PROBLEM SESSION: WORKSHOP ON GRAPHS, SEMIGROUPS, AND SEMIGROUP ACTS

Abstract. One session was dedicated to specifying and briefly discussing pending open problems. The following problems were brought up.

1. Planar Cayley graphs

A semigroup is called *planar* if it admits a generating system such that the resulting (right) Cayley graph is planar, that is, it admits a plane drawing. Finite planar groups were characterized by Maschke [3]. This characterization has been extended to planar right groups [2]. Some more progress towards this problem has been achieved in [4] where certain planar Clifford semigroups have been characterized. More generally, one may thus ask:

Problem 1 (Ulrich Knauer). *Characterize the planar completely regular semigroups.*

In the discussion it was observed that a semigroup $S \cong L_m \times G \times R_n$ is planar if and only if $G \times R_n$ is planar, since its Cayley graph consists of m disjoint copies of the Cayley graph of $G \times R_n$. Hence, such semigroups are characterized as a corollary of [2].

Analyzing the result of Maschke, one finds that the planar Cayley graphs of groups are exactly the graphs of the Archimedean and Platonic solids, (including prisms and the anti-prisms) with two exceptions: the Dodecahedron, the Icosidodecahedron. It has been shown in [1] that the Dodecahedron graph is an induced subgraph of a Brandt semigroup Cayley graph. However, the following is open:

Problem 2 (Kolja Knauer). *Are the graph of the Dodecahedron and the Icosidodecahedron the (undirected) Cayley graph of a semigroup?*

2. Cayley posets

Given a semigroup (right) act of S on A one can define for $a, a' \in A$ that $a \leq_S a'$ if there is $s \in S$ such that $as = a'$. If $\leq_S$ is an order-relation, we say that the obtained poset $P(A, S)$ is the Cayley poset of the act. It is easy to show that every poset is isomorphic to the Cayley poset of an act. One can thus restrict the definition to the setting where A is a supersemigroup of S and the S-act is just right-multiplication. If one furthermore requires that A and S are monoids, it can be shown that not all posets are Cayley posets of monoids. However, the following is open:

Problem 3 (Kolja Knauer). *Is there for every poset P a pair of semigroups $S < A$ such that P is isomorphic to the Cayley poset $P(A, S)$?*

References

[1] Yifei Hao, Xing Gao, and Yanfeng Luo, *On the Cayley graphs of Brandt semigroups*, Communications in Algebra **39** (2011), no. 8, 2874–2883.

[2] Kolja Knauer and Ulrich Knauer, *On planar right groups.*, Semigroup Forum **92** (2016), no. 1, 142–157 (English).

[3] Heinrich Maschke, *The Representation of Finite Groups, Especially of the Rotation Groups of the Regular Bodies of Three-and Four-Dimensional Space, by Cayley's Color Diagrams*, Amer. J. Math. **18** (1896), no. 2, 156–194.

[4] Xia Zhang, *Clifford semigroups with genus zero*, Semigroups, Acts and Categories with Applications to Graphs, Proceedings, Tartu 2007, Mathematics Studies, vol. 3, Estonian Mathematical Society, Tartu, 2008, pp. 151–160.