CONCEPTS OF DIMENSION FOR CONVEX GEOMETRIES

KOLJA KNAUER AND WILLIAM T. TROTTER

ABSTRACT. Let X be a finite set. A family P of subsets of X is called a *convex geometry* with ground set X if (1) $\emptyset, X \in P$; (2) $A \cap B \in P$ whenever $A, B \in P$; and (3) if $A \in P$ and $A \neq X$, there is an element $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$. As a non-empty family of sets, a convex geometry has a well defined VC-dimension. In the literature, a second parameter, called convex dimension, has been defined expressly for these structures. Partially ordered by inclusion, a convex geometry is also a poset, and four additional dimension parameters have been defined for this larger class, called Dushnik-Miller dimension, Boolean dimension, local dimension, and fractional dimension, respectively. For each pair of these six dimension parameters, we investigate whether there is an infinite class of convex geometries on which one parameter is bounded and the other is not.

1. STATEMENT OF RESULTS

The primary goal of this paper is to investigate concepts of dimension for a special class of posets called convex geometries, that are defined as inclusion orders of certain set systems on a finite ground set. The concepts of dimension that we consider are called convex dimension, VC-dimension, Dushnik-Miller dimension, Boolean dimension, fractional dimension, and local dimension, denoted $\text{cdim}(P)$, $\text{vcdim}(P)$, $\dim(P)$, $\text{bdim}(P)$, $\text{ldim}(P)$, and $\text{fdim}(P)$, respectively. We will also study a related poset parameter, called standard example number, denoted $\text{se}(P)$. When P is a convex geometry, the following inequalities are known or easily seen to hold:

(1) $\text{cdim}(P) \geq \dim(P) \geq \max\{\text{vcdim}(P), \text{bdim}(P), \text{fdim}(P), \text{ldim}(P), \text{se}(P)\}$, see Proposition 3.9,
(2) $\text{se}(P) \geq \text{vcdim}(P)$ unless $\text{vcdim}(P) = 2$ and $\text{se}(P) = 1$, see Observation 3.4.

For readers who are familiar with concepts of dimension *and* convex geometries, we state here the results of this paper. Motivation, definitions, and essential preliminary material will be provided in Sections 2 and 3. Proofs are given in Sections 4 through 8, and we close with some comments on open problems that remain in Section 9.

Our first result separates Dushnik-Miller dimension and convex dimension.

Theorem 1.1. *If $n \geq 3$, there is a convex geometry P_n such that $\dim(P_n) = 3$, and $\text{cdim}(P_n) = n + 1$.*

The next result shows that the bound on Dushnik-Miller dimension in the preceding theorem cannot be improved.

Theorem 1.2. *If P is a convex geometry and $\dim(P) \leq 2$, then $\operatorname{cdim}(P) = \dim(P)$.*

Although there are convex geometries with VC-dimension 2 and standard example number 1, we show that these two parameters are essentially the same.

Theorem 1.3. *If P is a convex geometry, then $\operatorname{vcdim}(P) = \operatorname{se}(P)$ unless $\operatorname{vcdim}(P) = 2$ and $\operatorname{se}(P) = 1$.*

The next result collapses the exceptional case in the preceding theorem.

Theorem 1.4. *If P is a convex geometry and $\operatorname{se}(P) = 1$, then $\operatorname{cdim}(P) \leq 2$.*

The next result separates convex dimension, Dushnik-Miller dimension, Boolean dimension and local dimension from VC-dimension and fractional dimension. In stating this result, we use the abbreviation $[n]$ for $\{1, \ldots, n\}$ for an integer n, where for later purposes in the paper we set $[n] = \emptyset$ if $n < 1$. Also, we denote the base 2 logarithm of n by $\lg n$, while the natural logarithm of n is denoted $\log n$.

Theorem 1.5. *If k and n are integers with $1 \leq k \leq n - 2$, there is a convex geometry $P(k, n)$ with ground set $[n]$ such that:*
  (1) $\operatorname{vcdim}(P(k, n)) = \operatorname{se}(P(k, n)) = k + 1$.
  (2) $\operatorname{fdim}(P(k, n)) < 2^{k+1}$.
  (3) *For fixed $k \geq 1$, $\operatorname{bdim}(P(k, n)) \to \infty$ as $n \to \infty$.*
  (4) *For fixed $k \geq 1$, $\operatorname{ldim}(P(k, n)) \to \infty$ as $n \to \infty$.*
  (5) $\dim(P(1, n)) = 1 + \lfloor \lg n \rfloor$ *and* $\dim(P(k, n)) \leq (k + 1)2^{k+2} \log n$.
  (6) $\operatorname{cdim}(P(k, n)) = \binom{n-1}{k}$.
In particular, for fixed $k \geq 1$, $\dim(P(k, n)) \to \infty$ as $n \to \infty$ but $\operatorname{se}(P(k, n))$ is constant, i.e., convex geometries are not $\dim$-bounded.

The most interesting and best-understood case in Theorem 1.5 is $P(1, n)$. However, we believe that it is of general interest to see $P(1, n)$ as a member of a larger family $P(k, n)$. In particular it shows that the properties of $P(1, n)$ are not an effect of low VC-dimension or standard-example number.

2. Convex Geometries and Partially Ordered Sets

A *partially ordered set* (we prefer the short form *poset*) is a set P equipped with a binary relation $\leq$ that is reflexive, antisymmetric and transitive. We will assume that readers are familiar with basic concepts for posets including: covering relation, order diagrams (also called Hasse diagrams); chains and antichains; subposets; maximal and minimal points; isomorphic posets; and linear extensions. We will also assume that readers are familiar with the basics of finite lattices, including zeroes, ones, meets and joins.

In [15], Edelmann and Jamison introduce a class of posets they call *convex geometries*. As noted in [15], posets in this class have been studied by various authors, and many combinatorial objects naturally carry the order structure of a convex geometry. Examples include subtrees of a tree [12], convex subsets of a poset [5], convex subgraphs of an acyclic digraph [31], transitively oriented subgraphs of a transitively oriented digraph [7], convex sets of oriented matroids [26], and many more as discussed in [15].

As pointed out in [15], the notation and terminology in the literature for convex geometries is not uniform. As just one example, they have also been called

antimatroids, e.g., by Korte, Lovász, and Schrader [25]. Accordingly, a unifying framework for convex geometries is developed in [15], and we will follow to a large degree their framework.

Let X be a finite non-empty[1] set. A *convex geometry* with ground set X is a family P of subsets of X satisfying the following three properties:

(1) $\emptyset, X \in P$.   (*Base Property*)

(2) If A and $B \in P$, then $A \cap B \in P$.   (*Intersection Property*)

(3) If $A \in P$, and $A \neq X$, there exists $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$. (*Extension Property*)

We refer the reader to [15] and [16] for additional background information on convex geometries, including several equivalent definitions.

There are two special cases of convex geometries which are of particular interest in this paper. As a first example, when L is a linear order on a finite non-empty set X, we obtain a convex geometry P from L, with $|P| = 1 + |X|$, by taking P as the set of all initial segments of L, i.e., if $|X| = n$ and $L = \alpha_1 < \cdots < \alpha_n$, then P consists of the empty set together with the subsets of X of the form $\{\alpha_1, \ldots, \alpha_i\}$, where $i \in [n]$. These simple examples are called *linear geometries*.

As a second example, when X is a finite non-empty set, and P consists of *all* subsets of X, then P is a convex geometry. This special case has been called various names in the literature, with *Boolean algebra* and *subset lattice* being two popular choices. To emphasize the relationship between $|X|$ and $|P|$, we will denote this special case as $P = \mathbf{2}^n$, where $n = |X|$.

As a family of sets, a convex geometry is partially ordered by inclusion, and is therefore a poset. When P is a convex geometry, P is closed under intersections, but in general, P is not closed under unions. Nevertheless, a convex geometry is a lattice. The empty set is the zero, while the ground set X is the one. For sets $A, B \in P$, we have:

$$A \wedge B = A \cap B, \quad \text{and}$$
$$A \vee B = \cap\{C \in P : A \cup B \subseteq C\}.$$

Although we will reference results that involve posets in general, our particular focus is on the class of convex geometries.

3. Concepts of Dimension

When P is a poset, we will sometimes use the short form $a \leq_P b$ as a substitute for $a \leq b$ in P. Also, we will write $a \parallel_P b$ when a and b are distinct incomparable elements of P. As our emphasis is combinatorial in the main, and all parameters we study have the same value for two posets that are isomorphic, we will say that $P = Q$ when P and Q are isomorphic posets. In the same spirit, we say P *contains* Q when there is a subposet of P that is isomorphic to Q.

As it serves to motivate the definitions for the subclass of convex geometries, we elect to begin with four concepts of dimension for the broader class of posets. The first concept of dimension we will investigate is the classic parameter defined by Dushnik and Miller [14]. When $t \geq 1$, a sequence $(L_1, \ldots, L_t)$ of linear extensions of a poset P is a *realizer of* P if for all $x, y \in P$, $x \leq_P y$ if and only if $x \leq y$

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[1]Some authors allow the ground set X to be the empty set, leading to the family $\{\emptyset\}$ to be a convex geometry. Very little of interest can be said about this special case, so we only consider non-empty ground sets.

in L_i for all $i \in [t]$. The *Dushnik-Miller dimension* of P, denoted $\dim(P)$, is the least positive integer t such that P has a realizer of size t. Following the traditions of the extensive literature on this parameter, it will henceforth simply be called *dimension*.

For each $t \geq 2$, the *standard example* S_t is a height 2 poset with minimal elements $a_1, \ldots, a_t$, maximal elements $b_1, \ldots, b_t$, and order relation $a_i < b_j$ in S_t when $1 \leq i, j \leq t$ and $i \neq j$. As noted in [14], $\dim(S_t) = t$ for all $t \geq 2$. The *standard example number* of a poset P, denoted $\mathrm{se}(P)$, is set to be 1 when P does not contain the standard example S_2; otherwise $\mathrm{se}(P)$ is the largest $t \geq 2$ for which P contains the standard example S_t. Evidently, $\dim(P) \geq \mathrm{se}(P)$ for all posets P.

