Distributive Lattices, Polyhedra, and Generalized Flows

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Abstract

A D-polyhedron is a polyhedron P such that if x, y are in P then so are their componentwise max and min. In other words, the point set of a D-polyhedron forms a distributive lattice with the dominance order. We provide a full characterization of the bounding hyperplanes of D-polyhedra.

Aside from being a nice combination of geometric and order theoretic concepts, D-polyhedra are a unifying generalization of several distributive lattices which arise from graphs. In fact with a D-polyhedron we associate a directed graph with arc-parameters, such that points in the polyhedron correspond to a vertex potentials on the graph. Alternatively, an edge-based description of the points of a D-polyhedron can be given. In this model the points correspond to the duals of generalized flows, i.e., duals of flows with gains and losses.

These models can be specialized to yield distributive lattices that have been previously studied. Particular specializations are: flows of planar digraphs (Khuller, Naor and Klein), α-orientations of planar graphs (Felsner), c-orientations (Propp) and Δ-bonds of digraphs (Felsner and Knauer). As an additional application we identify a distributive lattice structure on generalized flow of breakeven planar digraphs.

1 Introduction

A polyhedron $P \subseteq \mathbb{R}^n$ is called **distributive** if *distributive*

$$x, y \in P \Longrightarrow \min(x, y), \max(x, y) \in P$$

where minimum and maximum are taken componentwise. Distributive polyhedra are abbreviated **D-polyhedra**. *D-polyhedra*

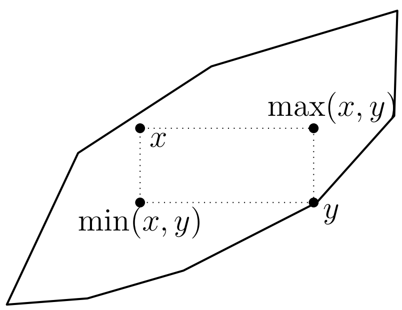

Figure 1: A distributive polytope in $\mathbb{R}^2$.

Denote by $\leq_{\text{dom}}$ the **dominance order** on $\mathbb{R}^n$, i.e., *dominance order*

$$x \leq_{\text{dom}} y \iff x_i \leq y_i \quad \text{for all } 1 \leq i \leq n.$$

The dominance order is a distributive lattice on $\mathbb{R}^n$. Join and meet in the lattice are given by the componentwise max and min. We call a subset $S \subseteq \mathbb{R}^n$ a sublattice of the dominance order, if it is a lattice with respect to the dominance order and its join and meet are max and min. Since distributive lattices can be defined only in terms of join and meet, we note the following:

Fact 1 *A subset $S \subseteq \mathbb{R}^n$ is a distributive sublattice of the dominance order if and only if it is closed with respect to* max *and* min.

By Birkhoff's Fundamental Theorem of Finite Distributive Lattices [2] every *finite* distributive lattice is isomorphic to a subset $S \subseteq \mathbb{Z}^n$ with the dominance order, see e.g. [6]. The name *distributive* polyhedron is justified by the following:

Observation 2 *A polyhedron $P \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if it is a distributive sublattice of the dominance order.*

In Section 2 we prove a characterization of D-polyhedra in terms of their description as intersection of halfspaces. In particular we obtain distributivity for known classes of polytopes, e.g. *order-polytopes* [26] and more generally *polytropes* [14], also called *alcoved polytopes* [17]. Moreover a new combinatorial description of these is derived, see Remark 11.

In Section 3 we use the geometric characterization of D-polyhedra to give a combinatorial description in terms of vertex-potentials of arc-parameterized digraphs. Moreover we provide a family of objects in the arc-space of an arc-parameterized digraph – called generalized bonds. They also correspond to the points of a distributive polyhedron. Hence they carry a distributive lattice structure.

In Section 4 we consider the special case of distributive polyhedra coming from ordinary digraphs (without arc-parameters). Indeed, this case arose from our considerations in [9] and inspired us to investigate distributive polyhedra in general. In the first subsection we prove that in this case even the *integral* generalized bonds carry a distributive lattice structure. In the second subsection we describe how integral generalized bonds correspond to the Δ-bonds of a directed graph. As was shown in [9], the distributive lattice on Δ-bonds generalizes distributive lattices on flows of planar digraphs [15], α-orientations of planar graphs [8], and c-orientations of graphs [22]. For more examples see the list in the beginning of Subsection 4.2. Our results imply that these objects correspond to the integral points of integral polyhedra.

In Section 5 the case of general arc-parameterized digraphs is considered. We give a combinatorial description of the generailzed bonds of a parameterized digraph. Our main theorem may be seen as a characterization of arc-space objects which carry a distributive lattice structure coming from a D-polyhedron.

Section 6 contains a new application of the theory. We prove a distributive lattice structure on a class of generalized flow of planar digraphs.

We conclude in Section 7 with final remarks and open problems.

2 Geometric Characterization

We want to find a geometric characterization of distributive polyhedra. As a first ingredient we need the basic

Observation 3 *The property of being a D-polyhedron is invariant under translation, scaling, and intersection.*

In order to give a description of D-polyhedra in terms of bounding halfspaces we will pursue the following strategy. We start by characterizing distributive affine subspaces of $\mathbb{R}^n$. Then we provide a characterization of the orthogonal complements of distributive affine spaces. Finally we show that D-polyhedra are exactly those polyhedra that have a representation as the intersection of distributive halfspaces.

2.1 Distributive Affine Space

For a vector $x \in \mathbb{R}^n$ we call $\mathrm{supp}(x) := \{i \in [n] \mid x_i \neq 0\}$ the **support** of x. Also let $x^+ := \max(\mathbf{0}, x)$ and $x^- := \min(\mathbf{0}, x)$. Call a set of vectors $B \subseteq \mathbb{R}^n$ **NND (non-negative disjoint)** if the elements of B are non-negative and have pairwise disjoint supports. Note that a NND set of non-zero vectors is linearly independent.

NND (non-negative disjoint)

Lemma 1 *Let $I \cup \{x\} \subset \mathbb{R}^n$ be linearly independent, then $I \cup \{x^+\}$ or $I \cup \{x^-\}$ is linearly independent.*

Proof. Suppose there are linear combinations $x^+ = \sum_{b \in I} \mu_b b$ and $x^- = \sum_{b \in I} \nu_b b$, then $x = \sum_{b \in I}(\mu_b + \nu_b)x_b$, which proves that $I \cup \{x\}$ is linearly dependent – contradiction. □

Proposition 1 *A linear subspace $A \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if it has a non-negative disjoint basis B.*

Proof. "$\Longleftarrow$": Let $x, y \in A$ and let $x = \sum_{b \in B} \mu_b b$ and $y = \sum_{b \in B} \nu_b b$ be their representations with respect to a NND basis B of A. Since the supports of vectors in B are disjoint $x_i < y_i$ is equivalent to $x_i = \mu_b b_i < \nu_b b_i = y_i$ for the unique $b \in B$ with $i \in \mathrm{supp}(b)$. Since every $b \in B$ is non-negative this is equivalent to $\mu_b < \nu_b$. This implies $\max(x_i, y_i) = \max(\mu_b, \nu_b) b_i$ and:

$$\max(x, y) = \max(\sum_{b \in B} \mu_b b, \sum_{b \in B} \nu_b b) = \sum_{b \in B} \max(\mu_b, \nu_b) b.$$

The analog holds for minima, hence $x, y \in A \Longrightarrow \max(x, y), \min(x, y) \in A$, i.e., A is distributive.

"$\Longrightarrow$": Let A be distributive and $I \subset A$ a NND set of support-minimal non-zero vectors. If I is not a basis of A, then there is $x \in A$ with:

(1) $I \cup \{x\}$ is linearly independent.

(2) $\exists_{i \in [n]} : x_i > 0$.

(3) $\mathrm{supp}(x)$ is minimal among the vectors with (1) and (2).

Claim: $I \cup \{x\}$ is NND.

If x is not non-negative, then x^+ and $-x^-$ are non-negative and have smaller support than x. By Lemma 1 one of $I \cup \{x^+\}$ and $I \cup \{-x^-\}$ is linearly independent – a contradiction to the support-minimality of x.

