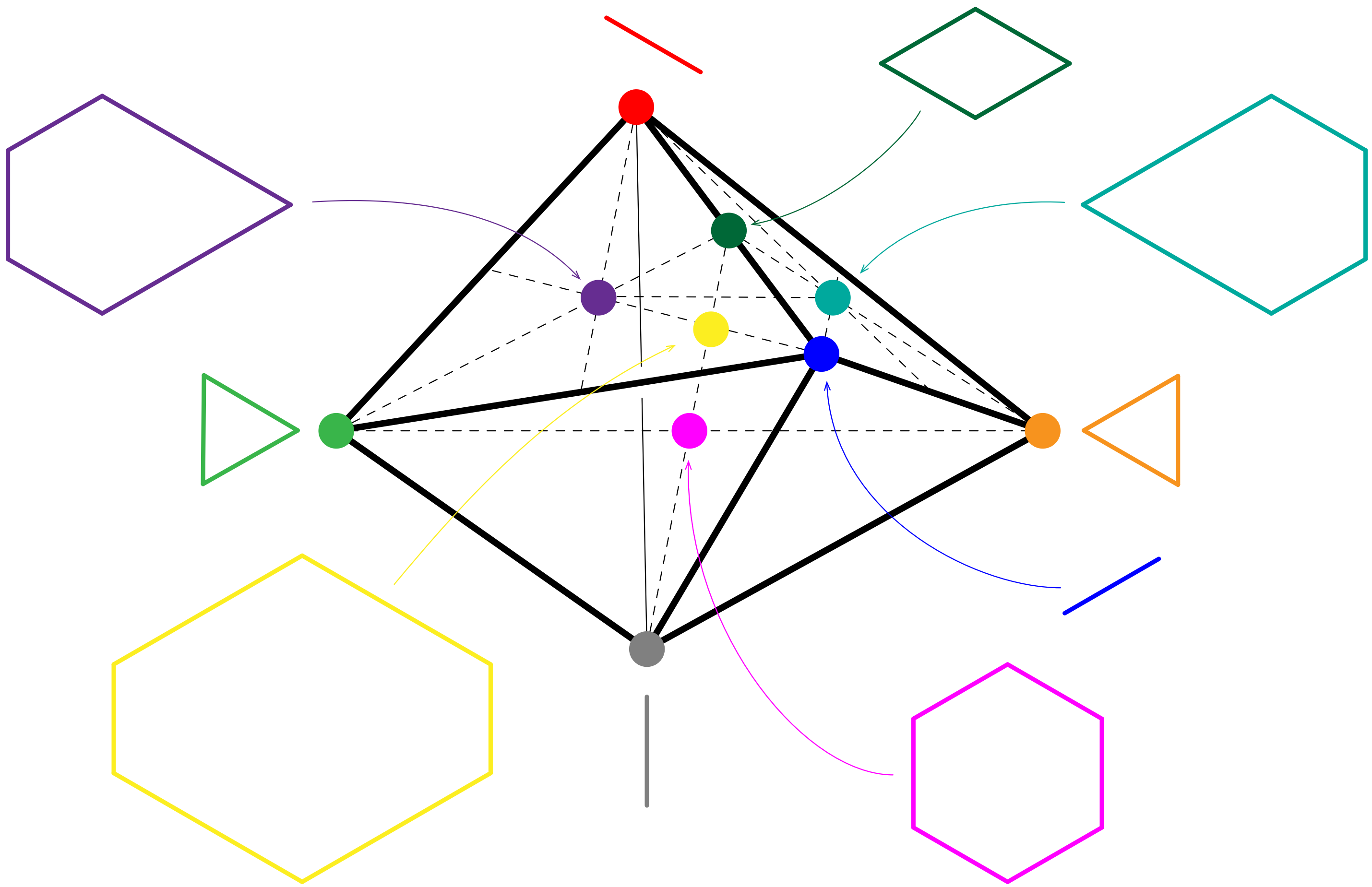
ALGEBRAIC COMBINATORICS IN DEFORMATION CONES



V. PILAUD (Universitat de Barcelona)

FPSAC'26 Seattle, Monday July 13th, 2026

slides available at: https://www.ub.edu/comb/vincentpilaud/documents/presentations/FPSAC26.pdf

"The biggest lesson I learned from Richard Stanley's work is,
combinatorial objects want to be partially ordered! [...]"

"A related lesson that Stanley has taught me is,
combinatorial objects want to belong to polytopes! [...]"

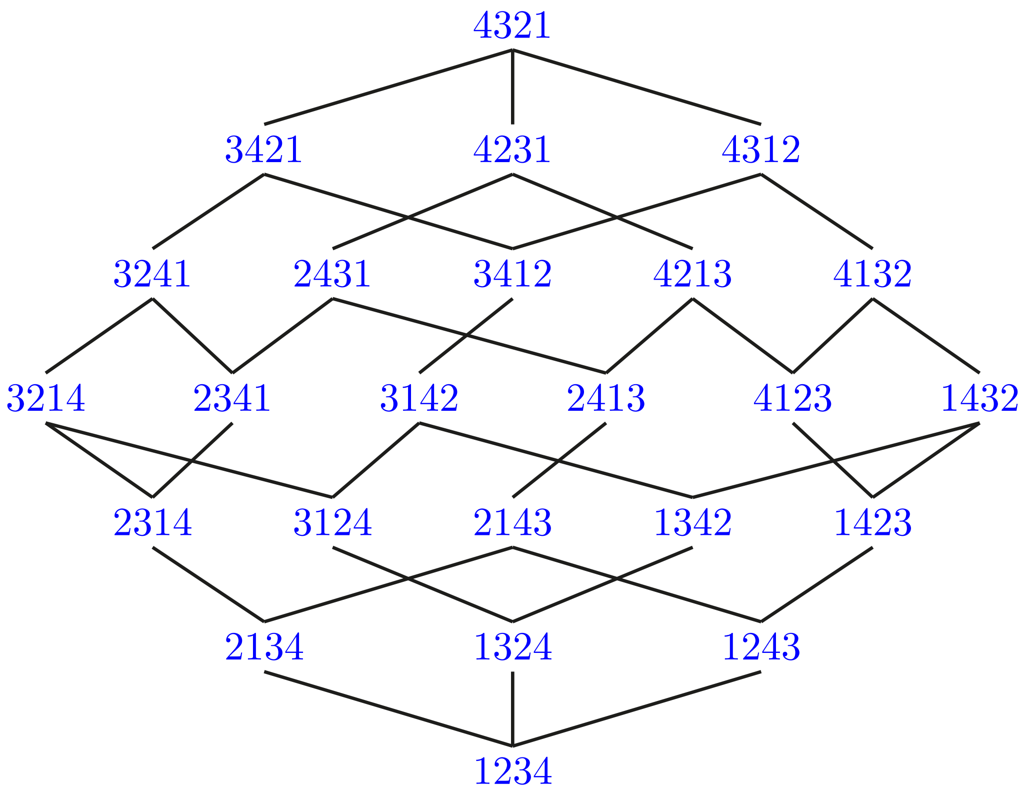
Propp, *Lessons I Learned from Richard Stanley* ('15)

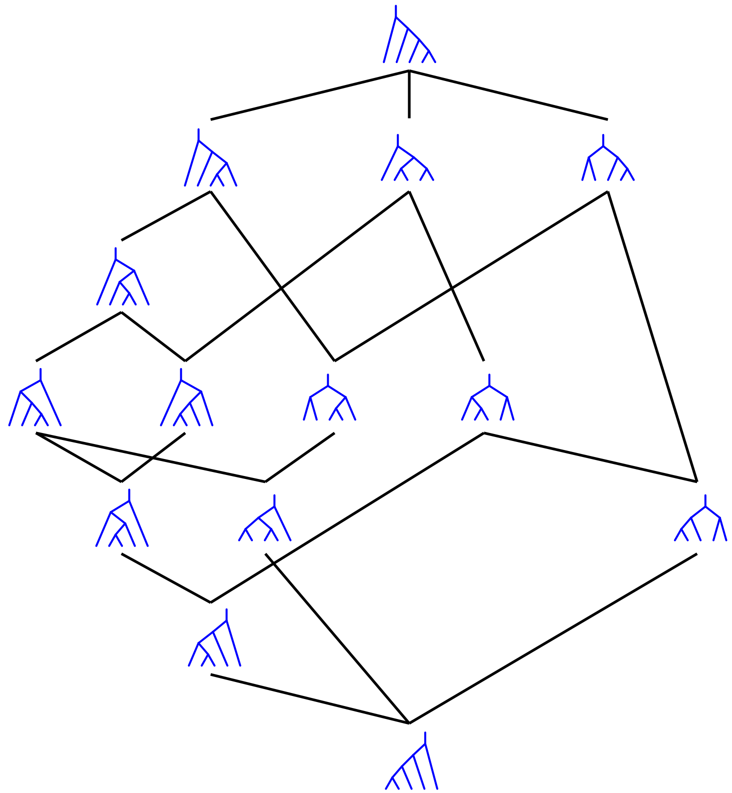
PERMUTAHEDRON & ASSOCIAHEDRON

Ceballos–Santos–Ziegler, *Many non-equivalent realizations of the associahedron* ('15)
P.–Santos–Ziegler, *Celebrating Loday's associahedron* ('23)

LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice $=$ partially ordered set L where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$





weak order $=$ permutations of $[n]$
ordered by paths of simple transpositions

Tamari lattice $=$ binary trees on $[n]$
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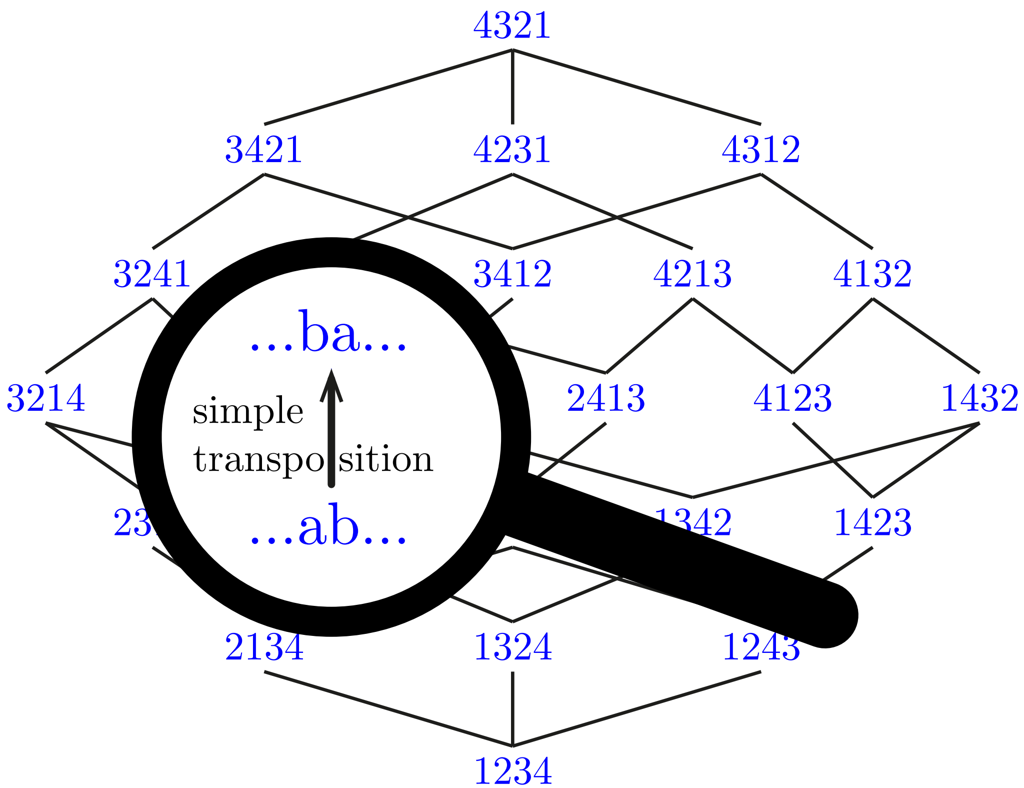

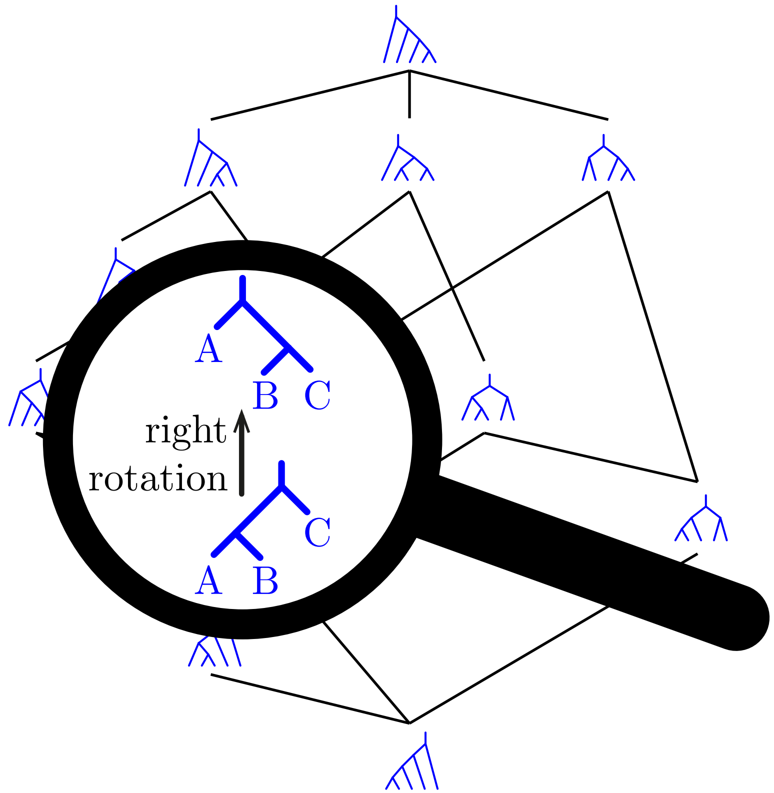

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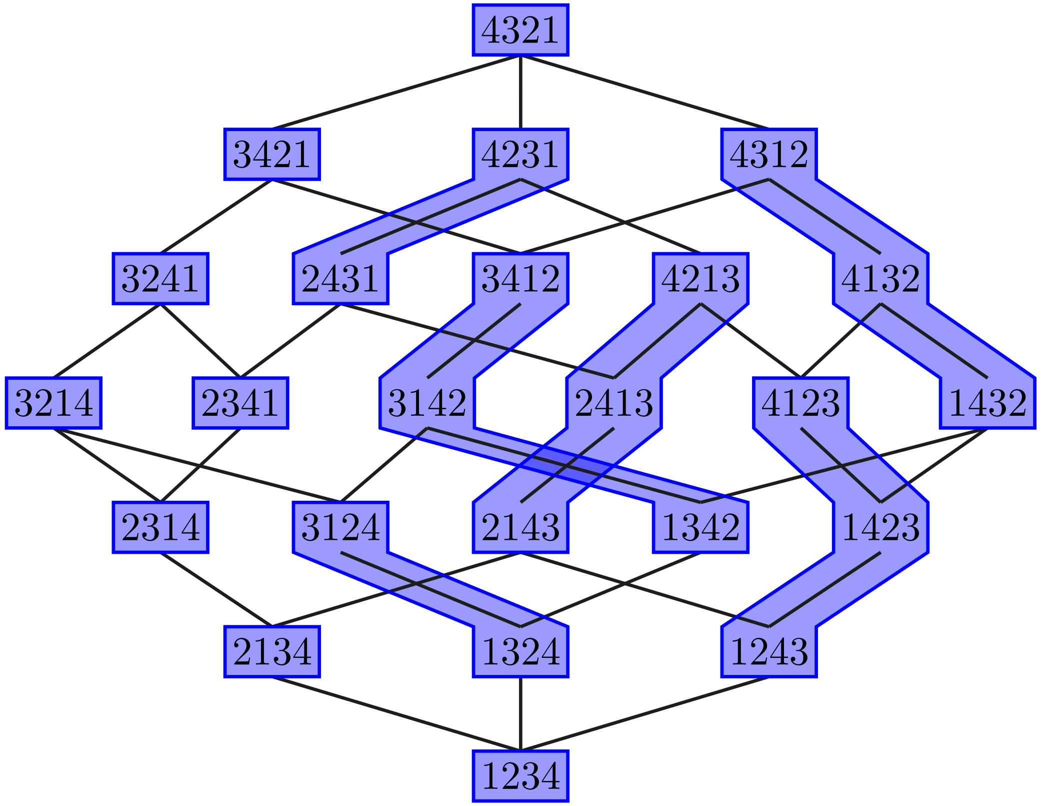

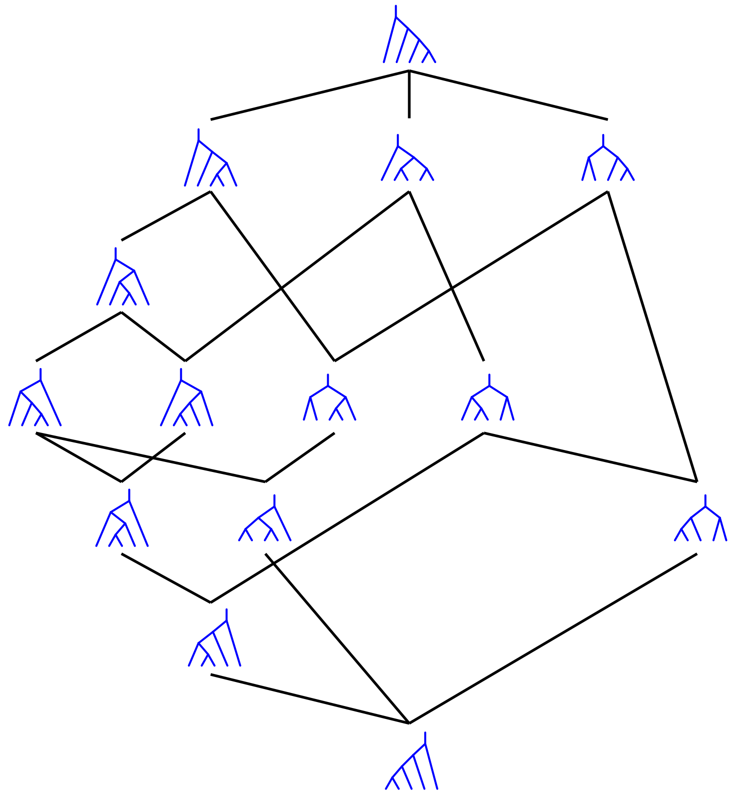

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$=$ rewriting rule $UacVbW \equiv_{\mathrm{sylv}} UcaVbW$ with $a < b < c$

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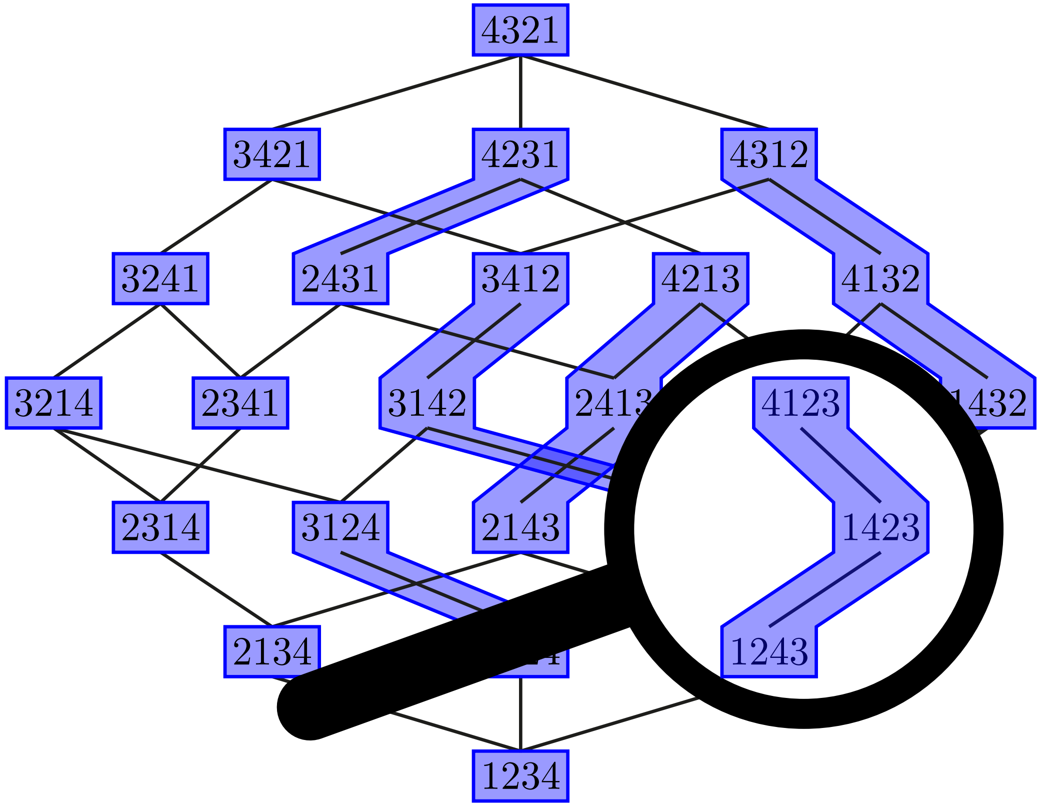

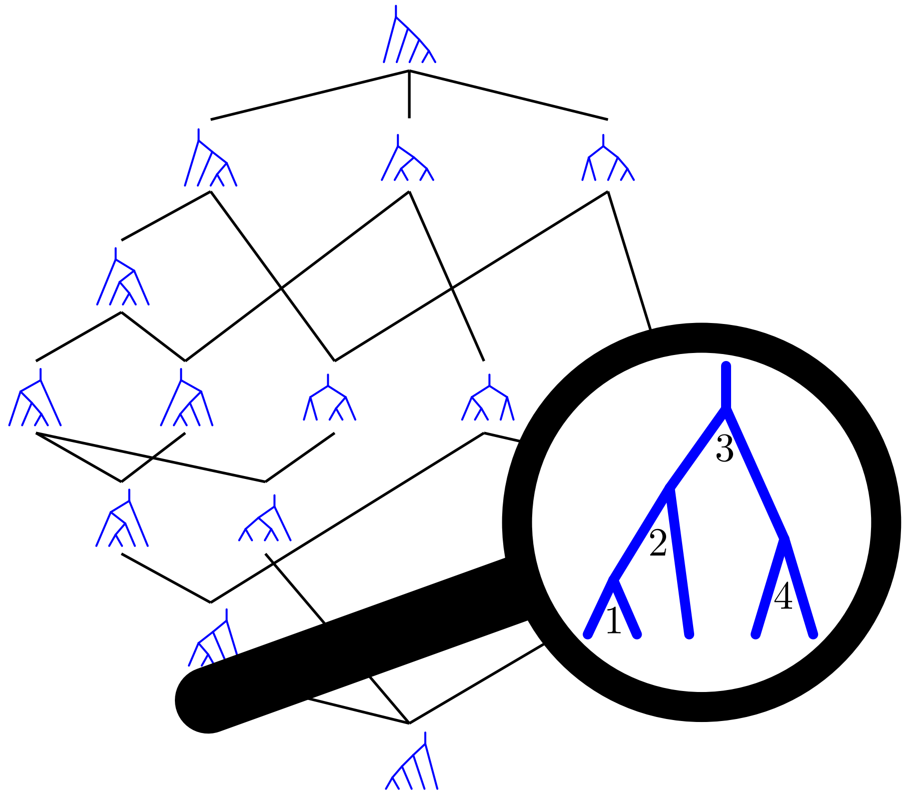

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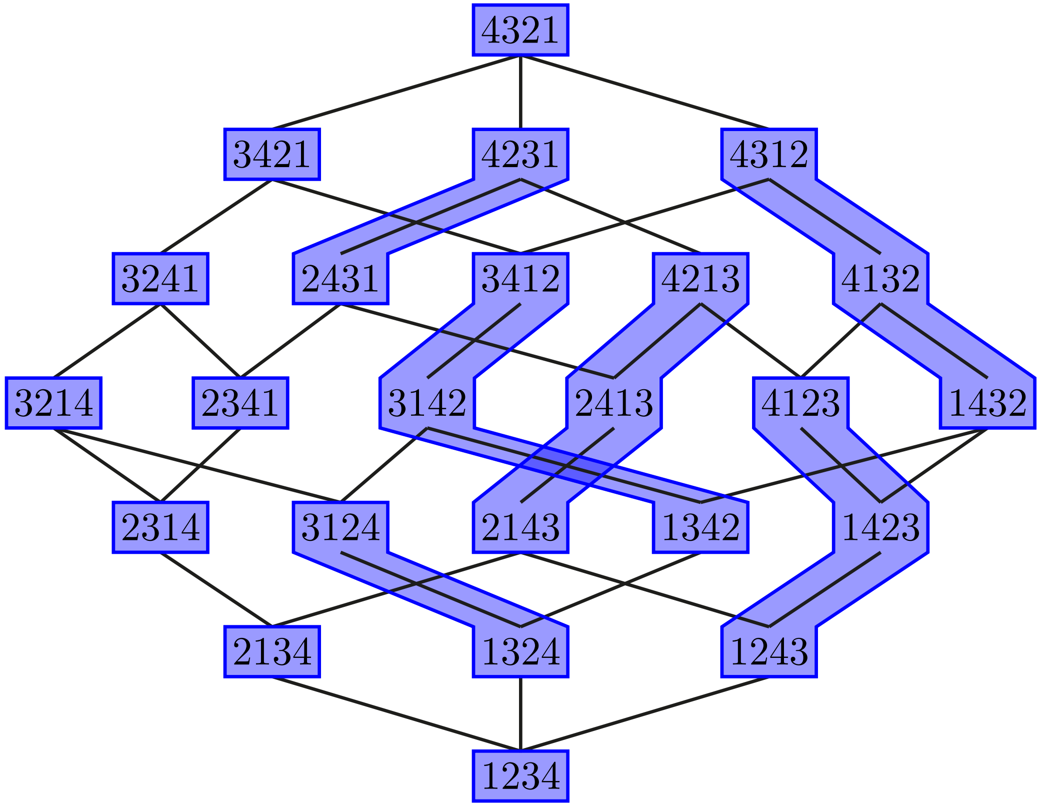