On the one hand, the inequality $\dim(P) \geq \mathrm{se}(P)$ can be far from tight, as among the class of posets which have standard example number 1 (this is the well studied class of interval orders [19, 36]), there are posets that have arbitrarily large dimension. At the other extreme, when $P = \mathbf{2}^n$ is a Boolean algebra and $n > 2$, then we have $n = \dim(P) = \mathrm{se}(P)$. Indeed, S_n is induced in P by all subsets of size 1 and $n - 1$. On the other hand for $i \in [n]$ define L_i as the lexicographic ordering of $\mathbf{2}^n$ with respect to the ordering $i < i + 1 < \cdots < n < 1 < \cdots < i - 1$, then it is easy to see that $(L_1, \ldots, L_n)$ is a realizer of P.

A class $\mathbb{P}$ of posets is said to be dim-*bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $P \in \mathbb{P}$, $\dim(P) \leq f(\mathrm{se}(P))$. We point out the analogous problem in graph theory. Although there are triangle-free graphs with arbitrarily large chromatic number, there are interesting classes of graphs where the chromatic number is bounded in terms of maximum clique size. Such classes are said to be χ-*bounded*.

Blake, Hodor, Micek, Seweryn and Trotter [10] have just resolved a conjecture that is more than 40 years old by showing that the class of posets that have planar cover graphs is dim-bounded. In time, readers will sense how their result prompts many of the questions we address in this paper. For readers who are new to the concept of dimension for posets, compact summaries are given in several recent research papers including [9, 20, 29], and the survey paper [37].

When P is a poset, we let $\mathrm{Inc}(P)$ denote the set of all pairs (a, b) of distinct elements of P with $a \parallel_P b$. Trivially, the following statements are equivalent: (1) $\mathrm{Inc}(P) = \emptyset$; (2) P is a chain; and (3) $\dim(P) = 1$. When $\mathrm{Inc}(P) \neq \emptyset$, and $t \geq 2$, a sequence $(L_1, L_2, \ldots, L_t)$ of linear extensions of P is a realizer of P if and only if for every $(a, b) \in \mathrm{Inc}(P)$, there is some $i \in [t]$ such that $a > b$ in L_i.

When S is a non-empty subset of $\mathrm{Inc}(P)$, we say S is *reversible* if there is a linear extension L of P such that $a > b$ in L whenever $(a, b) \in S$. Accordingly, when $\mathrm{Inc}(P) \neq \emptyset$, $\dim(P)$ is the least integer $t \geq 2$ such that $\mathrm{Inc}(P)$ can be covered by t reversible sets.

Now let P be a poset that is not a chain. A pair $(a, b) \in \mathrm{Inc}(P)$ is called a *critical pair* if (1) $x <_P b$ whenever $x <_P a$; and (2) $a <_P y$ whenever $b <_P y$. The set of all critical pairs of P is denoted $\mathrm{crit}(P)$. A sequence $(L_1, \ldots, L_t)$ of linear extensions of P is a realizer of P if and only if for every $(a, b) \in \mathrm{crit}(P)$, there is some $i \in [t]$ such that $a > b$ in L_i. Accordingly, $\dim(P)$ is the least integer $t \geq 2$ such that $\mathrm{crit}(P)$ can be covered with t reversible sets.

Again, let P be a poset that is not a chain, and let $k \geq 2$. A sequence $((a_1, b_1), \ldots, (a_k, b_k))$ of pairs from $\mathrm{Inc}(P)$ is called an *alternating cycle* in P if $a_i \leq_P b_{i+1}$ for all $i \in [k]$. As suggested by the terminology, this requirement holds

cyclically, i.e., we also require $a_k \leq_P b_1$. An alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ is said to be *strict* if for all $i, j \in [k]$, $a_i \leq_P b_j$ if and only if $j = i + 1$. As is also well known, a non-empty subset $S \subseteq \text{Inc}(P)$ is reversible if and only if there are no strict alternating cycles in P consisting entirely of pairs from S. Also, we note that when $((a_1, b_1), \ldots, (a_k, b_k))$ is a strict alternating cycle, the sets $\{a_i : i \in [k]\}$ and $\{b_i : i \in [k]\}$ are both k-element antichains.

Let P be a poset, and let $\mathcal{R} = (L_1, \ldots, L_t)$ be a sequence of linear orders on the ground set of P (these linear orders need not be linear extensions). For a pair (x, y) of distinct elements of P, determine a 0–1 *query string* $q = q(x, y, \mathcal{R})$ of length t by setting $q(i) = 1$ if $x < y$ in L_i; otherwise, set $q(i) = 0$. The sequence $\mathcal{R}$ is called a *Boolean realizer for P* if there is a set τ of bit strings of length t such that for each pair (x, y) of distinct elements of P, $x <_P y$ if and only if $q(x, y, \mathcal{R}) \in \tau$. As defined by Nešetřil and Pudlák in [30], the *Boolean dimension* of P, denoted $\text{bdim}(P)$, is the least positive integer t for which P has a Boolean realizer of size t. For every poset P, we always have $\text{bdim}(P) \leq \dim(P)$, since a realizer $\mathcal{R} = (L_1, \ldots, L_t)$ is a Boolean realizer with $\tau = \{q\}$, where q is a bit string of length t with $q(i) = 1$ for all $i \in [t]$. On the other hand, it is an easy exercise to verify that if $t \geq 2$ and S_t is a standard example, then $\text{bdim}(S_2) = 2$; $\text{bdim}(S_3) = 3$; and $\text{bdim}(S_t) = 4$ for all $t \geq 4$. We refer readers to [28], [18] and [38] for recent results on Boolean dimension.

Let P be a poset. A linear order M is called a *partial linear extension*, abbreviated ple, of P if M is a linear extension of a subposet of P. A sequence $(M_1, \ldots, M_m)$ of ple's of P is called a *local realizer of P* if (1) whenever $x \leq_P y$, there is some $i \in [m]$ with $x \leq y$ in M_i; and (2) whenever $(x, y) \in \text{Inc}(P)$, there is some $j \in [m]$ with $x > y$ in M_j. As defined by Ueckerdt[2], the *local dimension* of P, denoted $\text{ldim}(P)$, is the least r for which there is a local realizer $(M_1, \ldots, M_m)$ of P such that for each $x \in P$, there are at most r different values of $i \in [m]$ with $x \in M_i$. Again, it is clear that $\text{ldim}(P) \leq \dim(P)$ since a realizer is a local realizer. Also, it is an easy exercise to show that if $t \geq 2$ and S_t is the standard example, then $\text{ldim}(S_2) = 2$ and $\text{ldim}(S_t) = 3$ for all $t \geq 3$. We refer readers to [11], [23] and [2] for recent results on local dimension. In particular, [2] contrasts the three notions of dimension we have defined thus far.

It is worth noting that neither of Boolean dimension and local dimension is bounded in terms of the other, as the following results are proved in [38]:

(1) There is a constant C such that if $\text{ldim}(P) = 3$, then $\text{bdim}(P) \leq C$.
(2) For every $t \geq 4$, there is a poset P with $\text{ldim}(P) = 4$ and $\text{bdim}(P) \geq t$.
(3) For every $t \geq 3$, there is a poset P with $\text{bdim}(P) = 3$ and $\text{ldim}(P) \geq t$.

Let P be a poset, and let $\mathcal{E}(P)$ be the set of all linear extensions of P. A *fractional realizer of P* is a function f which assigns non-negative real numbers to the linear extensions in $\mathcal{E}(P)$ such that whenever $(x, y) \in \text{Inc}(P)$, $\sum\{f(L) : L \in \mathcal{E}(P), x > y$ in $L\} \geq 1$. As defined by Felsner and Trotter [17], the *fractional dimension* of P, denoted $\text{fdim}(P)$, is the least positive real number t, with $t \geq 1$, such that there is a fractional realizer f with $\sum\{f(L) : L \in \mathcal{E}(P)\} = t$. Now we have $\text{fdim}(P) \leq \dim(P)$ as we can take f to the 0–1 function assigning value 1 to linear extensions in a realizer of P, while assigning value 0 to all other linear extensions. It is an easy exercise to verify that $\text{fdim}(S_t) = t$ for all $t \geq 2$. On the other hand,

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[2]T. Ueckerdt proposed the notion of local dimension at the 2016 Order and Geometry Workshop held in Gułtowy, Poland.

it is a nice exercise to show that if $\mathrm{se}(P) = 1$, then $\mathrm{fdim}(P) < 4$. We refer readers to [17] and [6] for additional information and results on fractional dimension.

For the two special cases of convex geometries discussed in the preceding section, we first observe that if P is a linear geometry, then $\dim(P) = \mathrm{bdim}(P) = \mathrm{ldim}(P) = \mathrm{fdim}(P) = \mathrm{se}(P) = 1$. The situation with Boolean algebras is more complex. If $P = \mathbf{2}^n$ is a Boolean algebra, then

(1) $\dim(P) = \mathrm{fdim}(P) = n$, see [27],
(2) $\mathrm{se}(P) = n$ unless $n = 2$, and in this case $\mathrm{se}(P) = 1$,
(3) $\mathrm{bdim}(P) = \Omega(n/\log n)$ and $\mathrm{bdim}(P) < n$, see [13].
(4) $\mathrm{ldim}(P) = \Omega(n/\log n)$ and it is open if $\mathrm{ldim}(P) < n$, see [23].

3.1. VC-Dimension.
We now discuss two additional dimension parameters defined for convex geometries but not for posets in general. The first of the two is defined for any non-empty family of sets.

