-polynomials of single delay equations to treat the case of two delays. Yanchuk and Giacomelli [40] consider a scalar system with two large delays and show that this system can be described by a complex Ginzburg-Landau equation in the neighborhood of an equilibrium.

Finally, we want to mention studies of Hopfield neural networks with delayed connections where many different delays are taken into account [14, 41, 37, 24]. Here, the primary object of investigation are sufficient conditions for global convergence of the system towards a single steady state in order to obtain a well-defined method of input classification.

In this paper, the focus of our interest is a componentwise timeshift transformation (CTT) which allows to change the interaction delays while the dynamical properties of the system remain the same. Apart of providing an intuition about the functionality of delays, the transformation proves to be particularly useful in cases where it is possible to achieve a smaller or in some sense preferable set of delays in the transformed system.

This is especially apparent for the case when the nodes are coupled in a single unidirectional ring. The corresponding special case of the CTT was utilized more or less explicitly already in [2, 27, 31, 32]. Therefore, let us illustrate the CTT for this example. In a general form, a ring of N unidirectionally delay-coupled systems can be written as

$$\frac{\mathrm{d}}{\mathrm{dt}}x_j(t) = f_j\left(x_j(t), x_{j-1}(t-\tau_{j-1})\right),\ j = 1, ..., N, \tag{1.1}$$

where j is considered modulo N and $\tau_j \geq 0$ are N possibly different delays. Given a solution $x_j(t)$, $j = 1, ..., N$, we introduce new variables

$$y_j\left(t\right) := x_j\left(t + \eta_j\right), \tag{1.2}$$

by shifting the nodes independently in time by certain amounts $\eta_j \geq 0$. Formally, we find that the dynamics of the variables $y_j(t)$ obey

$$\frac{\mathrm{d}}{\mathrm{dt}}y_j(t) = f_j\left(y_j(t), y_{j-1}(t-\tilde{\tau}_{j-1})\right),\ j = 1, ..., N, \tag{1.3}$$

with new delays $\tilde{\tau}_j = \tau_j + \eta_j - \eta_{j+1}$. In other words, finding timeshifts η_j for given delays $\tilde{\tau}_j$ is equivalent to solving a system of linear equations.

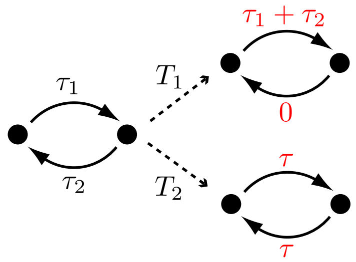


Figure 1. *Two simple examples of CT-Transformations for a ring of two delay-coupled systems. The transformation T_1 reduces the number of delays from two to one and T_2 reveals a hidden symmetry in the system by adjusting the two delays to the same value $\tau = (\tau_1 + \tau_2)/2$.*

For the case of a ring with only two nodes, which is depicted in Fig. 1, we illustrate the possible effect of the timeshift (1.2) in Fig. 2. The plots (a)–(c) show an initial piece of a solution and its change under the CTT (two delay-coupled Mackey-Glass systems [26] were used to create this example). The process of transformation is indicated by the symbols T and $\tilde{T}$, which are given a precise meaning in §4. Each transformation corresponds to a particular choice of the timeshifts in (1.2). Here, T converts (1.1) to (1.3) and $\tilde{T}$ describes the reverse transformation from (1.3) to (1.1). In (a) and (c), the arrows between the timetraces of both components indicate the delayed dependence of $\dot{x}_1(t)$ on $x_2(t-\tau_2)$ and $\dot{x}_1(t)$ on $x_2(t-\tau_2)$, similar in (b) for y_1 and y_2 for the transformed delays $\tilde{\tau}_1$ and $\tilde{\tau}_2$. From (1.2) it follows that the timetraces of the single components have exactly the same form in both systems. Both solutions only differ in the relative timeshifts between their components.

Baldi and Atiya [2] discovered that by choosing timeshifts $\eta_j = \sum_{k=1}^{j-1} \tau_k$ one obtains the transformed delays

$$\tilde{\tau}_j = \begin{cases} 0, & \text{for } 1 \le j \le N-1, \\ \sum_{k=1}^{N} \tau_k, & \text{for } j = N. \end{cases} \tag{1.4}$$

They used this simplified form to predict oscillatory behavior and bifurcations for a neural circuit exhibiting delayed excitatory and inhibitory connections. Mallet-Paret [27] utilized the same transformation to prove a Poincaré-Bendixson Theorem for monotone cyclic feedback systems with delays. A slightly different set of timeshifts can be found in [31] and [32], where the choice $\eta_j = (N-j)\bar{\tau} + \sum_{k=1}^{j} \tau_k$, with the mean delay $\bar{\tau} = \frac{1}{N}\sum_k \tau_k$, was proposed. This leads to identical transformed delays

$$\tilde{\tau}_j \equiv \bar{\tau}. \tag{1.5}$$

For a ring of identical nodes, i.e. $f_j \equiv f$ for all j, it turns out that the obtained system is much more tractable due to its rotational symmetry. One may say that this hitherto hidden symmetry was revealed by the CTT. See also Fig. 1 for an illustration of the two above mentioned timeshifts in a ring of two nodes.

For both variants, (1.4) and (1.5), it is evident that if one is interested in the changes of dynamic behavior with respect to the different delays τ_j it suffices to vary a single parameter, i.e. the mean delay $\bar{\tau} = \frac{1}{N}\sum_k \tau_k$, instead of the N different parameters τ_j, $j = 1, ..., N$. In

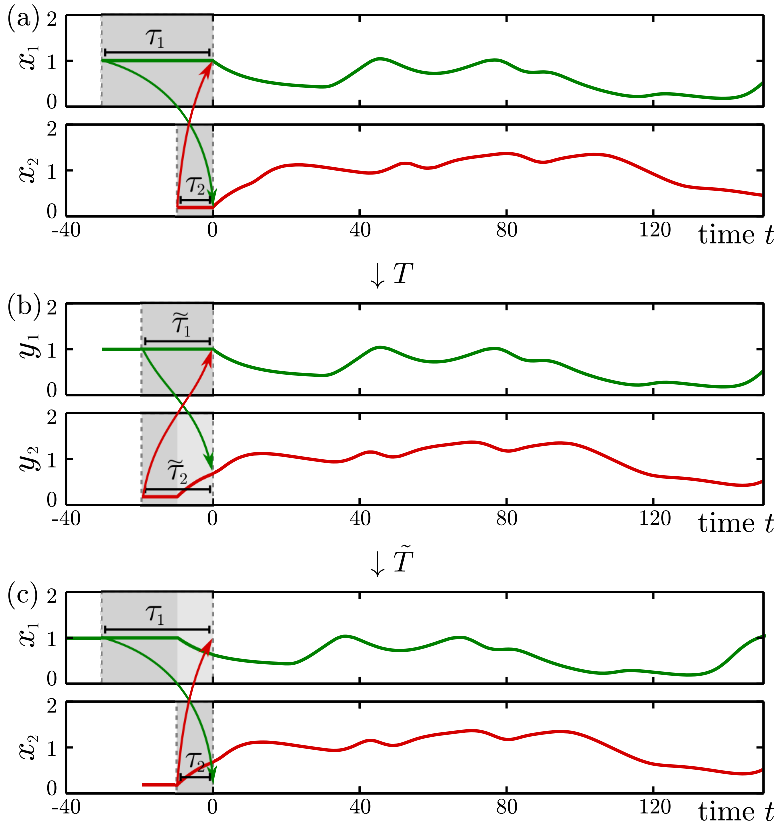


Figure 2. *CT-Transformation for a pair of bidirectionally coupled Mackey-Glass systems [26], cf. Fig. 1. Plot (a) shows the original solution $\boldsymbol{x}(t) = (x_1(t), x_2(t))$ of (2.1). Plot (b) shows the transformed solution $\boldsymbol{y}(t) = (y_1(t), y_2(t))$ of (2.2), which is shifted to the left by amounts $\eta_1 = 0$ and $\eta_2 = 10$. Plot (c) shows the solution $\boldsymbol{x}(t + \eta_1)$ obtained after a reverse transformation of $\boldsymbol{y}(t)$ [see §4 for details].*

other words, the parameter space dimension is much smaller than it might have appeared at first glance. Similar as for a ring, the CTT allows to identify a canonical set of delay parameters in a general network as we explain in §5.