If there is $b \in I$ such that $\mathrm{supp}(x) \cap \mathrm{supp}(b) \neq \emptyset$, then choose a maximal $\mu \in \mathbb{R}$ such that for some coordinate j we have $x_j = \mu b_j$. We distinguish two cases.

If $\mathrm{supp}(x) \subseteq \mathrm{supp}(b)$, then $\emptyset \neq \mathrm{supp}(\mu b - x) \subsetneq \mathrm{supp}(b)$ contradicts the support-minimality in the choice of $b \in I$.

If $\mathrm{supp}(x) \not\subseteq \mathrm{supp}(b)$, then since $I \cup \{\mu b - x\}$ is linearly independent one of $I \cup \{(\mu b - x)^+\}$ and $I \cup \{(x - \mu b)^-\}$ is linearly independent by Lemma 1. This again contradicts support-minimality in the choice of x. □

Proposition 2 *An NND basis is unique up to scaling.*

Proof. Suppose $A \subseteq \mathbb{R}^n$ has NND bases B and B'. Suppose there are $b \in B$ and $b' \in B'$ such that $\emptyset \neq \mathrm{supp}(b) \cap \mathrm{supp}(b') \neq \mathrm{supp}(b'), \mathrm{supp}(b)$. By Proposition 1 we have $\min(b, b') \in A$ but $\mathrm{supp}(\min(b, b'))$ is strictly contained in the supports of b and b'. Since B and B' are NND the vector $\min(b, b')$ can neither be linearly combined by B nor by B'.

So the supports of vectors in B and B' induce the same partition of $[n]$. Since they are NND, the vectors $b \in B$ and $b' \in B'$ with $\mathrm{supp}(b) = \mathrm{supp}(b')$ must be scalar multiples of each other. □

Define a basis of an affine space as a basis of the linear space obtained by translating to the origin. Since the involved properties are invariant under translation, we have:

Proposition 3 *An affine subspace $A \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if it has a non-negative disjoint basis B. Moreover B is unique up to scaling.*

The next step is to define a class of network matrices of arc-parameterized digraphs such that an affine space A is distributive if and only if there is a network matrix N_Λ in the class such that $A = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p = c\}$.

An **arc-parameterized digraph** is a tuple $D_\Lambda = (V, A, \Lambda)$, where $D = (V, A)$ is a directed multi-graph, i.e., D may have loops, parallel, and anti-parallel arcs. We call D the **underlying digraph** of D_Λ. Moreover for convenience we set $V = [n]$ and $|A| = m$. Now Λ is a non-negative vector in $\mathbb{R}^m_{\geq \mathbf{0}}$ with the property that $\lambda_a = 0$ only if a is a loop. For emphasis we repeat: All arc-weights λ_a are non-negative.

arc-parameterized digraph

Given an arc parameterized digraph D_Λ we define its **generalized network-matrix** to be the matrix $N_\Lambda \in \mathbb{R}^{n \times m}$ with a column $e_j - \lambda_a e_i$ for every arc $a = (i, j)$ with parameter λ_a. Here e_k denotes the vector, which has a 1 in the kth entry and is 0 elsewhere.

generalized network-matrix

Proposition 4 *Let $A \subseteq \mathbb{R}^n$ be a non-empty affine subspace. Then A is distributive if and only if $A = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p = c\}$, where N_Λ is some generalized network-matrix. Moreover, N_Λ can be chosen such that the corresponding arc parameterized digraph D_Λ is a disjoint union of trees and loops.*

Proof. Since the properties involved are invariant under translation, we can assume A to be linear, hence $c = \mathbf{0}$.

"$\Longrightarrow$": By Proposition 3 the distributive A has a NND basis B. We construct an arc-parameterized digraph D_Λ, such that the columns of its generalized network-matrix N_Λ form a basis of the orthogonal complement of A.

For every $b \in B$ choose some arbitrary directed spanning tree on $\mathrm{supp}(b)$. For every $i \notin \bigcup_{b \in B} \mathrm{supp}(b)$ insert a loop $a = (i, i)$. To an arc $a = (i, j)$ with $i, j \in \mathrm{supp}(b)$ we associate the arc parameter $\lambda_a := b_j / b_i > 0$. For loops we set $\lambda_a := 0$. Collect the λ_a of all the arcs in a vector $\Lambda \in \mathbb{R}^m_{\geq \mathbf{0}}$. The resulting arc-parameterized digraph D_Λ is a disjoint union of loops and directed trees.

Denote by $\mathrm{col}(N_\Lambda)$ the set of columns of N_Λ. If $b \in B$ and $z_a \in \mathrm{col}(N_\Lambda)$, then either $\mathrm{supp}(b) \cap \mathrm{supp}(z_a) = \emptyset$ or $\langle b, z_a \rangle = b_j - \lambda_a b_i = b_j - (b_j / b_i) b_i = 0$ for $a = (i, j)$. Therefore,

$\mathrm{col}(N_\Lambda)$ is orthogonal to A. Since the underlying digraph of D_Λ consists of trees and loops only, $\mathrm{col}(N_\Lambda)$ is linearly independent. To conclude that $\mathrm{col}(N_\Lambda)$ generates $A^\perp$ in $\mathbb{R}^n$ we calculate

$$|B|+|\mathrm{col}(N_\lambda)| = |B|+\sum_{b\in B}(|\mathrm{supp}(b)|-1)+|[n]\setminus\bigcup_{b\in B}\mathrm{supp}(b)| = \sum_{b\in B}|\mathrm{supp}(b)|+n-|\bigcup_{b\in B}\mathrm{supp}(b)|.$$

Since the supports in B are mutually disjoint this equals n.

"$\Longleftarrow$": Let D_Λ be an arc parameterized digraph such that $N_\Lambda^\top p = \mathbf{0}$ has a solution. If $a = (i,j)$ is an arc, then $p_i - \lambda_a p_j = 0$, hence to know p it is enough to know p_i for one vertex i in each connected component of D_Λ. Therefore, the affine space of solutions is unaffected by deleting an edge from a cycle of D_Λ. This shows that there is no loss of generality in the assumption that the underlying digraph D of D_Λ is a disjoint union of trees and loops. Under this assumption we construct a NND basis of $\{p \in \mathbb{R}^n \mid N_\Lambda^\top p = \mathbf{0}\}$: For every tree-component T of D define a vector b with $\mathrm{supp}(b) = V(T)$ as follows: choose some $i \in V(T)$ and set $b_i := 1$, for any other $j \in V(T)$ consider the (i,j)-walk W in T. Define $b_j := \Pi_{a\in W^+}\lambda_a \Pi_{a\in W^-}\lambda_a^{-1}$ where W^+ and W^- are the sets of forward and backward arcs on W. Since arc-weights are non-negative this procedure yields an NND set B of non-zero vectors. Note that B is orthogonal to $\mathrm{col}(N_\Lambda)$ and that $\mathrm{col}(N_\Lambda)$ is a linearly independent set with as many vectors as there are arcs in $A(D)$. Denote by k the number of tree-components of D. To see that B is spanning, we calculate

$$|B| + |\mathrm{col}(N_\Lambda)| = k + |A(D)| = k + n - k = n.$$

Hence, $\mathrm{span}(B) = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p = \mathbf{0}\}$. □

2.2 Distributive Polyhedra

For a polyhedron P we define $F \subseteq P$ to be a **face** if there is $A = \{p \in \mathbb{R}^n \mid \langle z,p\rangle = c\}$ such that $F = P \cap A$ and P is contained in the **induced halfspace** $A^+ := \{p \in \mathbb{R}^n \mid \langle z,p\rangle \le c\}$.