Let X be a finite set, and let F be a non-empty family of subsets of X. The *VC-dimension* of F, denoted $\mathrm{vcdim}(F)$, is defined as follows. Set $\mathrm{vcdim}(F) = 0$ if there is no element $\alpha \in X$ for which there are sets $A, A' \in F$ with $\alpha \in A$ and $\alpha \notin A'$; otherwise, set $\mathrm{vcdim}(F)$ to be the largest $t \geq 1$ for which there is a t-element subset $\{\alpha_1, \ldots, \alpha_t\}$ of X such that for every set $S \subset [t]$, there is a set $A = A(S) \in F$ such that $\alpha_i \in A$ if and only if $i \in S$. We note that if P is a linear convex geometry, then $\mathrm{vcdim}(P) = 1$. Also, if P is the Boolean algebra $\mathbf{2}^n$, then $\mathrm{vcdim}(P) = n$.

The next two theorems, both proved in [15], make clear the essential role Boolean algebras play in a discussion of convex geometries. For the statement, recall that for elements x, y of a poset P their *interval* is $[x, y] = \{z \in P \mid x \leq z \leq y\}$.

Theorem 3.1 (Boolean property). *Let P be a convex geometry, let X and Y be distinct elements of P such that (1) $Y \neq \emptyset$; and (2) X is the intersection of all sets in P that are covered by Y in P. Then the interval $[X, Y]$ of P is isomorphic to the Boolean algebra $\mathbf{2}^m$, where m is the number of sets that are covered by Y in P.*

Now let Q be a finite lattice. It is natural to ask whether there is a convex geometry P such that $P = Q$. The next theorem, proved in [15], provides an answer.

Theorem 3.2. *Let Q be a finite lattice. Then there is a convex geometry P such P is isomorphic to the poset Q if and only if for every y in Q with $y \neq 0$, if x is the meet of all elements covered by y in Q, then the interval $[x, y]$ in Q is isomorphic to the Boolean algebra $\mathbf{2}^m$, where m is the number of elements of Q that are covered by y.*

In the literature, a finite lattice Q is said to be *meet-distributive* if it satisfies the property in Theorem 3.2. Accordingly, the study of convex geometries can be recast as the study of meet-distributive lattices. One useful property of these lattices is that they are *graded*, i.e., all maximal chains between two comparable elements have the same length.

Motivated by the preceding discussion, we make the following definitions. Let Q be a finite poset, and let $y \in Q$. We denote by $\mathrm{dd}(y, Q)$ the *down degree* of y in Q, i.e., $\mathrm{dd}(y, Q)$ is the number of elements of Q that are covered by y in Q. Of course $\mathrm{dd}(y, Q) = 0$ if and only if y is a minimal element of Q. In turn, we let $\mathrm{maxdd}(Q)$ denote the maximum value of $\mathrm{dd}(y, Q)$ taken over all elements $y \in Q$.

When P is a convex geometry, if $d = \mathrm{maxdd}(P)$, then Theorem 3.1 now implies that P contains the Boolean algebra $\mathbf{2}^d$. Therefore, $\dim(P) \geq \mathrm{vcdim}(P) \geq \mathrm{maxdd}(P)$. The next theorem, proved in [1] shows that this inequality is tight.

Theorem 3.3. *When P is a convex geometry,* $\mathrm{vcdim}(P) = \mathrm{maxdd}(P)$.

Because some of our proofs make direct use of the diagram and the specific value of maximum down degree and for the balance of the paper we make the following choice:

> From now on we will phrase results in terms of maximum down degree instead of VC-dimension, asking readers to keep in mind that when working with convex geometries, maximum down degree is *exactly* the same as VC-dimension.

Dually, when Q is a finite poset, and $x \in Q$, we define the *up degree* of x in Q, denoted $\mathrm{ud}(x, Q)$ as the number of elements of Q that cover x. Also, $\mathrm{maxud}(X)$ is the maximum value of $\mathrm{ud}(x, Q)$ taken over all $x \in Q$. As we will soon see, there is no bound on the value of maximum up degree, even when maximum down degree is 2.

We finish with a quick observation announced in the introduction:

Observation 3.4. *Let P be a convex geometry, then* $\mathrm{se}(P) \geq \mathrm{maxdd}(P)$ *unless* $\mathrm{maxdd}(P) = 2$ *and* $\mathrm{se}(P) = 1$.

Proof. If $\mathrm{maxdd}(P) = d \geq 3$, then by Theorem 3.1 there is a Boolean algebra $\mathbf{2}^d$ contained in P. Hence, by $d = \mathrm{se}(\mathbf{2}^d) \leq \mathrm{se}(P)$. $\square$

3.2. **Convex Dimension.** The second dimension parameter for convex geometries borrows from the set up for (Dushnik-Miller) dimension. When X is the ground set of a convex geometry P, $|X| = n$, and $L = \alpha_1 < \cdots < \alpha_n$ is a linear order on X, we say L is *compatible* when $\{\alpha_1, \ldots, \alpha_i\}$ is a set in P for each $i \in [n]$. There is a natural 1–1 correspondence between compatible linear orders on X and maximal chains in P.

Let X be a finite set, let $t \geq 1$, and let $(P_1, \ldots, P_t)$ be a sequence of convex geometries each of which has ground set X. Define a family P of subsets of X by:

$$P = \{A_1 \cap \cdots \cap A_t : A_i \in P_i \text{ for all } i \in [t]\}.$$

Then P is a convex geometry, and we denote this by writing $P = P_1 \vee \cdots \vee P_t$. Furthermore, given a convex geometry P, if we define the sequence $(P_1, \ldots, P_t)$ by taking the linear geometries associated with the maximal chains in P, then $P = P_1 \vee \cdots \vee P_t$. These observations give rise to the following definitions.

Let P be a convex geometry, and let X be the ground set of P. A sequence $(P_1, \ldots, P_t)$ of linear geometries, each with ground set X, is called a *convex realizer of P* if $P = P_1 \vee \cdots \vee P_t$. The *convex dimension* of P, denoted $\mathrm{cdim}(P)$, is the least positive integer t for which P has a convex realizer $(P_1, \ldots, P_t)$ of size t.

We pause to make an elementary observation. Let $n \geq 2$, let $L_1 = 1 < 2 < \cdots < n$, and $L_2 = n < \cdots < 2 < 1$. If Q_1 and Q_2 are the linear geometries with ground set $[n]$ determined by L_1 and L_2, respectively, and $P = Q_1 \vee Q_2$, then $\mathrm{cdim}(P) = \dim(P) = \mathrm{maxdd}(P) = 2$, while $\mathrm{maxud}(P) = n$.

In a finite lattice Q, an element y that satisfies $\mathrm{ud}(y, Q) = 1$ is called a *meet-irreducible element*. This terminology reflects the property that if $y = w \wedge z$, then

one of w and z must be y. The following theorem, which provides a *very* useful alternative definition of convex dimension, is proved in [15].

Theorem 3.5. *Let P be a convex geometry. Then $\mathrm{cdim}(P)$ is the width of the subposet of P determined by the meet-irreducible elements of P.*

Next, we present a brief series of elementary results that support more substantive arguments to follow.

Proposition 3.6. *Let Q be a finite lattice. If $(x, y) \in \mathrm{crit}(Q)$, then $\mathrm{dd}(x, Q) = \mathrm{ud}(y, Q) = 1$.*

Proof. We prove that $\mathrm{dd}(x, Q) = 1$. The argument to show that $\mathrm{ud}(y, Q) = 1$ is dual. Suppose to the contrary that $\mathrm{dd}(x, Q) \geq 2$. Then there are distinct elements w and z of Q such that x covers both w and z in Q. Since (x, y) is a critical pair, and $w <_Q x$, we know $w <_Q y$. Similarly, we have $z <_Q x$, so $z <_Q y$. Since Q is a lattice and $\{x, y\}$ is an antichain, we know that $\{w, z\} \leq_Q w \vee z \leq_Q x \wedge y <_Q \{x, y\}$. However, these inequalities imply that neither of w and z is covered by x. The contradiction completes the proof. $\square$

When Q is a finite lattice, and y is a meet-irreducible element of Q, we let $\mathrm{uc}(y, Q)$ denote the unique element of Q that covers y. An element x of Q is called a *join-irreducible element of Q* if $\mathrm{dd}(x, Q) = 1$. In this case, we let $\mathrm{dc}(x, Q)$ denote the unique element of Q covered by x. The next result shows that in a convex geometry P, there is a natural 1–1 correspondence between the set of meet-irreducible elements of P and the set of critical pairs of P.

Proposition 3.7. *Let P be a convex geometry, let B be a meet-irreducible element of P, let $Y = \mathrm{uc}(B, P)$, and let $\{\alpha\} = Y - B$. Then the following statements are equivalent:*

(1) $A = \cap \{U \in P : \alpha \in U\}$.
(2) $(A, B) \in \mathrm{crit}(P)$.

Proof. Assume that $A = \cap \{U \in P : \alpha \in U\}$. We show that (A, B) is a critical pair. First, since $\alpha \in A$, and $\alpha \notin B$, we know $A \not\subset B$. If there is an element $\beta \in A - Y$, then $A \cap Y \subsetneq A$, which contradicts the definition of A. We conclude that $A \subset Y$. Furthermore, if $V \subsetneq A$ for some $V \in P$, then $\alpha \notin V$. Since $Y = B \cup \{\alpha\}$, we conclude that $V \subset B$. Now suppose that B is a proper subset of some $W \in P$. Then since B is a meet-irreducible element, we have $Y \subseteq W$. This implies $A \subset W$. These observations imply that (A, B) is a critical pair.