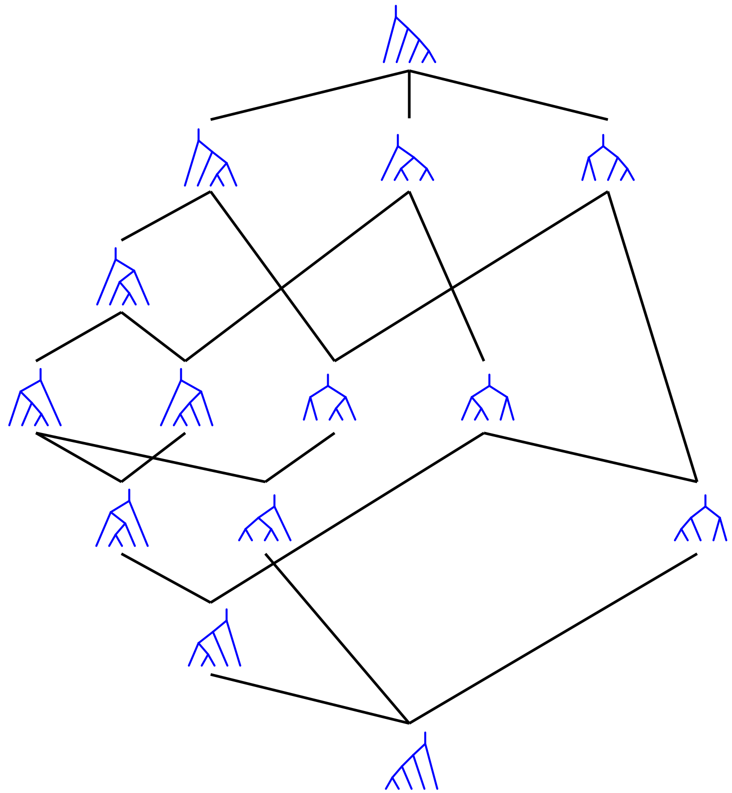

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lattice congruence $=$ equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \implies x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

quotient lattice $=$ lattice on classes with $X \leq Y \iff \exists\, x \in X,\ y \in Y, x \leq y$

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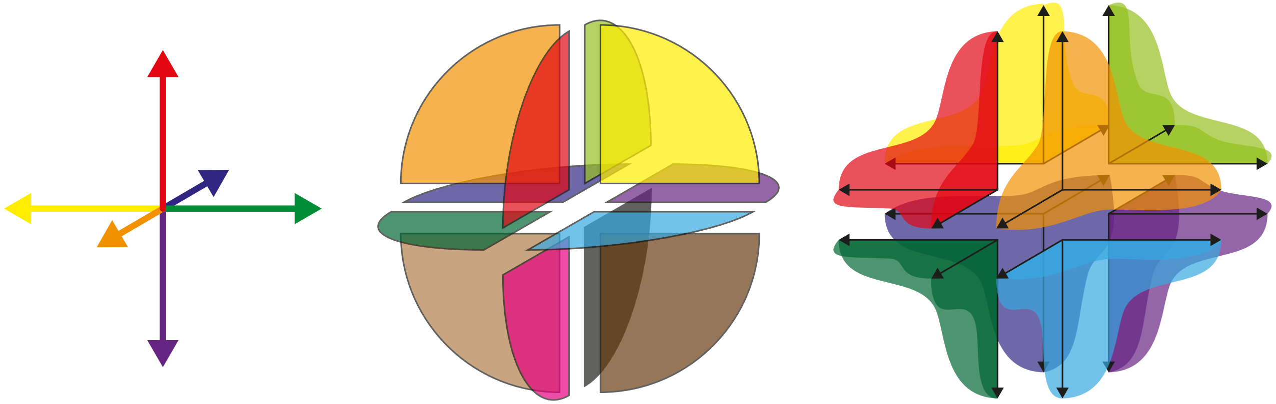
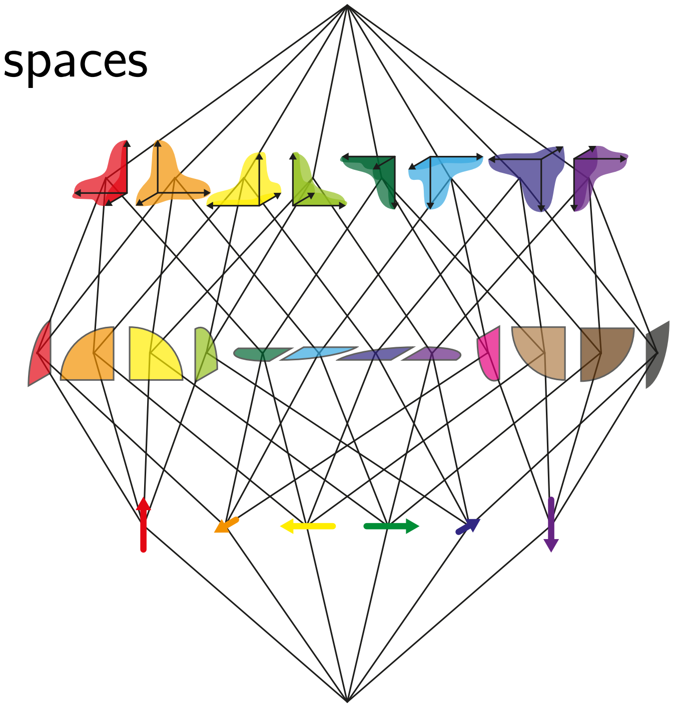
polyhedral cone = positive span of a finite set of vectors
= intersection of a finite set of linear half-spaces

fan = collection of polyhedral cones closed by faces
and where any two cones intersect along a face





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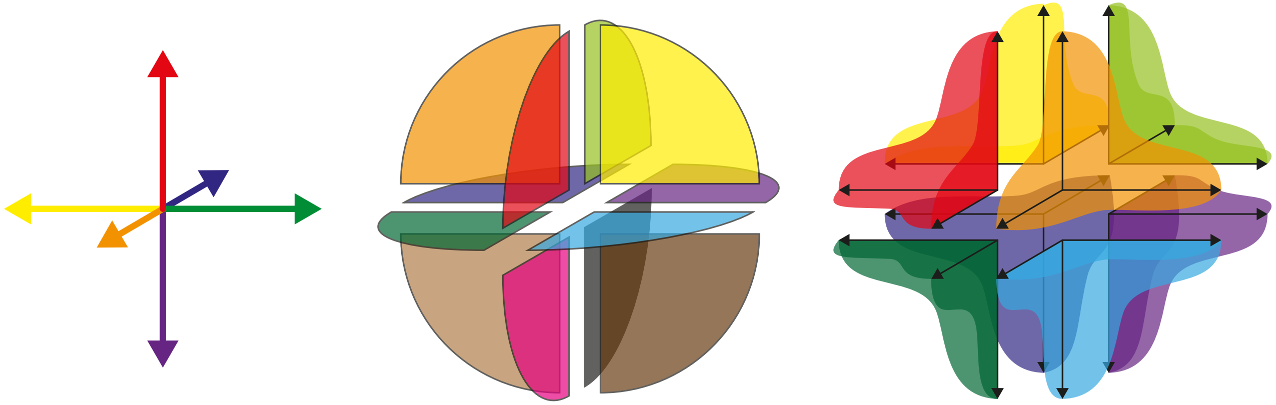
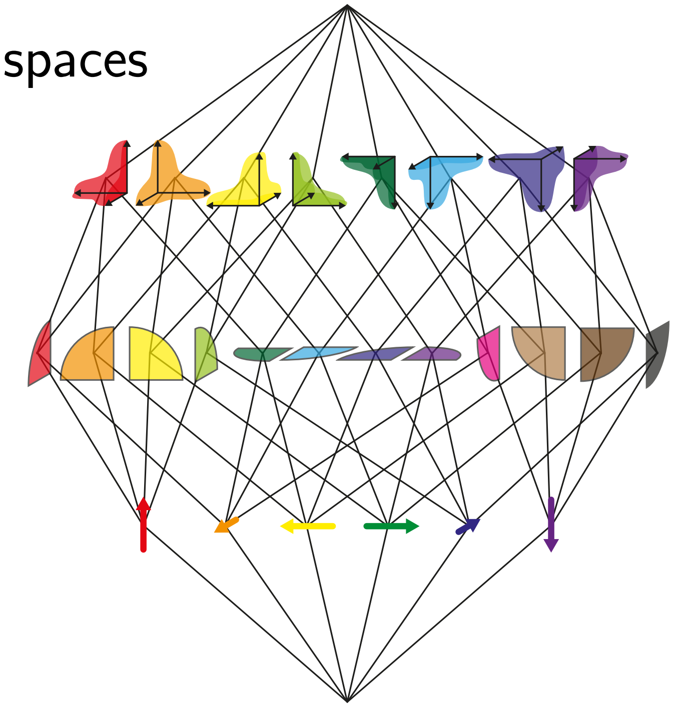
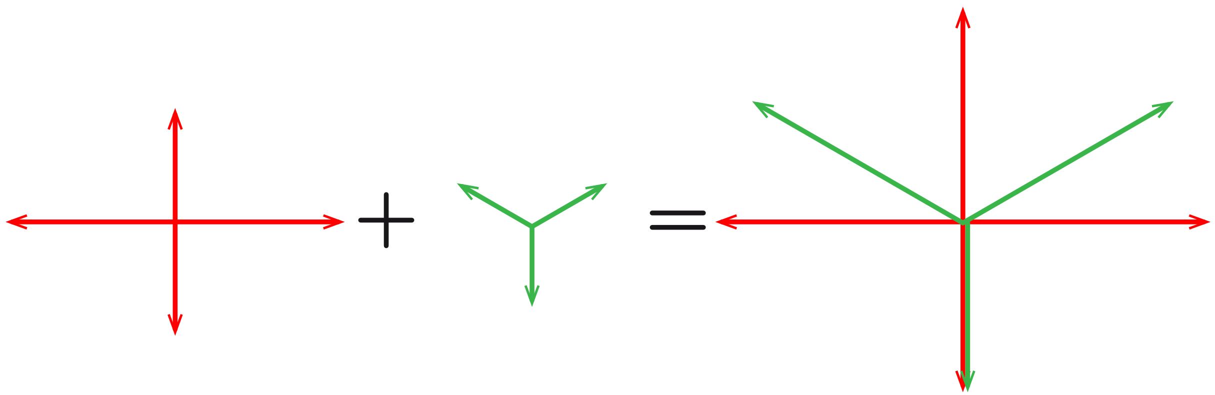




Common refinement $\mathcal{F} + \mathcal{G} = \{$non-empty intersections of cones of $\mathcal{F}$ and $\mathcal{G}\}$



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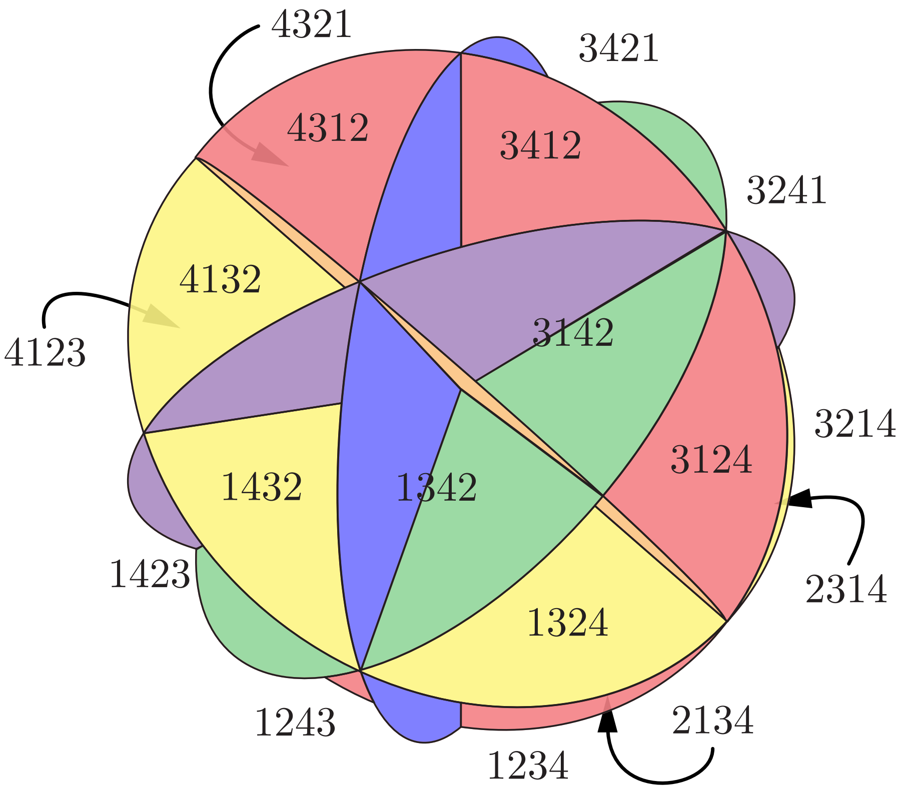

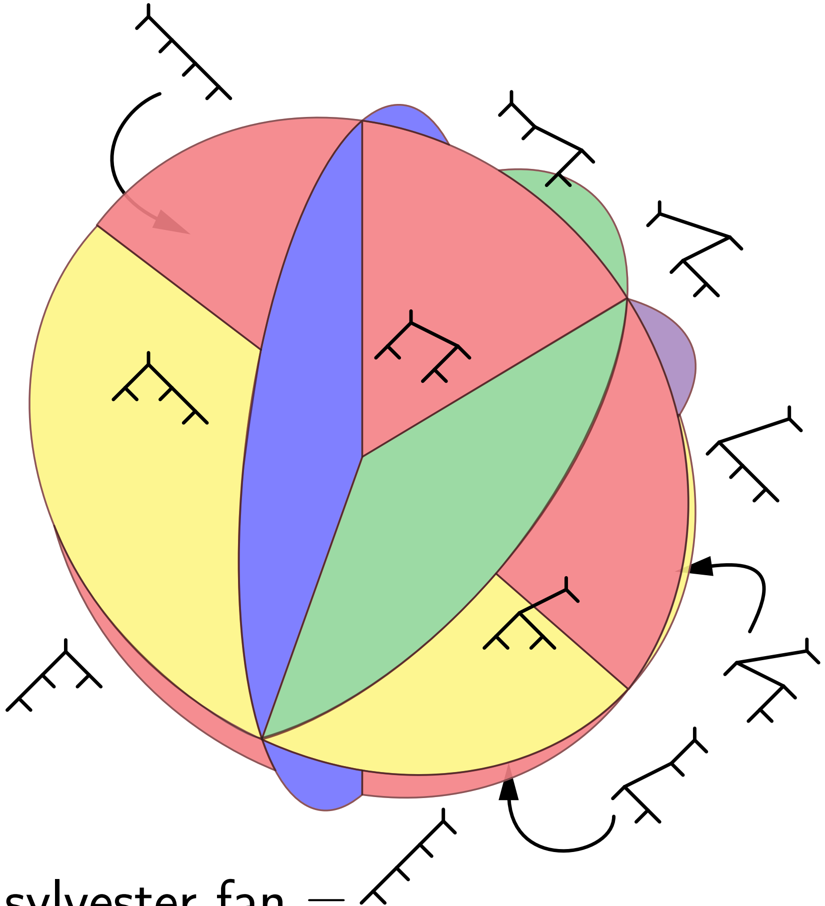

braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$

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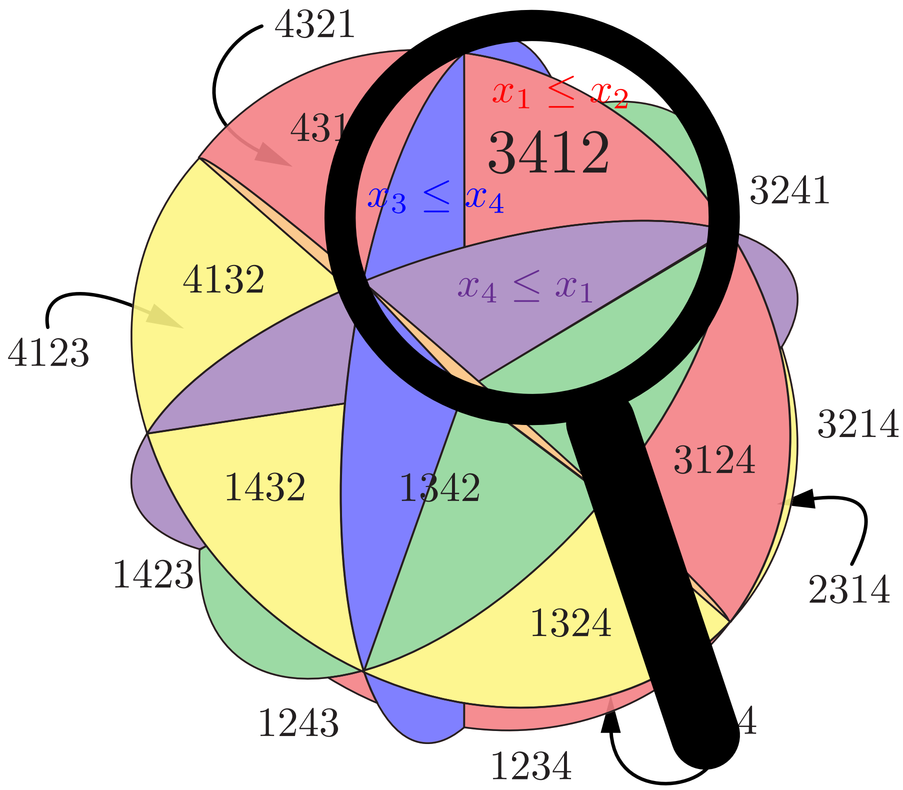

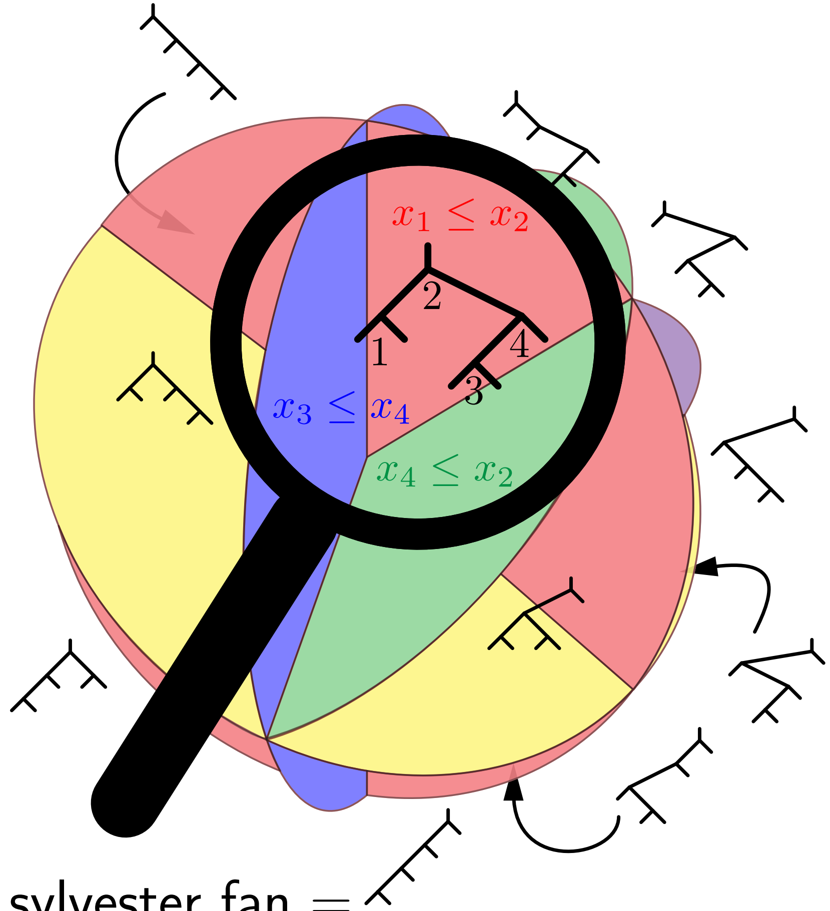

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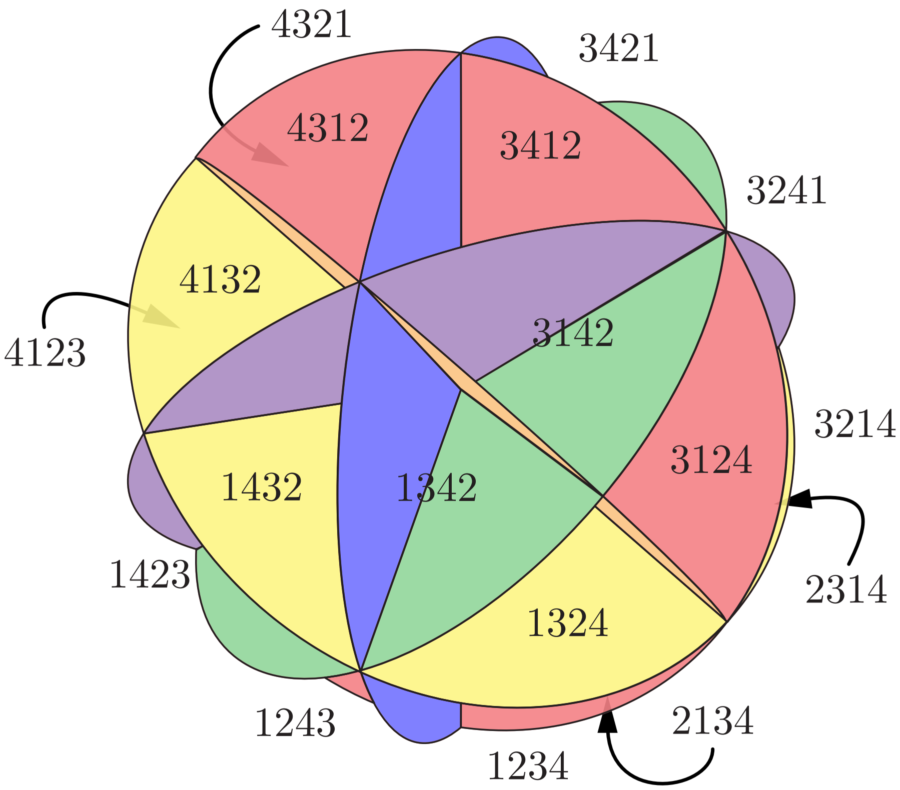

