Polynomial Time Recognition of Uniform Cocircuit Graphs

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Abstract

We present an algorithm which takes a graph as input and decides in polynomial time if the graph is the cocircuit graph of a uniform oriented matroid. In the affirmative case the algorithm returns the set of signed cocircuits of the oriented matroid.

Keywords: Oriented matroid, cocircuit graph, recognition algorithm, polynomial algorithm.

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1 Introduction

The cocircuit graph is a natural combinatorial construction associated with oriented matroids. In the case of pseudoline-arrangements, i.e., uniform rank 3 oriented matroids, its vertices are the intersection points of the lines and two points share an edge if they are connected by a line segment which does not intersect other lines. More generally, the Topological Representation Theorem of Folkman and Lawrence [4] says that every oriented matroid can be represented as an arrangement of pseudospheres. The cocircuit graph is the 1-skeleton of this arrangement.

For the uniform case, Montellano-Ballesteros and Strausz [6] provide a graph theoretical characterization of cocircuit graphs. Cordovil, Fukuda and Guedes de Oliveira [2] show that a uniform oriented matroid is basically determined by its cocircuit graph (up to isomorphism and reorientation).

After introducing basic notions of oriented matroids we provide an algorithm inspired by [2] which given a graph G decides in polynomial time if G is the cocircuit graph of a uniform oriented matroid. In the affirmative case the algorithm returns the set of signed cocircuits of the oriented matroid.

Given a finite ground set E we define a *signed set* $X \subseteq E$ as an *underlying set* $\underline{X}$ with a bipartition (X^+, X^-) into a *positive* and a *negative* part. The parts may be empty. By X^0 we refer to the *zero-support* $E\backslash\underline{X}$. *Reorienting* X on $A \subseteq E$ yields the signed set ${}_AX := ((X^+\backslash A)\cup(X^-\cap A), (X^-\backslash A)\cup(X^+\cap A))$. By $-X$ we refer to ${}_EX$. Given signed sets X, Y we denote by $S(X, Y) := (X^+ \cap Y^-) \cup (X^- \cap Y^+)$ their *separator*.

We define an *oriented matroid* as a pair $\mathcal{M} = (E, \mathcal{C}^*)$ of a ground set E and a multiset $\mathcal{C}^*$ of signed sets called *cocircuits* satisfying the following axioms:

(C1) $\emptyset \notin \mathcal{C}^*$

(C2) $X \in \mathcal{C}^* \Rightarrow -X \in \mathcal{C}^*$

(C3) $X, Y \in \mathcal{C}^*$ and $\underline{X} = \underline{Y} \Rightarrow X = \pm Y$

(C4) $X, Y \in \mathcal{C}^*$ and $\underline{X} \neq \underline{Y}$ and $e \in S(X, Y) \Rightarrow$ there is $Z \in \mathcal{C}^*$ with $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$ and $e \in Z^0$.

The *cocircuit graph* $G_{\mathcal{M}}$ of an oriented matroid $\mathcal{M} = (E, \mathcal{C}^*)$ has as vertex set $\mathcal{C}^*$ and $X, Y \in \mathcal{C}^*$ share an edge if and only if $S(X, Y) = \emptyset$ and the symmetric difference $|X^0 \Delta Y^0| = 2$.

A *uniform oriented matroid* $\mathcal{M} = (E, \mathcal{C}^*)$ of *order* n and *rank* r is an oriented matroid with $|E| = n$ and $\mathcal{C}^*$ having exactly the subsets of size $n - r + 1$ as underlying sets. For more about oriented matroids, see [1].

2 The Algorithm

We are given a (simple, connected) graph $G = (V, A)$ and want to test in polynomial time whether G is the cocircuit graph of a uniform oriented matroid. In the affirmative case we assign a signed set $X(v)$ to every vertex v such that $X(V)$ is the set of cocircuits of $\mathcal{M}$ with $G = G_{\mathcal{M}}$. The algorithm has the following structure:

(A) Determine the parameters n and r

(B) Construct the set of great cycles $\mathcal{G}$ (defined below)

(C) Assign zero-supports X^0 to vertices and great cycles

(D) Assign signed sets $X(v) = (X^+(v), X^-(v))$ to vertices

(E) Check the cocircuit axioms

Each of these steps works if G is the cocircuit graph of a uniform oriented matroid and outputs that G is not, otherwise. The main idea of step (C) is contained in [2].