Now suppose that (A, B) is a critical pair. We show that $A = \cap \{U \in P : \alpha \in U\}$. Since $B \subsetneq Y$, we must have $A \subseteq Y$. However, $Y = B \cup \{\alpha\}$, and we conclude that $\alpha \in A$. Now suppose that there is a set $U \in P$ with $\alpha \in U$ and $U \subsetneq A$. Then $U \not<_P B$, which contradicts the assumption that (A, B) is a critical pair. We conclude that $A = \cap \{U \in P : \alpha \in U\}$. $\square$

Proposition 3.8. *Let Q be a finite poset. If $t = \mathrm{se}(Q) \geq 2$, then Q contains a copy of the standard example S_t labeled such that $(a_i, b_i) \in \mathrm{crit}(Q)$ for all $i \in [t]$.*

Proof. Of all copies of the standard example S_t contained in Q, choose one for which the following sum is minimum:

$$\sum_{i \in [t]} |D_Q[a_i]| + |U_Q[b_i]|.$$

Clearly, (a_i, b_i) is a critical pair in Q, for each $i \in [t]$. □

Proposition 3.9. *Let P be a convex geometry. Then*

$$\mathrm{cdim}(P) \geq \dim(P) \geq \max\{\mathrm{maxdd}(P), \mathrm{se}(P), \mathrm{bdim}(P), \mathrm{fdim}(P), \mathrm{ldim}(B)\}.$$

Proof. As noted in Section 3, the inequalities $\dim(P) \geq \mathrm{se}(P), \mathrm{bdim}(P), \mathrm{fdim}(P),$ $\mathrm{ldim}(B)$ hold for every poset P. For $\dim(P) \geq \mathrm{maxdd}(P)$ note that by Theorem 3.2 the poset P contains a Boolean algebra of dimension $\mathrm{maxdd}(P)$. Let now J be the set of meet-irreducible elements in P, i.e., J is the set of all sets B for which $\mathrm{ud}(B, P) = 1$. From Theorem 3.5, we know that $\mathrm{cdim}(P)$ is the width of the subposet J. If $\mathrm{cdim}(P) = s$, it follows that there are chains $\mathcal{C}_1, \ldots, \mathcal{C}_s$ in P that cover all the sets in J. For each $i \in [s]$, let S_i consist of all critical pairs (A, B) such that $B \in \mathcal{C}_i$. Each set S_i of this form is reversible since it cannot contain a strict alternating cycle. It follows that $\dim(P) \leq s$. □

4. SEPARATING DIMENSION AND CONVEX DIMENSION

Proposition 3.9 asserts that if P is a convex geometry, then $\mathrm{cdim}(P) \geq \dim(P)$. If P is a linear geometry or the Boolean algebra $\mathbf{2}^n$, then $\mathrm{cdim}(P) = \dim(P)$. More generally, this equality even holds for distributive lattices, see [16]. However, the inequality is not always tight: Figure 1 in [16] illustrates a convex geometry with convex dimension 4 but (Dushnik-Miller) dimension 3.

It is natural to ask whether dimension and convex dimension can be *separated*, i.e., is there a family of convex geometries for which dimension is bounded while convex dimension is not. We now proceed to give an affirmative answer. The remainder of this section constitutes the proof of Theorem 1.1.

For $n \geq 3$ denote by Q_n the sets of the form $[i] \times [j] \times [k]$ for $i \in [2]$ and $j, k \in [n]$. It is easy to see that the dimension of Q_n is at most 3. Pick lexicographic orderings L_1, L_2, L_3 of Q_n where each orders Q_n lexicographically with respect to a different cyclic ordering of the three coordinates.

Denote by P_n the subposet of Q_n consisting of the sets of the form $[2] \times [j] \times [k]$ for all $j, k \in [n]$ but $[1] \times [j] \times [k]$ only for those $j, k \in [n]$ such that $j + k \leq n + 1$. As a subposet of Q_n, also $\dim(P_n) \leq 3$.

Note that P_n contains the empty set as well as the full set $[2] \times [n] \times [n]$. Further, it clearly is intersection closed and every element can be either extended by 2 or otherwise by any other element α missing on the first or second coordinate. We conclude that P_n forms a convex geometry. Since $\mathrm{maxdd}(P_n) = 3$, we know $\dim(P_n) \geq 3$ by Proposition 3.9. Therefore, $\dim(P_n) = 3$.

Finally, note that $u(Y, P_n) = 1$ exactly for the subsets of the form $[2] \times [n] \times [k]$, $[2] \times [k] \times [n]$ for $1 \leq k < n$ as well as $[1] \times [j] \times [k]$ for $j + k = n+1$. And furthermore the latter type of sets forms a maximum antichain of size $n + 1$. Theorem 3.5 we have $\dim(P_n) = n + 1$.

We conclude that we have an infinite family $\{P_n : n \geq 3\}$ of convex geometries with $\dim(P_n) = 3$, and $\mathrm{cdim}(P_n) = n+1$. Accordingly, we have separated dimension and convex dimension for the class of convex geometries.

We show in Figure 1 an order diagram for P_6 as a subposet of Q_6.

5. CONVEX GEOMETRIES WITH DIMENSION 2

The family constructed in the preceding section suggests the following question. Among convex geometries with dimension 2, is convex dimension bounded? In
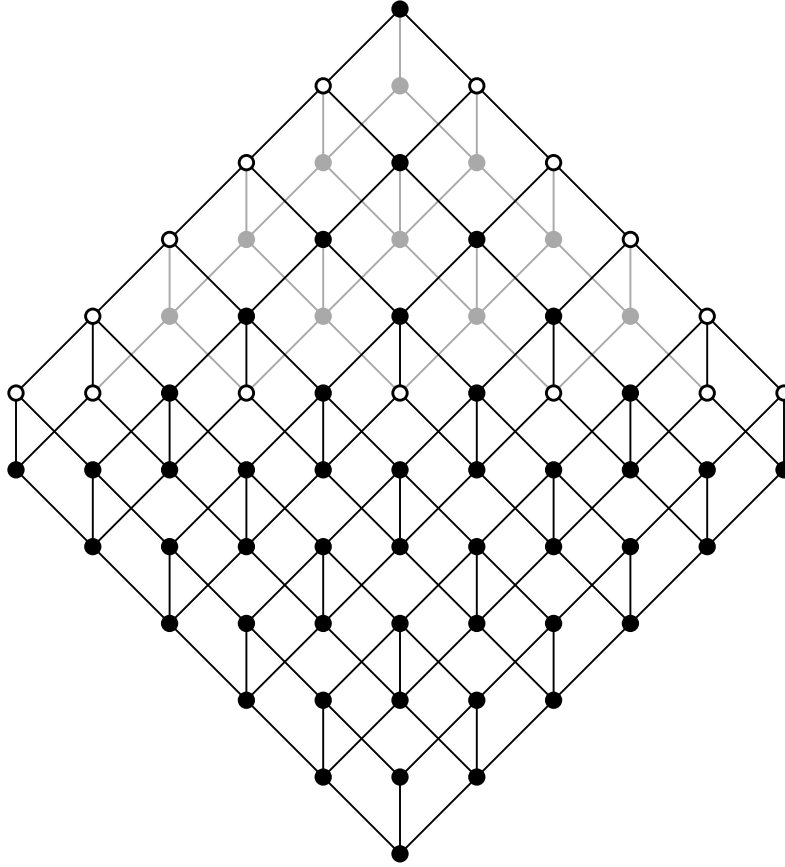
FIGURE 1. For $n \geq 3$, we illustrate a 3-dimensional poset Q_n and
its subposet P_n obtained by removing the grey points. Then P_n
is a convex geometry, and the white points constitute the set of
meet-irreducible elements of P_n.

this section, we give an affirmative answer. In fact, we show that when $\dim(P) \leq 2$, $\operatorname{cdim}(P) = \dim(P)$, and the argument constitutes the proof of Theorem 1.2. The statement holds trivially if $\dim(P) = 1$, so we fix a convex geometry P with $\dim(P) = 2$.

Our argument requires an unpublished but by now quite well known result due to K. Baker: A finite lattice has Dushnik-Miller dimension at most 2 if and only if its order diagram is planar. Recall that a poset is said to be planar if its order diagram can be drawn (following all rules for diagrams) without edge crossings in the plane. We show on the left side of Figure 2 a 2-dimensional convex geometry with a planar order diagram.

Using the result of Baker, we take a planar drawing of the order diagram of P. Since P is a lattice, there is a well defined maximal chain in P that constitutes the left boundary in the drawing. Also, there is a well defined maximal chain in P that constitutes the right boundary in the two drawings. These two boundary chains
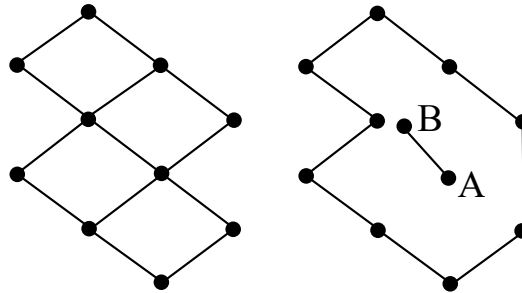
FIGURE 2. On the left, we show a convex geometry with a planar order diagram. Note that maximum down degree is 2, and every interior face is a diamond. On the right, we suggest how an element in the interior would appear if it had up degree 1. We will show that this is impossible.

can intersect. Regardless, any element of P that is not on either of these two chains is in the *interior* of the drawing.

Claim 5.1. *The interior faces in the drawing are diamonds.*

Proof. Since P is a lattice, each face F has a unique maximum element which we denote 1_F. Also, F has a unique minimum element which we denote 0_F. Let A, B be the vertices of F covered by 1_F. By the Boolean property, we know that element $A \cap B$ is covered by A and B, so F is a diamond formed by the elements of $\{1_F, A, B, A \cap B\}$. □

Claim 5.2. *If A is in the interior of the drawing, then* $\mathrm{ud}(A, P) \geq 2$,

Proof. Referring to the illustration on the right side of Figure 2, suppose to the contrary that A is in the interior of the drawing, and $\mathrm{ud}(A, P) = 1$. Let B be the unique element of P that covers A. Also, let e be the edge in the drawing having A and B as its end points. All the points in the plane that belong to e, except possibly B, are in the interior of the drawing. This implies that e is a boundary edge of two interior faces F and F' that have no points in common other than the end points of the edge e.