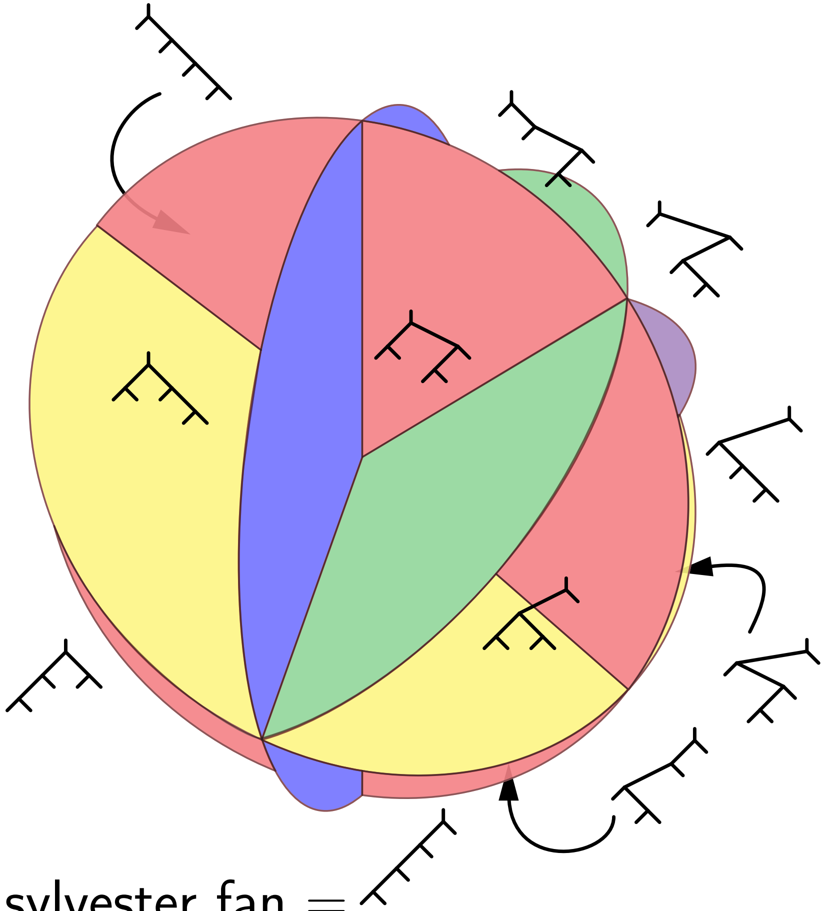

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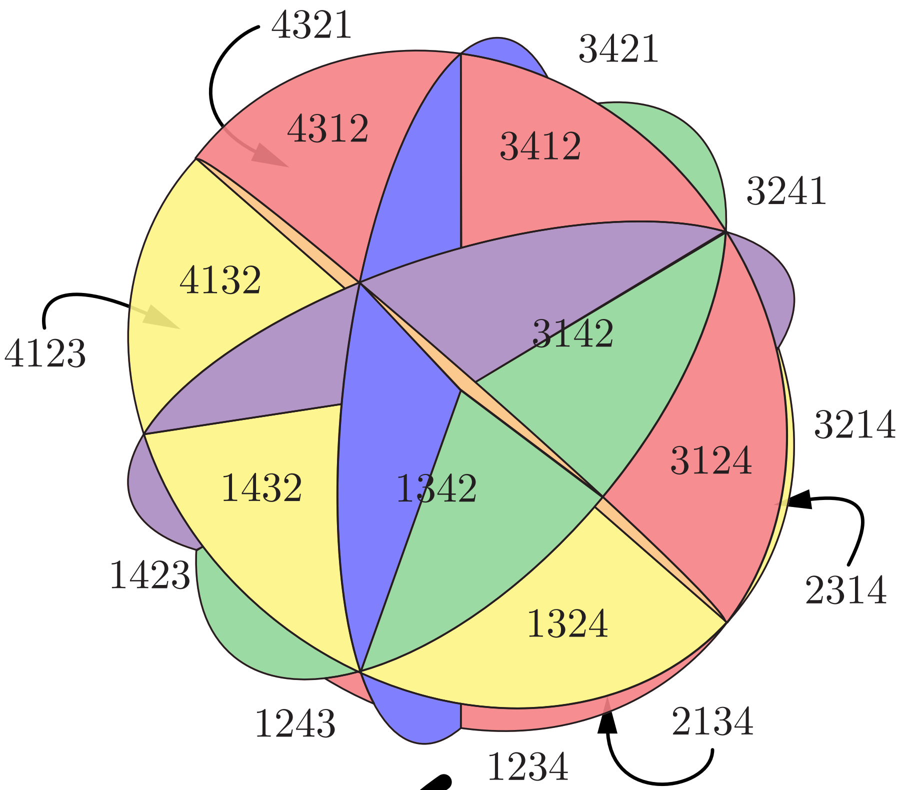
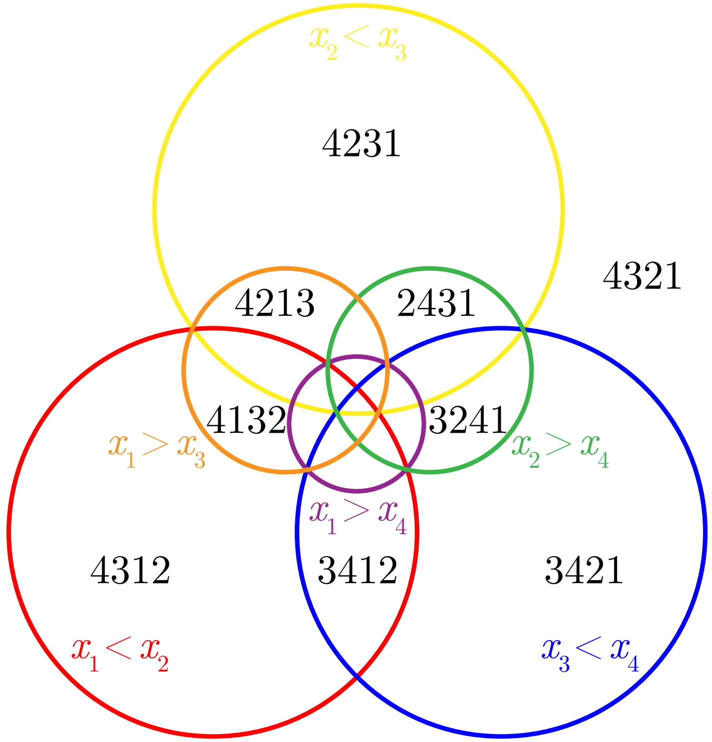
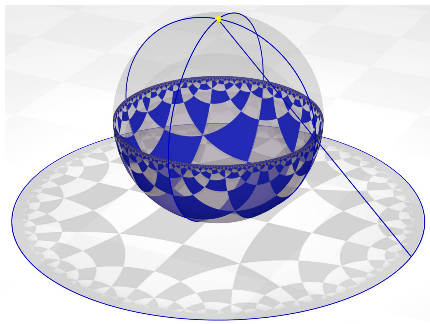




stereographic projection =



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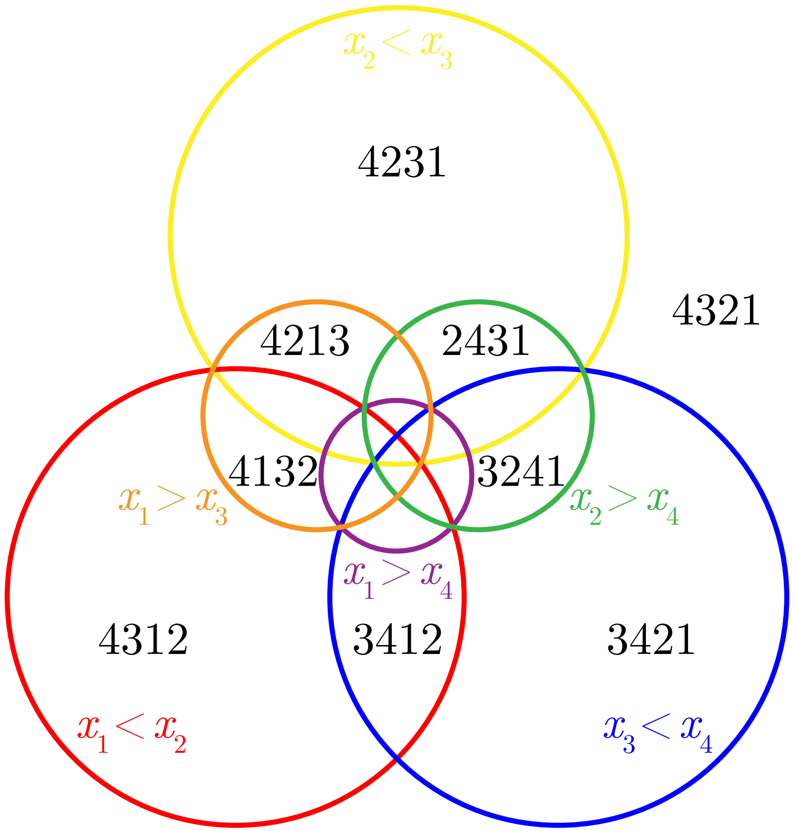

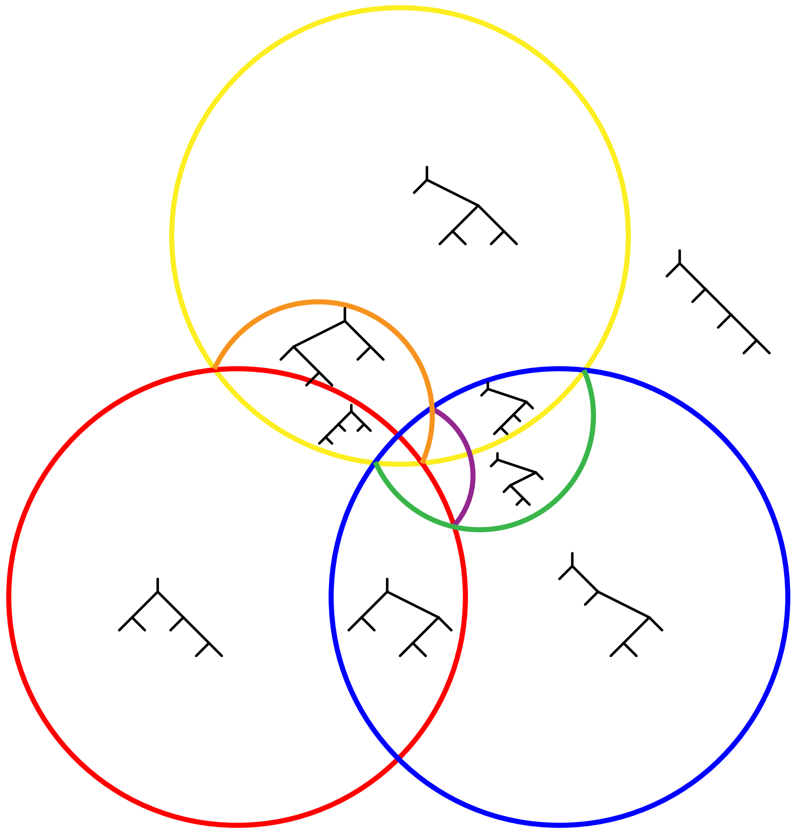

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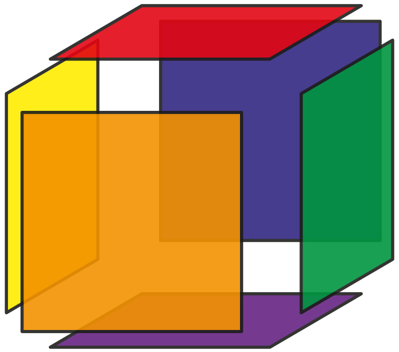
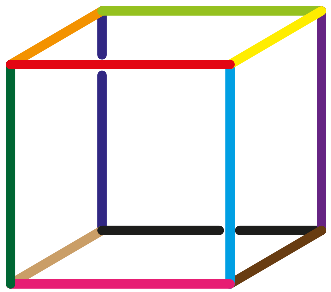
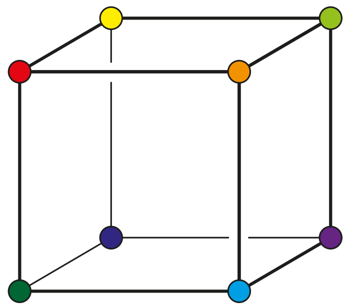
= bounded intersection of a finite set of affine half-spaces

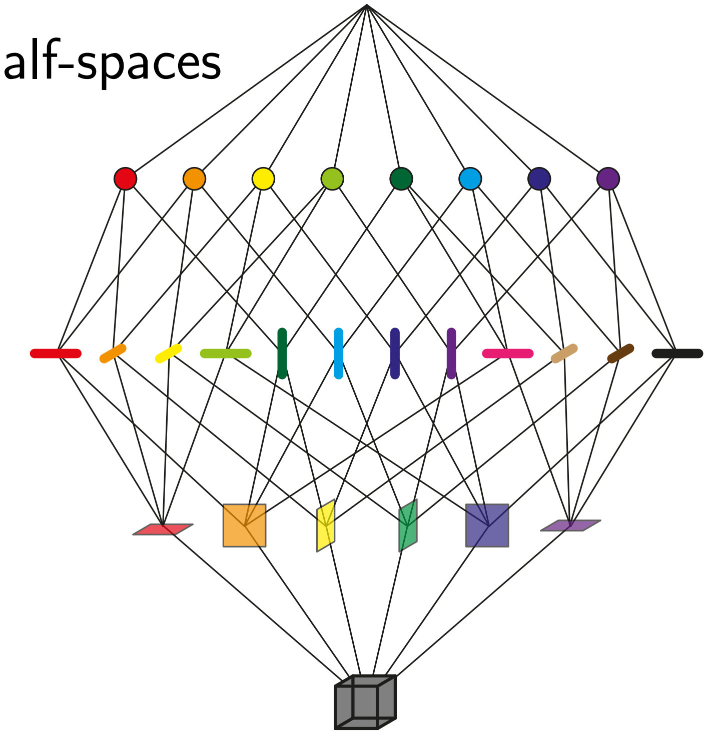
face = intersection with a supporting hyperplane

face lattice = all the faces with their inclusion relations









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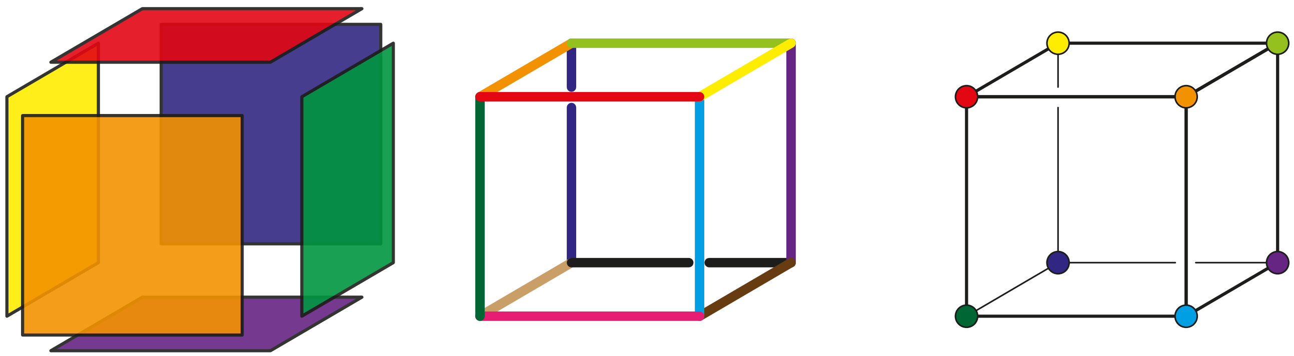
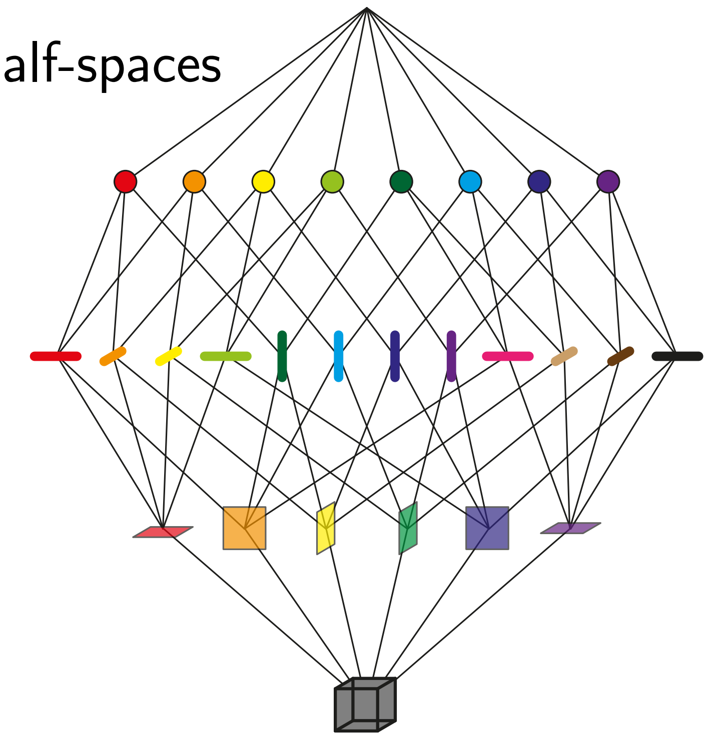
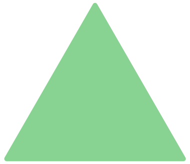




Minkowski sum $\mathbb{P} + \mathbb{Q} = \{\boldsymbol{p} + \boldsymbol{q} \mid \boldsymbol{p} \in \mathbb{P},\ \boldsymbol{q} \in \mathbb{Q}\}$

$+$



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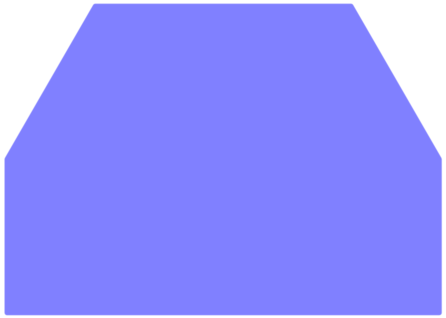

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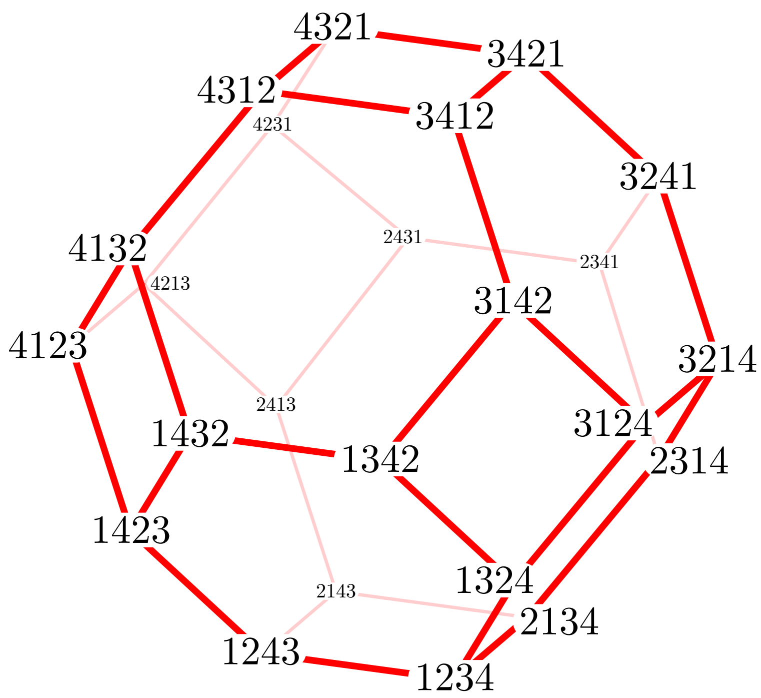
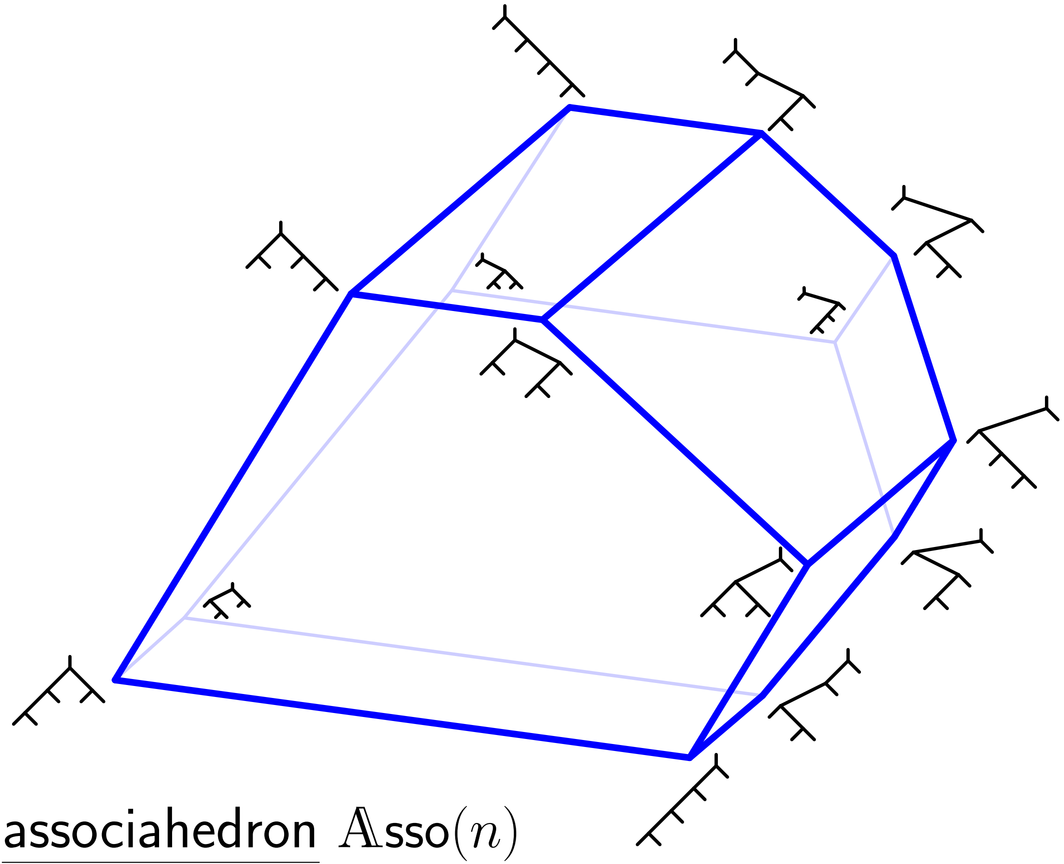


permutahedron $\mathbb{P}\mathrm{erm}(n)$



associahedron $\mathbb{A}\mathrm{sso}(n)$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)
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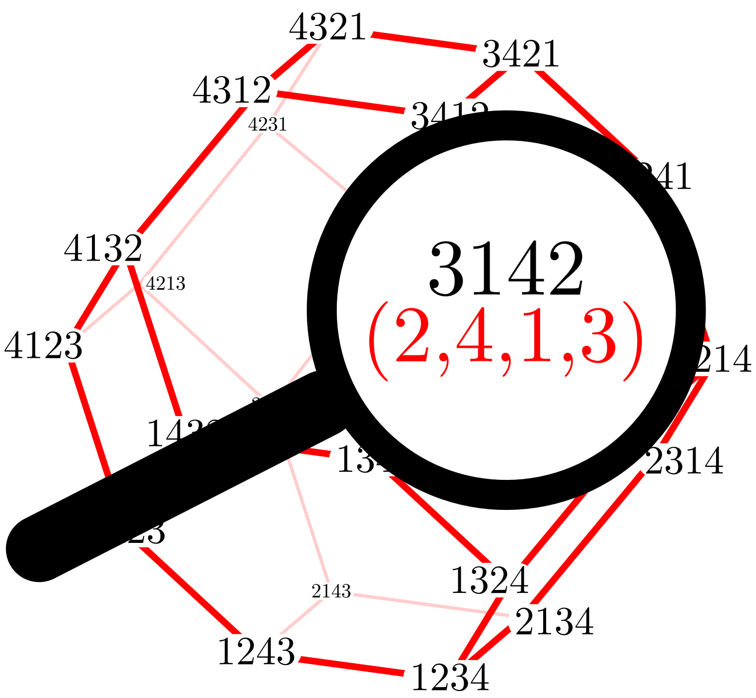