Lemma 2 *Faces of D-polyhedra are D-polyhedra.*

Proof. Let P be a D-polyhedron such that $P \subseteq A^+ = \{p \in \mathbb{R}^n \mid \langle z,p\rangle \le c\}$ and let $F = P \cap A$ be a face. Suppose that there are $x, y \in F$ such that $\max(x,y) \notin F$. Since $\max(x,y) \in P$ this means $\langle z, \max(x,y)\rangle < c$. Since $2c = \langle z, x+y\rangle = \langle z, \max(x,y)\rangle + \langle z, \min(x,y)\rangle$ this implies $\min(x,y) \notin P$ – contradiction. □

Lemma 3 *The affine hull of a D-polyhedron is distributive.*

Proof. Let P be a D-polyhedron and $x, y \in \mathrm{aff}(P)$. Scale P to P' such that $x, y \in P' \subseteq \mathrm{aff}(P)$. Since scaling preserves distributivity $\min(x,y), \max(x,y) \in P' \subseteq \mathrm{aff}(P)$. □

Lemma 4 *Let $z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. If $A = \{p \in \mathbb{R}^n \mid \langle z,p\rangle = c\}$ is distributive, then the* **halfspace** *$A^+ = \{p \in \mathbb{R}^n \mid \langle z,p\rangle \le c\}$ is distributive as well.* *halfspace*

Proof. Suppose that $x, y \in A^+$ such that $\max(x,y) \notin A^+$. The line segments $[x, \max(x,y)]$ and $[y, \max(x,y)]$ contain points $x', y' \in A$ such that $\max(x',y') = \max(x,y)$. This contradicts the distributivity of A. □

Theorem 4 *A polyhedron $P \subseteq \mathbb{R}^n$ is a D-polyhedron if and only if*

$$P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \le c\}$$

for some generalized network-matrix N_Λ and $c \in \mathbb{R}^m$.

Proof. "$\Longrightarrow$": By Lemma 2 every face F of P is distributive. Lemma 3 ensures that $\mathrm{aff}(F)$ is distributive. Proposition 4 yields $\mathrm{aff}(F) = \{p \in \mathbb{R}^n \mid N(F)_{\Lambda(F)}^\top p = c(F)\}$ for a generalized network-matrix $N(F)_{\Lambda(F)}$. In particular this holds for $\mathrm{aff}(P)$. Now if F is a facet of P every row z of $N(F)_{\Lambda(F)}^\top$ is a generalized network-matrix as well. Choose a row z_F such that $A_F^+ := \{p \in \mathbb{R}^n \mid \langle z_F, p\rangle \le c_F\}$ is a facet-defining halfspace for F.

By the Representation Theorem for Polyhedra [29] we can write

$$P = \Big(\bigcap_{F \text{ facet}} A_F^+\Big) \cap \mathrm{aff}(P).$$

The above chain of arguments yields

$$P = \Big(\bigcap_{F \text{ facet}} \{p \in \mathbb{R}^n \mid \langle z_F, p\rangle \le c_F\}\Big) \cap \{p \in \mathbb{R}^n \mid N(P)_{\Lambda(P)}^\top p = c(P)\}.$$

Here the single matrices involved are generalized network-matrices. Glueing all these matrices horizontally together one obtains a single generalized network-matrix N_Λ and a vector c such that $P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \leqq c\}$. It remains to show, that we can replace defining equalities by inequalities, while preserving a network-matrix representation. We distinguish two cases.

(1) If $\lambda_a \neq 0$, then we have $\langle e_j - \lambda_a e_i, p\rangle = c_a \Leftrightarrow (\langle e_j - \lambda_a e_i, p\rangle \le c_a$ and $\langle e_i - \lambda_a^{-1} e_j, p\rangle \le -\lambda_a^{-1} c_a)$.

(2) If $\lambda_a = 0$, then since $a = (i, i)$ must be a loop of D_Λ we have $\langle e_i - 0e_i, p\rangle = c_a$, which can be replaced by $(\langle e_i - 0e_i, p\rangle \le c_a$ and $\langle e_i - 2e_i, p\rangle \le -c_a)$.

In each of the cases a single arc with an equality-constraint is replaced by a pair of oppositely oriented arcs. This shows that we can write P as $P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \le c\}$, for some generalized network-matrix N_Λ.

"$\Longleftarrow$": If $P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \le c\}$, then P is the intersection of bounded halfspaces, which are distributive by Lemma 4, because their defining hyperspaces are distributive by Proposition 4. Since intersection preserves distributivity, P is a D-polyhedron. □

Remark 5 *From the proof it follows that the system $N_\Lambda^\top p \leqq c$ with equality- and inequality-constraints defines a D-polyhedron whenever N_Λ is a generalized network-matrix.*

Remark 6 *Generalized network matrices are not the only matrices that can be used to represent D-polyhedra.*

To see this observe that scaling columns of N_λ and entries of c simultaneously preserves the polyhedron but may destroy the property of the matrix. There may, however, be representations of different type. Consider e.g., the D-polyhedron consisting of all scalar multiples of $(1, 1, 1, 1)$ in $\mathbb{R}^4$, it can be described by the six inequalities $\sum_{i \in A} x_i - \sum_{i \notin A} x_i \le 0$, for A a 2-subset of $\{1, 2, 3, 4\}$.

3 Towards a Combinatorial Model

We have shown that a D-polyhedron P is completely described by an arc-parameterized digraph D_Λ and an arc-capacity vector $c \in \mathbb{R}^m$. This characterization suggests to consider the points of P as 'graph objects'. A **potential** for D_Λ is a vector $p \in \mathbb{R}^n$, which assigns a real number p_i to each vertex i of D_Λ, such that the inequality $p_j - \lambda_a p_i \leq c_a$ holds for every arc $a = (i, j)$ of D_Λ. The points of the D-polyhedron $P(D_\Lambda)_{\leq c} = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \leq c\}$ are exactly the potentials of D_Λ. Theorem 4 then can be rewritten *potential*

Theorem 7 *A polyhedron is distributive if and only if it is the set of potentials of an arc-parameterized digraph D_Λ.*

Interestingly there is a second class of graph objects associated with the points of a D-polyhedron. While potentials are weights on vertices this second class consists of weights on the arcs of D_Λ. We define $\mathcal{B}(D_\Lambda)$ to be the space $\mathrm{Im}(N_\Lambda^\top)$. Given a D-polyhedron $P = \{p \in \mathbb{R}^n \mid N_\Lambda^\top p \leq c\}$ with capacity constraints we look at $\mathcal{B}(D_\Lambda)_{\leq c} := \{x \in \mathbb{R}^m \mid x \leq c \text{ and } x \in \mathrm{Im}(N_\Lambda^\top)\}$. Note that $p \in P$ if and only if $N_\Lambda^\top p \in \mathcal{B}(D_\Lambda)_{\leq c}$, i.e., $\mathcal{B}(D_\Lambda)_{\leq c} = N_\Lambda^\top P$.

In the spirit of the terminology of *generalized flow*, c.f. [1], we call the elements of $\mathcal{B}(D_\Lambda)$ the **generalized bonds** of D_Λ. Restricting to $\mathcal{B}(D_\Lambda)_{\leq c}$ ewe say generalized bonds within the capacity constraints c. *generalized bonds*

Theorem 8 *Let D_Λ be an arc-parameterized digraph with capacities $c \in \mathbb{R}^m$. The set $\mathcal{B}(D_\Lambda)_{\leq c}$ inherits the structure of a distributive lattice from a bijection $P' \to \mathcal{B}(D_\Lambda)_{\leq c}$, where P' is a D-polyhedron that can be obtained from $P = P(D_\Lambda)_{\leq c}$ by intersecting P with some hyperplanes of type $H_i = \{x \mid x_i = 0\}$.*

Proof. Since $\mathcal{B}(D_\Lambda)_{\leq c} = N_\Lambda^\top P$ for the D-polyhedron P of feasible vertex-potentials of D_Λ, and $N_\Lambda^\top$ is a linear map, the set of generalized bonds is a polyhedron.

If $N_\Lambda^\top$ is bijective on P, then the set $\mathcal{B}(D_\Lambda)_{\leq c}$ inherits the distributive lattice structure from P. This is not always the case. Later we show that we can always find a D-polyhedron $P' \subseteq P$ such that $N_\Lambda^\top$ is a bijection from P' to $\mathcal{B}(D_\Lambda)_{\leq c}$.

From Proposition 4 we know that $\mathrm{Ker}(N_\Lambda^\top)$ is a distributive space. By Proposition 3 there is a NND basis B of $\mathrm{Ker}(N_\Lambda^\top)$. For every $b \in B$ fix an arbitrary element $i(b) \in \mathrm{supp}(b)$. Denote the set of these elements by $I(B)$. Define $A := \mathrm{span}(\{e_i \in \mathbb{R}^n \mid i \in [n] \backslash I(B)\})$.

(1) A is distributive:
By definition A has a NND basis, i.e. is distributive by Proposition 3.