We note that we cannot have $B = 1_F$, and $B = 1_{F'}$, as this would require $\mathrm{dd}(B, P) \geq 3$. Without loss of generality, we may assume $B \neq 1_F$. Since F is a diamond, this forces $B = 0_F$. In turn, this forces $\mathrm{ud}(A, P) \geq 2$. The contradiction completes the proof of the claim. □

To complete the proof of the theorem, we simply observe that the width of the subposet of P consisting of all elements that have up degree 1 in P is at most 2, since all these elements are on the two boundary chains in the drawing.

6. Standard Example Number and Maximum Down Degree

We have already noted that when P is a convex geometry with ground set X, then $\mathrm{se}(P) \geq \mathrm{maxdd}(P)$, except possibly when $\mathrm{maxdd}(P) = 2$. If $\mathrm{maxdd}(P) = 2$, we can have $\mathrm{se}(P) = 1$, as is the case when P is the Boolean algebra $\mathbf{2}^2$. This raises the question as to whether there is a convex geometry P for which $\mathrm{se}(P) > \mathrm{maxdd}(P)$.

We answer this question in the negative. The results of this section constitute the proof of Theorem 1.3.

We argue by contradiction and assume that P is a convex geometry with $\mathrm{se}(P) > \mathrm{maxdd}(P)$. Then $\mathrm{se}(P) \geq 2$. Choose a standard example S_n in P with the elements labeled as $\{A_1, \ldots, A_n\} \cup \{B_1, \ldots, B_n\}$, so that for all $i, j \in [n]$, $A_i \subseteq B_j$ if and only if $i \neq j$. By Proposition 3.8, we may further assume that for each $i \in [n]$, (A_i, B_i) is a critical pair. From Proposition 3.6, it follows that for each $i \in [n]$, $\mathrm{dd}(A_i, P) = \mathrm{ud}(B_i, P) = 1$.

For each $i \in [n]$, let Y_i be the unique element covering B_i in P. Since $|Y_i - B_i| = 1$, there is a unique element $\alpha_i \in X$ such that $Y_i = B_i \cup \{\alpha_i\}$. Since (A_i, B_i) is a critical pair, and $B_i \subsetneq Y_i$, we know $A_i \subset Y_i$. Since $A_i \parallel_P B_i$, this forces $\alpha_i \in A_i - B_i$.

Now let X_i be the unique set in P that is covered by A_i. Since (A_i, B_i) is a critical pair, we know $X_i \subset B_i$. This now requires $X_i = A_i - \{\alpha_i\}$. i.e., α_i is the only element of A_i that is not in B_i.

Now set $Y_0 = A_1 \vee \cdots \vee A_n$. Also, for each $i \in [n]$, let $B_i' = Y_0 \cap B_i$. Then B_i' is an element of P, and $\alpha_i \in A_i - B_i'$. Furthermore, for each $j \in [n]$ with $j \neq i$, A_j is a subset of B_i and Y_0. Therefore, $A_j \subset B_i'$. It follows that the sets in $\{A_1, \ldots, A_n\} \cup \{B_1', \ldots, B_n'\}$ determine a subposet of P which is isomorphic to S_n. In particular, the sets in $\{B_1', \ldots, B_n'\}$ form an n-element antichain in P.

For each $i \in [n]$, let $Y_i' = B_i' \cup \{\alpha_i\}$. We then have:

(1) $Y_i' = B_i' \cup \{\alpha_i\} = (Y_0 \cap B_i) \cup \{\alpha_i\} = (Y_0 \cup \{\alpha_i\}) \cap (B_i \cup \{\alpha_i\}) = Y_0 \cap Y_i.$

Since Y_0 and Y_i are elements of P, it follows that Y_i' is an element of P. Also, from its definition, it is clear that Y_i' covers B_i'.

Claim 6.1. *If $i, j \in [n]$, then $A_j \subset Y_i'$.*

Proof. If $j \neq i$, then $A_j \subset B_i' \subset Y_i'$. If $j = i$, since $A_i \subset Y_i$ and $A_i \subset Y_0$, it follows from (1) that $A \subset Y_i'$. $\square$

Claim 6.2. *If $i \in [n]$, then $Y_0 = Y_i'$.*

Proof. The first claim implies that $Y_0 \subseteq Y_i'$ for all $i \in [n]$. If $i \in [n]$, the definition of B_i' implies that $B_i' \subseteq Y_0$. Since Y_i' covers B_i', we conclude that $Y_0 = Y_i'$. $\square$

We have now shown the the element Y_0 covers all elements of $\{B_1', \ldots, B_n'\}$. This implies $\mathrm{maxdd}(P) \geq \mathrm{dd}(Y_0, P) \geq n$. In turn, we have $\mathrm{maxdd}(P) \geq n$. With this observation, the proof is complete.

7. Convex Geometries with Standard Example Number 1

We now know that if P is a convex geometry, then $\mathrm{se}(P) = \mathrm{maxdd}(P)$, except possibly when $\mathrm{se}(P) = 1$ and $\mathrm{maxdd}(P) = 2$. In this section, we show that if P is a convex geometry, and $\mathrm{se}(P) = 1$, then $\dim(P) = \mathrm{cdim}(P) \leq 2$. This section constitutes the proof of Theorem 1.4.

We fix a convex geometry P with $\mathrm{se}(P) = 1$. If $\mathrm{maxdd}(P) = 1$, then P is a chain, and the conclusion holds trivially. So we may assume that $\mathrm{maxdd}(P) = 2$ and hence $\dim(P) \geq 2$.

Let J denote the subposet of P determined by the meet-irreducible elements. Then $\mathrm{cdim}(P) = \mathrm{width}(J)$. If the proposition fails, there is a 3-element antichain $\{B_1, B_2, B_3\}$ in J. For each $i \in [3]$, let Y_i be the unique element of P that covers B_i. Since $\mathrm{maxdd}(P) = 2$, we may assume that $Y_1 \neq Y_2$. Since P does not contain

S_2, either $B_1 <_P Y_2$ or $B_2 <_P Y_1$. Note that since P is a lattice, it cannot be the case that both inequalities hold, so we may assume without loss of generality that $B_1 <_P Y_2$ and $B_2 \parallel_P Y_1$.

Then since Y_1 is the unique element covering B_1, we have $Y_1 <_P Y_2$, and there is an element Z_1 covered by Y_2 such that $Y_1 \leq_P Z_1$. The Boolean property then implies that there is an element X covered by Z_1 and B_2. Furthermore, since P is graded and $B_1 \parallel_P B_2$, if $Y_1 <_P Z_1$, then $B_1, Y_1 \parallel_P X$, and $\{B_1, Y_1, B_2, X\}$ determines S_2. We conclude that $Y_1 = Z_1$.

We note that $\{B_3, Y_3\} \cap \{B_1, Y_1, B_2, Y_2, X\} = \emptyset$, as any common point implies a comparability that does not exist or a down degree larger than two. Furthermore, note that Y_1, Y_2, Y_3 must form a chain in some order: otherwise if for $i = 1, 2$ we would have $Y_3 \parallel_P Y_i$, then since Y_i covers B_i, we get $Y_3 \parallel_P B_i$ yielding a S_2 together with B_3. Now, since Y_2 covers Y_1 and Y_1 has already down degree two, we must have $Y_2 < Y_3$. Let $Y_2 \leq Z_2 < Y_3$ be covered by Y_3. The Boolean property then implies that there is an element W covered by Z_2 and B_3. Since Y_2 already has down degree two, and $\{B_1, B_2, B_3\}$ is an antichain, we force $Y_2 < Z_2$. Since P is graded, Y_2 cannot be larger than W. Since $\{B_1, B_2, B_3\}$ is an antichain, B_3 cannot be larger than B_2. Hence $\{Y_2, B_2, B_3, W\}$ induce a copy of S_2. The contradiction completes the proof.

8. Separating Three Parameters from the Other Four

We have noted that when P is a convex geometry, then $\dim(P) \geq \mathrm{maxdd}(P)$. We note that if P is the Boolean algebra $\mathbf{2}^n$, or a linear geometry, then $\dim(P) = \mathrm{maxdd}(P)$. Note that these parameters also agree on the convex geometries discussed in the preceding two sections. Now it is natural to ask whether there is a class of convex geometries on which maximum down degree is bounded but dimension is not. We now proceed to give an affirmative answer to this question. In fact, we will construct an infinite family of convex geometries for which (1) maximum down degree, fractional dimension, and standard example number are bounded; while (2) convex dimension, dimension, Boolean dimension, and local dimension are unbounded.

The remainder of this section constitutes the proof of Theorem 1.5. For the readers convenience, we give nine statements including the ones of the theorem. Let k and n be integers with $1 \leq k \leq n - 2$. We show there is a family $P(k, n)$ of subsets of $[n]$ satisfying the following properties:

1. $P(k, n)$ is a convex geometry with ground set $[n]$.
2. If $1 \leq k < k' \leq n - 2$, then $P(k, n)$ is a subposet of $P(k', n)$.
3. $\mathrm{cdim}(P(k, n)) = \binom{n-1}{k}$.
4. $\mathrm{maxdd}(P(k, n)) = k + 1$.
5a. $\dim(P(1, n)) = 1 + \lfloor \lg n \rfloor$,
5b. $\dim(P(k, n)) \leq (k + 1)2^{k+2} \log n$.
6. $\mathrm{se}(P(k, n)) = k + 1$.
7. For fixed $k \geq 1$, $\mathrm{bdim}(P(k, n)) \to \infty$ as $n \to \infty$.
8. For fixed $k \geq 1$, $\mathrm{ldim}(P(k, n)) \to \infty$ as $n \to \infty$.
9. $\mathrm{fdim}(P(k, n)) < 2^{k+1}$.