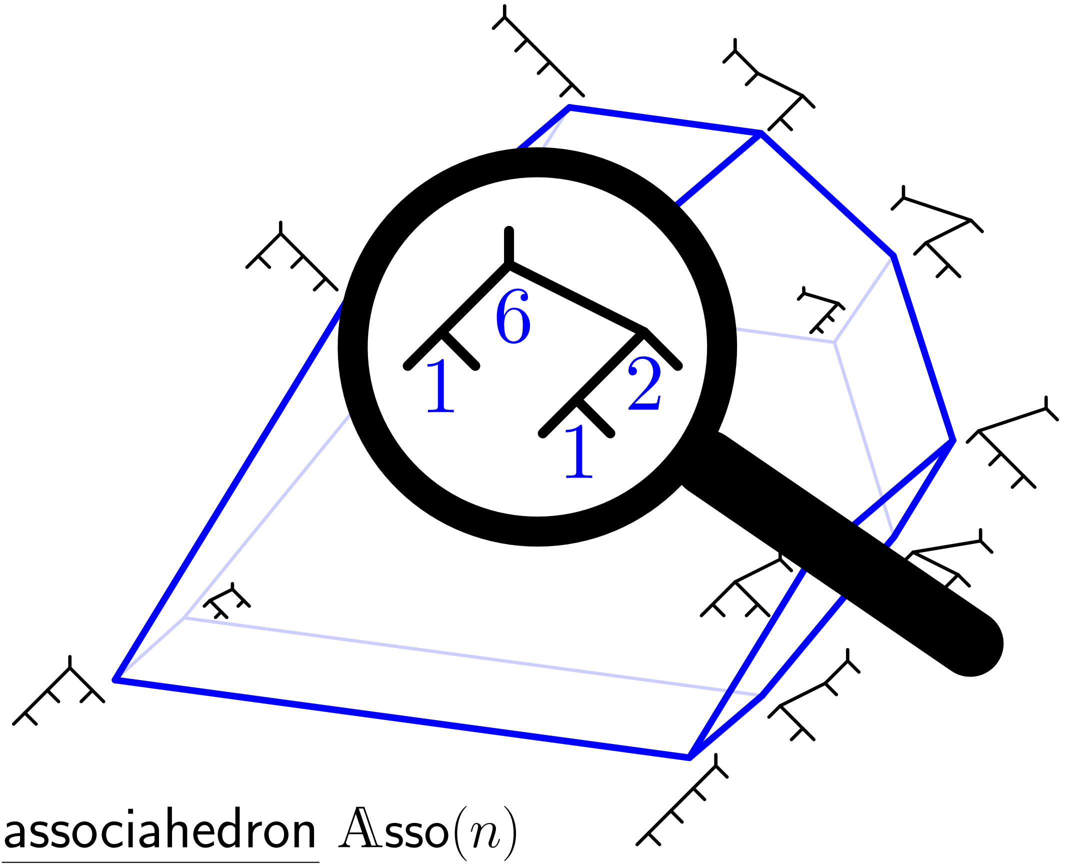

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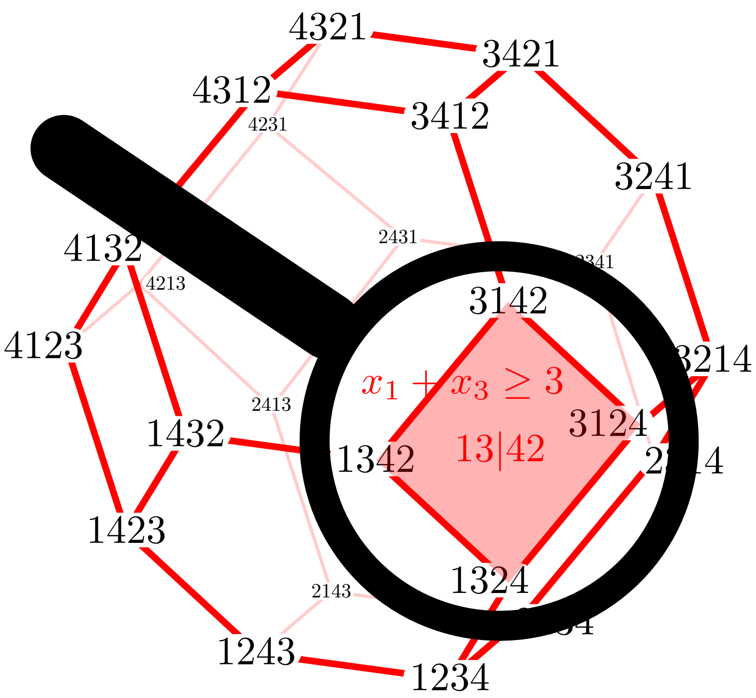

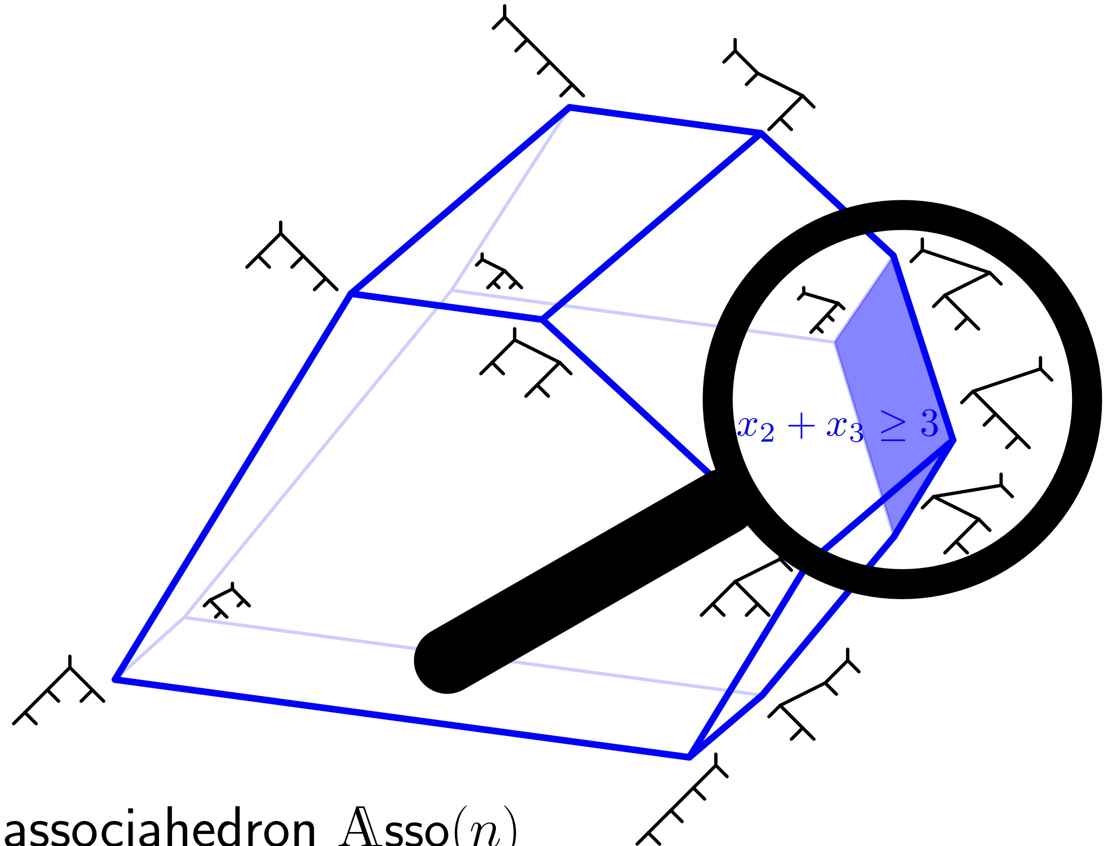

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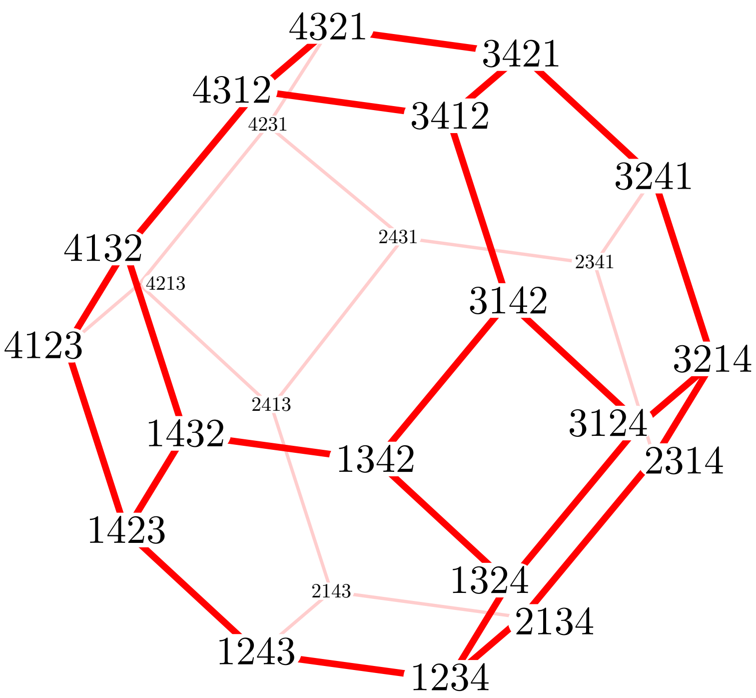

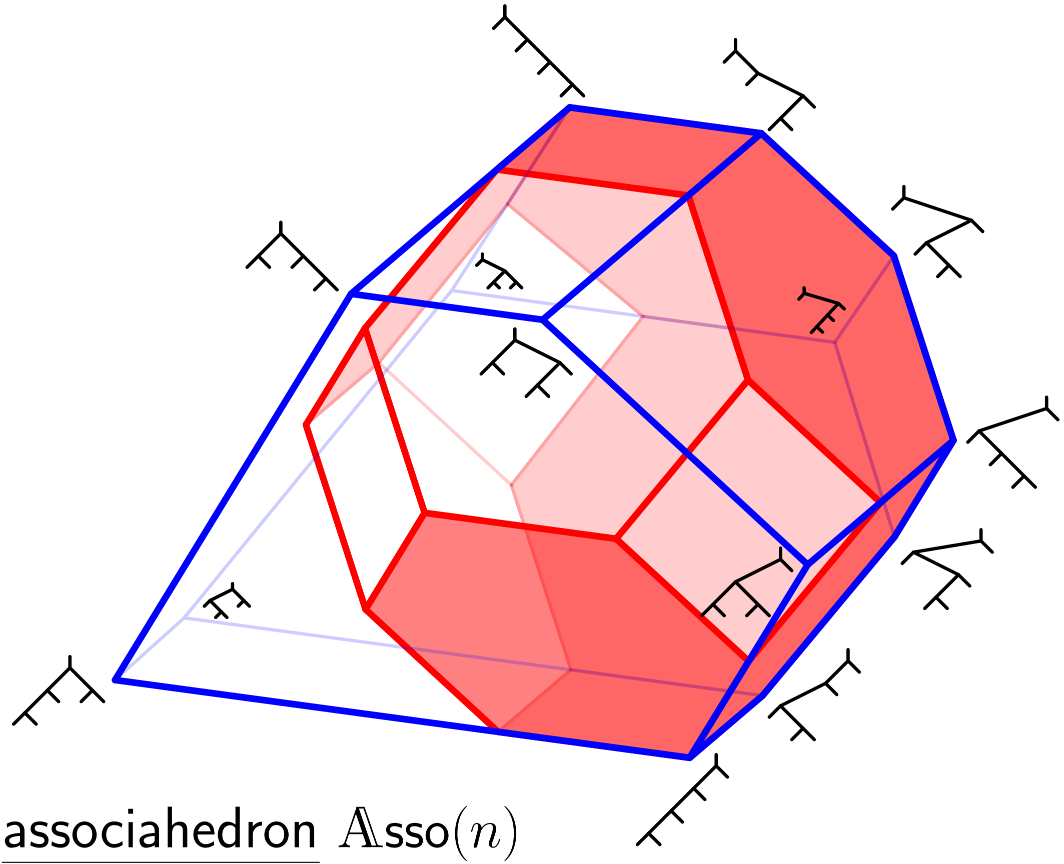

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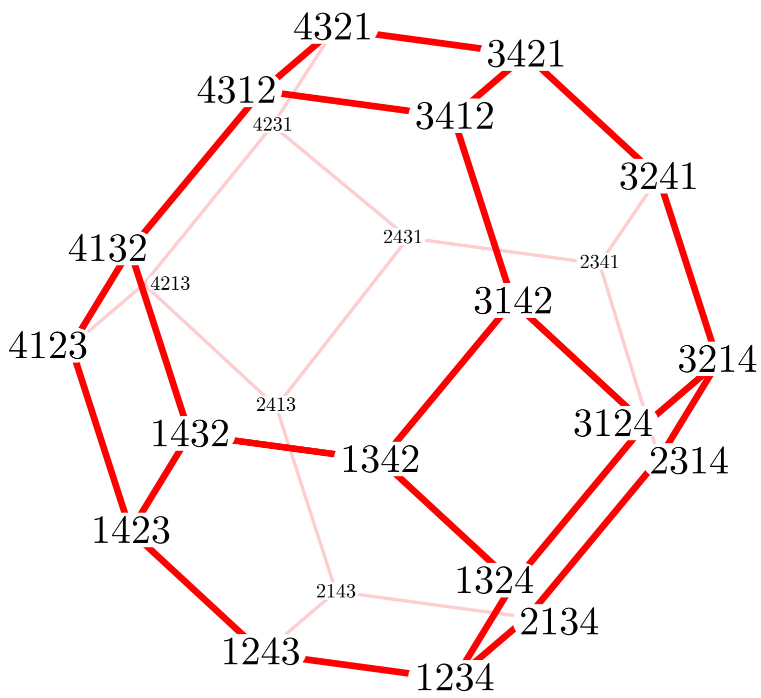

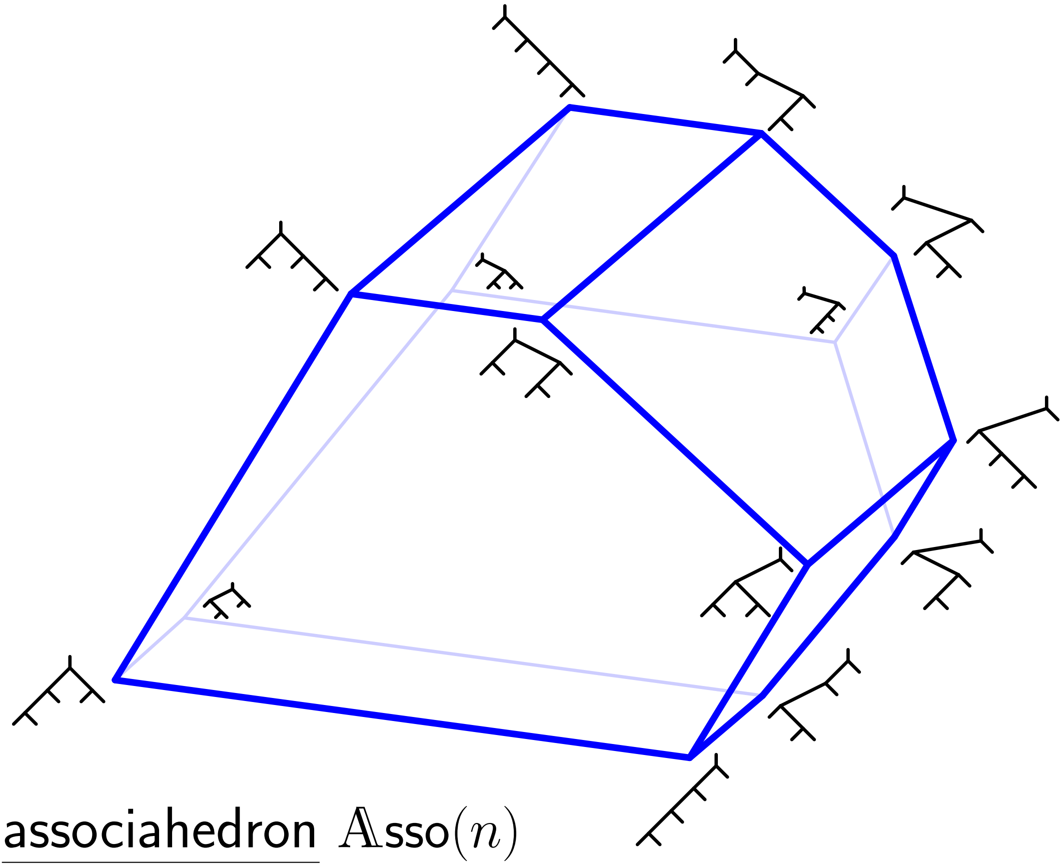

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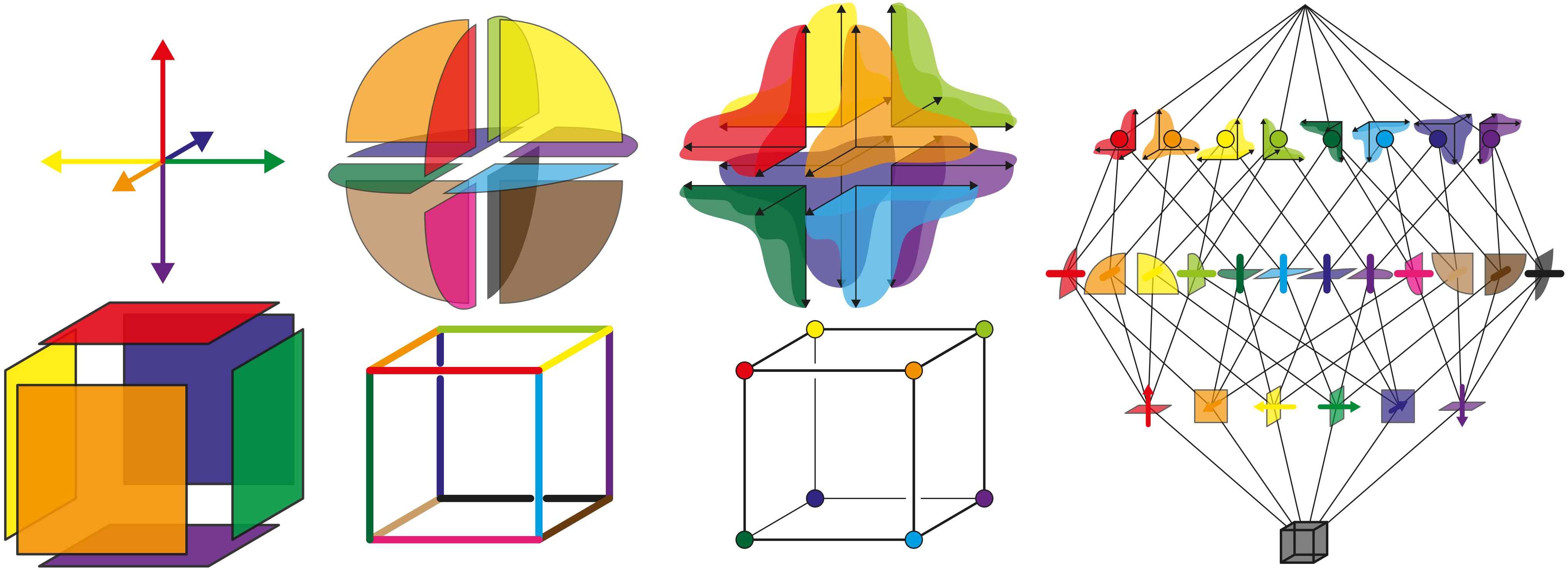
Postnikov ('09)

LATTICES – FANS – POLYTOPES

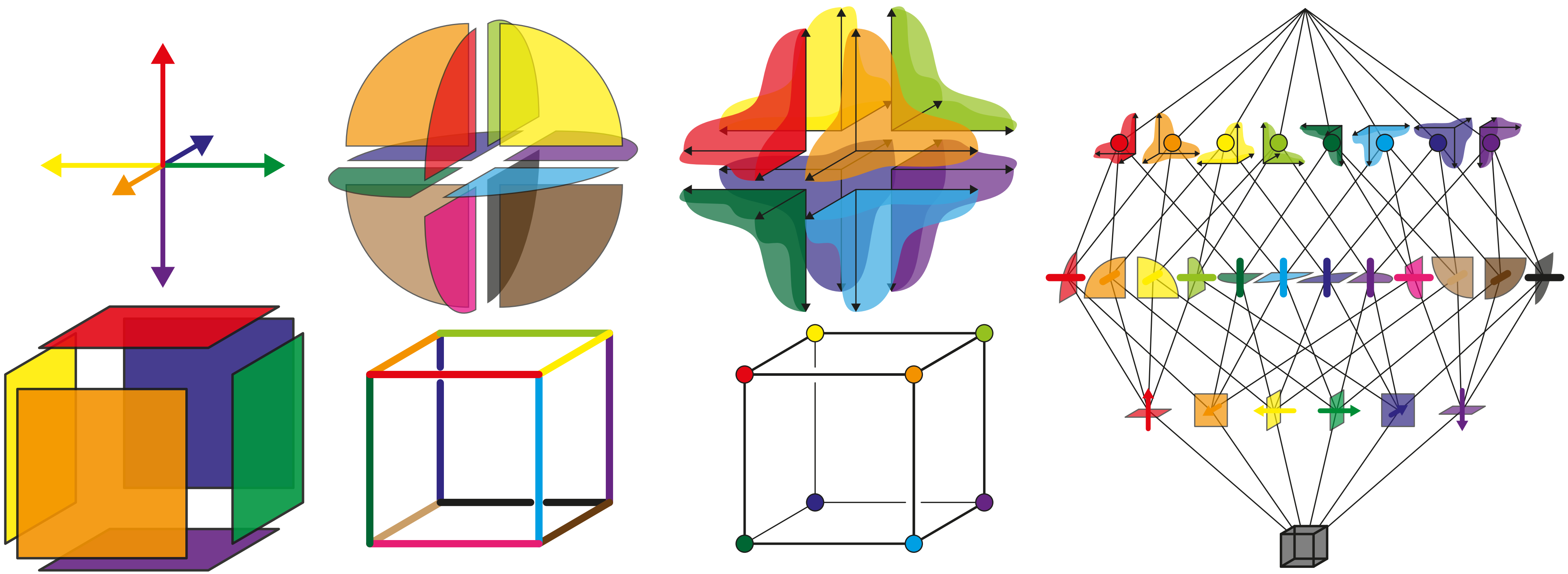


face $\mathbb{F}$ of polytope $\mathbb{P}$

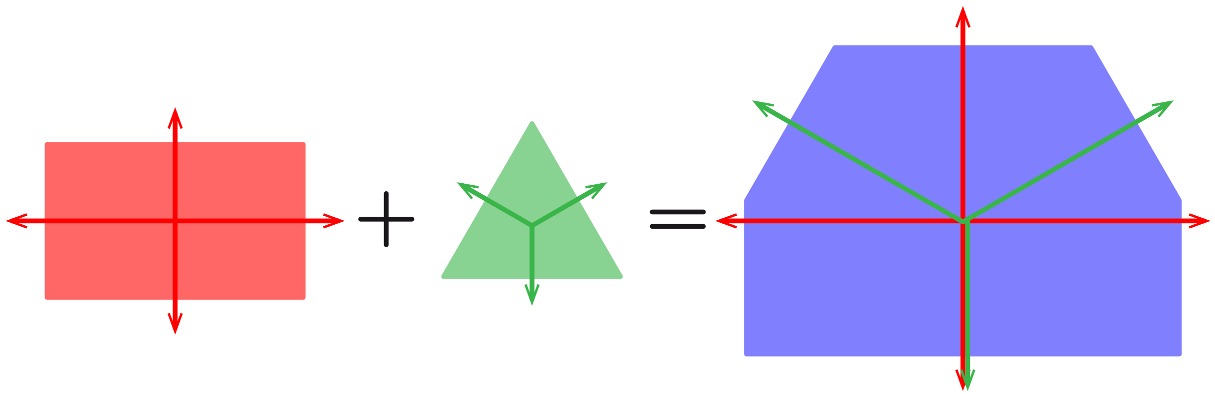
normal cone of $\mathbb{F}$ = positive span of the outer normal vectors of the facets containing $\mathbb{F}$

normal fan of $\mathbb{P}$ = $\{$ normal cone of $\mathbb{F}$ $\mid$ $\mathbb{F}$ face of $\mathbb{P}$ $\}$