(2) $\mathcal{B}(D_\Lambda) = N_\Lambda^\top A$:
Let $N_\Lambda^\top p = x \in \mathcal{B}(D_\Lambda)$. Define $p' := p - \sum_{b \in B}(\frac{p_{i(b)}}{b_i} b)$. Since $\sum_{b \in B}(p_{i(b)} b) \in \mathrm{Ker}(N_\Lambda^\top)$ we have $N_\Lambda^\top p' = x$. Moreover $p'_i = 0$ for all $i \in I(B)$, i.e. $p' \in A$.

(3) $N_\Lambda^\top : A \hookrightarrow \mathcal{B}(D_\Lambda)$ is injective:
Suppose there are $p, p' \in A$ such that $N_\Lambda^\top p = N_\Lambda^\top p'$. Then $p - p' \in \mathrm{Ker}(N_\Lambda^\top) \cap A$. But by the definition of A this intersection is trivial, i.e., $p = p'$.

We have shown that $N_\Lambda^\top$ is an isomorphism from A to $\mathcal{B}(D_\Lambda)$ and that A is distributive. Thus $P' := P \cap A$ is a D-polyhedron such that the map of the matrix $N_\Lambda^\top$ is a bijection from P' to $\mathcal{B}(D_\Lambda)_{\leq c}$. □

The intersection of P with H_i can be modeled by adding a loop $a = (i, i)$ with capacity $c_a = 0$ to the digraph. Hence, with Remark 5 the preceding theorem says that for every

$\mathcal{B}(D_\Lambda)_{\leq c}$ we can add some loops to yield a graph $D'_{\Lambda'}$ and capacities c' such that

$$\mathcal{B}(D_\Lambda)_{\leq c} = \mathcal{B}(D'_{\Lambda'})_{\leq c'} \cong P(D'_{\Lambda'})_{\leq c'} = P'.$$

In the following we will always assume that generalized bonds $\mathcal{B}(D_\Lambda)_{\leq c}$ have the property that $\mathcal{B}(D_\Lambda)_{\leq c} \cong P(D_\Lambda)_{\leq c}$. In this case we call (D_Λ, c) **reduced**. *reduced*

Note that $\mathcal{B}(D_\Lambda)_{\leq c}$ can be far from being a D-polyhedron, but it inherits the distributive lattice structure via an isomorphism from a D-polyhedron.

In the following we investigate generalized bonds, i.e., the elements of $\mathcal{B}_\Lambda(D)$, as objects in their own right. Since $\mathcal{B}_\Lambda(D) = \mathrm{Im}(N_\Lambda^\top) = \mathrm{Ker}(N_\Lambda)^\perp$ we have $\langle x, f\rangle = 0$ for all $x \in \mathcal{B}_\Lambda(D)$ and $f \in \mathrm{Ker}(N_\Lambda)$. Understanding the elements of $\mathrm{Ker}(N_\Lambda)$ as objects in the arc space of D_Λ is vital to our analysis. In Section 4 we review the case of *ordinary* bonds, which leads to a description closely related to the definition of Δ-bonds in Subsection 4.2. This definition is based on the notion of *circular balance*. In Section 5 we then are able to describe the generalized bonds of D_Λ as capacity-respecting arc values, which satisfy a *generalized circular balance condition* around elements of $\mathrm{Ker}(N_\Lambda)$, see Theorem 14.

4 Special Parameters and Applications

In this section we present a special case of distributive polytopes with particularly nice properties. In fact, the results presented in the following two subsections gave rise to the idea of distributive polyhedra in general.

4.1 Bonds

Consider the case where D is a digraph and $\Lambda \in \{0,1\}^m$, i.e., $\lambda_a = 0$ if a is a loop and $\lambda_a = 1$ otherwise. In this case N_Λ is the network-matrix N of D, i.e., $N \in \mathbb{R}^{n\times m}$ consists of columns $e_j - e_i$ for every non-loop arc $a = (i,j)$ and e_i for a loop $a = (i,i)$.

The elements of $\mathrm{Ker}(N) =: \mathcal{F}(D)$ are the flows of D, i.e., those real arc values $f \in \mathbb{R}^m$ which respect flow-conservation at every vertex of D. Moreover, each support-minimal element of $\mathcal{F}(D)$ is a scalar multiple of the **signed incidence vector** $\overrightarrow{\chi}(C)$ of a cycle C of D, where $\overrightarrow{\chi}(C)_a$ is $+1$ if a is a forward arc of C, and -1 if a is a backward arc, and 0 otherwise. We denote by C^+ and C^- the sets of forward and backward arcs of C, repsectively. The set $\mathcal{B}(D)$ of generalized bonds of D consists of those $x \in \mathbb{R}^m$ with $\langle x, f\rangle = 0$ for all flows f. This is equivalent to $\langle x, \overrightarrow{\chi}(C)\rangle = 0$ for all $C \in \mathcal{C}$. In this particular case of $\Lambda \in \{0,1\}^m$ we refer to generalized bonds as **bonds**. *signed incidence vector* *bonds*

Theorem 8 yields a distributive lattice structure on the set of bonds $\mathcal{B}_{\leq c}(D)$ by identifying it with a distributive polytope $P(D)$.

The particular form of Λ allows us to show a distributive lattice structure on the *integral* bonds $\mathcal{B}_{\leq c}(D) \cap \mathbb{Z}^m$ of D. To this end we first make the following

Observation 9 *The intersection of a D-polytope $P \subseteq \mathbb{R}^n$ and any other (particularly finite) distributive sublattice L of $\mathbb{R}^n$ yields a distributive lattice $P \cap L$.*

So if $P \subseteq \mathbb{R}^n$ is a D-polyhedron then $P \cap \mathbb{Z}^n$ is a distributive lattice. Since by Theorem 8 we can assume $N^\top$ to be bijective on P we obtain a distributive lattice structure on $N^\top(P \cap \mathbb{Z}^n)$.

However, we want a distributive lattice on integral bonds, i.e., on $\mathcal{B}(D)_{\leq c} \cap \mathbb{Z}^m$. Luckily N is a totally unimodular matrix, which yields $\mathcal{B}(D)_{\leq c} \cap \mathbb{Z}^m = N^\top(P \cap \mathbb{Z}^n)$, see [24].

We have proven

Theorem 10 *The set of integral bonds $\mathcal{B}(D)_{\leq c} \cap \mathbb{Z}^m$ carries a distributive lattice structure.*

In the next subsection we will use this theorem to prove a distributive lattice structure on a large number of combinatorial sets.

4.2 Applications

Many researchers have constructed distributive lattices on sets of combinatorial objects, e.g.,

- domino and lozenge tilings of plane regions (Rémila [23] and others based on Thurston [27])
- planar spanning trees (Gilmer and Litherland [11])
- planar bipartite perfect matchings (Lam and Zhang [16])
- planar bipartite d-factors (Felsner [8], Propp [22])
- Schnyder woods of planar 3-connected graphs (Brehm [4])
- Eulerian orientations of planar graphs (Felsner [8])
- α-orientations of planar graphs (Felsner [8], de Mendez [7])
- circular integer flows in planar graphs (Khuller, Naor and Klein [15])
- higher dimensional rhombic tilings (Linde, Moore, and Nordahl [18])
- c-orientations of graphs (Propp [22])

All these objects can easily be modeled as special instances of *integral Δ-bonds* of a directed graph, see [9, 8]. In [9] we proved that the set of integral Δ-bond forms a distributive lattice. In this subsection we will recover this result as an application of Theorem 10. Moreover we obtain a polytopal structure on these objects.

The set $\mathcal{B}_\Delta(D, c_\ell, c_u)$ of **integral Δ-bonds** is defined by a directed multi-graph $D = (V, A)$ with upper and lower integral arc-capacities $c_u, c_\ell : A \to \mathbb{Z}$ and a number Δ_C for each cycle $C \in \mathcal{C}$. For a map $x : A \to \mathbb{Z}$ and $C \in \mathcal{C}$ denote by *integral Δ-bonds*

$$\delta(C, x) := \sum_{a \in C^+} x(a) - \sum_{a \in C^-} x(a)$$

the **circular balance*** of x around C. Note that fixing Δ on a basis of the cycle space of D suffices to determine it on $\mathcal{C}$. A map $x : A \to \mathbb{Z}$ is in $\mathcal{B}_\Delta(D, c_\ell, c_u)$ if *circular balance*

(B$_1$) $c_\ell(a) \leq x(a) \leq c_u(a)$ for all $a \in A$. (capacity constraints)

(B$_2$) $\Delta_C = \delta(C, x)$ for all $C \in \mathcal{C}$. (circular Δ-balance conditions)

The crucial observation is that if we allow a change of arc-capacities we can assume $\Delta = \mathbf{0}$.