We begin with the definition of $P(k, n)$. Then we proceed to prove that the nine statements of the theorem are satisfied.

Let k and n be integers with $1 \le k \le n - 2$. A set $A \subseteq [n]$ belongs to $P(k, n)$ if and only if the following property is satisfied:

Membership.   If $|A| = k + i - 1$, then $[i - 1] \subseteq A$.

With our convention that $[i - 1] = \emptyset$ when $i - 1 \le 0$, we note that $A \in P(k, n)$ whenever $A \subset [n]$ and $|A| \le k$. On the other hand, $\{1, 2, 3, 6, 11\}$ and $\{1, 2, 6, 10, 11\}$ belong to $P(3, 12)$ while $\{1, 3, 6, 10, 11\}$ does not.

The next result is statement 1.

Proposition 8.1. *The family $P(k, n)$ is a convex geometry with ground set $[n]$.*

Proof. By definition $P(k, n)$ is a family of subsets of $[n]$. We now show that $P(k, n)$ satisfies the Base, Intersection and Extension Properties. First, we observe that $\emptyset$ and $[n]$ satisfy the Membership requirement, so both are in $P(k, n)$. Therefore, $P(k, n)$ satisfies the Base Property.

Let $A, B \in P(k, n)$. We show that $A \cap B \in P(k, n)$. This holds trivially if one of A and B is a subset of the other, so we may assume that neither is contained in the other. The conclusion that $A \cap B \in P(k, n)$ also holds trivially if $|A \cap B| \le k$, so we may assume that $|A \cap B| = k + i - 1$ for some i with $2 \le i \le n - k + 1$. Then $|A| \ge k + i - 1$ and $|B| \ge k + i - 1$, so $[i - 1] \subset A$ and $[i - 1] \subset B$. Therefore, $[i - 1] \subset A \cap B$. Hence, $A \cap B \in P(k, n)$. With these observations, we have shown that $P(k, n)$ satisfies the Intersection Property.

Finally, let $A \in P(k, n)$ with $A \ne X$. Suppose first that $|A| \le k$. If $1 \notin A$, then $A \cup \{1\} \in P(k, n)$. If $1 \in A$, then $A \cup \{\alpha\} \in P(k, n)$ for every $\alpha \in [n] - A$. Now suppose that $|A| = k + i - 1$ for some i with $2 \le i \le n - k + 1$. Then $[i - 1] \subset A$. Since $A \ne [n]$, we know $i \le n - k$. If $i \notin A$, then $A \cup \{i\} \in P(k, n)$. If $i \in A$, then $A \cup \{\alpha\} \in P(k, n)$ for every $\alpha \in [n] - A$. With this observation, we have shown that $P(k, n)$ satisfies the Extension Property. Therefore $P(k, n)$ is a convex geometry with ground set $[n]$. □

The subfamily $\{P(1, n) : n \ge 3\}$ will play an important role in arguments to follow. We illustrate $P(1, 5)$ in Figure 3 and note that $P(1, 4)$ is the same convex geometry illustrated in Figure 2 in [16].

The next result is statement 2, and it helps to explain our interest in the subfamily $\{P(1, n) : n \ge 3\}$.

Proposition 8.2. *If $1 \le k < k' \le n - 2$, then $P(k, n)$ is a subposet of $P(k', n)$.*

Proof. Let $A \in P(k, n)$ and suppose that $|A| = k' + i - 1$. Then $|A| = k + (k' - k) + i - 1$. Since $A \in P(k, n)$, we must have $[k' - k + i - 1] \subset A$. Since $k' - k > 0$, this implies $[i - 1] \subset A$. Therefore, $A \in P(k', n)$. □

Let $J(k, n)$ denote the family of all sets of the form $[i - 1] \cup B$ where i and B satisfy the following requirements:

  (1) $i \in [n]$ and $B \subset [n]$.
  (2) $i < j$ for every $j \in B$.
  (3) $|B| \le k$.
  (4) If $|B| < k$, then $B = \{j \in [n] : i + 1 \le j \le n\}$.

Proposition 8.3. *If $1 \le k \le n - 2$, then $J(k, n)$ is the set of all meet-irreducible elements of $P(k, n)$.*
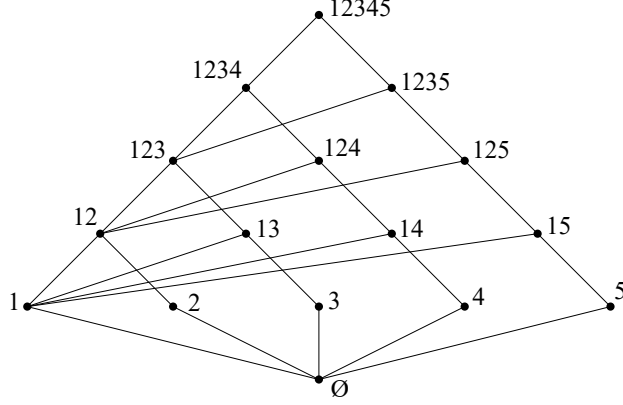
FIGURE 3. We illustrate the convex geometry $P(1,5)$. In the drawing, sets are indicated without braces and commas. Although these facts will follow from the more general arguments given below, readers may enjoy verifying that $\operatorname{cdim}(P(1,5)) = 4$, $\dim(P(1,5)) = \operatorname{bdim}(P(1,5)) = \operatorname{fdim}(P(1,5)) = 3$, and $\operatorname{se}(P(1,5)) = \operatorname{maxdd}(P(1,5)) = 2$.

Proof. Clearly, all sets in $J(k,n)$ satisfy the membership requirement and belong to $P(k,n)$. Note further that all sets in $J(k,n)$ have size of at least k.

We observe that if $A = [i-1] \cup B$ is in $J(k,n)$ and $|B| < k$. Then $|A| = n-1$, and $[n]$ is the only element of $P(k,n)$ that covers A. On the other hand, if $A = [i-1] \cup B$ is in $J(k,n)$ and $|B| = k$, then the only element of P covering A is $A' = [i] \cup B$. With these observations, we have verified that all elements of $J(k,n)$ are meet-irreducible.

For the converse, suppose A is a meet-irreducible element of $P(k,n)$. Clearly $n > |A|$ since the entire set is not a meet-irreducible element. If $|A| = n - 1$, say $A = [n] - \{i\}$, then $A = [i-1] \cup B$, where $B = \{j \in [n] : i+1 \le j \le n\}$. Since the membership requirement of $P(k,n)$ forces $[n-k-1] \subset A$ we have $i \ge n-k$ and $|B| \le k$. This implies $A \in J(k,n)$.

If $k \le |A| \le n-2$, write $|A| = k + i - 1$ where $1 \le i \le n-k-1$. Then by the membership requirement of $P(k,n)$ we have $[i-1] \subseteq A$. If $i \notin A$, then $A = [i-1] \cup B$ where $|B| = k$. Hence, $A \in J(k,n)$. On the other hand, if $i \in A$, then A is covered by $A \cup \{j\}$ for all $j \in [n] - A$. The assumption that $|A| \le n-2$ then implies that $\operatorname{ud}(A, P(k,n)) \ge 2$, contradicting that A was a meet-irreducible element.

Finally note that $|A| \ge k$, for if $|A| < k$, then A is covered by $A \cup \{\alpha\}$ for every $\alpha \in [n-k]$. Since by definition $k \le n-2$, we get $\operatorname{ud}(A, P(k,n)) \ge 2$, contradicting that A was a meet-irreducible element. Hence, we have considered all the cases and the proof is complete. $\qquad\square$

For the balance of this section, when we say that $A = [i-1] \cup B$ is meet-irreducible, we also mean that the requirements for membership in $J(k,n)$ are satisfied by i and B.

The next result is statement 3.

Proposition 8.4. *If $1 \le k \le n-2$, then $\operatorname{cdim}(P(k,n)) = \binom{n-1}{k}$.*

Proof. Let $J = J(k,n)$. Then set $w = \binom{n-1}{k}$. The set $\mathcal{A}$ of all meet-irreducible elements of $P(k,n)$ of size k is an antichain in J. Note that a k-element set $A \subset [n]$ belongs to J if and only if $1 \notin A$. It follows that the width of J is at least $|\mathcal{A}| = w$.

To show that $\text{width}(J) \leq w$, we construct an explicit covering of J using w chains. Let B be a k-element subset of $[n]$ with $1 \notin B$, and let $j = \min(B)$. Then $j \geq 2$, and the sets in $\{[i-1] \cup B : 1 \leq i \leq j-1\}$ form a chain of size $j-1$ in $J(k,n)$. Clearly, every element of $J(k,n)$ is contained in a unique chain of this form. $\square$

The next result implies statement 4.