(A) Determine the parameters. Cocircuit graphs are regular and antipodal. These properties are checked here.

- Check if G is regular. If so, let δ denote the degree and set $r := \frac{\delta}{2} + 1$.
- Check if for every $v \in V$ there exists a unique $v^- \in V$ with $\mathrm{dist}(v, v^-) = \mathrm{diam}(G)$. In the affirmative case set $n := \mathrm{diam}(G) + r - 1$.
- Check if $|V| = 2\binom{n}{r-1}$.
- If any of the above tests results negative, G is not a cocircuit graph.

All the checked properties are necessary conditions for G to be a cocircuit graph of some uniform $\mathcal{M}$. Moreover n and r must be the order and the rank of the oriented matroid of $\mathcal{M}$, respectively. The runtime here is bounded from above by calculating a shortest path matrix for every starting vertex in the second step. This can be solved in $\mathcal{O}(V^3)$ by applying Dijkstra's Algorithm [3] $|V|$ times.

(B) Construct the set of great cycles A set $C \subseteq \mathcal{C}^*$ is called a *great cycle* with zero-support $X^0(C)$ if it consists of the cocircuits whose zero-support contains the $(r-1)$-set $X^0(C)$. The set of great cycles is denoted by $\mathcal{G}$. The following part of the algorithm computes $\mathcal{G}$.

- For every $v \in V(G)$ and every neighbor $w \in N(v)$. if the edge $\{v, w\}$ is not contained in any $C \in \mathcal{G}$ then

- · compute a shortest path P from w to v^-,
  · set $C' := (\{v, w\}, P, \{v^-, w^-\}, P^-)$,
  · add C' to $\mathcal{G}$.
- If $\mathcal{G}$ is no partition of the edge set then G is not a cocircuit graph.

For the correctness of this step we need:

Lemma 2.1 *The set of great cycles $\mathcal{G}$ of a cocircuit graph G partitions the edge set. Moreover the edge $\{v, w\}$ is contained in the cycle C' constructed in the algorithm.*

The lemma tells us in particular that $|\mathcal{G}||C| = |A| = \frac{\delta}{2}|V| = 4(r-1)\binom{n}{r-1}$ and that $|C| = 2\text{diam}(G) = 2(n-r+1)$. Hence the runtime $|V|\delta|C||V|^2$ is bounded by $4(r-1)(n-r+1)|V|^3 \in \mathcal{O}(|V|^4)$.

(C) Assign zero-supports to vertices and great cycles. In this part we assign zero suports to great cycles and vertices. We use the incidence structure on $\mathcal{G}$, i.e., let J be the graph with vertex set $\mathcal{G}$ where C and C' share an edge if they have a vertex in common in G. Note that J is isomorphic to the *Johnson Graph* $J(n, n-r+2)$, see [5] for a definition. During the algorithm we denote by $\mathcal{P}$ the set of cycles with already assigned zero-support. Cycles with all their vertices having a zero-support already are in $\mathcal{T}$. We assign supports starting from a vertex v, then all cycles containing v. These cycles form a clique in J. By $\mathcal{G}_i, \mathcal{P}_i, \mathcal{T}_i$ we denote the respective sets at distance i from this clique. We write $[k]$ for $\{1, \ldots, k\}$.

- Take any $v \in V$ and set $X^0(v) := [r-1]$.
- For every C_i in the set $\{C_1, \ldots, C_{r-1}\}$ of great cycles containing v,
  · set $X^0(C_i) := [r-1]\backslash\{i\}$,
  · add C_i to $\mathcal{P}_0$.
- Take any $\widetilde{C} = (v_1, \ldots, v_{2(n-r+1)})$ from $\mathcal{P}_0$.
- Set $X^0(v_{(n-r+1)+i}) := X^0(v_i) := X^0(C) \cup \{i\}$ for every $i \in [n-r+1]$.
- Add $\widetilde{C}$ to $\mathcal{T}_0$ and $\mathcal{T}_{-1}$.
- Set $i := 0$ and until all V is labelled repeat:
  · T-loop: For every $C \in \mathcal{P}_i$ take $\widetilde{C} \in \mathcal{T}_{i-1}$ such that $C \cap \widetilde{C} \neq \emptyset$.
    For every $v \in C$ find a $C' \in \mathcal{G}$ with $v \in C'$ and $C' \cap \widetilde{C}$ containing a w:
    set $X^0(v) := X^0(C) \cup (X^0(w)\backslash X^0(\widetilde{C}))$,
    add C to $\mathcal{T}_i$.
  · P-loop: For every $\widetilde{C} \in \mathcal{T}_i$ and $v \in \widetilde{C}$:
    If there exists an unlabeled great cycle C containing v look for another

$w \in C$ with known $X^0(w)$,
set $X^0(C) := X^0(v) \cap X^0(w)$,
add C to $\mathcal{P}_{i+1}$.