Before doing so let us pose a question. What does the knowledge about dynamical features of the transformed system (1.3) really tell us about the dynamical features of the original system (1.1)? It might seem intuitive that they are the same since the timeshift $y_j(t) \mapsto y_j(t - \eta_j)$ inverses (1.2). This is true for the particular solution (1.2) but, in general, the expression $y_j(t - \eta_j)$ (with $\eta_j > 0$) may not be defined at all. This is because, in general, solutions of delay differential equations (DDEs) cannot be continued backwards. Another subtlety arises if one considers timeshifts η_j leading to anticipating arguments, that is negative delays. In this case (1.3) has fundamentally different properties from an ordinary DDE with positive delays. For instance, initial value problems are ill-posed in general [18]. Even though we will restrict ourselves to the situation where transformed delays are positive, a proper treatment of the above question should introduce state spaces and flows to formulate and compare the dynamical properties of the original and the transformed system. Such a rigorous treatment of the CTT was not given in any of the above mentioned works [2, 27, 31, 32]. We will do this in a quite general setting using semidynamical systems in §4 where we also give a rigorous definition of the CTT. In particular, the structural similarity of the state spaces and the stability of invariant sets is studied. In §2 we introduce notations to describe general networks of delay coupled dynamical systems and in §3 we study the special cases of equilibria and periodic orbits of DDEs where the equivalence of the dynamics of the original and the transformed system is relatively easy to show.

2. Networks of delay coupled dynamical systems. To describe a general network of N coupled systems we choose a framework which enables us to account for multiple links between two nodes holding different delays. That is, the coupling structure of the network is assumed to be represented by a *multidigraph*. This is a set of node indices, $\mathcal{N} = \{1, ..., N\}$, and a set $\mathcal{E}$ of *directed links*. Throughout the whole article we assume that $(\mathcal{N}, \mathcal{E})$ is *weakly connected*. This means that each node can be reached from any other node by traversing a sequence of links, where each link may be traversed in arbitrary direction. For networks with several connected components our results can be applied separately to each component. For each link $\ell \in \mathcal{E}$ the functions $s, t : \mathcal{E} \to \mathcal{N}$ assign its source $s(\ell)$ and its target $t(\ell)$. This means that the link ℓ connects the node $x_{s(\ell)}$ to the node $x_{t(\ell)}$. The delay time of ℓ is denoted by $\tau(\ell)$. Note that there may indeed exist two links ℓ_1 and ℓ_2 in $\mathcal{E}$ with $s(\ell_1) = s(\ell_2)$ and $t(\ell_1) = t(\ell_2)$ but $\tau(\ell_1) \neq \tau(\ell_2)$. For node x_j, we introduce the set of its incoming links as

$$I_j = \{\ell \in \mathcal{E} : t(\ell) = j\}.$$

Then, the dynamics of x_j can be written as

$$\frac{\mathrm{d}}{\mathrm{dt}} x_j(t) = f_j\left(\left(x_{s(\ell)}\left(t - \tau\left(\ell\right)\right)\right)_{\ell \in I_j}\right) \in \mathbb{R},\ j = 1, ..., N, \tag{2.1}$$

where we assume $x_j(t) \in \mathbb{R}$ without loss of generality. This notation allows to include a self-dependency of $x_j(t)$ via a link ℓ with $s(\ell) = t(\ell) = j$. It is also possible to have an instantaneous dependence by setting $\tau(\ell) = 0$. For (2.1), the introduction of new variables $y_j(t)$ as in (1.2) leads to the transformed system

$$\frac{\mathrm{d}}{\mathrm{dt}} y_j(t) = f_j\left(\left(y_{s(\ell)}\left(t - \tilde{\tau}\left(\ell\right)\right)\right)_{\ell \in I_j}\right),\ j = 1, ..., N, \tag{2.2}$$

with modified delays

$$\tilde{\tau}(\ell) = \tau(\ell) - \eta_{t(\ell)} + \eta_{s(\ell)}. \tag{2.3}$$

3. Spectrum of equilibria and periodic orbits. An equilibrium point $\bar{\boldsymbol{x}} = (\bar{x}_1, ..., \bar{x}_N) \in \mathbb{R}^N$ of (2.1) is a point which satisfies

$$f_j\left(\left(\bar{x}_{s(\ell)}\right)_{\ell \in I_j}\right) = 0,\ j = 1, ..., N. \tag{3.1}$$

Obviously $\bar{\boldsymbol{x}}$ is also an equilibrium of (2.2). In the following we assume that $f_j \in C^1$. Then a characteristic exponent λ of $\bar{\boldsymbol{x}}$ in (2.1) corresponds to an exponential solution $\boldsymbol{\xi}(t) = e^{\lambda t}\boldsymbol{\xi}_0 \in$

$\mathbb{C}^N$ of the variational equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\xi_j(t) = \sum_{\ell\in I_j} \partial_{s(\ell)} f_j\left(\left(\bar{x}_{s(\ell')}\right)_{\ell'\in I_j}\right)\xi_{s(\ell)}(t-\tau(\ell)). \tag{3.2}$$

If all of the characteristic exponents of $\bar{\boldsymbol{x}}$ possess negative real parts this assures that $\bar{\boldsymbol{x}}$ is stable, if at least one has positive real part then $\bar{\boldsymbol{x}}$ is unstable [18]. In the transformed system (2.2) the timeshifted variation $\boldsymbol{\chi}(t)$, given by $\chi_j(t) = \xi_j(t+\eta_j)$, is a solution of the corresponding variational equation of $\bar{\boldsymbol{x}}$

$$\frac{\mathrm{d}}{\mathrm{dt}}\chi_j(t) = \sum_{\ell\in I_j} \partial_{s(\ell)} f_j\left(\left(\bar{x}_{s(\ell')}\right)_{\ell'\in I_j}\right)\chi_{s(\ell)}(t-\tilde{\tau}(\ell)). \tag{3.3}$$

Hence, the characteristic exponents of $\bar{\boldsymbol{x}}$ are the same in (2.1) and (2.2).

Similarly, the stability of a periodic solution $\bar{\boldsymbol{x}}(t) = \bar{\boldsymbol{x}}(t+T)$ is determined by its Floquet exponents if their real parts are different from zero. Each exponent λ corresponds to a solution $\boldsymbol{\xi}(t)$ of the variational equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\xi_j(t) = \sum_{\ell\in I_j} \partial_{s(\ell)} f_j\left(\left(\bar{x}_{s(\ell')}\left(t-\tau(\ell')\right)\right)_{\ell'\in I_j}\right)\xi_{s(\ell)}(t-\tau(\ell)) \tag{3.4}$$

which has the form $\boldsymbol{\xi}(t) = e^{\lambda t}\boldsymbol{p}(t)$ with a periodic function $\boldsymbol{p}(t) = \boldsymbol{p}(t+T)$. Again, the timeshifted solution $\boldsymbol{\chi}(t)$ with $\chi_j(t) = \xi_j(t+\eta_j)$ fulfills the variational equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\chi_j(t) = \sum_{\ell\in I_j} \partial_{s(\ell)} f_j\left(\left(\bar{y}_{s(\ell')}\left(t-\tilde{\tau}(\ell')\right)\right)_{\ell'\in I_j}\right)\chi_{s(\ell)}(t-\tilde{\tau}(\ell)) \tag{3.5}$$

of the corresponding periodic solution $\bar{y}_j(t) = \bar{x}_j(t+\eta_j)$ in the transformed system. Let us summarize this section in a theorem:

Theorem 3.1. *The following statements hold true:*

(i) *A fixed point $\bar{\boldsymbol{x}}$ possesses the same characteristic exponents in (2.1) and (2.2).*

(ii) *A periodic solution $\bar{\boldsymbol{x}}(t)$ possesses the same Floquet exponents in (2.1) as the corresponding transformed solution $\bar{\boldsymbol{y}}(t)$ of (2.2).*

4. The componentwise timeshift transformation.

4.1. Definitions. In this section we introduce a rigorous formulation of the idea which underlies the change of variables (1.2). We define the CTT in terms of the underlying infinite dimensional phase spaces of (2.1) and (2.2).

First recall that a *semidynamical system* (or a *semiflow*) is a mapping

$$\begin{aligned}\Phi : [0,\infty)\times X &\to X,\\ (t,\boldsymbol{x}) &\mapsto \Phi_t(\boldsymbol{x}),\end{aligned}$$

on a Banach space X which fulfills:

(i) $\Phi_0 = Id$
(ii) $\Phi_{t+s} = \Phi_t\circ\Phi_s$, $\forall t,s\geq 0$
(iii) $\Phi_t : X\to X$ is continuous for all $t\geq 0$.