Lemma 5 *For every D, c_ℓ, c_u, Δ there are c'_ℓ, c'_u such that $\mathcal{B}_\Delta(D, c_\ell, c_u) \cong \mathcal{B}_\mathbf{0}(D, c'_\ell, c'_u)$*

Proof. Fix a spanning tree T of D. Let $f : \mathbb{Z}^A \to \mathbb{Z}^A$ be defined as follows: $f(z)_a := z_a$ if a is an arc of T and $f(z)_a := z_a - \Delta_{C(T,a)}$ otherwise. Here $C(T, a)$ denotes the fundamental cycle of T induced by a with the cyclic orientation that makes a a forward arc.

Applying the translation f to Δ-bonds and capacity constraints yields a bijection

$$f : \mathcal{B}_\Delta(D, c_\ell, c_u) \to \mathcal{B}_\mathbf{0}(D, f(c_\ell), f(c_u)).$$

□

Now we are ready to state a main result of [9] as a corollary of Theorem 10.

*In previous work on the topic [22, 8] this term was sometimes called the *circular flow difference*. Since bonds are not flows but orthogonal to flows that name may cause confusion.

Corollary 1 *The set of integral Δ-bonds of a digraph D within capacities c_ℓ, c_u carries the structure of a distributive lattice.*

Proof. First by Lemma 5 we can look at an isomorphic set $\mathcal{B}_{\mathbf{0}}(D, c'_\ell, c'_u)$ of integral $\mathbf{0}$-bonds. Now note that $\delta(C, x) = \langle x, \overrightarrow{\chi}(C)\rangle$, hence

$$\mathcal{B}_{\mathbf{0}}(D, c'_\ell, c'_u) = \mathcal{B}(D)_{\leq c'_u} \cap \mathcal{B}(D)_{\geq c'_\ell} \cap \mathbb{Z}^m.$$

The latter carries a distributive lattice structure by Theorem 10. $\square$

The result that the set $\mathcal{B}_\Delta(D, c_\ell, c_u)$ is the set of integral points of a polytope was not obtained in [9].

Remark 11 *Distributive polytopes with $\Lambda, c \in \{0,1\}^m$ are exactly order polytopes [26]. The distributive lattice on the integral bonds is isomorphic to the lattice of ideals of the defining poset. With parameters $\Lambda \in \{0,1\}^m$ and $c \in \mathbb{Z}^m$ one obtains the more general class of alcoved polytopes, which has proven to model a variety of combinatorial objects [17]. It has been shown in [14] that this class coincides with polytropes, which are important in tropical convexity.*

Theorem 4 tells us that alcoved polytopes are distributive. Moreover Theorem 10 characterizes the integer point sets of alcoved polytopes, i.e., P is an alcoved polytope iff $N^T P \cap \mathbb{Z}^m$ corresponds to the integral bonds of directed graph.

In the next section we characterize those real-valued subsets of the arc space of parameterized digraphs, which can be proven to carry a distributive lattice structure by the above method as *generalized* Δ-bonds. The generalization of Corollary 1 to generalized Δ-bonds is stated in Theorem 15.

5 General Parameters

Let us now look at the case of general bonds of an arc-parameterized digraph D_Λ. The aim of this section is to describe $\mathcal{B}(D_\Lambda)_{\leq c}$ as the orthogonal complement of $\mathrm{Ker}(N_\Lambda)$ within the capacity bounds given by c. For $f \in \mathbb{R}^m$ and $j \in V$ we define the **excess** of f at j as *excess*

$$\omega(j, f) := \Big(\sum_{a=(i,j)} f_a\Big) - \Big(\sum_{a=(j,k)} \lambda_a f_a\Big).$$

Since $f \in \mathrm{Ker}(N_\Lambda)$ means $\omega(v, f) = 0$ for all $v \in V$ we think of f as an edge-valuation satisfying a *generalized flow-conservation.* We call the elements of $\mathrm{Ker}(N_\Lambda)$ the **generalized flows** of D_Λ. *generalized flows*

Generalized flows were introduced by Dantzig [5] in the sixties and there has been much interest in related algorithmic problems. For surveys on the work, see [1, 28]. The most efficient algorithms known today have been proposed in [10].

We will denote $\mathcal{F}(D_\Lambda) := \mathrm{Ker}(N_\Lambda)$ and call it the **generalized flow space**. Let $\mathcal{C}(D_\Lambda)$ be the set of support-minimal vectors of $\mathcal{F}(D_\Lambda)\backslash\{\mathbf{0}\}$, i.e., $f \in \mathcal{C}(D_\Lambda)$ if and only if $\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$ implies $\mathrm{supp}(g) = \mathrm{supp}(f)$ for all $g \in \mathcal{F}(D_\Lambda)\backslash\{\mathbf{0}\}$. Elements of $\mathcal{C}(D_\Lambda)$ will be called **generalized cycles**. Since the support-minimal vectors $\mathcal{C}(D_\Lambda)$ span the entire space $\mathcal{F}(D_\Lambda)$; the generalized bonds of D_Λ are already determined by being orthogonal to $\mathcal{C}(D_\Lambda)$, i.e., to all generalized cycles. *generalized cycles*

In the following we answer the question what generalized cycles look like as subgraphs of D_Λ. After some definitions and technical lemmas we give a combinatorial characterization of generalized cycles, see Theorem 13.

For a loop-free oriented arc set S of D_Λ define its **multiplier** as *multiplier*

$$\lambda(S) := \prod_{a\in S} \lambda_a^{\overrightarrow{\chi}(S)_a},$$

where $\overrightarrow{\chi}(S)_a = \pm 1$ depending on the orientation of a in S.

A cycle C in the underlying graph with a cyclic orientation will be called **lossy** if $\lambda(C) < 1$, *lossy* and **gainy** if $\lambda(C) > 1$, and **breakeven** if $\lambda(C) = 1$. A **bicycle** is an oriented arc set that *gainy* can be written as $C \cup W \cup C'$ with a gainy cycle C, a lossy cycle C' and a (possibly trivial) *breakeven* oriented path W from C to C'; moreover, the intersection of C and C' is a (possibly empty) *bicycle* interval of both and W is minimal as to make the bicycle connected. In addition we require that C and C' are equally oriented on common arcs. See Fig. 2 for two generic examples.

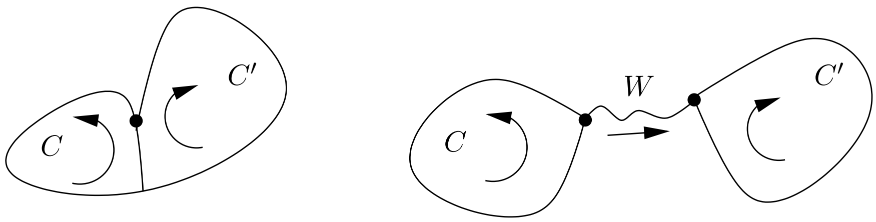

Figure 2: Bicycles with $W = \emptyset$ and $W \neq \emptyset$.

Lemma 6 *A bicycle does not contain a breakeven cycle.*

Proof. The cycles C and C' of a bicycle $H = C \cup W \cup C'$ are not breakeven. If H contains an additional cycle $\widetilde{C}$, then the support of $\widetilde{C}$ must equal the symmetric difference of supports of C and C'. Let $x := \lambda(C \backslash C')$, $y := \lambda(C \cap C')$, and $z := \lambda(C' \backslash C)$, where orientations are taken according to C and C', respectively. We have $xy = \lambda(C) > 1 > \lambda(C') = zy$. Hence $\lambda(\widetilde{C}) = (zx^{-1})^{\pm 1}$, but $zx^{-1} = zy(xy)^{-1} < 1$. That is, $\widetilde{C}$ cannot be breakeven. $\square$

We call the set of bicycles and breakeven cycles of D_Λ the **combinatorial support** for the set $\mathcal{C}(D_\Lambda)$ of generalized cycles and denote it by $\underline{\mathcal{C}}(D_\Lambda)$. Recall that for $x \in \mathbb{R}^m$ the support was defined as $\mathrm{supp}(x) := \{i \in [m] \mid x_i \neq 0\}$. We define the **signed support** *signed* $\mathrm{sign}(x)$ of x as the partition (X^+, X^-) of $\mathrm{supp}(x)$, where $X^+ := \{i \in \mathrm{supp}(x) \mid x_i > 0\}$ and *support* $X^- := \{i \in \mathrm{supp}(x) \mid x_i < 0\}$. For $i \in \mathrm{supp}(x)$ we write $i = \pm 1$ if $i \in X^\pm$, respectively.