LATTICES – FANS – POLYTOPES



Normal fan of $\mathbb{P} + \mathbb{Q}$ = common refinement of normal fans of $\mathbb{P}$ and $\mathbb{Q}$



permutahedron $\mathbb{P}\mathsf{erm}(n)$

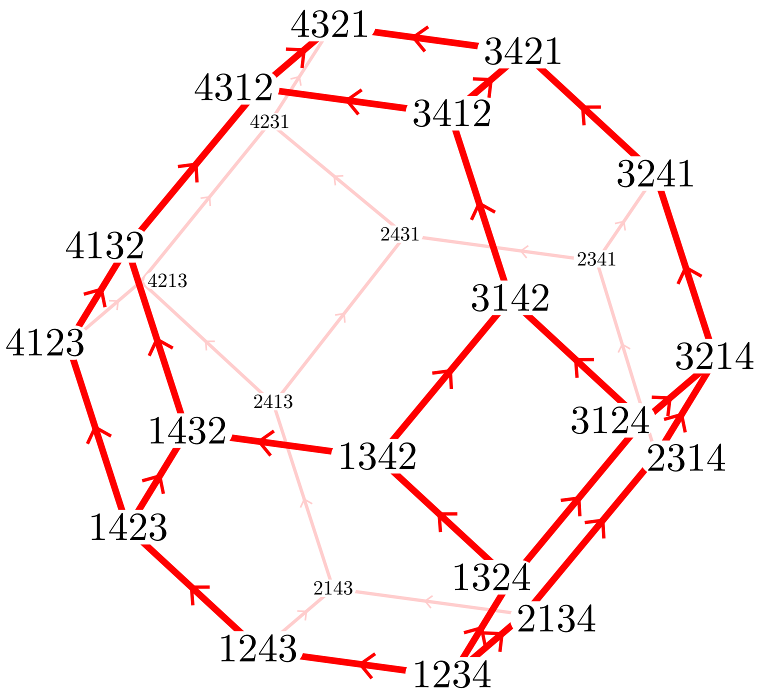
$\Longrightarrow$ braid fan

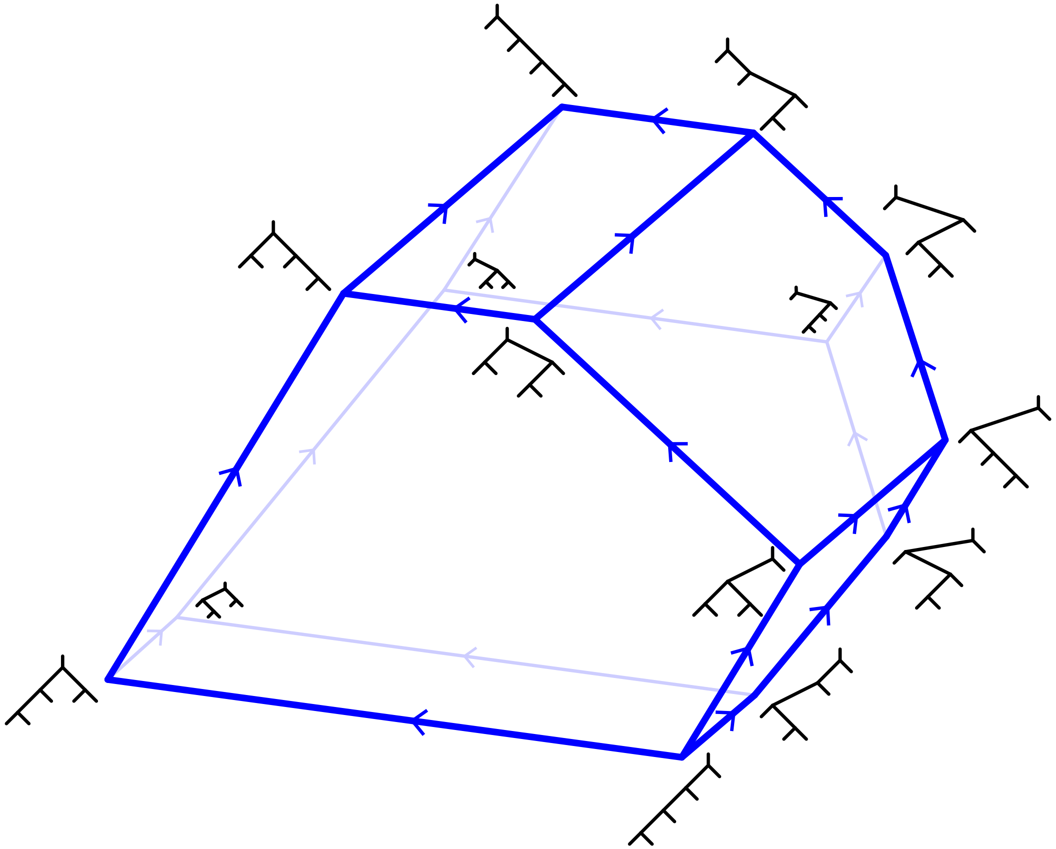
$\Longrightarrow$ weak order on permutations

associahedron $\mathbb{A}\mathsf{sso}(n)$

$\Longrightarrow$ Sylvester fan

$\Longrightarrow$ Tamari lattice on binary trees





LATTICES – FANS – POLYTOPES

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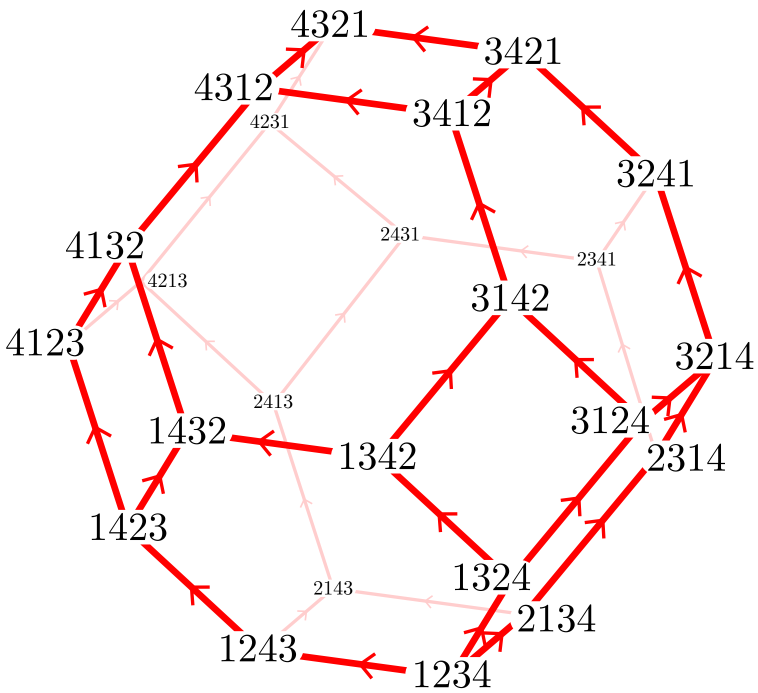
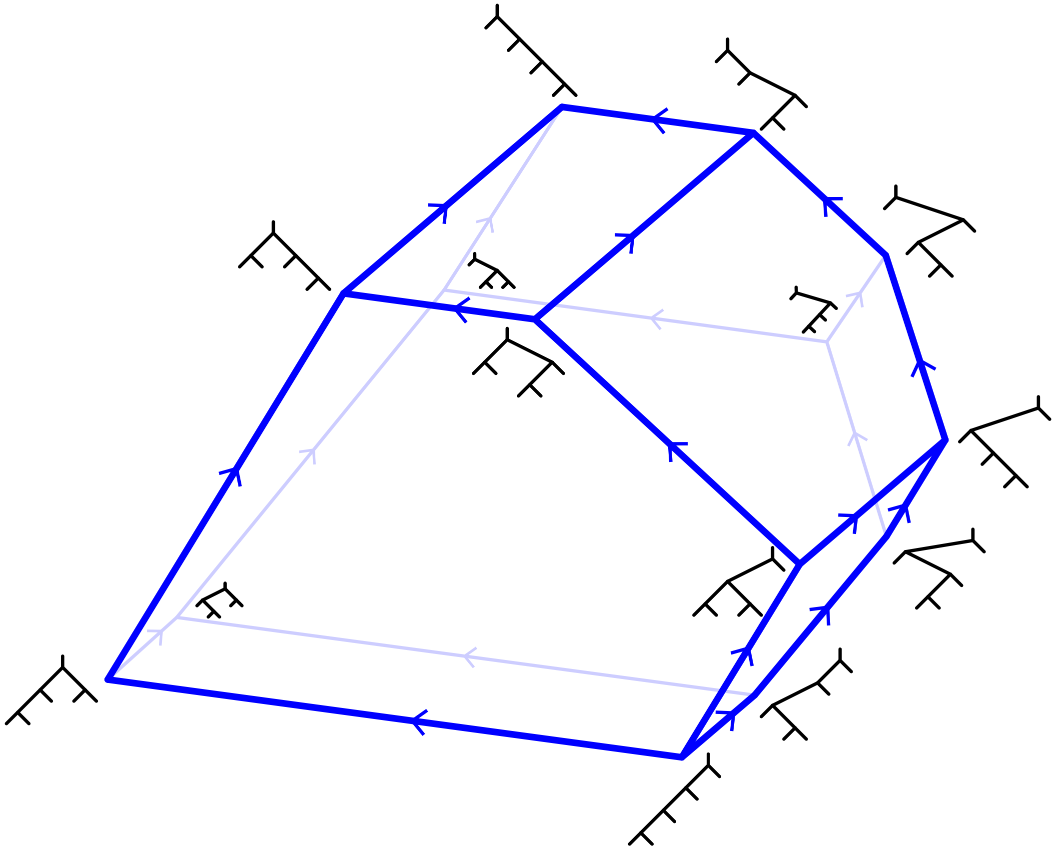
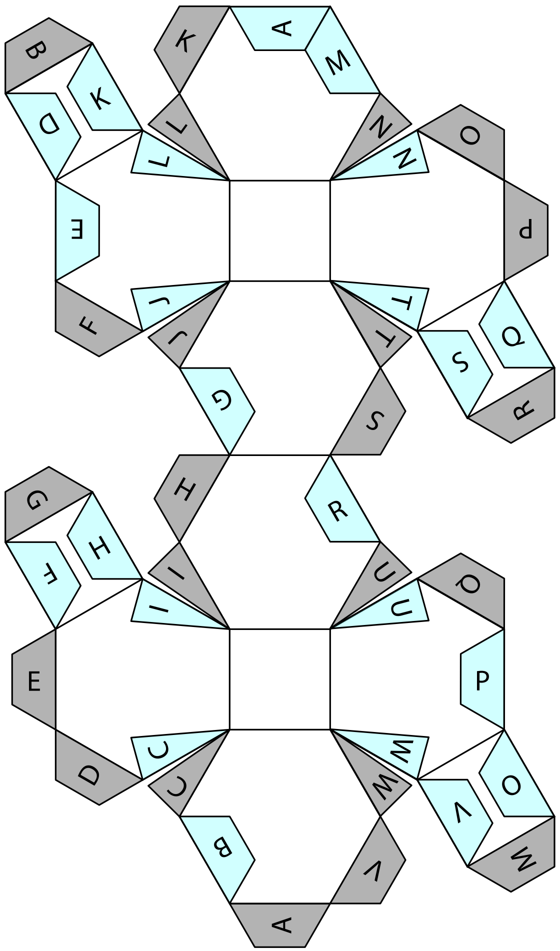
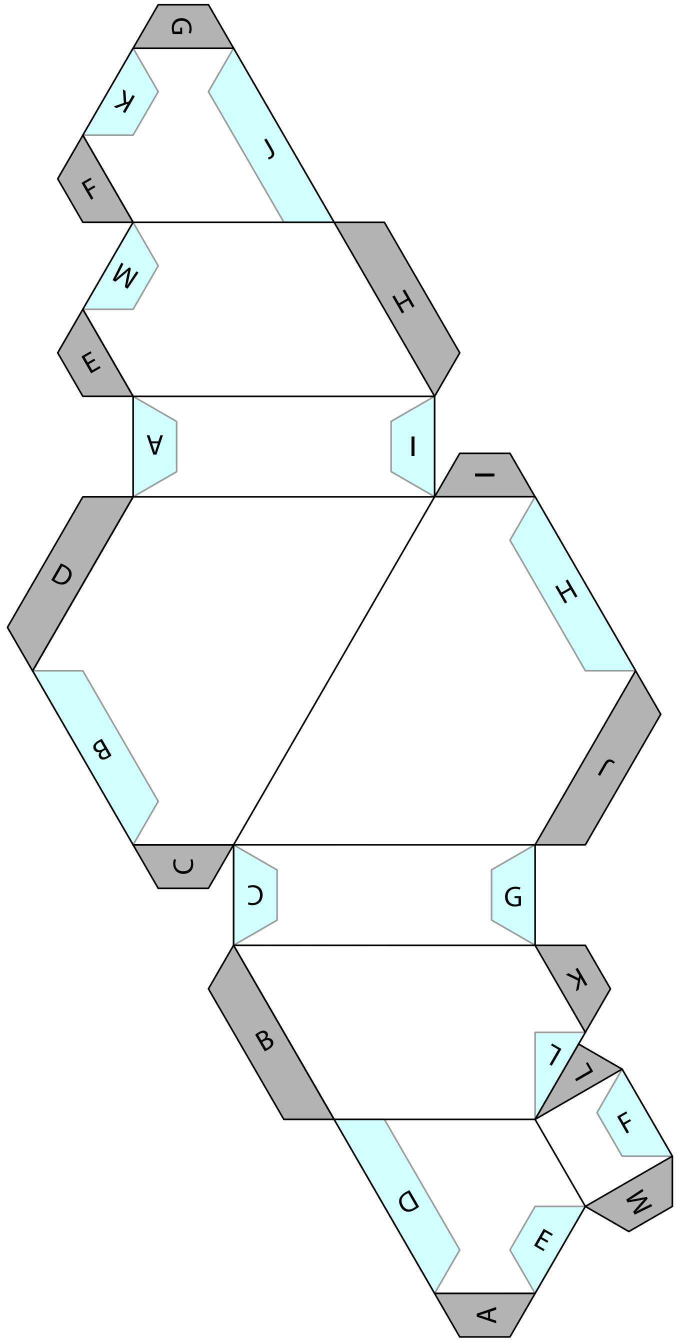




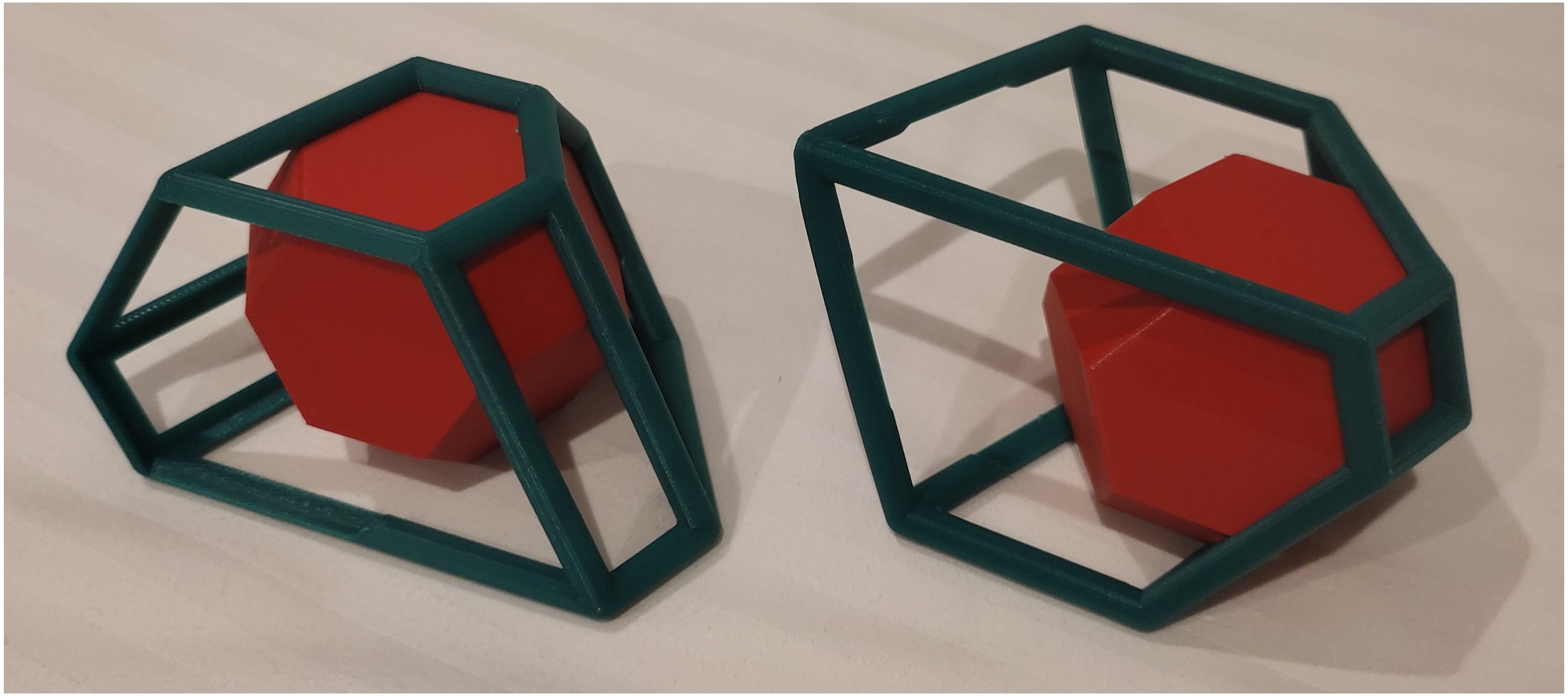
TAKE HOME MESSAGE: Permutahedron and associahedron are nice polytopes...

TAKE HOME MESSAGE...





TAKE HOME MESSAGE...



https://www.ub.edu/comb/vincentpilaud/documents/3dprint/

DEFORMED PERMUTAHEDRA

Postnikov, *Permutohedra, associahedra, and beyond* ('09)

Postnikov–Reiner–Williams, *Faces of generalized permutohedra* ('08)

DEFORMED POLYTOPES

deformation of a polytope $\mathbb{P}$ = polytope $\mathbb{Q}$ such that

- $\mathbb{Q}$ is obtained from $\mathbb{P}$ by moving its vertices such that edge directions are preserved
- $\mathbb{Q}$ is obtained from $\mathbb{P}$ by translating its inequalities without passing through a vertex
- the normal fan of $\mathbb{P}$ refines the normal fan of $\mathbb{Q}$
- $\mathbb{Q}$ is a weak Minkowski summand of $\mathbb{P}$, i.e. there is $\mathbb{R}$ and $\lambda > 0$ such that $\lambda\mathbb{P} = \mathbb{Q} + \mathbb{R}$

POLYWOOD

deformed permutahedron = polymatroid = generalized permutahedron

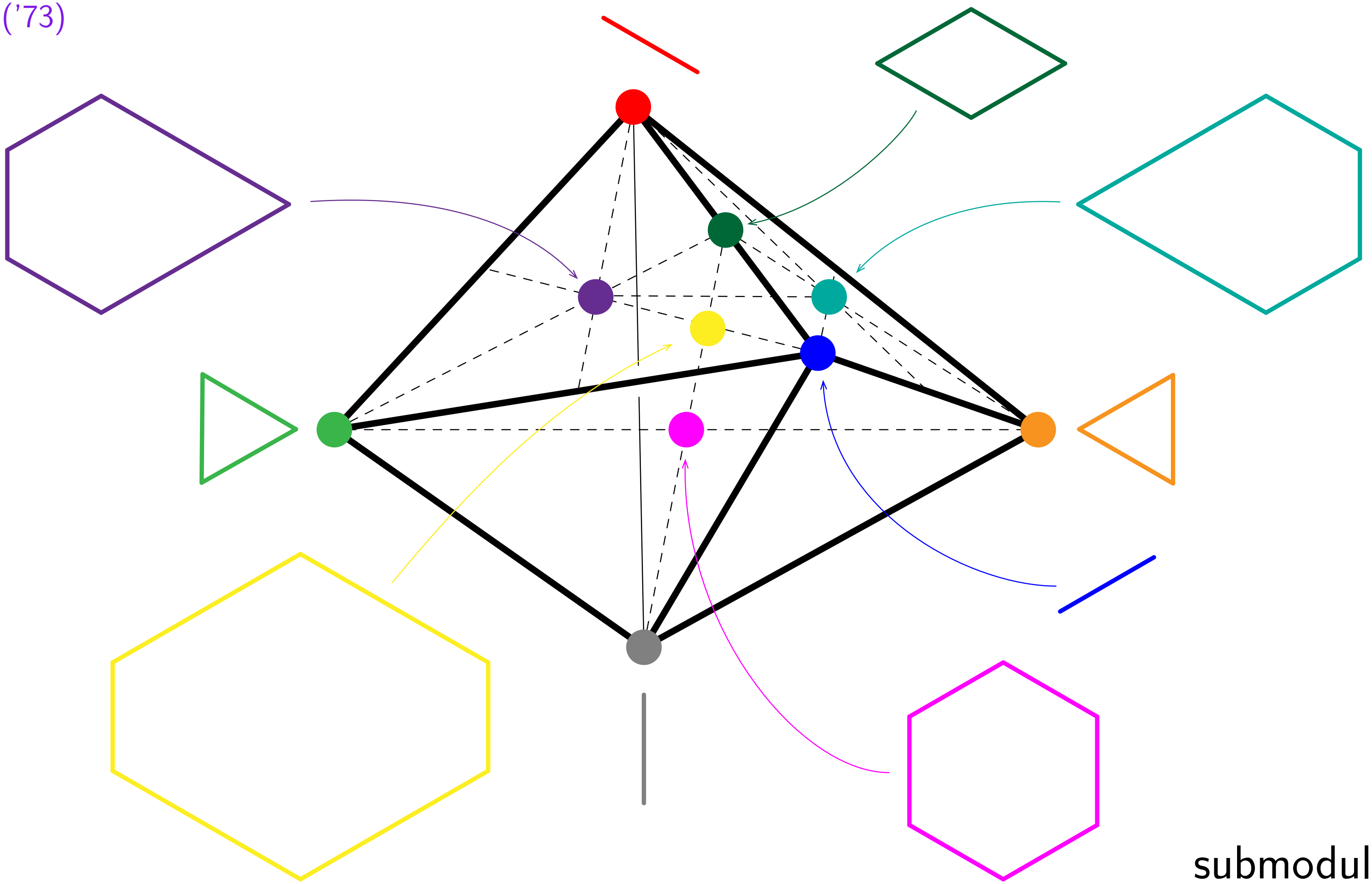
Edmonds ('70) Postnikov ('09)

DEFORMATION CONE

deformation of a polytope $\mathbb{P}$ = polytope $\mathbb{Q}$ such that $\lambda\mathbb{P} = \mathbb{Q} + \mathbb{R}$ for some $\mathbb{R}$ and $\lambda > 0$

deformation cone of $\mathbb{P}$ = all deformations of $\mathbb{P}$ (under dilations and Minkowski sums)