The state spaces for DDEs contain segments of functions, which represent the history of the solution curve $\boldsymbol{x}(t)$. For instance, in Fig. 2 the shaded part of the timetrace in (a) corresponds to the initial segment of the depicted solution. For the original system (2.1), we choose the state space to be

$$\mathcal{C} = \prod_{j=1}^{N} C\left([-r_j, 0]\,;\,\mathbb{R}\right),\ r_j = \max_{\ell \in O_j} \tau\left(\ell\right),$$

where $O_j = \{\ell \in \mathcal{E} :\ s(\ell) = j\}$ is the set of all outgoing links from the j-th node. Hence, the value r_j is the largest delay time on outgoing links of the node j. This definition ensures, that the history for the j-th component is available for all delayed arguments appearing on the right hand side of (2.1). Similarly, we choose

$$\tilde{\mathcal{C}} = \prod_{j=1}^{N} C\left([-\tilde{r}_j, 0]\,;\,\mathbb{R}\right),\ \tilde{r}_j = \max_{\ell \in O_j} \tilde{\tau}\left(\ell\right)$$

as state space for the transformed system (2.2). Assuming that solutions exist for all future times (e.g. if f_j are Lipschitz continuous), there exist semiflows

$$\begin{aligned}\Phi &: [0,\infty) \times \mathcal{C} \to \mathcal{C},\ \ (t, \boldsymbol{x}) \mapsto \Phi_t\left(\boldsymbol{x}\right),\\ \Psi &: [0,\infty) \times \tilde{\mathcal{C}} \to \tilde{\mathcal{C}},\ \ (t, \boldsymbol{y}) \mapsto \Psi_t\left(\boldsymbol{y}\right),\end{aligned}$$

for (2.1) and (2.2). Now let us formulate (1.2) as a state space transformation $T : \mathcal{C} \to \tilde{\mathcal{C}}$. We define T componentwise for $j = 1, ..., N$, and pointwise for $\boldsymbol{x}_0 \in \mathcal{C}$ and $t \in [-\tilde{r}_j, 0]$, as

$$T_j\left[\boldsymbol{x}_0\right](t) = \begin{cases} \left[\boldsymbol{x}_0\left(t + \eta_j\right)\right]_j, & \text{for } t \in \left[-\tilde{r}_j, -\eta_j\right], \\ \left[\Phi_{t+\eta_j}\left(\boldsymbol{x}_0\right)\right]_j(0), & \text{for } t \in \left[-\min\left\{\eta_j, \tilde{r}_j\right\}, 0\right]. \end{cases} \tag{4.1}$$

For illustration see Fig. 2(b), where the shading marks the segment $\boldsymbol{y}_0 = T[\boldsymbol{x}_0]$ for the initial segment $\boldsymbol{x}_0$ indicated in (a). The lighter shading indicates the part defined by the second case of (4.1). Let us show that (4.1) is well-defined. For this we need to assure that $\boldsymbol{y}_0 = T\left[\boldsymbol{x}_0\right] \in \tilde{\mathcal{C}}$. This only requires that the term $[\boldsymbol{x}_0(t - \eta_j)]_j$ appearing in (4.1) is defined for all $t + \eta_j$ with $t \in [-\tilde{r}_j, -\eta_j]$. For the case in which $[-\tilde{r}_j, -\eta_j]$ is non-empty, this is equivalent to $t+\eta_j \geq -r_j$ for all $t \in [-\tilde{r}_j, -\eta_j]$. In order to show this, choose a link $\ell \in O_j$ with maximal delay $\tilde{\tau}(\ell) = \tilde{r}_j$. Then,

$$r_j - \tilde{r}_j \geq \tau\left(\ell\right) - \tilde{\tau}\left(\ell\right) = \tau\left(\ell'\right) - \left(\tau\left(\ell'\right) - \eta_{t(\ell')} + \eta_j\right) \geq -\eta_j.$$

Therefore, $t + \eta_j \geq -\tilde{r}_j + \eta_j \geq -r_j$ for $t \in [-\tilde{r}_j, -\eta_j]$ and (4.1) is well-defined.

Furthermore, one easily checks that

$$T \circ \Phi_t = \Psi_t \circ T,\ \text{for all } t \geq 0. \tag{4.2}$$

That is, T transforms solutions of (2.1) into solutions of (2.2). As inherited from the semiflow Φ, the transformation T is neither injective nor surjective in general. Therefore, one cannot expect a dynamical equivalence of (2.1) and (2.2) in the strict form of topological conjugacy, i.e. $\Psi = h \circ \Phi \circ h^{-1}$ for some homeomorphism h. Moreover, since T is not surjective, (4.2) does not even signify that Ψ is properly semiconjugate to Φ. However, this "weak semiconjugacy" is mutual as we show in the following, and this fact implies a strong equivalence as well.

4.2. CT-equivalence. Let us find a reverse transformation from (2.2) to (2.1). It should transform $\tilde{\tau}(\ell)$ back to $\tau(\ell)$ which leads to the natural definition of reverse timeshifts

$$\tilde{\eta}_j = \bar{\eta} - \eta_j \geq 0, \text{ with } \bar{\eta} = \max_{1\leq j\leq N} \eta_j.$$

Then, the reverse transformation $\tilde{T} : \tilde{\mathcal{C}} \to \mathcal{C}$ is given as

$$\tilde{T}_j[\boldsymbol{y}_0](t) = \begin{cases} \left(\Psi_{t+\tilde{\eta}_j}(\boldsymbol{y}_0)\right)_j(0), & t \in [-\min\{\tilde{\eta}_j, r_j\}, 0], \\ \left(\boldsymbol{y}_0(t+\tilde{\eta}_j)\right)_j(0), & t \in [-r_j, -\tilde{\eta}_j]. \end{cases} \tag{4.3}$$

Analogously to (4.2), we have

$$\tilde{T} \circ \Psi = \Phi \circ \tilde{T}, \tag{4.4}$$

and additionally T and $\tilde{T}$ are reverse in the sense that

$$\tilde{T} \circ T = \Phi_{\bar{\eta}} \text{ and } T \circ \tilde{T} = \Psi_{\bar{\eta}}. \tag{4.5}$$

See, for example, Fig. 2(c), where the shaded region indicates the initial segment $(\tilde{T} \circ T)[\boldsymbol{x}_0]$ of the solution $(\tilde{T} \circ T)[\boldsymbol{x}](t) = \boldsymbol{x}(t+\bar{\eta})$, with $\bar{\eta} = 10$. The lighter shaded region indicates where the second case of (4.3) takes effect.

The equations (4.2), (4.4) and (4.5) constitute a notion of equivalence which can be applied not only to semidynamical systems which stem from systems of DDEs, but as well to systems such as delay coupled PDEs or more generally to abstract evolution equations. Therefore we define in a more abstract fashion:

Definition 4.1. *Two semidynamical systems $\Phi : [0,\infty) \times X \to X$ and $\Psi : [0,\infty) \times Y \to Y$ are called CT-equivalent if there exist mappings $T : X \to Y$, $\tilde{T} : Y \to X$ and $\bar{\eta} \geq 0$ such that*

$$T \circ \tilde{T} = \Psi_{\bar{\eta}},\ T \circ \Phi_t = \Psi_t \circ T,$$
$$\tilde{T} \circ T = \Phi_{\bar{\eta}},\ \tilde{T} \circ \Psi_t = \Phi_t \circ \tilde{T}.$$

In the following, the analysis is carried out in the general setting on Banach spaces X and Y but we keep in mind that the results apply to the case of DDEs on the spaces $X = \mathcal{C}$ and $Y = \tilde{\mathcal{C}}$ as introduced above. We still use the term CT-equivalence for the general formulation though, since most probably the equivalent systems Φ and Ψ will in practice be connected by a transformation, which resembles (4.1).