Note that $\mathrm{sign}(\mathcal{C}(D_\Lambda))$ is exactly the set of signed circuits of the *oriented matroid* induced by the matrix N_Λ, see [3]. We justify the name *combinatorial support* by proving $\underline{\mathcal{C}}(D_\Lambda) = \mathrm{sign}(\mathcal{C}(D_\Lambda))$ in Theorem 13. It turns out that oriented matroids arising as the combinatorial support of an arc-parameterized digraph are oriented versions of a combination of a classical *cycle matroid* and a *bicircular matroid*. The latter were introduced in the seventies [19, 25]. Active research in the field can be found in [12, 13, 20]. We feel that oriented matroids of generalized network matrices are worth further investigation.

Let $W = (a(0), \ldots, a(k))$ be a walk in D, i.e., W may repeat vertices and arcs. We abuse notation and identify W with its **signed support** $\mathrm{sign}(W)$, which is defined as the signed *signed* support of the signed incidence vector of W, i.e, $\mathrm{sign}(W) := \mathrm{sign}(\overrightarrow{\chi}(W))$. Even more, we *support* write W_i and $W_{a(i)}$ for the same sign, namely the orientation of the arc $a(i)$ in W. Note that cycles and bicycles can be regarded to be walks; these will turn out to be the most interesting cases in our context.

A vector $f \subseteq \mathbb{R}^m$ is an **inner flow** of W if $\text{sign}(f) = \pm\text{sign}(W)$ and f satisfies the generalized flow conservation law between consecutive arcs of W. *inner flow*

Lemma 7 *Let $W = (a(0), \ldots, a(k))$ be a walk in D_Λ and f an inner flow of W. Then*

$$f_{a(k)} = K\lambda(W)^{-1} f_{a(0)}$$

where the 'correction term' K is given by $K = W_0 W_k \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(k)}^{\min(0,W_k)}$. In particular the space of inner flows of W is one-dimensional.

Proof. We proceed by induction on k. If $k = 0$, then

$$\begin{aligned} & W_0 W_0 \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(0)}^{\min(0,W_0)} \lambda(W)^{-1} f_{a(0)} \\ = \;& \lambda_{a(0)}^{W_0} \lambda(W)^{-1} f_{a(0)} \\ = \;& \lambda_{a(0)}^{W_0} \lambda_{a(0)}^{-W_0} f_{a(0)} \\ = \;& f_{a(0)}. \end{aligned}$$

If $k = 1$, then our walk consisting of two arcs has a middle vertex, say i. Since f is an inner flow $\omega(i, f) = 0$. This can be rewritten as $W_0 \lambda_{a(0)}^{-\min(0,W_0)} f_{a(0)} = W_1 \lambda_{a(1)}^{\max(0,W_1)} f_{a(1)}$. Now we can transform

$$\begin{aligned} & W_0 W_1 \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(1)}^{\min(0,W_1)} \lambda(W)^{-1} f_{a(0)} \\ = \;& W_0 \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(0)}^{-W_0} f_{a(0)} W_1 \lambda_{a(1)}^{\min(0,W_1)} \lambda_{a(1)}^{-W_1} \\ = \;& W_1 \lambda_{a(1)}^{\max(0,W_1)} f_{a(1)} W_1 \lambda_{a(1)}^{\min(0,W_1)} \lambda_{a(1)}^{-W_1} \\ = \;& f_{a(1)}. \end{aligned}$$

If $k > 1$, then we can look at two overlapping walks $W' = (a(0), \ldots, a(\ell))$ and $W'' = (a(\ell), \ldots, a(k))$. Clearly f restricted to W' and W'' respectively satisfies the preconditions for the induction hypothesis. By applying the induction hypothesis to W'' and W' we obtain

$$\begin{aligned} f_{a(k)} &= W_\ell W_k \lambda_{a(\ell)}^{\max(0,W_\ell)} \lambda_{a(k)}^{\min(0,W_k)} \lambda(W'')^{-1} f_{a(\ell)} \quad \text{and} \\ f_{a(\ell)} &= W_0 W_\ell \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(\ell)}^{\min(0,W_\ell)} \lambda(W')^{-1} f_{a(\ell)}. \end{aligned}$$

Substitute the second formula into the first and observe that $W_\ell W_\ell = 1$, and that from the product of four terms $\lambda_{a(\ell)}$ with different exponents the single $\lambda_{a(\ell)}^{-W_\ell}$ needed for $\lambda(W)^{-1}$ remains. This proves the claimed formula for $f_{a(k)}$. $\square$

Lemma 8 *Let $W = (a(0), \ldots, a(k))$ be a simple path from v to v' in D_Λ. If f is an inner flow of W with $\text{sign}(f) = \text{sign}(W)$, then $\omega(v, f) < 0$ and $\omega(v', f) > 0$.*

Proof. By definition $\omega(v,f) = -W_0\lambda_{a(0)}^{\max(0,W_0)} f_{a(0)}$. Since $\lambda_{a(0)} > 0$ and $\mathrm{sign}(f_{a(0)}) = W_0$ we conclude $\omega(v,f) < 0$. For the second inequality we use Lemma 7:

$$\begin{aligned}
\omega(v',f) &= W_k\lambda_{a(k)}^{-\min(0,W_k)} f_{a(k)} \\
&= W_k\lambda_{a(k)}^{-\min(0,W_k)} W_0 W_k \lambda_{a(0)}^{\max(0,W_0)} \lambda_{a(k)}^{\min(0,W_k)} \lambda(W)^{-1} f_{a(0)} \\
&= W_0\lambda_{a(0)}^{\max(0,W_0)} \lambda(W)^{-1} f_{a(0)}.
\end{aligned}$$

Since $\lambda_{a(0)}, \lambda(W)^{-1} > 0$ and $\mathrm{sign}(f_{a(0)}) = \mathrm{sign}(W_{a(0)}) = W_0$ we conclude $\omega(v',f) > 0$. □

Lemma 9 *Let $C = (a(0),\dots,a(k))$ be a cycle in D_Λ and f an inner flow of C with* $\mathrm{sign}(f) = \mathrm{sign}(C)$. *Then the excess $\omega(v,f)$ at the initial vertex v satisfies* $\mathrm{sign}(\omega(v,f)) = \mathrm{sign}(1-\lambda(C))$.

Proof. Reusing the computations from Lemma 7 we obtain

$$\begin{aligned}
\omega(v,f) &= C_k\lambda_{a(k)}^{-\min(0,C_k)} f_{a(k)} - C_0\lambda_{a(0)}^{\max(0,C_0)} f_{a(0)} \\
&= C_0\lambda_{a(0)}^{\max(0,C_0)} \lambda(C)^{-1} f_{a(0)} - C_0\lambda_{a(0)}^{\max(0,C_0)} f_{a(0)} \\
&= C_0\lambda_{a(0)}^{\max(0,C_0)} f_{a(0)} (\lambda(C)^{-1} - 1).
\end{aligned}$$

Since $\lambda_{a(0)} > 0$ and $\mathrm{sign}(f_{a(0)}) = C_0$ we conclude $\mathrm{sign}(\omega(v,f)) = \mathrm{sign}(\lambda(C)^{-1} - 1)$. Finally observe that $\mathrm{sign}(\lambda(C)^{-1} - 1) = \mathrm{sign}(1 - \lambda(C))$. □

Theorem 12 *Given a bicycle or breakeven cycle H of D_Λ, the set of flows f with* $\mathrm{sign}(f) = \pm\mathrm{sign}(H)$ *is a 1-dimensional subspace of $\mathcal{F}(D_\Lambda)$.*

Proof. Given $H \in \underline{\mathcal{C}}(D_\Lambda)$ we want to characterize those $f \in \mathcal{F}(D_\Lambda)$ with $\mathrm{sign}(f) = \pm\mathrm{sign}(H)$. Lemma 7 implies that the dimension of the inner flows of H is at most one. Hence, it is enough to identify a single nontrivial flow on H.