· If any vertex receives several different labels G is not a cocircuit graph.
· Increase i by one.

The runtime is bounded by the T-loop $|\mathcal{G}|(|C|\delta + |C|\delta|C||C|) \in \mathcal{O}(V^3)$. For the existence of C' in the T-loop we need:

Lemma 2.2 *Let $C \in \mathcal{G}_i$ and $C' \in \mathcal{G}_{i-1}$ be intersecting. Then for every $v \in C$ there is a $C'' \in \mathcal{G}_i$ that contains v and intersects with C'.*

For the existence of w in the P-loop we need:

Lemma 2.3 *For every $C \in \mathcal{G}_i$ there are at least two $C', C'' \in \mathcal{G}_{i-1}$ which intersect with C.*

(D) Assign signed sets to vertices. We will label the vertices with signed sets. As in (C) we start with cycles $\mathcal{S}_0$ which form a clique in J. Then we label cycles at increasing distance of $\mathcal{S}_0$. Since we calculated zero-supports in (C) we set $\underline{X}(v) := E \backslash X^0(v)$ for every $v \in V$.

- Take any $v \in V$ and set $X(v) := (\underline{X}(v), \emptyset)$.
- For all great cycles $C \ni v$ and $w, w' \in N(v) \cap C$ set $X(w) := (X^+(v) \cap \underline{X}(w), \underline{X}(w) \backslash X^+(v))$ and $X(w') := (\underline{X}(w), \emptyset)$.
- Add all $C \in \mathcal{G}$ containing v to $\mathcal{S}_0$.
- Set $i := 0$ and repeat until all V is signed:
  · For every great cycle $C \in \mathcal{S}_i$
    get two signed non-antipodal vertices $v, w \in C$ and take $u \in N(v) \cap C$ such that $0 < d(u, w) < d(u, w^-)$, i.e., u, v, w *lie on the same half* of C, sign u identically to v on $\underline{X}(v) \cap \underline{X}(u)$ and take the signing of $X(w)$ on $e = \underline{X}(u) \backslash \underline{X}(v)$.
    This way we sign all vertices in C.
    If a vertex receives two different labels G is not a cocircuit graph.
    Add all unsigned cycles $C' \cap C \neq \emptyset$ to $\mathcal{S}_{i+1}$.
  · Increase i by one.

The runtime is $|\mathcal{G}||C|^2 \in \mathcal{O}(|V|^4)$. We need Lemma 2.3 for the existence of $v, w \in C$. For the correctness of the signing we have:

Lemma 2.4 *Take a great cycle C of a uniform cocircuit graph, $X, Y \in C$, $\underline{X} \neq \underline{Y}$ and let $(X = X_0, \ldots, X_k = Y)$ be the shortest (X, Y)-path in C and $e \in \underline{X} \cap \underline{Y}$. Then $e \in S(X, Y)$ if and only if there is $X_i^0 \ni e$.*

(E) Check the cocircuit axioms. Here we check if the signed sets we assigned to the vertices satisfy (C4) of the cocircuit axioms. The other axioms are satisfied by construction.

- For every two non-antipodal vertices $u, v \in V$ and every element in the separator of their signed labels $e \in S(X(u), X(v))$ check if there exists a $w \in V$ with $X(w)^+ \subseteq X(u)^+ \cup X(v)^+$ and $X(w)^- \subseteq X(u)^- \cup X(v)^-$ and $e \in X(w)^0$.

Here the runtime is $|V|^3(n-r+1) \in \mathcal{O}(|V|^4)$. The correctness is obvious. The last step of the algorithm could also be done by checking whether the constructed $\mathcal{C}^*$ is a certain metrical and antipodal embedding into the $(r-1)$-dual of the n-cube, see [6].
Thus we have the following result:

Theorem 2.5 *The preceeding algorithm checks if a given graph $G = (V, A)$ is the cocircuit graph of a uniform oriented matroid $\mathcal{M}$ and constructs such $\mathcal{M}$ in the affirmative case in time $\mathcal{O}(|V|^4)$.*

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