4.3. Dynamical invariants of CT-equivalent semidynamical systems. In this section, we derive some properties of CT-equivalent systems in the sense of Definition 4.1. We show that their state spaces hold a structure of corresponding strongly invariant sets. Then, assuming that the CTTs T and $\tilde{T}$ are Lipschitz continuous, we prove that stability properties are preserved as well. For the DDE systems (2.1) and (2.2) this is the case if all f_j are Lipschitz continuous. Throughout, we assume that X and Y are Banach spaces. Before stating the main results of this section (Theorems 4.2 and 4.4), we give some definitions:

A set $A \subseteq X$ is called *positively invariant* under Φ if for any $t \geq 0$:

$$\Phi_t(A) \subseteq A.$$

$A \subseteq X$ is called *invariant* under Φ if for any $t \geq 0$:

$$\Phi_t(A) = A.$$

For any set $A \subseteq X$, we define its *strongly invariant hull* $H_\Phi(A)$ as the set

$$H_\Phi(A) := \{\boldsymbol{x} \in X \mid \exists t_1, t_2 \geq 0, \hat{\boldsymbol{x}} \in A : \Phi_{t_1}(\hat{\boldsymbol{x}}) = \Phi_{t_2}(\boldsymbol{x})\}.$$

A *strongly invariant set* $A \subseteq X$ is a set that coincides with its strongly invariant hull, i.e.

$$H_\Phi(A) = A.$$

The class of strongly invariant sets is denoted by

$$\mathrm{sis}(\Phi) = \{A \subseteq X \mid A = H_\Phi(A)\}.$$

Note that positive invariance is implied by both, invariance and strong invariance. But between invariance and strong invariance there holds no implication. Of course a strongly invariant set always contains a maximal invariant set which might be empty.

Theorem 4.2. *Let $\Phi : [0,\infty) \times X \to X$ and $\Psi : [0,\infty) \times Y \to Y$ be CT-equivalent. Then,*

(i) *for each (positively) Φ-invariant set $A \in X$, the set $T[A] \in Y$ is (positively) Ψ-invariant,*

(ii) *there is a one-to-one correspondence between (strongly) invariant sets of Φ and Ψ.*

Proof. Ad (i): Let A be positively invariant, $\boldsymbol{x} \in A$ and $\boldsymbol{y} = T[\boldsymbol{x}]$. Then, $\Psi_t(\boldsymbol{y}) = T[\Phi_t(\boldsymbol{x})] \in T[A]$. Hence $T[A]$ is positively invariant.
If A is invariant, then for each $\boldsymbol{x} \in A$ and each $t \in [0,\infty)$ there is an $\boldsymbol{x}_{-t} \in A$ such that $\Phi_t(\boldsymbol{x}_{-t}) = \boldsymbol{x}$ and correspondingly for each $\boldsymbol{y} = T[\boldsymbol{x}] \in T[A]$ there is $\boldsymbol{y}_{-t} = T[\boldsymbol{x}_{-t}]$ with

$$\Psi_t(\boldsymbol{y}_{-t}) = T[\Phi_t(\boldsymbol{x}_{-t})] = T[\boldsymbol{x}] = \boldsymbol{y}.$$

Ad (ii): Let $A \in \mathrm{sis}(\Phi)$. We define a corresponding set $\tilde{A} = H_\Psi(T[A]) \in \mathrm{sis}(\Psi)$. Correspondence for strongly invariant sets is proven via

$$(H_\Phi \circ \tilde{T}) \circ (H_\Psi \circ T) = \mathrm{id}_{\mathrm{sis}(\Phi)} \text{ and } (H_\Psi \circ T) \circ (H_\Phi \circ \tilde{T}) = \mathrm{id}_{\mathrm{sis}(\Psi)},$$

where, by symmetry, it suffices to show only one equality. Let $\boldsymbol{y} \in \tilde{A}$, then there exist $t_1, t_2 \geq 0$ and $\hat{\boldsymbol{y}} = T[\hat{\boldsymbol{x}}] \in T[A]$ such that $\Psi_{t_2}(\boldsymbol{y}) = \Psi_{t_1}(\hat{\boldsymbol{y}}) \in \tilde{A}$. Thus,

$$\begin{aligned}
\Phi_{t_2}\left(\tilde{T}[\boldsymbol{y}]\right) &= \tilde{T}[\Psi_{t_2}(\boldsymbol{y})] = \tilde{T}[\Psi_{t_1}(\hat{\boldsymbol{y}})] \\
&= \tilde{T}[\Psi_{t_1}(T[\hat{\boldsymbol{x}}])] = \tilde{T}[T[\Phi_{t_1}(\hat{\boldsymbol{x}})]] \\
&= \Phi_{t_1+\bar{\eta}}(\hat{\boldsymbol{x}}) \in H_\Phi(A) = A.
\end{aligned}$$

Hence, $\tilde{T}[\boldsymbol{y}] \in A$. That is, $\tilde{T}\left[\tilde{A}\right] \subseteq A$ and $H_\Phi\left(\tilde{T}\left[\tilde{A}\right]\right) \subseteq A$. We have also shown that for $\hat{\boldsymbol{x}} \in A$, $\Phi_{t_1+\bar{\eta}}(\hat{\boldsymbol{x}}) \in \tilde{T}\left[\tilde{A}\right]$ and therefore $\hat{\boldsymbol{x}} \in H_\Phi\left(\tilde{T}\left[\tilde{A}\right]\right)$. This yields $A \subseteq H_\Phi\left(\tilde{T}\left[\tilde{A}\right]\right) \subseteq A$, i.e. $A = H_\Phi\left(\tilde{T}\left[\tilde{A}\right]\right)$ and $\left(H_\Phi \circ \tilde{T}\right) \circ (H_\Psi \circ T) = \mathrm{id}_{\mathrm{sis}(\Phi)}$.

For invariant sets, the one-to-one correspondence is mediated directly via T and $\tilde{T}$.■

Definition 4.3. *The maximal Lyapunov exponent (MLE) of a point $\boldsymbol{x} \in X$ with respect to a semidynamical system $\Phi : [0, \infty) \times X \to X$ on a Banach space X is defined as*

$$\lambda(\boldsymbol{x}) := \limsup_{t\to\infty} \limsup_{|\xi|\searrow 0} \frac{1}{t} \ln\left(\frac{|\Phi_t(\boldsymbol{x}+\xi) - \Phi_t(\boldsymbol{x})|}{|\xi|}\right) \in [-\infty, \infty].$$

The MLE of $\boldsymbol{x} \in A \subseteq X$ with respect to the set A is defined as

$$\begin{aligned} \lambda(x, A) &:= \limsup_{t\to\infty} \limsup_{|\xi|\searrow 0} \min_{a\in A} \frac{1}{t} \ln\left(\frac{|\Phi_t(\boldsymbol{x}+\xi) - a|}{|\xi|}\right) \\ &= \limsup_{t\to\infty} \limsup_{|\xi|\searrow 0} \frac{1}{t} \ln\left(\frac{\operatorname{dist}(\Phi_t(\boldsymbol{x}+\xi), A)}{|\xi|}\right). \end{aligned}$$

The MLE of a set $A \in X$ is defined as $\lambda(A) = \sup_{\boldsymbol{x}\in A} \lambda(\boldsymbol{x}, A)$.

Theorem 4.4. *Let the CT-transformations T and $\tilde{T}$ be Lipschitz-continuous and let X, Y be Banach spaces. Then,*

(i) *corresponding (strongly) invariant sets of Φ and Ψ possess the same maximal Lyapunov exponents (MLEs),*

(ii) *for each positively invariant set $A \in X$, the set $T[A] \in Y$ has the same type of stability.*

Claim (i) is a direct consequence of the following Lemma.

Lemma 4.5. *Let X, Y be Banach spaces and let T and $\tilde{T}$ be Lipschitz-continuous with constants L_T and $L_{\tilde{T}}$. Then, we have $\lambda(\boldsymbol{x}) \le \lambda(T[\boldsymbol{x}]) \le \lambda(\Phi_{\bar{\eta}}(\boldsymbol{x}))$ for all $\boldsymbol{x} \in X$.*

Proof. For each $\boldsymbol{x}, \boldsymbol{\chi} \in \mathcal{C}$, $t \ge \bar{\eta}$:

$$\begin{aligned} &\frac{1}{t} \ln\left(\frac{|\Phi_t(\boldsymbol{x}) - \Phi_t(\boldsymbol{x}+\boldsymbol{\chi})|}{|\boldsymbol{\chi}|}\right) \\ &= \frac{1}{t} \ln\left(\frac{\left|\Phi_{t-\bar{\eta}}\left(\tilde{T}\circ T[\boldsymbol{x}]\right) - \Phi_{t-\bar{\eta}}\left(\tilde{T}\circ T[\boldsymbol{x}+\boldsymbol{\chi}]\right)\right|}{|\boldsymbol{\chi}|}\right) \\ &= \frac{1}{t} \ln\left(\frac{\left|\tilde{T}\circ\Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]) - \tilde{T}\circ\Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]+\boldsymbol{\xi})\right|}{|\boldsymbol{\chi}|}\right), \end{aligned}$$

with $|\boldsymbol{\xi}| = |T[\boldsymbol{x}+\boldsymbol{\chi}] - T[\boldsymbol{x}]| \le L_T |\boldsymbol{\chi}|$,

$$\begin{aligned} \ldots &\le \frac{1}{t} \ln\left(\frac{L_{\tilde{T}} |\Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]) - \Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]+\boldsymbol{\xi})|}{(|\boldsymbol{\xi}|/L_T)}\right) \\ &= \frac{1}{t} \ln\left(\frac{|\Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]) - \Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]+\boldsymbol{\xi})|}{|\boldsymbol{\xi}|}\right) + \frac{1}{t}\ln\left(L_T L_{\tilde{T}}\right) \end{aligned}$$

