

Planar cubic graphs of small diameter

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Abstract

Cubic planar n -vertex graphs with faces of length at most 6, e.g. fullerene graphs, have diameter in $\Omega(\sqrt{n})$. It has been suspected, that a similar result can be shown for cubic planar graphs with faces of bounded length. This note provides a family of cubic planar n -vertex graphs with faces of length at most 7 and diameter in $O(\log n)$, thus refuting the above suspicion.

A *fullerene graph* is a cubic 3-connected planar graph all of whose faces are of length 5 or 6. They model fullerene molecules, i.e., molecules consisting only of carbon atoms other than graphite and diamonds. The search for correlations of graph invariants and chemical stability motivates the study of fullerenes in chemical graph theory, see e.g. [2, 4].

The diameter of a planar cubic graph can be logarithmic in the number of vertices. However, the diameter of a fullerene graph on n vertices is at least $\frac{1}{6}\sqrt{24n-15}-\frac{1}{2}$, as proved in [1]. Indeed, the proof of this lower bound extends to planar cubic graphs whose faces are of size at most 6. It has been suspected, that a similar lower bound holds for the diameter of cubic planar graphs with faces of bounded length, see [1, 2]. This is false as shown by the following family $\{G_k \mid k \geq 2\}$.

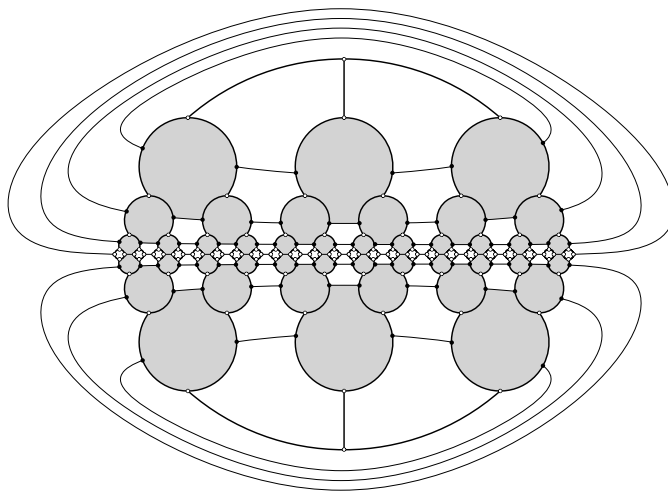


Figure 1: The graph G_5 . The vertices of the ternary trees are white and the original tree edges are bold. The gray faces are of length 7, all the others are shorter.

Let $k \geq 2$. Denote by T_k the rooted tree all of whose internal vertices are of degree 3 and all leaves are at distance k from the root. To define G_k , first take two copies of T_k and glue them by identifying corresponding leaves, see Figure 1. Now, subdivide all edges except those incident to leaves or roots. In Figure 1, the new vertices are black, while the original vertices of the trees are white. Fix a plane embedding of the obtained graph such that both roots are on the outer face. Note that, for each tree the vertices in the tree of a given distance d to its root are naturally ordered from left to right. Note that all the subdivision vertices and all the leaves are exactly the vertices at even distance to the root of their tree. Now, for each of the two trees and for every even integer d with $2 \leq d \leq 2k - 2$, let v_0, \dots, v_ℓ be the vertices at distance d from the root in the tree ordered from left to right. Note that ℓ is odd as $d \geq 2$. Put edge $v_i v_{i+1}$ for every even i (where $i + 1$ is taken modulo $\ell + 1$). This completes the definition of G_k .

The added edges between leaves and subdivision vertices did not destroy planarity, see Figure 1. After gluing the two trees together and subdividing the edges the only vertices of degree 2 were the leaves and the subdivision vertices. Each of them got one new incident edge, thus G_k is cubic.

The faces of G_k have lengths 4, 5, 6 or 7. Indeed, every face contains a tree vertex (white) thus it is enough to analyze faces around all the tree vertices. Let v be a vertex of one of the underlying trees in G_k . If v is the root, then all faces adjacent to v are of length 5. If v is an internal vertex, then it is adjacent to a 6-face and two 7-faces with an exception that when v is adjacent to leaves of its tree, then one of the 7-faces degenerates to a 4-face. If v is a leaf, then it is adjacent to a 4-face and two 6-faces. See Figure 2.

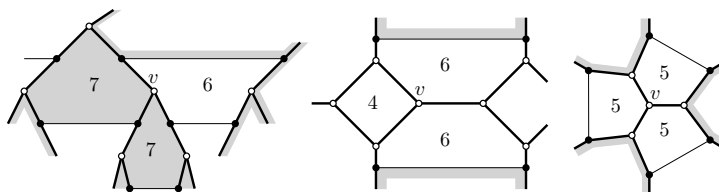


Figure 2: From left to right: face sizes around v as an inner vertex, leaf, and root, respectively.

Clearly, $|V(G_k)| = n \geq 2^k$. On the other hand, every two vertices in G_k can be joined by a path of length at most $3k$. Therefore the diameter of G_k is at most $3k \leq 3 \log(n)$.

Let us finally mention that in [3] fullerene graphs of diameter $\sqrt{\frac{4n}{3}}$ have been constructed. Maybe this is the smallest diameter a fullerene graph can have.

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