McMullen ('73)



submodular cone

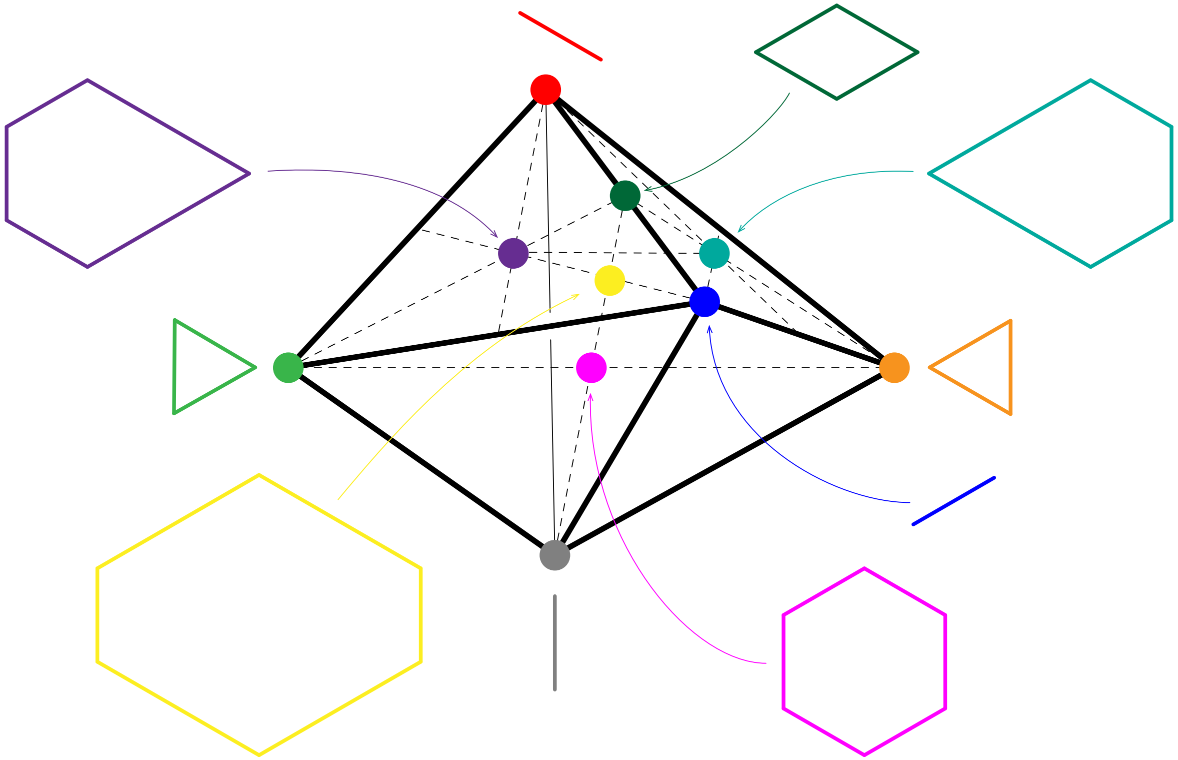
= deformation cone of $\mathbb{P}\mathrm{erm}(3)$

FACETS AND RAYS OF DEFORMATION CONE

THM. The deformation cone of the permutahedron $\mathbb{P}\mathrm{erm}(n)$ is (isomorphic to) the set of submodular functions $h : 2^{[n]} \to \mathbb{R}_{\geq 0}$ satisfying $h(\varnothing) = h([n]) = 0$ and the submodular inequalities $h(I) + h(J) \geq h(I \cap J) + h(I \cup J)$ for all $I, J \subseteq [n]$

THM. The facets correspond to submodular inequalities where $|I \smallsetminus J| = |J \smallsetminus I| = 1$

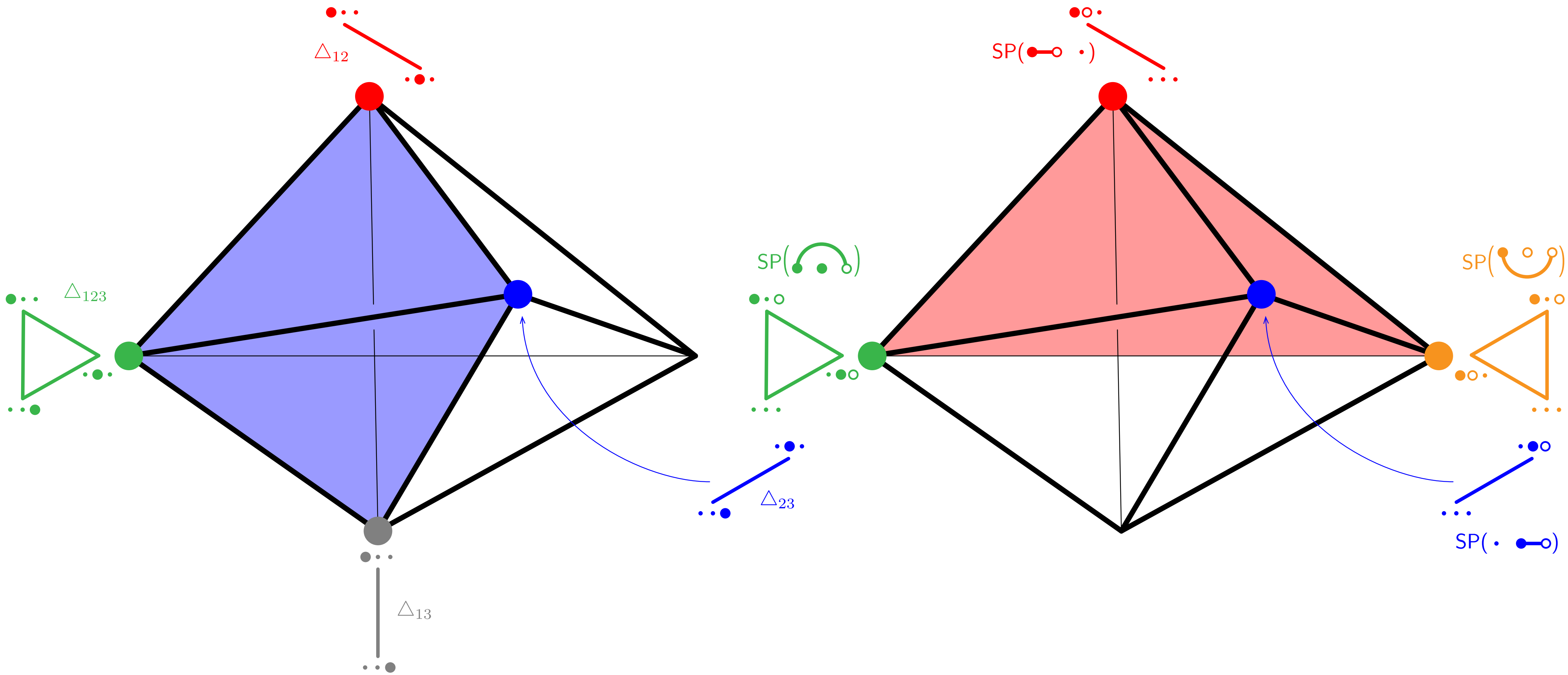
PROB. Describe (or count) the rays of the submodular cone Edmonds ('70)



LINEAR BASES OF THE SUBMODULAR CONE

PROP. Two relevant linear bases of the submodular cone:

- $\{\triangle_I \mid I \subseteq [n]\}$ = faces of the standard simplex $\triangle_{[n]}$ $\qquad \triangle_I = \mathrm{conv}\,\{\boldsymbol{e}_i \mid i \in I\}$
- $\{\mathbb{SP}(i,j,A,B) \mid 1 \le i < j \le n,\; A \sqcup B = [n]\}$ = shard polytopes

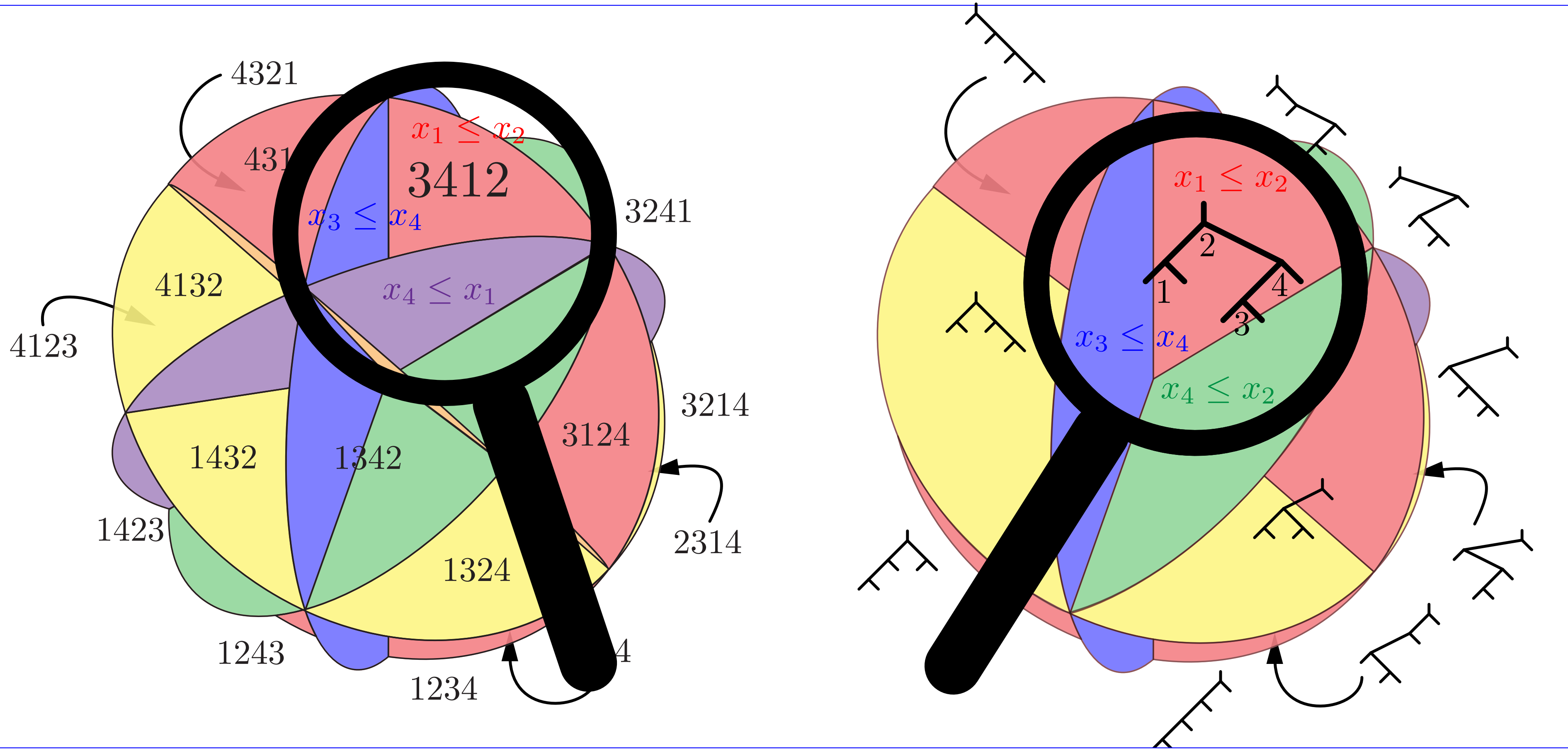


$\mathbb{P}$-POSETS

REM. poset $\preccurlyeq$ on $[n]$ $\longleftrightarrow$ full-dim. cone $\mathbb{C}_{\prec} = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ for all } i \preccurlyeq j\}$

preposet $\preccurlyeq$ on $[n]$ $\longleftrightarrow$ cone $\mathbb{C}_{\prec} = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ for all } i \preccurlyeq j\}$

DEF. $\mathbb{P}$-posets = posets corresponding to maximal cones in the normal fan of $\mathbb{P}$

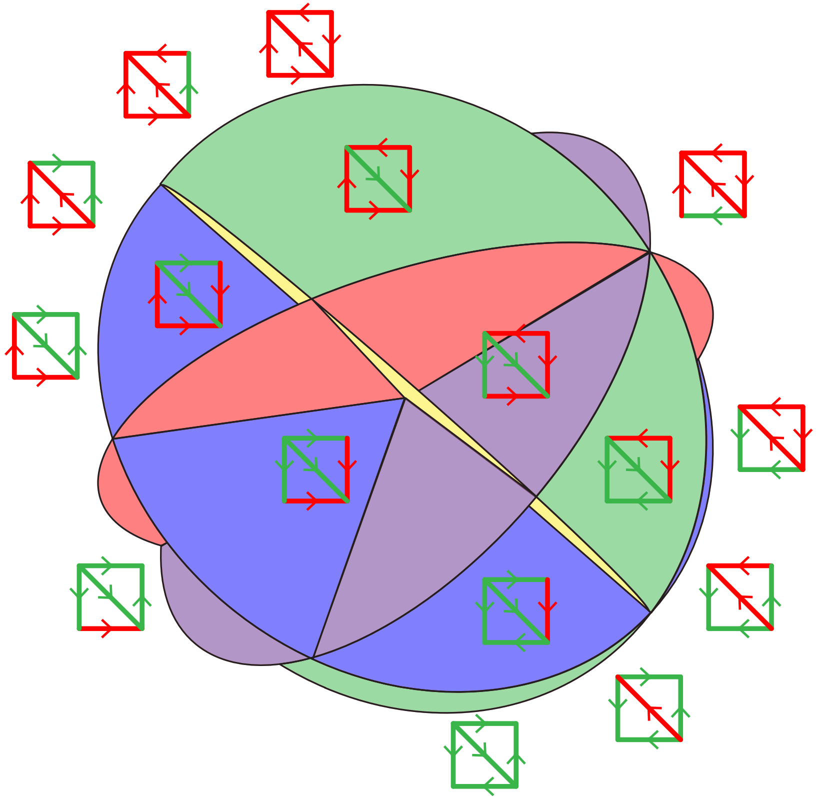


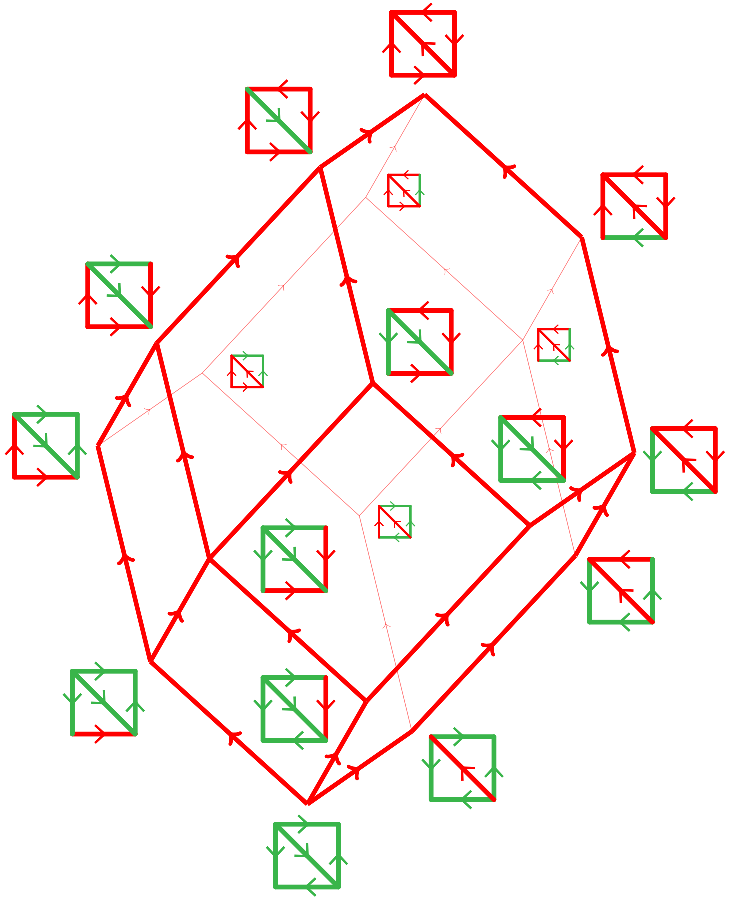
PROB. Describe the $\mathbb{P}$-posets of a deformed permutahedron $\mathbb{P}$, or of a family of DP

SIMPLE DEFORMED PERMUTAHEDRA

REM. TFAE:

- $\mathbb{P}$ is a simple polytope $=$ each vertex is incident to dimension many edges
- the normal fan of $\mathbb{P}$ is simplicial $=$ all cones are simplicial
- the Hasse diagram of any $\mathbb{P}$-poset is a forest





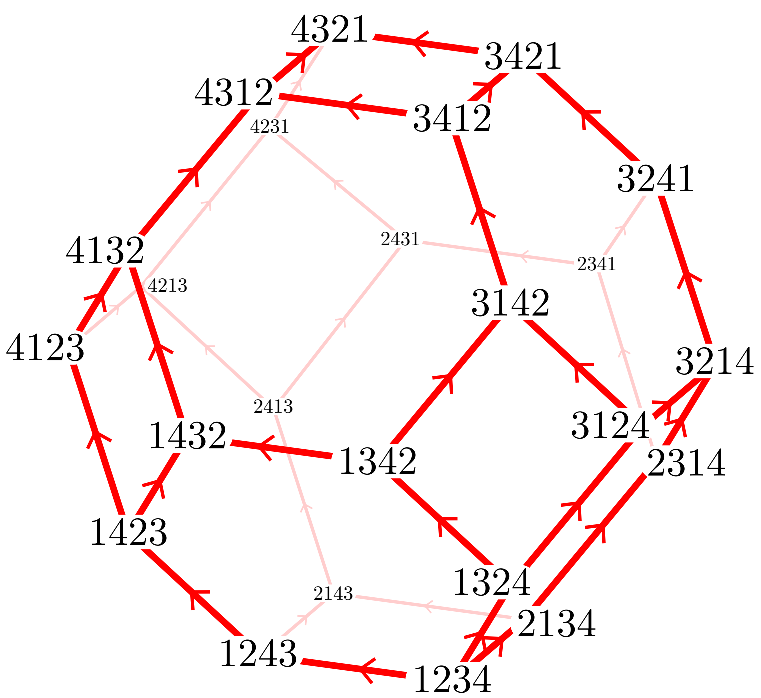
PROB. Characterize the simple polytopes in a given family of deformed permutahedra

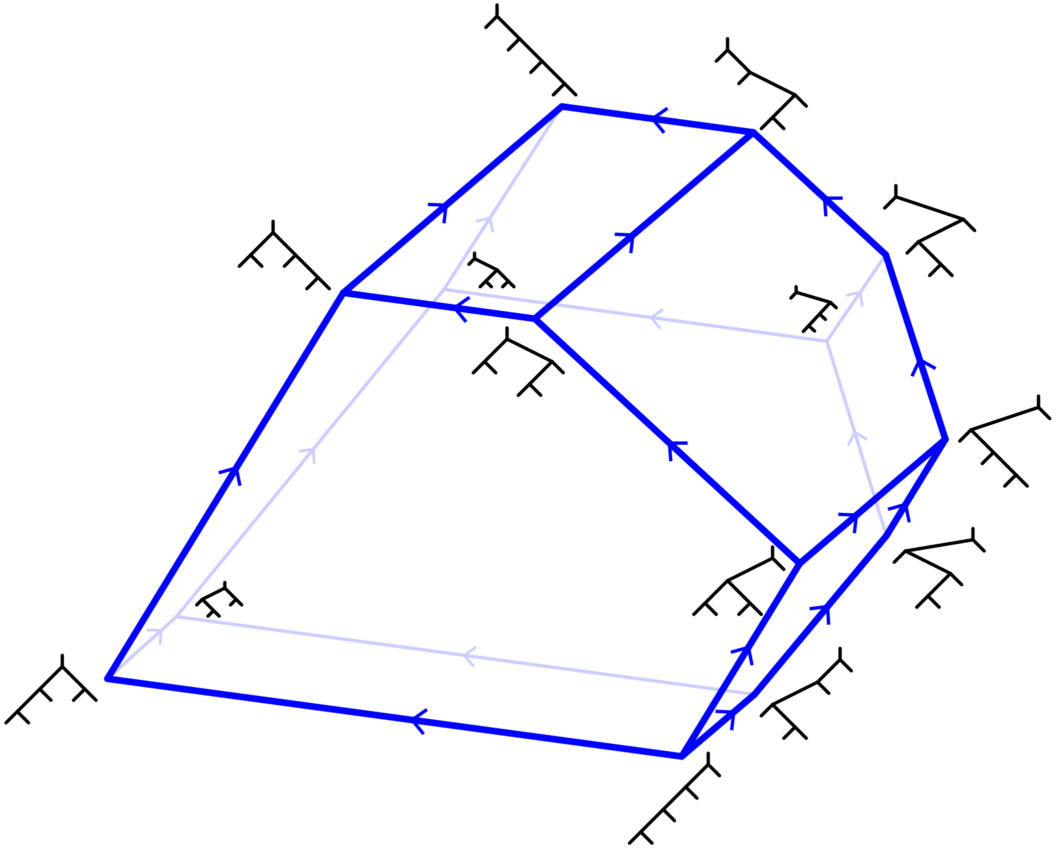
POSET ON $\mathbb{P}$-POSETS

DEF. $\mathbb{P}$ = deformed permutahedron

$\mathcal{P}_{\mathbb{P}}$ = transitive closure of the graph of $\mathbb{P}$ oriented in direction $(n, \dots, 2, 1) - (1, 2, \dots, n)$

$\mathcal{P}_{\mathbb{P}}$ is always a poset on $\mathbb{P}$-posets





PROB. When is $\mathcal{P}_{\mathbb{P}}$ a lattice? a distributive lattice? a semidistributive lattice?

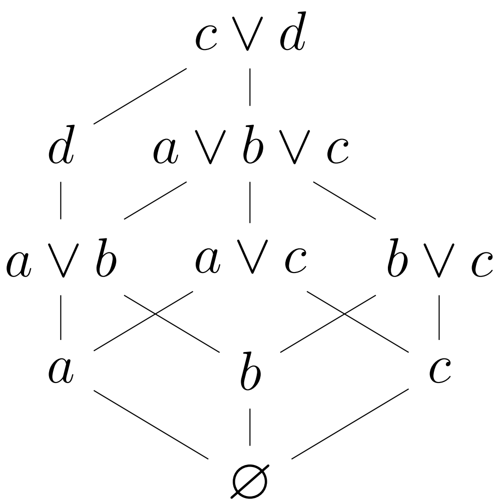
DISTRIBUTIVE & SEMIDISTRIBUTIVE LATTICES

$(L, \leq, \wedge, \vee)$ finite lattice is

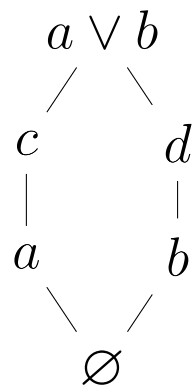
- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

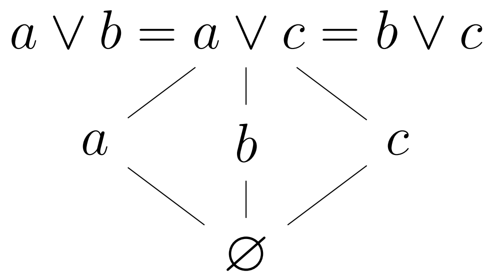
- semidistributive if both join and meet semidistributive



distributive



semidistributive



not semidistributive

DISTRIBUTIVE & SEMIDISTRIBUTIVE LATTICES

$(L, \leq, \wedge, \vee)$ finite lattice is

- distributive if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any $x, y, z \in L$

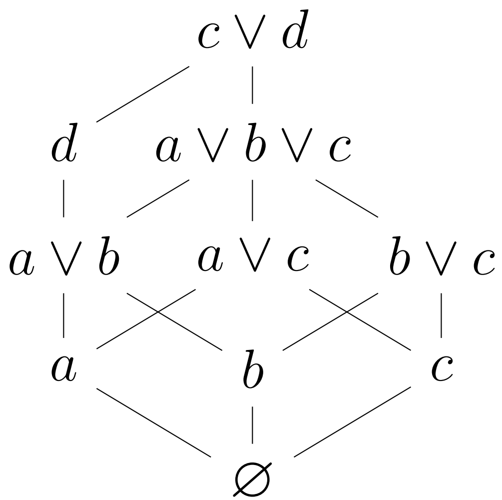
  $\Longrightarrow$ $L \simeq$ inclusion poset of lower ideals of $JI(L)$

  $\Longrightarrow$ canonical join representations $=$ antichains of join irreducibles

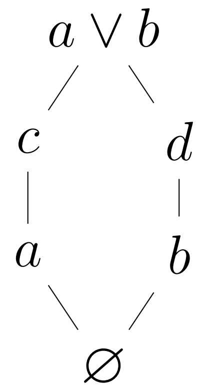
- join semidistributive if $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$

  $\Longrightarrow$ any $y \in L$ admits the canonical join representation $y = \bigvee_{x \lessdot y} k_\vee(x, y)$
  where $k_\vee(x, y)$ is the unique minimal element of $\{z \in L \mid x \vee z = y\}$

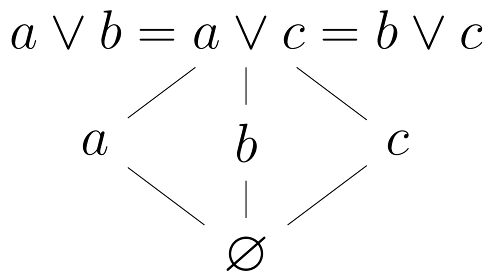
- semidistributive if both join and meet semidistributive



distributive



semidistributive



not semidistributive

SUMMARY...