Therefore,

$$\frac{1}{t} \ln\left(\frac{|\Phi_t(\boldsymbol{x}) - \Phi_t(\boldsymbol{x}+\boldsymbol{\chi})|}{|\boldsymbol{\chi}|}\right) \le \frac{1}{t} \ln\left(\frac{|\Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]) - \Psi_{t-\bar{\eta}}(T[\boldsymbol{x}]+\boldsymbol{\xi})|}{|\boldsymbol{\xi}|}\right),$$

and, thus,

$$\lambda(\boldsymbol{x}) \leq \lambda(T[\boldsymbol{x}]).$$

The same reasoning for $\boldsymbol{y} = T[\boldsymbol{x}]$ gives

$$\lambda(\boldsymbol{x}) \leq \lambda(T[\boldsymbol{x}]) \leq \lambda(\Phi_{\bar{\eta}}(\boldsymbol{x})).$$

■

Proof. (of Theorem 4.4) Lemma 4.5 implies claim (i). To see that consider two corresponding sets $A \in \operatorname{sis}(\Phi)$ and $\tilde{A} \in \operatorname{sis}(\Psi)$. Then it is impossible that $\lambda(A) > \lambda(\tilde{A})$ since for any $\boldsymbol{x} \in A$ there exists $\boldsymbol{y} = T[\boldsymbol{x}] \in \tilde{A}$ such that $\lambda(\boldsymbol{x}) \leq \lambda(\boldsymbol{y})$. The same holds vice versa and therefore $\lambda(A) = \lambda(\tilde{A})$. Similarly, one shows this for the case of invariant sets.

Ad (ii): Let $A \in \mathcal{C}$ be an (asymptotically) stable positively invariant set. We show that if A is (asymptotically) stable so is $T[A]$. Let $\varepsilon > 0$, $\boldsymbol{y}_0 = T[\boldsymbol{x}_0] \in T[A]$ and $\boldsymbol{\xi}_0 \in \tilde{\mathcal{C}}$ a small initial perturbation, i.e. $|\boldsymbol{\xi}_0| \leq \delta$ with $\delta = \delta(\varepsilon) > 0$ to be specified later. Define the perturbed solution

$$\tilde{\boldsymbol{y}}_t = \Psi_t(\boldsymbol{y}_0 + \boldsymbol{\xi}_0) = \boldsymbol{y}_t + \boldsymbol{\xi}_t,$$

where $\boldsymbol{y}_t = \Psi_t(\boldsymbol{y}_0)$ is the unperturbed solution. Consider

$$\begin{aligned}
\tilde{\boldsymbol{x}}_t := \tilde{T}[\tilde{\boldsymbol{y}}_t] &= \tilde{T}[\Psi_t(\boldsymbol{y}_0 + \boldsymbol{\xi}_0)] \\
&= \Phi_t\left(\tilde{T}[\boldsymbol{y}_0 + \boldsymbol{\xi}_0]\right) \\
&= \Phi_t\left(\tilde{T} \circ T[\boldsymbol{x}_0] + \left(\tilde{T}[\boldsymbol{y}_0 + \boldsymbol{\xi}_0] - \tilde{T}[\boldsymbol{y}_0]\right)\right) \\
&= \Phi_t\left(\boldsymbol{x}_{\bar{\eta}} + \boldsymbol{\chi}_{\bar{\eta}}\right),
\end{aligned}$$

with

$$\begin{aligned}
\left|\boldsymbol{\chi}_{\bar{\eta}}\right| &= \left|T[\boldsymbol{y}_0 + \boldsymbol{\xi}_0] - \tilde{T}[\boldsymbol{y}_0]\right| \\
&\leq L_{\tilde{T}} \cdot \delta,
\end{aligned}$$

where $L_{\tilde{T}}$ is the Lipschitz constant of $\tilde{T}$. Since $\boldsymbol{x}_{\bar{\eta}} \in A$ and A is (asymptotically) stable, we can find a $\delta = \delta(\varepsilon)$ such that

$$\mathrm{d}(\tilde{\boldsymbol{x}}_t, A) \leq \frac{\varepsilon}{L_T},$$

and, in case of asymptotic stability, such that

$$\tilde{\boldsymbol{x}}_t \to A.$$

This means, we can represent $\tilde{\boldsymbol{x}}_t$ as $\tilde{\boldsymbol{x}}_t = \boldsymbol{a}_t + \boldsymbol{\chi}_t$ with $\boldsymbol{a}_t \in A$, $|\boldsymbol{\chi}_t| \leq \frac{\varepsilon}{L_T}$ and, in case of asymptotic stability, $|\boldsymbol{\chi}_t| \to 0$, for $t \to \infty$. Note that $t \mapsto \boldsymbol{a}_t$ has not to be a solution. Define $\boldsymbol{b}_t = T[\boldsymbol{a}_t] \in T[A]$. Then,

$$\begin{aligned}
\left|\tilde{\boldsymbol{y}}_{t+\bar{\eta}} - \boldsymbol{b}_t\right| &= \left|T \circ \tilde{T}[\tilde{\boldsymbol{y}}_t] - T[\boldsymbol{a}_t]\right| \\
&= |T[\boldsymbol{a}_t + \boldsymbol{\chi}_t] - T[\boldsymbol{a}_t]| \\
&\leq L_T |\boldsymbol{\chi}_t| \leq \varepsilon
\end{aligned}$$

and $\left|\tilde{\boldsymbol{y}}_{t+\bar{\eta}} - \boldsymbol{b}_t\right| \to 0$, if A is asymptotically stable. This completes the proof. ■

5. Reduction of delay-parameters. The Theorems 4.2 and 4.4 show that CT-equivalence is indeed a very strong equivalence. For virtually all cases one is best advised to study the system which possesses the most convenient distribution of delays within the concerned equivalence class. However, it is difficult give a general identification of the appropriate distribution, since it strongly depends on the problem at hand. We can acknowledge two possible guiding principles, which correspond to the transformations of delays in a ring that were mentioned in the introduction [cf. (1.4) and (1.5)]. Firstly, a homogenization of delays can sometimes lead to a higher degree of symmetry and thereby allow for simplifications. In other cases a "concentration" of the delays on selected links may be useful.

In §5.1 we show, that it is always possible to find timeshifts η_j, $j = 1, ..., N$, such that the number of different delays in system (2.1) reduces to the cycle space dimension $C = L - N + 1$ of the network, where $L = \#\mathcal{E}$ is the number of links and $N = \#\mathcal{N}$ is the number of nodes in the network. Effectively, this means that no more than C delay-parameters have to be taken into account during investigation [see Fig. 1]. For a connected network this number cannot be reduced further as we show in §5.2.

5.1. Construction of an instantaneous spanning tree. In this section we construct a set of links, called a "spanning tree" (see Def. 5.2), on which all connection delays can be eliminated by componentwise timeshifts. Let us firstly introduce the necessary notions:

Definition 5.1. *A semicycle $c = (\ell_1, ..., \ell_k)$ is a closed path in the undirected graph which is obtained by dropping the orientation from all links from the multigraph $(\mathcal{N}, \mathcal{E})$.*

Definition 5.2. *A spanning tree of $(\mathcal{N}, \mathcal{E})$ is a set of links $S \subseteq \mathcal{E}$ which contains no semicycles but all nodes, that is $\{s(\ell), t(\ell)\}_{\ell \in S} = \mathcal{N}$.*

A spanning tree can be thought of as a "skeleton" of the graph. Following its links one can visit each node in the graph exactly once. For instance, the solid, red links in Figs. 4(c) form spanning trees. A spanning tree can also be characterized as a maximal cycle-free set of links. Consequently, adding another link which is not contained in the spanning tree creates a semicycle. For instance, if in Figs. 4(a) the link with delay τ_1 is added to the spanning tree the semicycle c_1 is created. A spanning tree of a connected network necessarily contains $N-1$ links, therefore the cycle space dimension C coincides with the number of links not contained in a spanning tree. In the context of the CTT, an important quantity for a semicycle is its delay sum.