Proposition 8.5. *Let A be a non-empty set in $P(k,n)$. Then $\text{dd}(A, P(k,n)) = 1$ if and only if A is a singleton. Furthermore, $\text{maxdd}(P(k,n)) = k + 1$.*

Proof. Evidently, a singleton set $A = \{i\}$ satisfies $\text{dd}(A, P(k,n)) = 1$ since $\emptyset$ is the only set covered by A. Now let A be a set in $P(k,n)$ with $|A| \geq 2$. If $|A| \leq k+1$, then A covers $A - \{\alpha\}$ for every $\alpha \in A$. Since the poset is graded A cannot cover any further elements. Thus, $\text{dd}(A, P(k,n)) = |A|$, which by assumption is between 2 and $k+1$. Now suppose $|A| = k + i - 1$ for some i with $3 \leq i \leq n - k + 1$. Then by the definition of $P(k,n)$ and the fact that it is graded A covers $A - \{\alpha\}$ if and only if $\alpha \in A$ with $\alpha \geq i - 1$. We conclude that $\text{dd}(A, P(k,n)) = k + 1$. With this observation, the proof of the first statement of the proposition is complete. Furthermore, we have shown that $\text{dd}(A, P(k,n)) \leq k+1$ for all A and this equality also attained. Hence, $\text{maxdd}(P(k,n)) = k + 1$. $\square$

With Propositions 3.6 and 3.7, we noted that there is a 1–1 correspondence between the set $J(k,n)$ of meet-irreducible elements of $P(k,n)$ and the set of critical pairs of $P(k,n)$. Using the fact that all singletons belong to $P(k,n)$, it follows that the critical pairs of $P(k,n)$ consist of all pairs of the form $(\{i\}, [i-1] \cup B)$ where $[i-1] \cup B$ is a meet-irreducible element of $P(k,n)$.

The following proposition is stated for emphasis. It is an immediate consequence of the rule for the 1–1 correspondence.

Proposition 8.6. *Let $A = [i-1] \cup B$ and $A' = [j-1] \cup C$ be meet irreducible elements of $P(k,n)$ with $i \leq j$. Also, let (X,Y) and (X',Y') be the critical pairs of $P(k,n)$ associated with A and A' respectively. Then $((X,Y),(X',Y'))$ is a strict alternating cycle of size 2 if and only if $i < j$ and $j \in B$.*

The next proposition is only marginally more complex.

Proposition 8.7. *Let $m \geq 2$, and let $(A_1, \ldots, A_m)$ be a sequence of meet-irreducible elements of $P(k,n)$. For each $j \in [m]$, let (X_j, Y_j) be the critical pair associated with A_j. If the sequence $(((X_1, Y_1), \ldots, (X_m, Y_m))$ is a strict alternating cycle, then then $m = 2$.*

Proof. For each $j \in [m]$, let $A_j = [i_j - 1] \cup B_j$. Then $(X_j, Y_j) = (\{i_j\}, A_j)$. Since the alternating cycle is strict, the elements of $\{i_1, \ldots, i_m\}$ are distinct. With a relabeling if necessary, we may assume that $i_1 < i_j$ for all $j = 2, \ldots, m$.

It follows that for each $j = 2, \ldots, m$, $X_1 = \{i_1\} \subset [i_j - 1] \subset A_j = Y_j$. Again, since the alternating cycle is strict, this now requires $m = 2$. $\square$

If $1 \leq k \leq n - 2$ and t is a positive integer, a sequence $(Y_1, \ldots, Y_n)$ of subsets of $[t]$ is said to be (k,n)-*distinguishing* if for every meet-irreducible element $A = [i-1] \cup B \in J(k,n)$, there is an element $\alpha \in Y_i$ such that $\alpha \notin Y_j$ whenever $j \in B$.

Proposition 8.8. *If $1 \le k \le n - 2$, $\dim(P(k, n))$ is the least positive integer t for which there is a (k, n)-distinguishing sequence $(Y_1, \ldots, Y_n)$ of subsets of $[t]$.*

Proof. Suppose first that $\dim(P(k, n)) = t$. We show that there is a sequence $(Y_1, \ldots, Y_n)$ of subsets of $[t]$ that is (k, n)-distinguishing. Let $S_1, \ldots, S_t$ be reversible sets covering all critical pairs of $P(k, n)$. For each $i \in [n]$, let Y_i consist of all $\alpha \in [t]$ such that there is a critical pair $(X, Y) = (\{i\}, [i-1] \cup B)$ assigned to S_α. We claim that $(Y_1, \ldots, Y_n)$ is (k, n)-distinguishing. To see this, let $[i - 1] \cup B$ be any meet-irreducible element of $P(k, n)$. Then there is some $\alpha \in [t]$ such that the critical pair $(X, Y) = (\{i\}, [i - 1] \cup B)$ is assigned to S_α. We claim that $\alpha \notin Y_j$ for all j with $j \in B$. If this fails, there is a meet-irreducible element $[j - 1] \cup C$ such that the critical pair $(X', Y') = (\{j\}, [j - 1] \cup C)$ is also assigned to S_α. However, $i < j$ and $j \in B$ imply that $((X, Y), (X', Y'))$ is a strict alternating cycle, contradicting the fact that S_α is reversible. We conclude that the sequence $(Y_1, \ldots, Y_n)$ is (k, n)-distinguishing as claimed.

For the converse, suppose that t is a positive integer, and there is a sequence $(Y_1, \ldots, Y_n)$ of subsets of $[t]$ that is (k, n)-distinguishing. We show $\dim(P(k, n)) \le t$. To accomplish this, for each $\alpha \in [t]$, we let S_α consist of all critical pairs $(X, Y) = (\{i\}, [i-1] \cup B)$ such that $\alpha \in Y_i$ and $\alpha \notin Y_j$ whenever $j \in B$. Using Propositions 8.6 and 8.7, for each $\alpha \in [t]$, the set S_α is reversible, so $\dim(P(k, n)) \le t$. $\square$

For each convex geometry in the family $\{P(1, n) : n \ge 3\}$, we now determine the value of the dimension *exactly*. The following result is statement 5a.

Proposition 8.9. *If $n \ge 3$, then $\dim(P(1, n)) = 1 + \lfloor \lg n \rfloor$.*

Proof. Let $t = \dim(P(1, n))$ and $s = 1 + \lfloor \lg n \rfloor$. Then there is a $(1, n)$-distinguishing sequence $(Y_1, \ldots, Y_n)$ of subsets of $[t]$. When $1 \le i < j \le n$, the meet-irreducible set $[i - 1] \cup \{j\}$ requires $Y_i - Y_j \ne \emptyset$. In particular, this requires $Y_i \ne Y_j$ and $Y_i \ne \emptyset$. The meet-irreducible set $[n - 1]$ requires $Y_n \ne \emptyset$. We have now shown that the sets in the sequence $(Y_1, \ldots, Y_n)$ are distinct *and* non-empty. This requires $t \ge s$.

For the converse, consider the family of all subsets of $[s]$ partially ordered by inclusion. Set $m = 2^s$, and consider an arbitrary linear extension L of the Boolean algebra $\mathbf{2}^s$. Then let $(Y_1, \ldots, Y_m)$ be the dual of L. Note that $Y_1 = [s]$ and $Y_m = \emptyset$. Since by assumption $n < 2^s = m$, the last set Y_n in the subsequence $(Y_1, \ldots, Y_n)$ is non-empty. It follows that this sequence is $(1, n)$-distinguishing. Indeed, for all $i \in [n]$ we have that Y_i has an element α (because $Y_i \ne \emptyset$) that is in no Y_j for $j > i$ (because Y_j does not contain Y_i because L is linear extension). Therefore, $t \le s$ by Proposition 8.8. $\square$

The following result is statement 5b. The proof is an elementary application of the probabilistic method.

Proposition 8.10. *For $2 \le k \le n - 2$, $\dim(P(k, n)) \le (k + 1)2^{k+2} \log n$.*

Proof. Let $t = \lfloor (k + 1)2^{k+2} \log n \rfloor$. Then consider a sequence $\sigma = (Y_1, Y_2, \ldots, Y_n)$ of random subsets of $[t]$, i.e. for each pair (i, α) with $i \in [n]$ and $\alpha \in [t]$, we assign α to the set Y_i with probability $1/2$. Assignments made for distinct pairs are independent. The probability that σ fails to be (k, n)-distinguishing is at most:

$$(2) \qquad\qquad n^{k+1}(1 - \frac{1}{2^{t+1}})^t.$$

Simple calculations show that for the specified value of t, the expression given in (2) is less than 1. It follows that a (k, n)-distinguishing sequence $\sigma = (Y_1, \ldots, Y_n)$ exists. This implies that $\dim(P(k, n)) \leq t$. $\qquad\square$

Statement 6 follows from Theorem 1.3. However, there is a direct proof which is more revealing of the structure of these geometries.

Proposition 8.11. *If* $1 \leq k \leq n - 2$, $\mathrm{se}(P(k, n)) = k + 1$.

Proof. For each $i \in [k+1]$, let (X_i, Y_i) be the critical pair $(\{i\}, [i-1] \cup \{i+1, \ldots, i+k\})$. Then the elements of $\{X_1, \ldots, X_{k+1}\}$ and $\{Y_1, \ldots, Y_{k+1}\}$ form the standard example S_{k+1}. This shows that $\mathrm{se}(P(k, n)) \geq k + 1$.

Now suppose that $P(k, n)$ contains a standard example of size $k + 2$. Then there is a sequence $((X_1, Y_1), \ldots, (X_{k+2}, Y_{k+2}))$ of critical pairs of $P(k, n)$ such that whenever $1 \leq i < j \leq k + 2$, the pairs (X_i, Y_i) and (X_j, Y_j) form a copy of the standard example S_2. For each $j \in [m]$, let $X_j = \{i_j\}$ and $Y_j = [i_j - 1] \cup B_j$. Then the elements of $\{i_1, \ldots, i_{k+2}\}$ are all distinct. After a relabelling, we may assume that $i_1 < i_2 < \cdots < i_{k+2}$. Then $i_j \in B_1$ for $j = 2, \ldots, k + 2$. This is impossible since $|B_1| \leq k$. The contradiction completes the proof. $\qquad\square$

The next two results are statements 7 and 8. In the proofs, when we write $\{i_1, i_2, \ldots, i_m\}$ is an m-element subset of $[n]$, we imply $i_1 < i_2 < \cdots < i_m$. Readers will note that our proofs use Ramsey theory and borrow from ideas in [2].

Proposition 8.12. *For fixed* $k \geq 1$, $\mathrm{bdim}(P(k, n)) \to \infty$ *as* $n \to \infty$.