deformation of a polytope $\mathbb{P}$ = the normal fan of $\mathbb{P}$ refines the normal fan of $\mathbb{Q}$
deformation cone of $\mathbb{P}$ = all deformations of $\mathbb{P}$ (under dilations and Minkowski sums)

deformed permutahedra = polymatroids = submodular functions

Edmonds ('70) Postnikov ('09)

Two relevant linear bases of the submodular cone:

- $\{\triangle_I \mid I \subseteq [n]\}$ = faces of the standard simplex $\triangle_n$
- $\{\mathbb{SP}(i, j, A, B) \mid 1 \le i < j \le n,\ A \sqcup B = [n]\}$ = shard polytopes

When you have a family of deformed permutahedra, always consider:

- vertex posets (and face preposets) of your polytopes
- simple polytopes in your family
- oriented skeleta of your polytopes

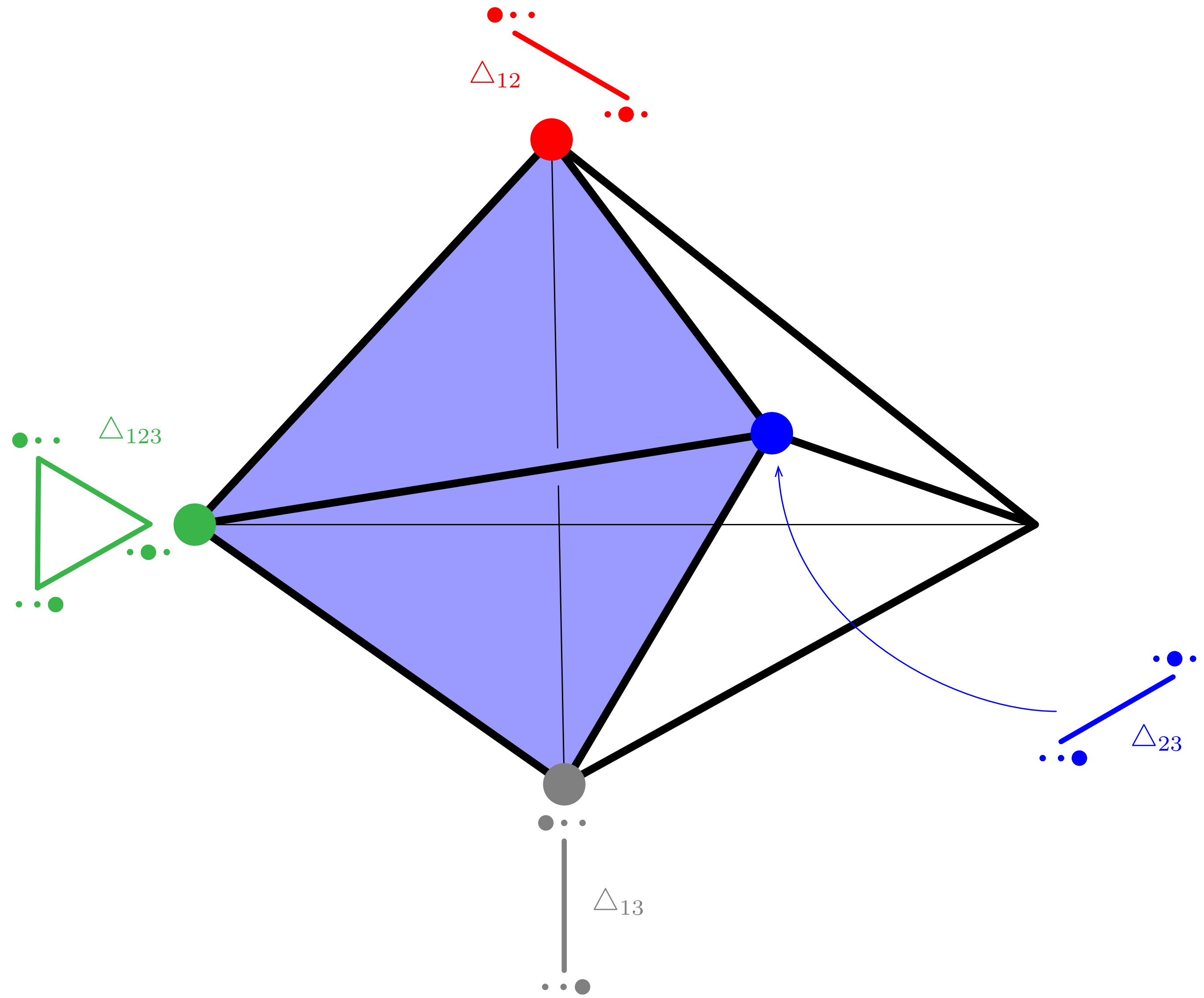
HYPERGRAPHIC POLYTOPES

Benedetti–Bergeron–Machacek, *Hypergraphic polytopes* ('19)
Bergeron–P., *Interval hypergraphic polytopes* ('25)
Bastidas–Gélinas–P.–Poullot–Sack–Tzanaki, *Interval hypergraphic polytopes (or deformed associahedra), Tamari interval posets, and weeping willows* ('26^{+})

STANDARD SIMPLICES FORM A BASIS

PROP. $\{\triangle_I \mid I \subseteq [n]\}$ linear basis of the submodular cone $\quad \triangle_I = \mathrm{conv}\{\boldsymbol{e}_i \mid i \in I\}$

Postnikov ('09) Ardila–Benedetti–Doker ('10)

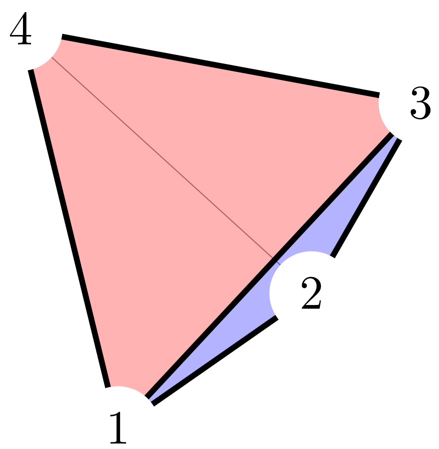


HYPERGRAPHIC POLYTOPES

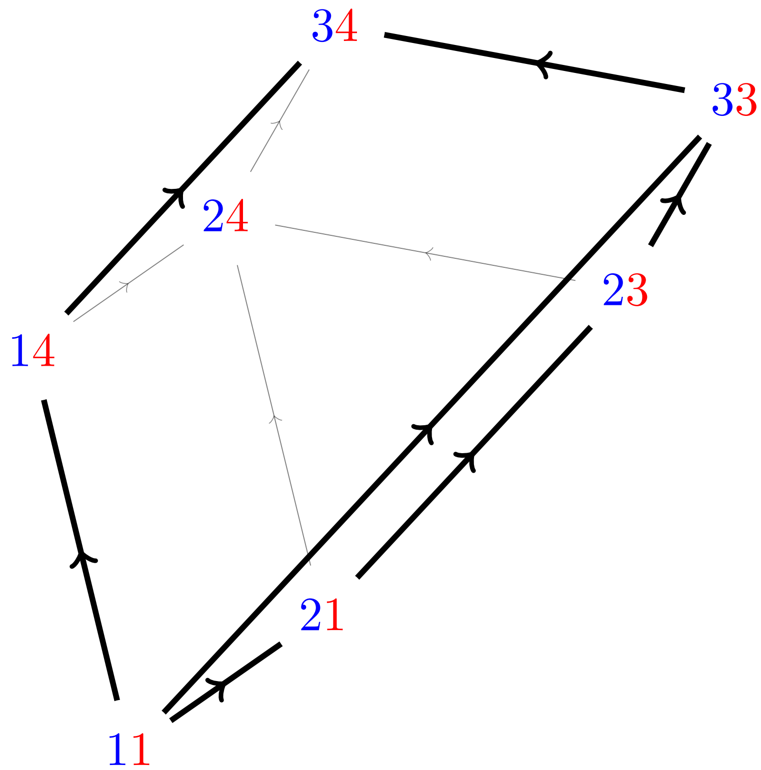
$\mathbb{H}$ = hypergraph on $[n]$

hypergraphic polytope $\triangle(\mathbb{H}) = \sum_{H \in \mathbb{H}} \triangle_H$ where $\triangle_H = \{\boldsymbol{e}_h \mid h \in H\}$

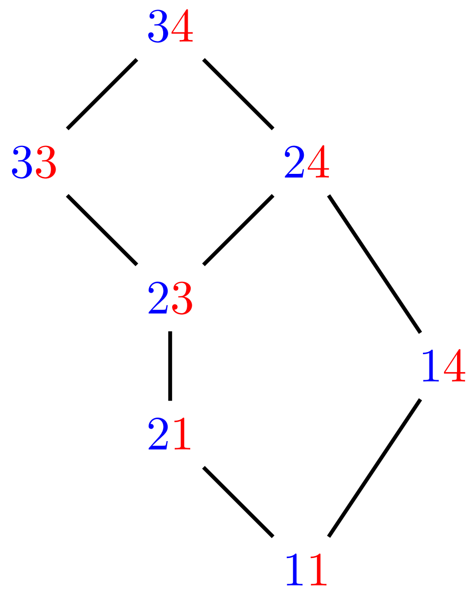
hypergraphic poset $P_{\mathbb{H}}$ = transitive closure of the graph of $\triangle(\mathbb{H})$ oriented in direction $\boldsymbol{\omega} = (n, \dots, 1) - (1, \dots, n)$



$\triangle_{123}$ and $\triangle_{134}$



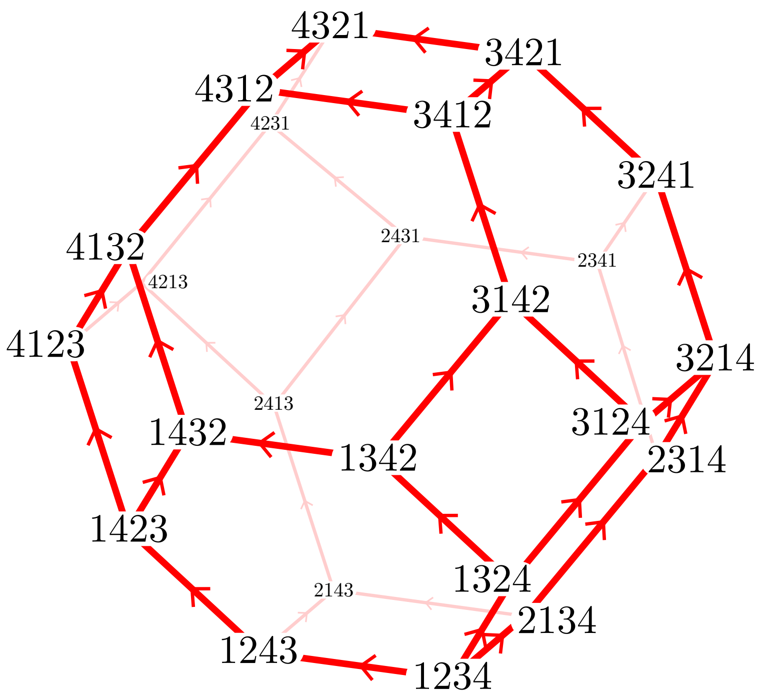
$\triangle(\mathbb{H}) = \triangle_{123} + \triangle_{134}$



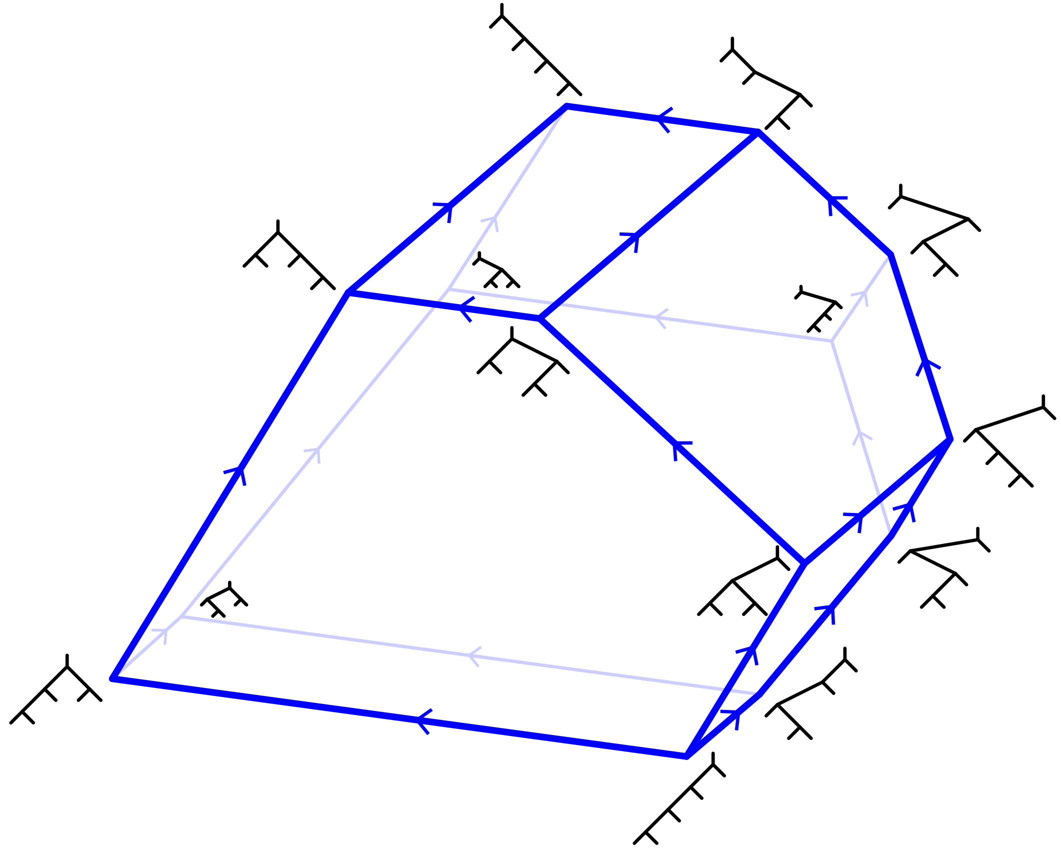
$P_{\mathbb{H}}$

PERMUTAHEDRON & ASSOCIAHEDRON

$\mathbb{H}$ = all 2-element subsets of $[n]$
$\triangle(\mathbb{H})$ = permutahedron $\mathbb{P}\mathsf{erm}(n)$
$P_{\mathbb{H}}$ = weak order on permutations



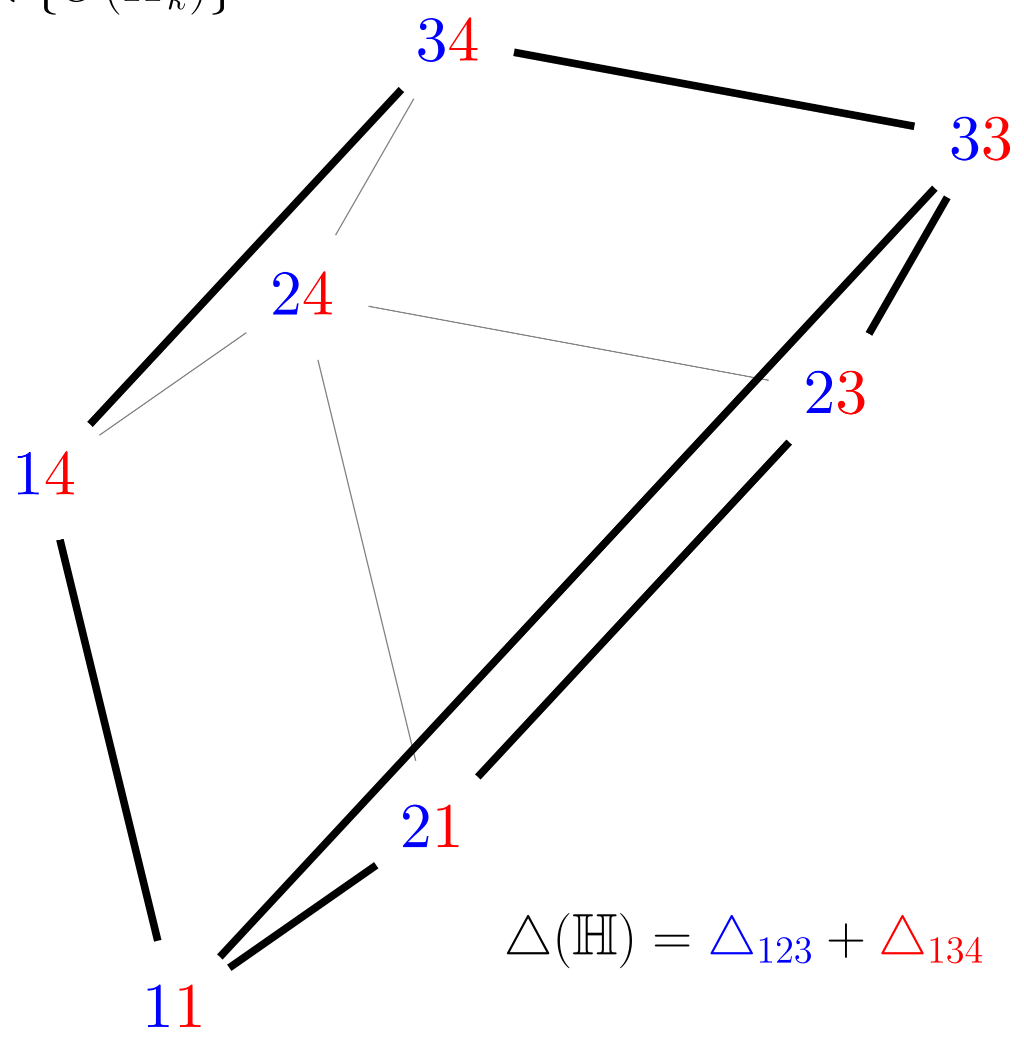
$\mathbb{H}$ = all intervals $[i, j]$ of $[n]$
$\triangle(\mathbb{H})$ = associahedron $\mathbb{A}\mathsf{sso}(n)$
$P_{\mathbb{H}}$ = Tamari lattice on binary trees



ACYCLIC ORIENTATIONS OF $\mathbb{H}$

vertices of $\triangle(\mathbb{H})$ = acyclic orientations of $\mathbb{H}$ = maps O from $\mathbb{H}$ to $[n]$ such that

- $O(H) \in H$ for all $H \in \mathbb{H}$
- there is no $H_1, \dots, H_k$ with $k \geq 2$ such that $O(H_{i+1}) \in H_i \smallsetminus \{O(H_i)\}$ for $i \in [k-1]$ and $O(H_1) \in H_k \smallsetminus \{O(H_k)\}$



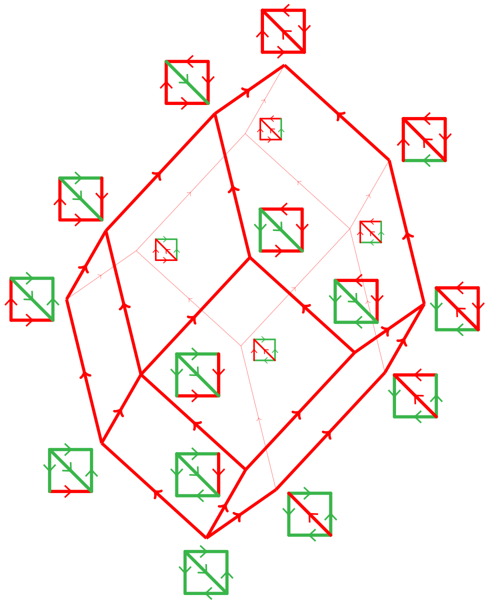
$\triangle(\mathbb{H}) = \triangle_{123} + \triangle_{134}$

LITTLE MORE IS KNOWN IN GENERAL

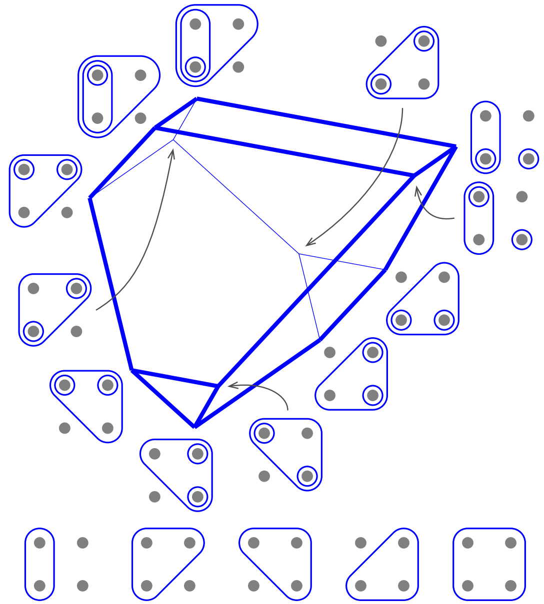
PROP. $\triangle_{\mathbb{H}}$-posets $=$ transitive closures of acyclic orientations of $\mathbb{H}$

PROB. Characterize hypergraphs $\mathbb{H}$ whose hypergraphic polytope $\triangle_{\mathbb{H}}$ is simple