Definition 5.3. *The delay sum of a semicycle $c = (\ell_1, ..., \ell_k)$ with respect to a delay distribution and an orienting link ℓ_1 is*

$$\Sigma(c) := \sum_{j=1}^{k} \sigma_j \tau(\ell_j),$$

where $\sigma_j \in \{\pm 1\}$ indicates whether the link ℓ_j points in the same direction as ℓ_1 ($\sigma_j = 1$) or not ($\sigma_j = -1$). The roundtrip of c is the modulus of its delay sum

$$\mathrm{rt}(c) := |\Sigma(c)|. \tag{5.1}$$

The following Lemma gives a characterization of timeshift-transformed delay distributions and describes their relation to the underlying graph structure. More specifically, it shows that the delay sums of semicycles are invariant under timeshifts.

Lemma 5.4. *Let $\tau, \tilde{\tau} : \mathcal{E} \to [0, \infty)$ be two delay distributions in a connected network. Then, the following statements are equivalent.*

(i) *There exist $\eta_j \geq 0$, $j = 1, ..., N$, such that $\tilde{\tau}(\ell) = \tau(\ell) - \eta_{t(\ell)} + \eta_{s(\ell)}$.*

(ii) *For all semicycles c of the network holds:*

$$\Sigma_\tau(c) = \Sigma_{\tilde{\tau}}(c).$$

Proof. (i) ⇒ (ii) Indeed, for any cycle $c = (\ell_1, ..., \ell_k)$ we have

$$\begin{aligned} \Sigma_{\tilde{\tau}}(c) &= \sum_{j=1}^{k} \sigma_j \tilde{\tau}(\ell_j) = \sum_{j=1}^{k} \sigma_j \left(\tau(\ell_j) - \eta_{t(\ell_j)} + \eta_{s(\ell_j)} \right) \\ &= \Sigma_\tau(c) + \sum_{j=1}^{k} \sigma_j \left(\eta_{s(\ell_j)} - \eta_{t(\ell_j)} \right) = \Sigma_\tau(c). \end{aligned}$$

(ii) ⇒ (i) Select an arbitrary spanning tree $S = (\ell_1, ..., \ell_{N-1})$. Select $\xi = (\xi_1, ..., \xi_N)$ and $\chi = (\chi_1, ..., \chi_N)$ such that $\tau(\ell_j) = \xi_{t(\ell_j)} - \xi_{s(\ell_j)}$ and $\tilde{\tau}(\ell_j) = \chi_{t(\ell_j)} - \chi_{s(\ell_j)}$ for $j = 1, ..., N-1$. (Note that both defining sets of equations are inhomogeneous linear systems of type $\mathbb{R}^{N \times (N-1)}$ and of rank $N-1$. Therefore they possess solutions.) The shifts ξ and χ define distributions (with possibly negative values)

$$\begin{aligned} \hat{\tau}_1(\ell) &= \tau(\ell) - \xi_{t(\ell)} + \xi_{s(\ell)}, \\ \hat{\tau}_2(\ell) &= \tilde{\tau}(\ell) - \chi_{t(\ell)} + \chi_{s(\ell)}. \end{aligned}$$

Both, $\hat{\tau}_1$ and $\hat{\tau}_2$, are instantaneous along S, i.e., $\hat{\tau}_1(\ell) = \hat{\tau}_2(\ell) = 0$ for $\ell \in S$. Each link ℓ which is not in S corresponds to a unique fundamental semicycle $c = c(S, \ell)$ which is created by adding ℓ to S. It is the only link in c which may hold a non-zero value $\hat{\tau}_1(\ell)$ or $\hat{\tau}_2(\ell)$, respectively. By the part (i) ⇒ (ii) we have

$$\hat{\tau}_1(\ell) = \Sigma_\tau(c) = \Sigma_{\tilde{\tau}}(c) = \hat{\tau}_2(\ell).$$

Hence $\hat{\tau}_1 = \hat{\tau}_2$. With $\eta_j := \xi_j - \chi_j$ this gives

$$\tilde{\tau}(\ell) = \tau(\ell) - \eta_{t(\ell)} + \eta_{s(\ell)}.$$

Note that η_j can always be chosen to be non-negative since a simultaneous shift $\eta_j \mapsto \eta_j + \bar{\eta}$ by some amount $\bar{\eta} \in \mathbb{R}$ does not change the resulting transformed distribution $\tilde{\tau}$. ■

Often, this lemma makes it straightforward to determine possible transformations, since it circumvents the explicit determination of timeshifts. For instance, in the case of the two coupled systems depicted in Fig. 1, it follows immediately that $\tau = (\tau_1 + \tau_2)/2$.

Now we can state the main result of this section, which is the construction of a transformation (1.2) such that the number of delays is minimized.

Theorem 5.5. *For every connected network with dynamics given by Eq. (2.1), there exists a spanning tree S and timeshifts η_j such that in the transformed system (2.2) all links $\ell \in S$ are instantaneous,*

$$\tilde{\tau}(\ell) = 0, \text{ for } \ell \in S,$$

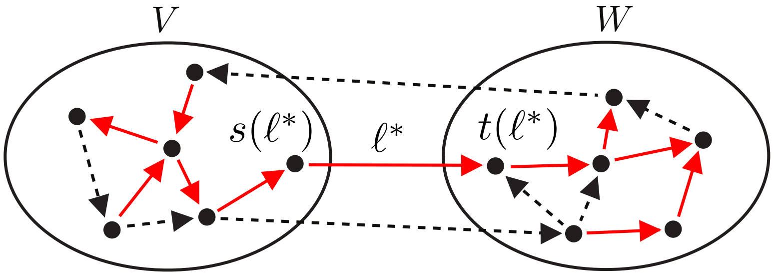


Figure 3. *Illustration of the fundamental cut corresponding to the link ℓ^* in the spanning tree S (solid red links), $s(\ell^*)$ and $t(\ell^*)$ denote the source and target of ℓ^*. Links which are not contained in S are indicated by dashed lines.*

and for each link ℓ outside the spanning tree the delay $\tilde{\tau}(\ell)$ equals the roundtrip $rt(c(\ell))$ [Eq. (5.1)] along the corresponding fundamental cycle $c(\ell)$,

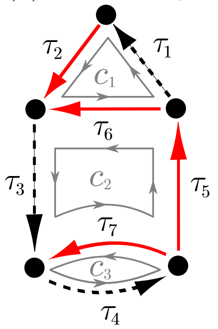
$$\tilde{\tau}(\ell) = T(c(\ell)), \text{ for } \ell \notin S.$$

Proof. The following algorithm describes the general procedure to find a reduced set of delays. For a more definite example see Fig. 4 and its description following this proof. The main idea of the algorithm is as follows. We construct spanning trees and timeshifts iteratively such that after each step the number of non-zero delays on the spanning tree has decreased by at least one. Therefore the algorithm finishes after at most $N-1$ iterations. Each step consists of two stages.

Stage (i). Construction of a spanning tree:
Select a spanning tree $S = \{\ell_1, ..., \ell_{N-1}\}$ in the following way. First, pick a link ℓ_1 with minimal delay, i.e. $\tau(\ell_1) = \min_{\ell \in \mathcal{L}} \{\tau(\ell)\} \in [0, \infty)$. Proceed picking links $\ell_2, ..., \ell_j$ with minimal delays (under all links except the ones already picked) as long as the set $S_j = \{\ell_1, ..., \ell_j\}$ contains no semicycles. If at the j-th step the chosen link would create a semicycle when added to S_{j-1}, do not add it but ignore it for the rest of this stage. Following this procedure, called Kruskal's algorithm [21], yields a spanning tree $S = \{\ell_1, ..., \ell_{N-1}\}$.