If $H = C \in \underline{\mathcal{C}}(D_\Lambda)$ is a breakeven cycle, which traverses the arcs $(a(0),\dots,a(k))$ starting and ending at vertex v, then by Lemma 9 we have $\mathrm{sign}(\omega(v,f)) = \mathrm{sign}(1-\lambda(C))$. Since C is breakeven $\lambda(C) = 1$, this implies generalized flow-conservation in v. Since by definition generalized flow-conservation holds for all other vertices we may conclude that f is a generalized flow, i.e., a nontrivial flow on H.

Let $H \in \underline{\mathcal{C}}(D_\Lambda)$ be a bicycle which traverses the arcs $(a(0),\dots,a(k))$ such that $C = (a(0),\dots,a(i)), W = (a(i+1),\dots,a(j-1))$ and $C' = (a(j),\dots,a(k))$. Let v and v' be the common vertices of C and W and W and C', respectively.

Consider the case where W is non-trivial. We construct $f \in \mathcal{F}(D_\Lambda)$ with $\mathrm{sign}(f) = \mathrm{sign}(H)$. First take any inner flow f_C of C with $\mathrm{sign}(f_C) = \mathrm{sign}(C)$. Since C is gainy Lemma 9 implies a positive excess at v. Let f_W be an inner flow of W with $\mathrm{sign}(f_W) = \mathrm{sign}(W)$. Lemma 8 ensures $\omega(v,f_W) < 0$. By scaling f_W with a positive scalar we can achieve $\omega(v, f_C + f_W) = 0$. From Lemma 8 we know that $f_C + f_W$ has positive excess at v'. Since C' is lossy any inner flow $f_{C'}$ of C' has negative excess at v' (Lemma 9). Hence we can scale $f_{C'}$ to achieve $\omega(v', f_{C'} + f_W) = 0$. Together we have obtained a generalized flow $f := f_C + f_W + f_{C'}$, i.e., a nontrivial flow on H.

If W is empty, then v and v' coincide. As in the above construction we can scale flows on C and C' such that $\omega(v,f) = 0$ holds for $f := f_C + f_{C'}$, i.e., f is a generalized flow. If C

and C' share an interval, then the sign vectors of C and C' coincide on this interval. From $\text{sign}(f_C) = \text{sign}(C)$ and $\text{sign}(f'_C) = \text{sign}(C')$ it follows that $\text{sign}(f) = \text{sign}(C \cup C') = \text{sign}(H)$. Hence f is a flow on H. $\square$

Theorem 13 *For an arc parameterized digraph D_Λ the set of supports of generalized cycles, i.e., of support-minimal flows, coincides with the set of bicycles and breakeven cycles. Stated more formally:* $\text{sign}(\mathcal{C}(D_\Lambda)) = \underline{\mathcal{C}}(D_\Lambda)$.

Proof. By Theorem 12 every $H \in \underline{\mathcal{C}}(D_\Lambda)$ admits a generalized flow f. To see support-minimality of f, assume that $H \in \underline{\mathcal{C}}(D_\Lambda)$ has a strict subset S which is support-minimal admitting a generalized flow. Clearly S cannot have vertices of degree 1 to admit a flow and must be connected to be support-minimal. Since $S \subset H \in \underline{\mathcal{C}}(D_\Lambda)$ this implies that is a cycle. Lemma 9 ensures that S must be a breakeven cycle. If H was a breakeven cycle itself, then it cannot strictly contain S. Otherwise if $H = C \cup W \cup C'$ is a bicycle then by Lemma 6 it contains no breakeven cycle.

For the converse consider any $S \in \text{sign}(\mathcal{C}(D_\Lambda))$, i.e., the signed support of some flow f. We claim that $\underline{S} := \text{supp}(f)$ contains a breakeven cycle or a bicycle. If it contains a breakeven cycle, then we are done. So we assume that it does not. Under this assumption it follows that there are two cycles C_1, C_2 in a connected component of $\underline{S}$. If C_1 and C_2 intersect in at most one vertex, then we can choose the orientations for these cycles such that $\lambda(C_1) > 1$ and $\lambda(C_2) < 1$. If $C_1 \cap C_2 = \emptyset$, then let W be an oriented path from C_1 to C_2. Now $C_1 \cup W \cup C_2$ is a bicycle contained in $\underline{S}$. The final case is that C_1 and C_2 share several vertices. Let B be a bow of C_2 over C_1, i.e, a consecutive piece of C_2 that intersects C_1 in its two endpoints v and w only. The union of C_1 and B is a theta-graph, i.e, it consists of three disjoint path B_1, B_2, B_3 joining v and w, see Fig. 3. Let the path B_i be oriented as shown in the figure and let $C = B_1 \cup B_2$ and $C' = B_2 \cup B_3$. If $C \cup C'$ is not a bicycle, then the cycles are either both gainy or both lossy. Assume that they are both gainy, i.e., $\lambda(C) > 1$ and $\lambda(C') > 1$. Consider the cycles $E = B_1 \cup B_3^{-1}$ and $E' = B_1^{-1} \cup B_3$, since $\lambda(E) = \lambda(B_1)\lambda(B_3)^{-1} = \lambda(E')^{-1}$ it follows that either E or E' is a lossy cycle. The orientation of E is consistent with C and the orientation of E' is consistent with C'. Hence either $C \cup E$ or $C' \cup E'$ is a bicycle contained in $\underline{S}$. This contradicts the support-minimality of f. $\square$

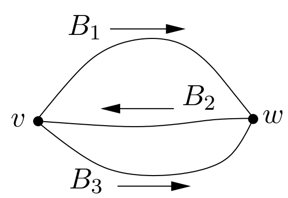

Figure 3: A theta graph and an orientation of the three path.

For $H \in \underline{\mathcal{C}}(D_\Lambda)$ we define $f(H)$ as the unique $f \in \mathcal{C}(D)$ with $\text{sign}(f(H)) = \text{sign}(H)$ and $\|f(H)\| = 1$. Let $x \in \mathbb{R}^m$ and $H \in \underline{\mathcal{C}}(D_\Lambda)$. Denote by $\delta(H, x) := \langle x, f(H) \rangle$ the **bicircular balance** of x on H.

bicircular balance

Theorem 14 *Let D_Λ be an arc-parameterized digraph and $x, c \in \mathbb{R}^m$. Then $x \in \mathcal{B}(D_\Lambda)_{\leq c}$ if and only if*

(1) $\quad x_a \leq c_a$ *for all $a \in A$.* (capacity constraints)

(2) $\delta(H,x)=0$ for all $H\in\underline{\mathcal{C}}(D_\Lambda)$. (bicircular balance conditions)

The theorem helps explain the name *generalized bonds*: usually a cocycle or bond is a set B of edges such that for every cycle C the incidence vectors are orthogonal, i.e., $\langle x_B, x_C\rangle = 0$. In our context the role of cycles is played by generalized cycles, i.e., by generalized flows f with $\operatorname{sign}(f)=\operatorname{sign}(H)$ for some $H\in\underline{\mathcal{C}}(D_\Lambda)$.

To make the statement of the theorem more general let D_Λ be an arc-parameterized digraph with arc capacities $c\in\mathbb{R}^m$ and a number Δ_H for each $H\in\underline{\mathcal{C}}(D_\Lambda)$. A map $x : A\to\mathbb{R}$ is called a **generalized Δ-bond** if

generalized Δ-bond

(B_1) $x(a)\le c_u(a)$ for all $a\in A$. (capacity constraints)

(B_2) $\delta(H,x)=\Delta_H$ for all $H\in\underline{\mathcal{C}}(D_\Lambda)$. (bicircular Δ-balance conditions)

Denote by $\mathcal{B}_\Delta(D_\Lambda)_{\le c}$ the set of generalized Δ-bonds of D_Λ. An argument as in the proof of Lemma 5 yields the real-valued generalization of Theorem 10 as a corollary of Theorem 8.