Proof. In view of Proposition 8.2, it suffices to prove the result when $k = 1$. We assume that there is an integer $t \in \mathbb{N}$ such that $\mathrm{bdim}(P(1, n)) \leq t$ and argue to a contradiction *provided* n is sufficiently large in terms of t. Let $\mathcal{R} = (L_1, \ldots, L_t)$ be a Boolean realizer for $P(1, n)$.

Now let $\{i_1, i_2, i_3\}$ be a 3-element subset of $[n]$. Then $A = [i_1 - 1] \cup \{i_2\}$ and $B = [i_2 - 1] \cup \{i_3\}$ are meet-irreducible elements of $P(k, n)$. Also, $A \parallel B$ in $P(1, n)$. The query string $q(A, B, \mathcal{R})$ is a bit string of length t, so we have defined a coloring of the 3-element subsets of $[n]$ using 2^t colors. It follows that if n is sufficiently large, then there is a 4-element subset $H = \{i_1, i_2, i_3, i_4\}$ of $[n]$ such that each of the 3-element subsets of H is assigned the same color. Let $A = [i_1 - 1] \cup \{i_2\}$, $B = [i_2 - 1] \cup \{i_3\}$ and $C = [i_3 - 1] \cup \{i_4\}$. It follows that $q(A, B, \mathcal{R}) = q(B, C, \mathcal{R})$. This implies that $q(A, C, \mathcal{R}) = q(A, B, \mathcal{R})$. This is a contradiction since by $i_1 < i_2 < i_3 < i_4$ we have $A \subseteq C$ hence $A < C$ in $P(1, n)$. $\quad\square$

Proposition 8.13. *For fixed* $k \geq 1$, $\mathrm{ldim}(P(k, n)) \to \infty$ *as* $n \to \infty$.

Proof. In view of Proposition 8.2, it suffices to prove the result when $k = 1$. We assume that there is an integer $t \in \mathbb{N}$ such that $\mathrm{ldim}(P(1, n)) \leq t$ and argue to a contradiction *provided* n is sufficiently large in terms of t. Let $\mathcal{R} = (L_1, \ldots, L_m)$ be a local realizer of $P(1, n)$ such that each element of $P(1, n)$ appears in at most t different extensions in this list.

For each 4-element subset $\{i_1, i_2, i_3, i_4\}$ of $[n]$, we consider the meet-irreducible element $A = [i_1 - 1] \cup \{i_3\}$ and the meet-irreducible element $B = [i_2 - 1] \cup \{i_4\}$. Then $A \parallel B$ in $P(1, n)$. Let s be the least integer in $[m]$ such that $A > B$ in L_s. Then there is a uniquely determined pair (r, r') of integers in $[t]$ such that occurrence r of A is in L_s and occurrence r' of B is in L_s.

Now we are coloring the 4-element subsets of $[n]$ with t^2 colors. It follows that if n is sufficiently large, then there is a 7-element subset $H = \{i_1, \ldots, i_7\}$ such that each 4-element subset of H is assigned the same color, say (r, r'), where $r, r' \in [t]$.

Let $A = [i_1-1] \cup \{i_4\}$, $B = [i_2-1] \cup \{i_5\}$, $C = [i_3-1] \cup \{i_6\}$, and $D = [i_5-1] \cup \{i_7\}$. Let s be the least integer in $[m]$ such that $A > B$ in L_s. Using the set $\{i_1, i_2, i_4, i_5\}$, we know that occurrence r of A is in L_s, and occurrence r' of B is in L_s. Using the set $\{i_1, i_3, i_4, i_6\}$, we know that occurrence r' of C is also in L_s with $A > C$ in L_s. Using the set $\{i_2, i_3, i_5, i_6\}$, we know that occurrence r of B is in L_s with $B > C$ in L_s. This forces $r = r'$.

Using the set $\{i_3, i_5, i_6, i_7\}$, we conclude that occurrence r of C is over occurrence r of D in L_s. This forces $A > D$ in L_s, which is a contradiction since $A < D$ in $P(1, n)$. The contradiction completes the proof. $\qquad\square$

The final result in this section is statement 9.

Proposition 8.14. *If $1 \leq k \leq n - 2$, then* $\mathrm{fdim}(P(k, n)) < 2^{k+1}$.

Proof. We show that $\mathrm{fdim}(P(k, n)) < 2^{k+1}$ by constructing an appropriate fractional realizer f.

Let $m = 2^n$, and let $\sigma = (Z_1, \ldots, Z_{m-1})$ be a listing of the non-empty subsets of $[n]$. The order of the sets in the list is arbitrary. Then for each $\alpha \in [m-1]$, there is a linear extension L_α of $P(k, n)$ that reverses all critical pairs of the form $(\{i\}, [i-1] \cup B)$, where $i \in Z_\alpha$ and $j \notin Z_\alpha$ whenever $i < j$ and $j \in B$. Namely, if $Z_\alpha = \{i_1 \ldots, i_\ell\}$ the ordering $[i_1 - 1] \cup B_{i_{1,1}}, \ldots, [i_1 - 1] \cup B_{i_{1,r}}, \{i_1\}, [i_2 - 1] \cup B_{i_{2,1}} \ldots \{i_\ell\}$ respects the comparabilities of $P(k, n)$ since the sets B are disjoint from Z_α. Hence, L_α can be taken to be any linear extension respecting this ordering.

Then we set $f(L_\alpha) = 2^{k+1}/m$ for each $\alpha \in [m - 1]$. Also, we set $f(L) = 0$ for all other linear extensions of $P(k, n)$.

We now show that f is a fractional realizer of $P(k, n)$. We need only to show that if $C, D \in P(k, n)$ and $C \parallel_{P(k,n)} D$, then the following inequality holds:

$$(3) \qquad s(C, D) = \sum \{f(L_\alpha) : \alpha \in [m - 1], C > D \text{ in } L_\alpha\} \geq 1.$$

Choose a critical pair (X, Y) such that $X \leq C$ in $P(k, n)$ and $D \leq Y$ in $P(k, n)$. If $\alpha \in [m - 1]$ and $X > Y$ in L_α, then $C > D$ in L_α. It follows that we need only show that inequality (3) holds for every critical pair.

Let $(X, Y) = (\{i\}, [i - 1] \cup B)$ be a critical pair. Then the number of sets in the sequence σ that contain i but do not contain j when $i < j$ and $j \in B$ is exactly $2^{n-|B|-1}$, which is at least 2^{n-k-1}. It follows that

$$s(X, Y) \geq 2^{n-k-1} \cdot \frac{2^{k+1}}{m} = 1$$

Further, we note that

$$\sum_{i=1}^{m-1} f(L_i) = 2^{k+1} \frac{m - 1}{m} < 2^{k+1}.$$

We conclude that $\mathrm{fdim}(P(k, n)) < 2^{k+1}$. $\qquad\square$

9. Open Problems

For posets in general, the standard examples have bounded Boolean dimension, bounded local dimension, and unbounded dimension. As noted previously, the constructions given in [38] show that neither of Boolean dimension and local dimension is bounded in terms of the other. However, for convex geometries, we have been unable to separate dimension from either of Boolean dimension and local dimension. Also, we have been unable to separate Boolean dimension and local dimension in either direction. We note that for the special class of distributive lattices, the dimension coincides with the convex dimension, see [15]. However large width of the poset of meet-irreducible elements of a distributive lattice forces a large Boolean subalgebra, which forces large Boolean dimension [13] and large local dimension [23]. Hence, separating these parameters is not possible on distributive lattices. On the other hand, many other special classes of convex geometries have been identified in the literature, and we wonder whether some of them exhibit properties that enable these questions to be answered.

For the family $\{P(k,n); 1 \leq k \leq n-2\}$ of convex geometries discussed in the preceding section, we determined the value of $\dim(P(1,n))$ exactly. When $k \geq 2$, we feel it is unlikely that we can obtain an exact formula for $\dim(P(k,n))$. However, a lower bound of the form $c2^k \log n$ where c is a positive constant should hold.

We suspect that the upper bound $\mathrm{fdim}(P(k,n)) < 2^{k+1}$ is asymptotically tight, i.e., for every $\epsilon > 0$, $\mathrm{fdim}(P(k,n)) > 2^{k+1} - \epsilon$ when n is sufficiently large.

Fractional dimension is the linear programming relaxation of dimension. Although not studied in the main body of this paper, there is a natural linear programming relaxation of local dimension, called *fractional local realizer*. We refer the reader to [33] for the precise definition. We wonder whether the fractional local dimension of the convex geometry $P(k,n)$ is always strictly less than its fractional dimension.

Convex geometries originally arise from abstracting convex subsets of finite point sets in Euclidean space and indeed every convex geometry P on $[n]$ can be represented by point sets in $\mathbb{R}^d$. The minimum such d is bounded by $\min(n, \mathrm{cdim}(P))$, see [22, 32]. Representations through hyperplane arrangements in $\mathbb{R}^d$ seem possible as well [4]. What is the relation of this *Euclidean dimension* to the other concepts?

Finally, we believe that the notion of VC-dimension could be extended to general posets P, where it can be defined as the minimum VC-dimension over all set systems whose inclusion order is isomorphic to P. How does this parameter related to the other concepts of dimension? The question relates to rich literature about poset embeddings into Boolean algebras, see [3, 35, 39]. By the Boolean property, for convex geometries this notion coincides with the VC-dimension as we defined it previously. Other classes in which a natural embedding into the Boolean algebra give a VC-dimension arise from oriented matroids by Björner, Edelman, and Ziegler [8] and more generally from tope graphs of complexes of oriented matroids (see [24, Chapter 7.6] for several related questions).

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Universitat de Barcelona, Departament de Mathemátiques i Informática, Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Spain
Email address: kolja.knauer@gmail.com

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, U.S.A.
Email address: wtt.math@gmail.com