Stage (ii). Construction of timeshifts:
Now consider the link $\ell^* \in S$ with minimal *positive* delay (in the very first step of the construction it equals ℓ_1 if $\tau(\ell_1) > 0$). The link induces a *fundamental cut* [10], i.e. it partitions the spanning tree S in two connected components: the source component V of ℓ^* and the target component W of ℓ^*, see Fig. 3. Since S is spanning, this is a partition of all nodes. Now let us define the timeshifts by $\eta_j = 0$ for $j \in V$ and $\eta_j = \tau(\ell^*)$ for all $j \in W$.
From the shifts η_j we obtain the new delay distribution $\tilde{\tau}(\ell) = \tau(\ell) + \eta_{s(\ell)} - \eta_{t(\ell)}$. For a link $\ell \neq \ell^*$ which connects nodes within one set of the partition, i.e. $s(\ell), t(\ell) \in V$ or $s(\ell), t(\ell) \in W$, we have $\eta_{s(\ell)} = \eta_{t(\ell)}$. This implies $\tilde{\tau}(\ell) = \tau(\ell)$. In particular this is the case for all links in $S \setminus \{\ell^*\}$ and therefore $\tau(\ell) = 0 = \tilde{\tau}(\ell)$ for all instantaneous links in S. For ℓ^* we have $\tilde{\tau}(\ell^*) = 0$ because $s(\ell^*) \in V$ and $t(\ell^*) \in W$, which means $\eta_{s(\ell^*)} - \eta_{t(\ell^*)} = -\tau(\ell^*)$.
Hence, the delay distribution $\tilde{\tau}$ reduced the number of delays on the spanning tree S by one compared to the distribution τ. We remind that $\tau(\ell^*)$ is the smallest positive delay not only on the spanning tree but over all the links which connect V and W. Therefore, $\tau(\ell) \geq 0$ for

(a) initial stage:



(b)

stage	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7
	4	3	10	8	9	2	5
(i) ς	4	[3]	10	8	[9]	[2]	[5]
(ii) ς	2	[3]	12	8	[9]	[0]	[5]
(i) ς	[2]	3	12	8	[9]	[0]	[5]
(ii) ς	[0]	5	12	8	[9]	[0]	[5]
(i) ς	[0]	5	12	8	[9]	[0]	[5]
(ii) ς	[0]	5	7	13	[9]	[0]	[0]
(i) ς	[0]	5	[7]	13	9	[0]	[0]
(ii) ς	[0]	5	[0]	13	16	[0]	[0]

(c) final stage:

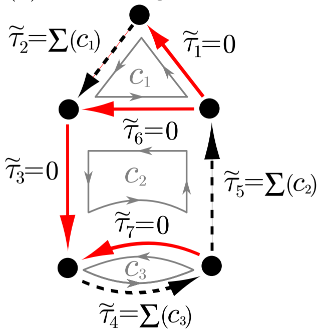


Figure 4. *Illustration of the delay reduction in a network of $N = 7$ delay-coupled systems with cycle space dimension $C = 3$ [cf. proof of Theorem. 5.5]. In (a) the initially selected spanning tree with original delays τ_j, $j = 1, ..., 7$, is indicated by red, solid links, Three semicycles c_1, c_2, and c_3 are indicated by the grey curves following the contained links; in (c) the final spanning tree with transformed delays $\tilde{\tau}_j$ is indicated. The table (b) shows the steps taken by the reduction algorithm. Bracketed, red values indicate the delay times on the currently selected spanning tree at each step.*

all $\ell \in \mathcal{L}$. Indeed, if there were a link between V and W with a smaller delay we must have included it in the spanning tree in step (i)

If a link ℓ exists in S with $\tilde{\tau}(\ell) \neq 0$ we repeat stage (i) starting from an initial set S_j which contains all j instantaneous links from the spanning tree constructed in the previous step. Then stage (ii) yields a spanning tree with a strictly larger number of instantaneous links. In this way we arrive at an instantaneous spanning tree S in at most $N - 1$ iterations. The statement that $\tilde{\tau}(\ell) = T(c(\ell))$ for $\ell \notin S$ follows from Lemma 5.4. ■

Fig. 4 illustrates how the algorithm described in the proof of Theorem. 5.5 determines a delay reduction in the depicted network of $N = 7$ coupled systems with cycle space dimension $C = 3$. We assume an initial delay-distribution as given in the first row of the table in Fig. 4(b). Then, the spanning tree in (a), indicated by solid, red links, is selected by the algorithm as a successive cycle-free collection of links with smallest delays. Let us denote the links of the example by ℓ_j with $\tau_j = \tau(\ell_j)$, $j = 1, ..., 7$. Note that in (a) ℓ_1 is not contained in the spanning tree although $\tau_1 < \tau_5$ and $\tau_1 < \tau_7$, and ℓ_5 and ℓ_7 are both contained. This is because if the links ℓ_2 and ℓ_6 with $\tau_2, \tau_6 < \tau_1$ are already selected, then an addition of ℓ_1 would lead to the inclusion of the semicycle c_1, which is not allowed. Each row of table (b) corresponds to a step of the algorithm, where the bracketed, red values correspond to the delay times of the currently selected spanning tree. Two stages are repeated alternatingly: In stage (i) a spanning tree with minimal delay sum is determined and in stage (ii) the minimal delay in this spanning tree is eliminated by a componentwise timeshift. In (c) the spanning tree is shown which is selected at the final stage. It contains only instantaneous links when the algorithm is completed. The remaining delays on links not contained in the final spanning tree, correspond to the delay sums of the corresponding semicycles (c_1, c_2, c_3), oriented as indicated in the figure, i.e., $\tilde{\tau}_2 = \Sigma(c_1) = \tau_1 + \tau_2 - \tau_6$; $\tilde{\tau}_5 = \Sigma(c_2) = \tau_3 + \tau_5 + \tau_6 - \tau_7$; and $\tilde{\tau}_4 = \Sigma(c_3) = \tau_4 + \tau_7$.

5.2. Genericity of the dimension of the delay parameter space. We call C the *essential number* of delays since it is the minimal number to which the number of different delays in a network can be reduced generically. Here "generically" means that the conditions which allow for further reduction of delays form a null set in the parameter space of delays $\mathbb{R}^L_{\geq 0} = \{(\tau(\ell))_{\ell \in \mathcal{E}} : \tau(\ell) \geq 0\}$ of the original system. The reducibility condition to m different delays is described by the L linear equations

$$\eta_{t(\ell)} - \eta_{s(\ell)} + \tilde{\tau}(\ell) = \tau(\ell), \ \ell \in I_j, \tag{5.2}$$

with the restriction for $\tilde{\tau}(\ell)$ to take one of m different values $\{\theta_1, ..., \theta_m\}$. System (5.2) can be equivalently written in the vector form $G_q\mathbf{v} = \boldsymbol{\tau}$, where $\mathbf{v}$ is the $(N+m)$-dimensional vector of unknowns $\mathbf{v} = (\eta_1, \dots, \eta_N, \theta_1, \dots \theta_m)$ and $\boldsymbol{\tau}$ is the L-dimensional vector of delays $\tau(\ell)$. For any fixed assignment $\tilde{\tau}(\ell) = \theta_{q(\ell)}$, $\ell \in I_j$, with $q : I_j \to \{0, ..., m\}$, and $\theta_0 := 0$, one obtains a different matrix $G_q \in \mathbb{R}^{L \times (N+m)}$. It can further be shown that the rank of the matrix G_q is smaller than $N-1+m$. This implies for $m < C = L-(N-1)$ that $N-1+m < L$. Hence, the number of equations in (5.2) is larger than the number of unknowns. Such equation cannot be solved generically, unless the given delays $\tau(\ell)$ satisfy some special condition of positive codimension.

6. Conclusion. In this article we have studied a componentwise timeshift transformation (CTT) for a general class of coupled differential equations with constant coupling delays. We have defined appropriate phase spaces such that the CTT conveys an equivalence for the semiflows of the original and the transformed system which is reminiscent to, but weaker than topological conjugacy. We have shown several dynamical invariants for the equivalent flows. Firstly, for delay differential equations, the characteristic exponents of equilibria and the Floquet exponents of periodic orbits are invariant under the CTT. More generally, for CT-equivalent semidynamical systems, we have shown that there exists a one-to-one correspondence between invariant and strongly invariant sets, respectively. As a main stability result we have shown that corresponding positively invariant sets have the same kind of stability, and invariant and strongly invariant corresponding sets possess the same maximal Lyapunov exponents. To sum up, the observable dynamics of the coupled units might change its relative timing in the transformed system, but qualitatively it remains the same. In particular attractors and their stability are invariant.

We have presented a constructive proof that there is always a CTT which reduces the delays to at most $C = L - N + 1$ different delays, where L is the number of links in the network and N is the number of nodes. The number of different delays C coincides with the cycle space dimension of the underlying graph. Furthermore, we have shown that the sum of delays along semicycles in the network is invariant under CTTs. This reveals a strong link between a coupled delay differential equation and the topology of the underlying graph. Although a CTT which reduces the number of different delays to a minimum is usually not unique, we have shown that the minimal number of different delays itself cannot be reduced in general.