Theorem 15 *Let D_Λ be an arc-parameterized digraph with capacities $c\in\mathbb{R}^m$ and $\Delta\in\mathbb{R}^{\underline{\mathcal{C}}(D_\Lambda)}$. The set $\mathcal{B}_\Delta(D_\Lambda)_{\le c}$ of generalized Δ-bonds carries the structure of a distributive lattice and forms a polyhedron.*

6 Planar Generalized Flow

The **planar dual** D^* of a non-crossing embedding of a planar digraph D in the sphere is an orientation of the planar dual G^* of the underlying graph G of D: Orient an edge $\{v,w\}$ of G^* from v to w if it appears as a backward arc in the clockwise facial cycle of D dual to v. Call an arc-parameterized digraph D_Λ **breakeven** if all its cycles are breakeven.

planar dual

breakeven

Theorem 16 *Let D_Λ be a planar breakeven digraph. There is an arc parameterization Λ^* of the dual D^* of D such that $\mathcal{F}(D_\Lambda)\cong\mathcal{B}(D^*_{\Lambda^*})$. More precisely, there is a vector $\sigma\in\mathbb{R}^m$ with positive components such that f is a generalized flow of D_Λ if and only if $x=S(\sigma)f$ is a generalized bond of $D^*_{\Lambda^*}$, where $S(\sigma)$ denotes the diagonal matrix with entries from σ.*

Proof. Let $C_1\ldots C_{n^*}$ be the list of clockwise oriented facial cycles of D. For each C_i let f_i be a generalized flow with $\operatorname{sign}(f_i)=\operatorname{sign}(C_i)$; since C_i is breakeven such an f_i exists by Lemma 9. Collect the flows f_i as rows of a matrix M. Columns of M correspond to edges of D and due to our selection of cycles each column contains exactly two nonzero entries. The orientation of the facial cycles and the sign condition implies that each column has a positive and a negative entry. For the column of arc a let $\mu_a>0$ and $\nu_a<0$ be the positive and negative entry. Define $\sigma_a := \mu_a^{-1}>0$ and note that scaling the column of a with σ_a yields entries 1 and $-\lambda^*_a=\nu_a\mu_a^{-1}<0$ in this column. Therefore, $N_{\Lambda^*}:=MS(\sigma)$ is a generalized network matrix. The construction implies that the underlying digraph of N_{Λ^*} is just the dual D^* of D.

Let $f\in\mathcal{F}(D_\Lambda)$ be a flow. Then f can be expressed as linear combination of generalized cycles. Since D_Λ is breakeven we know that the support of every generalized cycle is a simple cycle. The facial cycles generate the cycle space of D. Moreover, if C is a simple cycle and f_C is a flow with $\operatorname{sign}(f_C)=\operatorname{sign}(C)$, then f_C can be expressed as a linear combination of the flows f_i, $i=1,\ldots,n^*$. This implies that the rows of M are spanning for $\mathcal{F}(D_\Lambda)$, i.e., for every f there is a $q\in\mathbb{R}^{n^*}$ such that $f=M^\top q$. In other words $\mathcal{F}(D_\Lambda)=M^\top\mathbb{R}^{n^*}$.

A vector x is a bond for N_{Λ^*} if and only if x is in the row space of N_{Λ^*}, i.e., there is a potential $p \in \mathbb{R}^{n^*}$ with $x = N_{\Lambda^*}^\top p$. In other words $\mathcal{B}(D^*_{\Lambda^*}) = N_{\Lambda^*}^\top \mathbb{R}^{n^*} = (MS(\sigma))^\top \mathbb{R}^{n^*} = S(\sigma) M^\top \mathbb{R}^{n^*} = S(\sigma)\mathcal{F}(D_\Lambda)$. □

Corollary 2 *Let D_Λ be a planar breakeven digraph and $c \in \mathbb{R}^m$. The set $\mathcal{F}(D_\Lambda)_{\le c}$ carries the structure of a distributive lattice.*

Proof. The matrix $S(\sigma)$ is an isomorphism between $\mathcal{F}(D_\Lambda)$ and $\mathcal{B}(D^*_{\Lambda^*})$. Since σ is positive we obtain $\mathcal{F}(D_\Lambda)_{\le c} = S(\sigma)\big(\mathcal{B}(D^*_{\Lambda^*})_{\le S(\sigma)c}\big)$. Theorem 15 implies a distributive lattice structure on $\mathcal{B}(D^*_{\Lambda^*})_{\le S(\sigma)c}$ which can be pushed to $\mathcal{F}(D_\Lambda)$. □

In fact Theorem 15 even allows us to obtain a distributive lattice structure for planar generalized flows of breakeven digraphs with an arbitrarily prescribed excess at every vertex.

The reader may worry about the existence of non-trivial arc-parameterizations Λ of a digraph D such that D_Λ is breakeven. Here is a nice construction for such parameterizations. Let D be arbitrary and $x \in \mathbb{R}^m$ be a $\mathbf{0}$-bond of D, i.e., $\delta(C, x) := \sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a = 0$ for all oriented cycles C. Consider $\lambda = \exp(x)$ and note that $\lambda_a \ge 0$ for all arcs a and that $\lambda(C) = \big(\prod_{a \in C^+} \lambda_a\big)\big(\prod_{a \in C^-} \lambda_a\big)^{-1} = \exp(\delta(C, x)) = 1$ for all oriented cycles C. Hence weighting the arcs of D with λ yields a breakeven arc-parameterization of D. This construction is universal in the sense that application of the logarithm to a breakeven parameterization yields a $\mathbf{0}$-bond.

7 Conclusions and Open Questions

Old and New: In the present paper we have obtained a distributive lattice representation for the set of real-valued generalized Δ-bonds of an arc parameterized digraph. The proof is based on the bijection with potentials which allows us to push the obvious lattice structure based on componentwise max and min from potentials to generalized bonds. Consequently we obtain a distributive lattice on generalized bonds in terms of its join and meet. In [9] we obtained the distributive lattice structure on integral Δ-bonds, by showing that we can build the cover-graph of a distributive lattice by local vertex-push-operations and reach every Δ-bond this way. This qualitatively different distributive lattice representation was possible because we could assume the digraph to be *reduced* in a certain way.

Problem. Is there a way to reduce an arc-parameterized digraph such that the distributive lattice on its generalized bonds can be constructed locally by *pushing* vertices?

Order Theory: There is a natural finite distributive lattice associated to a D-polyhedron P. Start from the vertices of P and consider the closure of this set under join and meet. Let $L(P)$ be the resulting distributive *vertex lattice* of P. It would be interesting to know what information regarding P is already contained in $L(P)$.

Problem. What do the generalized bonds associated to the elements of $L(P)$ look like? In particular some special generalized bonds of $L(P)$ including join-irreducible, minimal and maximal elements are of interest.

Geometry: We have derived an $\mathcal{H}$-description of D-polyhedra.

Problem. What does a $\mathcal{V}$-description look like?
(This again asks for a special set of elements of the vertex lattice $L(P)$.)

In fact, the previous problem can be 'turned around': For every distributive lattice L there are integral D-polyhedra such that the integral points in the polyhedron form a lattice isomorphic to L.

Problem. Which subsets of L can arise as sets of vertices of such polyhedra?

Matroid Theory: In Section 5 we have related arc-parameterized digraphs to *bicircular oriented matroids*. On the other hand the face lattice of a D-polyhedron is a geometric lattice, hence encodes a simple matroid, see [21].

Problem. What is the relation between these two matroids? What do face lattices of D-polyhedra look like?

Optimization: There has been a considerable amount of research concerned with algorithms for generalized flows, see [1] for references. As far as we know it has never been taken into account that the LP-dual problem of a min-cost generalized flow is an optimization problem on a D-polyhedron. We feel that it might be fruitful to look at this connection. A special case is given by generalized flows of planar breakeven digraphs, where the flow-polyhedron also forms a distributive lattice (Corollary 2).

In particular, it would be interesting to understand the integral points of a D-polyhedron, which by Observation 9 form a distributive lattice. Related to this and to [9] is the following:

Problem. Find conditions on Λ and c such that the set of integral generalized bonds for these parameters forms a distributive lattice.

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