We believe that our results have a relevance for several applied areas (see also [25]). For example, in theoretical neuroscience it is an accepted fact that in the case of two mutually delay coupled units [30] (as in Fig. 1) or more generally, in unidirectionally coupled rings

[31], the delays can always be chosen identical for theoretical investigations. Moreover, in view of our results, the observations about the role of the greatest common divisor of loop-lengths in networks of delay coupled excitable systems with homogeneous delays [20, 33] can be formulated more generally in terms of the delay sums along the network's cycles. The CTT is an important tool for investigators working with delayed dynamical systems since it clarifies one aspect of the way in which different interaction delays in a coupled system work together.

Furthermore, the CTT can speed up the numerical simulation of a system by reducing the number of different delays or by reducing the maximal delay time. The computational advantages of the transformed system might be of interest especially in the field of delayed neural networks, where large scale simulations of networks with many different delay times are conducted.

REFERENCES

[1] F.M. Atay and A. Hutt, *Stability and bifurcations in neural fields with finite propagation speed and general connectivity*, SIAM Journal on Applied Mathematics, 65 (2004), pp. 644–666.
[2] P. Baldi and A. Atiya, *How delays affect neural dynamics and learning*, IEEE Transactions on Neural Networks, 5 (1994), pp. 1045–9227.
[3] J. Bélair and S.A. Campbell, *Stability and bifurcations of equilibria in a multiple-delayed differential equation*, SIAM Journal on Applied Mathematics, 54 (1994), pp. 1402–1424.
[4] R.E. Bellman and K.L. Cooke, *Differential-difference equations*, (1963).
[5] S.A. Campbell, *Time delays in neural systems*, in Handbook of brain connectivity, V. Jirsa and A. McIntosh, eds., Springer-Verlag Berlin Heidelberg, 2007, pp. 65–90.
[6] S.A. Campbell, I. Ncube, and J. Wu, *Multistability and stable asynchronous periodic oscillations in a multiple-delayed neural system*, Physica D: Nonlinear Phenomena, 214 (2006), pp. 101–119.
[7] S.A. Campbell, S. Ruan, and J. Wei, *Qualitative analysis of a neural network model with multiple time delays*, International Journal of Bifurcation and Chaos, 9 (1999), pp. 1585–1595.
[8] T. Danino, O. Mondragon-Palomino, L. Tsimring, and J. Hasty, *A synchronized quorum of genetic clocks*, Nature, 463 (2010), pp. 326–330.
[9] M. Dhamala, V.K. Jirsa, and M. Ding, *Enhancement of neural synchrony by time delay*, Physical Review Letters, 92 (2004).
[10] R. Diestel, *Graph Theory*, Springer-Verlag, Heidelberg, 2010.
[11] T. Erneux, *Applied Delay Differential Equations*, vol. 3 of Surveys and Tutorials in the Applied Mathematical Sciences, Springer, 2009.
[12] U. Ernst, K. Pawelzik, and T. Geisel, *Synchronization induced by temporal delays in pulse-coupled oscillators*, Physical Review Letters, 74 (1995), pp. 1570–1573.
[13] V.K. Jirsa G. Deco and R. McIntosh, *Emerging concepts for the dynamical organization of resting-state activity in the brain*, Nature Reviews Neuroscience, 12 (2011), pp. 43–56.
[14] K. Gopalsamy and X.-Z. He, *Stability in asymmetric hopfield nets with transmission delays*, Physica D: Nonlinear Phenomena, 76 (1994), pp. 344–358.
[15] A. Gosh, Y. Rho, A.R. McIntosh, R. Kötter, and V.K. Jirsa, *Noise during rest enables the exploration of the brain's dynamic repertoire*, PLoS Computational Biology, (2008).
[16] K. Gu, S.-I. Niculescu, and J. Chen, *On stability crossing curves for general systems with two delays*, Journal of Mathematical Analysis and Applications, 311 (2005), pp. 231–253.
[17] S. Guo, Y. Chen, and J. Wu, *Two-parameter bifurcations in a network of two neurons with multiple delays*, Journal of Differential Equations, 244 (2008), pp. 444–486.
[18] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.
[19] E.M. Izhikevich and G.M. Edelman, *Large-scale model of mammalian thalamocortical systems*, Pro-

ceedings of the national academy of sciences, 105 (2008), pp. 3593–3598.
[20] I. Kanter, M. Zigzag, A. Englert, F. Geissler, and W. Kinzel, *Synchronization of unidirectional time delay chaotic networks and the greatest common divisor*, EPL (Europhysics Letters), 93 (2011), p. 60003.
[21] J.B. Kruskal, Jr., *On the shortest spanning subtree of a graph and the traveling salesman problem*, Proceedings of the American Mathematical Society, 7 (1956), pp. 48–50.
[22] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, vol. 191 of Mathematics in science and engineering, Academic Press, 1993.
[23] J. Li and Y. Kuang, *Analysis of a model of the glucose-insulin regulatory system with two delays*, SIAM Journal on Applied Mathematics, 67 (2007), pp. 757–776.
[24] W. Lu and T. Chen, *Synchronization of coupled connected neural networks with delays*, IEEE Transactions on Circuits and Systems Part 1: Regular Papers, 51 (2004), pp. 2491–2503.
[25] L. Lücken, J.P. Pade, K. Knauer, and S. Yanchuk, *Reduction of interaction delays in networks*, EPL (Europhysics Letters), 103 (2013), p. 10006.
[26] M.C. Mackey and L. Glass, *Oscillation and chaos in physiological control systems*, Science, 197 (1977), pp. 287–289.
[27] J. Mallet-Paret and G.R. Sell, *The poincaré–bendixson theorem for monotone cyclic feedback systems with delay*, Journal of differential equations, 125 (1996), pp. 441–489.
[28] Y. Manor, C. Koch, and I. Segev, *Effect of geometrical irregularities on propagation delay in axonal trees*, Biophysical Journal, 60 (1991), pp. 1424–1437.
[29] R.D. Nussbaum, *Differential-delay equations with two time lags*, vol. 16, American Mathematical Society, 1978.
[30] A. Panchuk, D.P. Rosin, P. Hövel, and E. Schöll, *Synchronization of coupled neural oscillators with heterogeneous delays*, International Journal of Bifurcation and Chaos, 23 (2013).
[31] P. Perlikowski, S. Yanchuk, O.V. Popovych, and P.A. Tass, *Periodic patterns in a ring of delay-coupled oscillators*, Physical Review E, 82 (2010), p. 036208.
[32] O.V. Popovych, S. Yanchuk, and P.A. Tass, *Delay-and coupling-induced firing patterns in oscillatory neural loops*, Physical Review Letters, 107 (2011), p. 228102.
[33] D.P. Rosin, D. Rontani, D. J. Gauthier, and E. Schöll, *Control of synchronization patterns in neural-like boolean networks*, Physical Review Letters, 110 (2013), p. 104102.
[34] S. Ruan and J. Wei, *On the zeros of transcendental functions with applications to stability of delay differential equations with two delays*, Dynamics of Continuous, Discrete and Impulsive Systems Series A, 10 (2003), pp. 863–874.
[35] L.P. Shayer and S.A. Campbell, *Stability, bifurcation, and multistability in a system of two coupled neurons with multiple time delays*, SIAM Journal on Applied Mathematics, 61 (2000), pp. 673–700.
[36] M.C. Soriano, J. García-Ojalvo, C. R. Mirasso, and I. Fischer, *Complex photonics: Dynamics and applications of delay-coupled semiconductors lasers*, Reviews of Modern Physics, 85 (2013), pp. 421–470.
[37] P. Van Den Driessche and X. Zou, *Global attractivity in delayed hopfield neural network models*, SIAM Journal on Applied Mathematics, 58 (1998), pp. 1878–1890.
[38] J. Wu, *Introduction to neural dynamics and signal transmission delay*, vol. 6, Walter de Gruyter, 2001.
[39] H.-J. Wünsche, S. Bauer, J. Kreissl, O. Ushakov, N. Korneyev, F. Henneberger, E. Wille, H. Erzgräber, M. Peil, W. Elsässer, and I. Fischer, *Synchronization of delay-coupled oscillators: A study of semiconductor lasers*, Physical Review Letters, 94 (2005), p. 163901.
[40] S. Yanchuk and G. Giacomelli, *Pattern formation in systems with multiple delayed feedbacks*, Physical Review Letters, 112 (2014), p. 174103.
[41] H. Ye, A.N. Michel, and K. Wang, *Qualitative analysis of cohen-grossberg neural networks with multiple delays*, Physical Review E, 51 (1995), p. 2611.