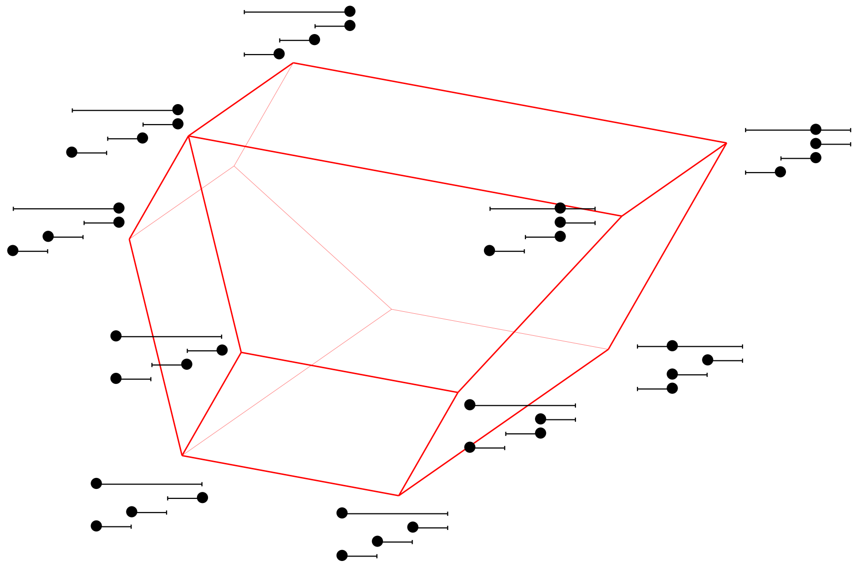
PROB. Characterize hypergraphs $\mathbb{H}$ whose oriented skeleton defines a lattice



graphical zonotopes



nestohedra



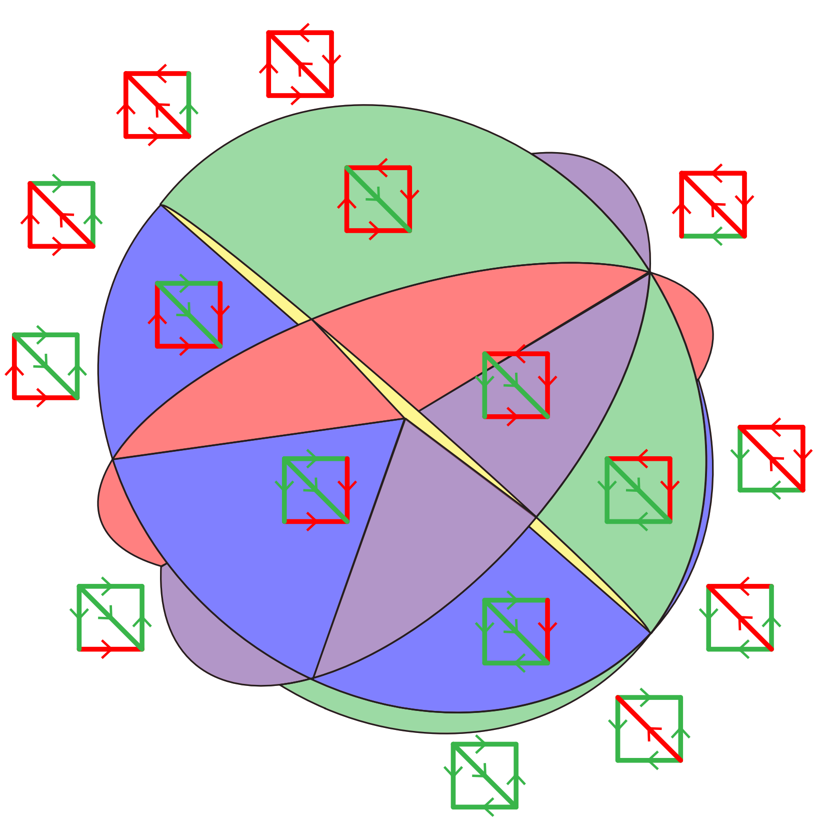
interval hypergraphic polytopes

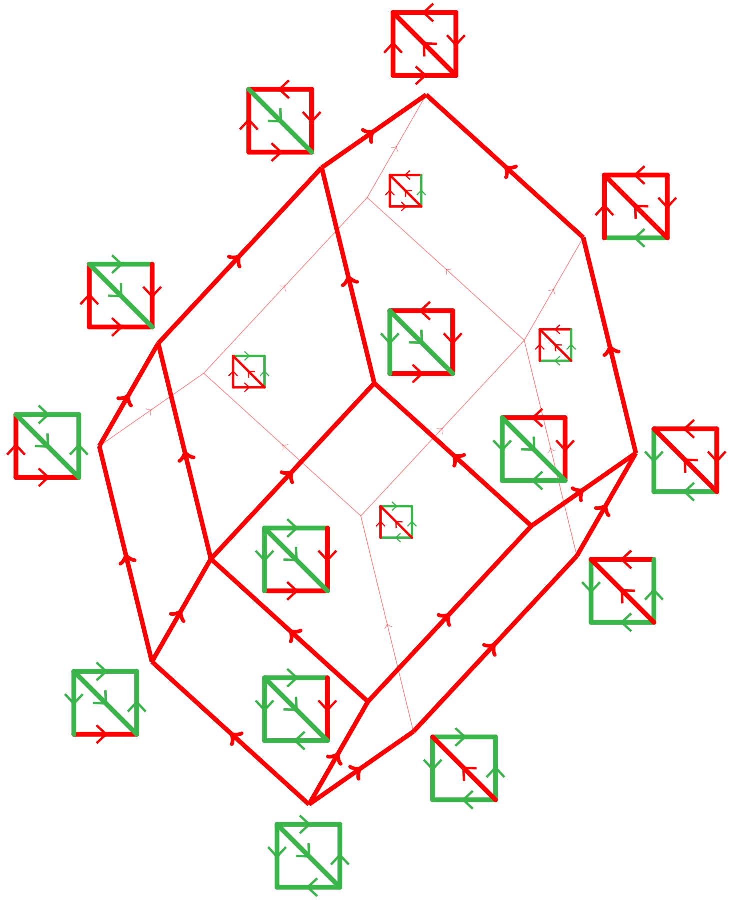
GRAPHICAL ZONOTOPES

D directed acyclic graph

graphical arrangement $\mathcal{H}_D$ = arrangement of hyperplanes $x_u = x_v$ for all arcs $(u, v) \in D$

graphical zonotope $\mathbb{Z}\mathrm{ono}(D)$ = Minkowski sum of $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all arcs $(u, v) \in D$





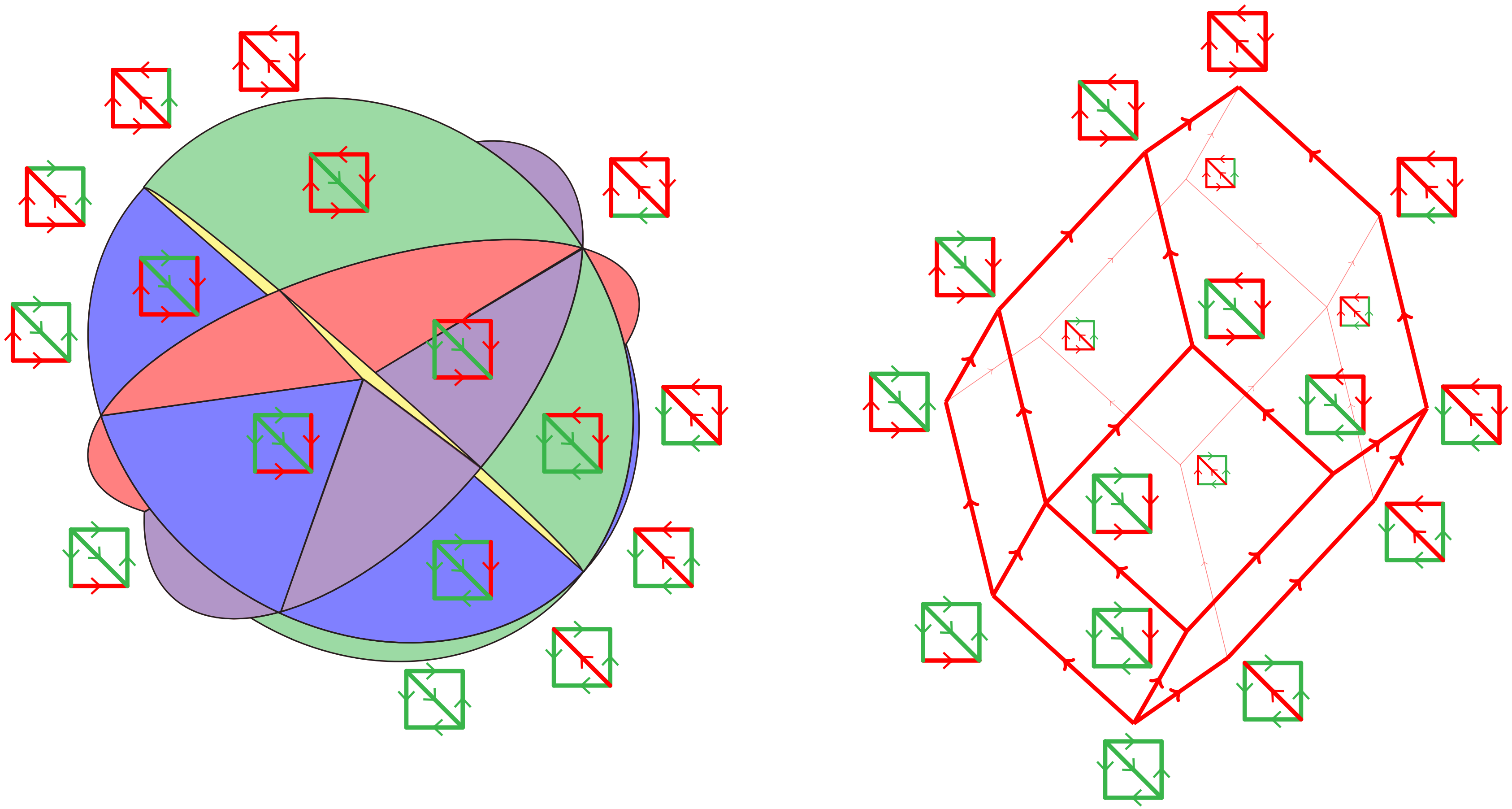
hyperplanes of $\mathcal{H}_D$	$\longleftrightarrow$	summands of $\mathbb{Z}\mathrm{ono}(D)$	$\longleftrightarrow$	arcs of D
regions of $\mathcal{H}_D$	$\longleftrightarrow$	vertices of $\mathbb{Z}\mathrm{ono}(D)$	$\longleftrightarrow$	acyclic reorientations of D
poset of regions of $\mathcal{H}_D$	$\longleftrightarrow$	oriented graph of $\mathbb{Z}\mathrm{ono}(D)$	$\longleftrightarrow$	acyclic reorientation poset of D

$\mathbb{Z}\text{ono}(D)$-POSETS

D directed acyclic graph

graphical arrangement $\mathcal{H}_D$ = arrangement of hyperplanes $x_u = x_v$ for all arcs $(u, v) \in D$

graphical zonotope $\mathbb{Z}\text{ono}(D)$ = Minkowski sum of $[\boldsymbol{e}_u, \boldsymbol{e}_v]$ for all arcs $(u, v) \in D$

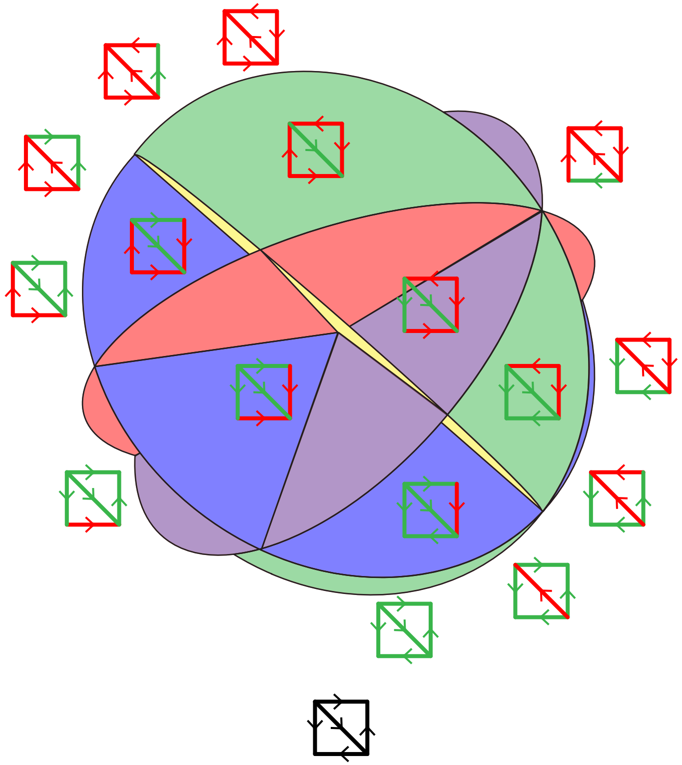


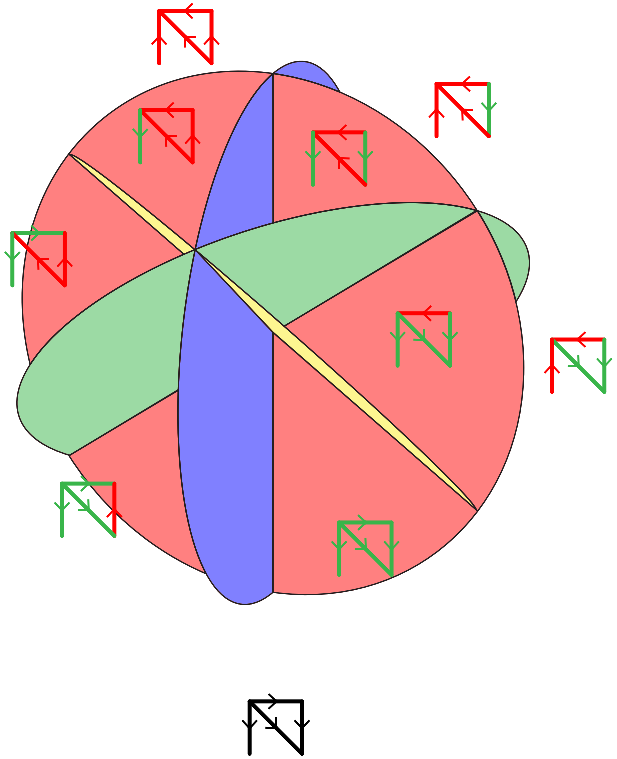
PROP. $\mathbb{Z}\text{ono}(D)$-posets = transitive closures of acyclic reorientations of D

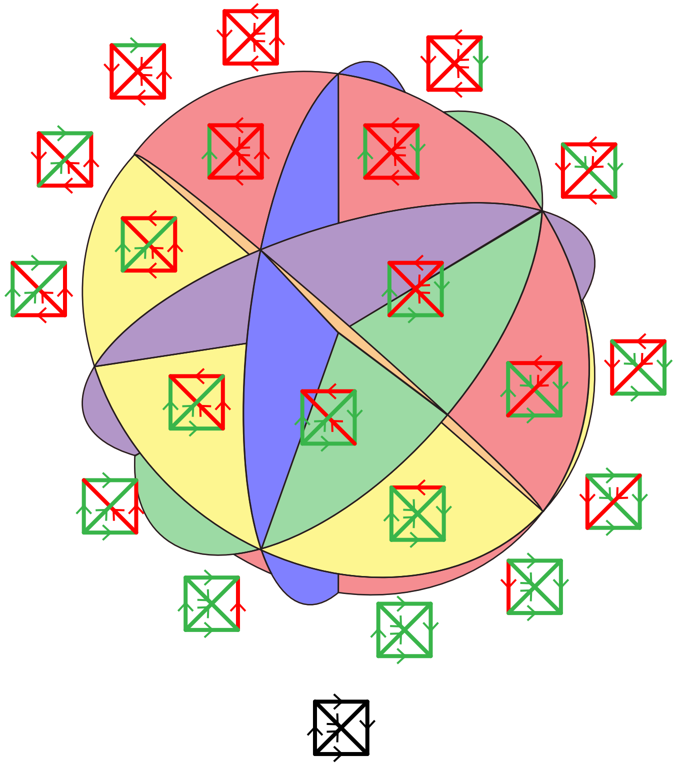
SIMPLE GRAPHICAL ZONOTOPES

PROP. TFAE:

- the graphical arrangement $\mathcal{H}_D$ is simplicial
- the graphical zonotope $\mathbb{Z}\mathrm{ono}(D)$ is simple
- the transitive reduction of any acyclic reorientation of D is a forest
- D is chordful (a.k.a. block graph) $=$ any cycle of D induces a clique


not simplicial


simplicial


simplicial

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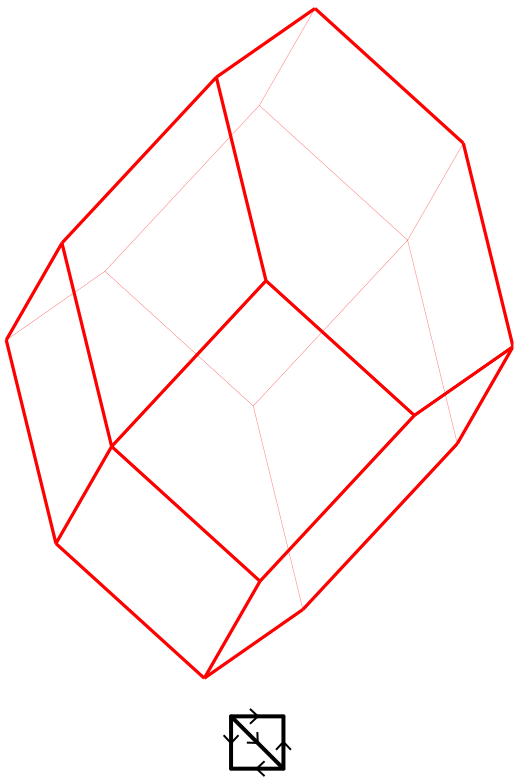
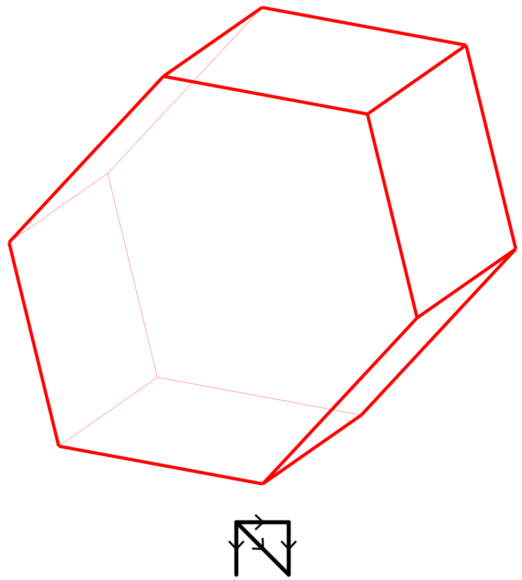
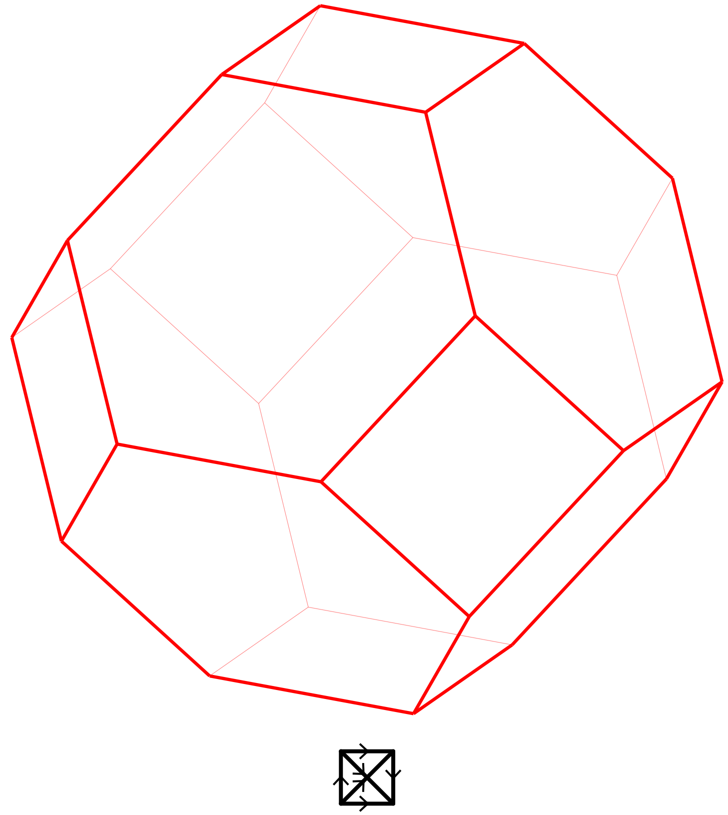


not simple



simple

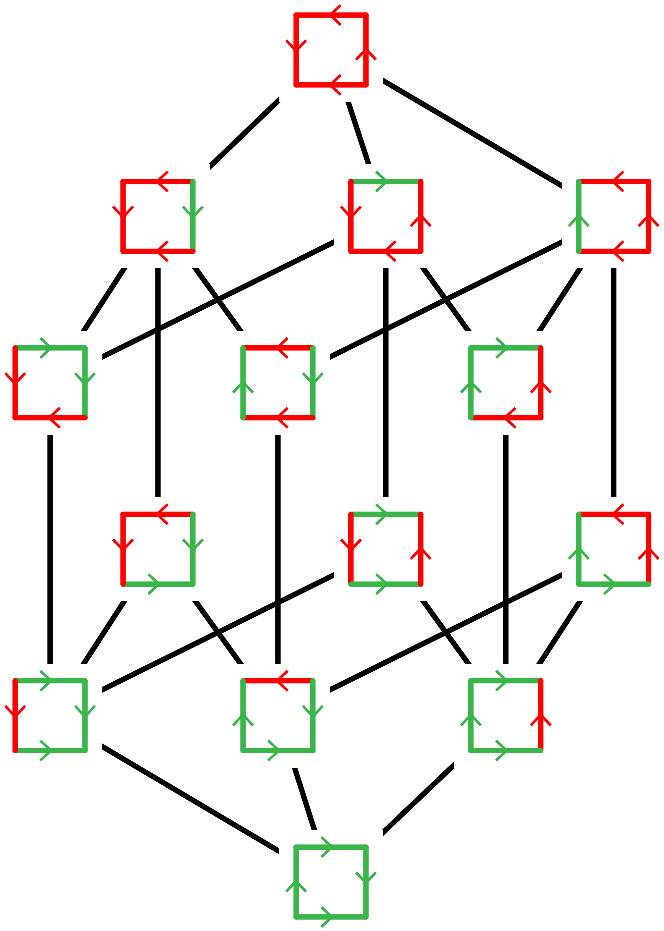


simple

ACYCLIC REORIENTATION POSETS

D directed acyclic graph

$\mathcal{AR}_D =$ all acyclic reorientations of D, ordered by inclusion of their sets of reversed arcs



minimal element D

maximal element $\bar{D}$

self-dual under reversing all arcs

cover relations = flipping a single arc

flippable arcs of E = transitive reduction of E

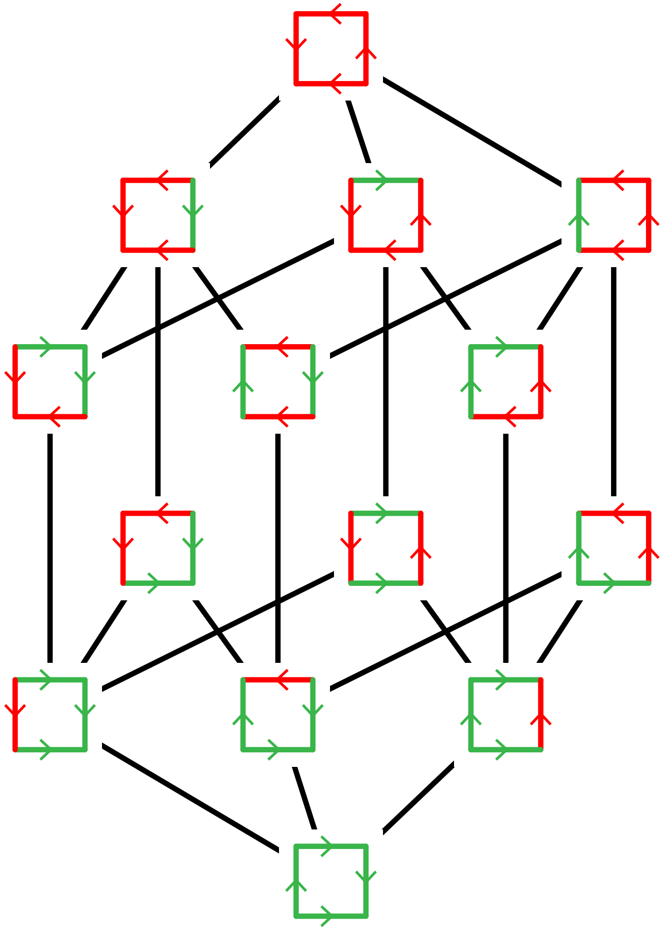
$= E \smallsetminus \{(u, v) \in E \mid \exists$ directed path $u \rightsquigarrow v$ in $E\}$

PROP. $\mathcal{AR}_D = \mathcal{P}_{\mathbb{Z}\text{ono}(D)}$

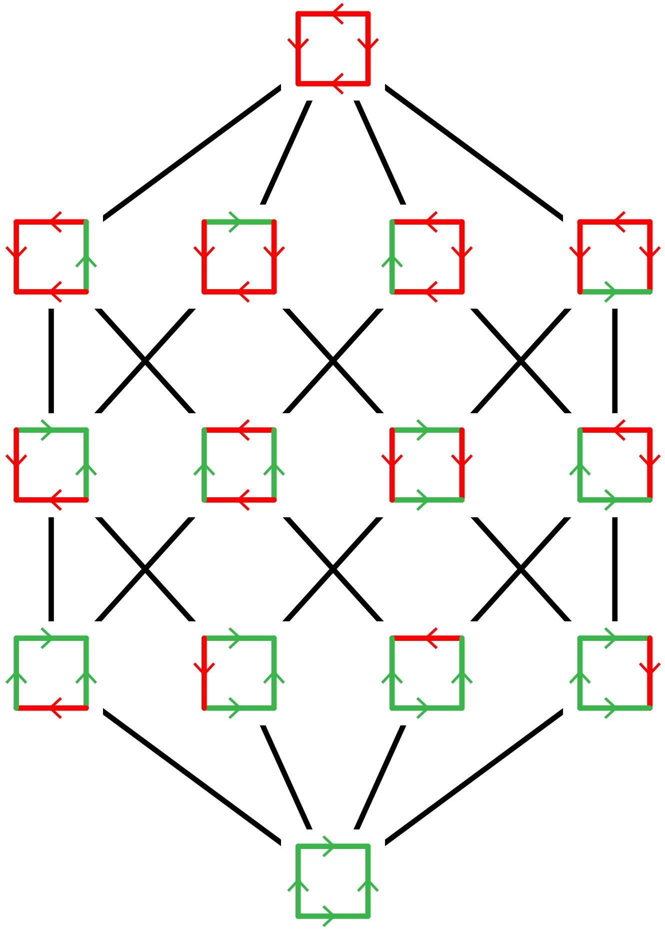
ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate P. ('25)



lattice



not lattice

ACYCLIC REORIENTATION LATTICES

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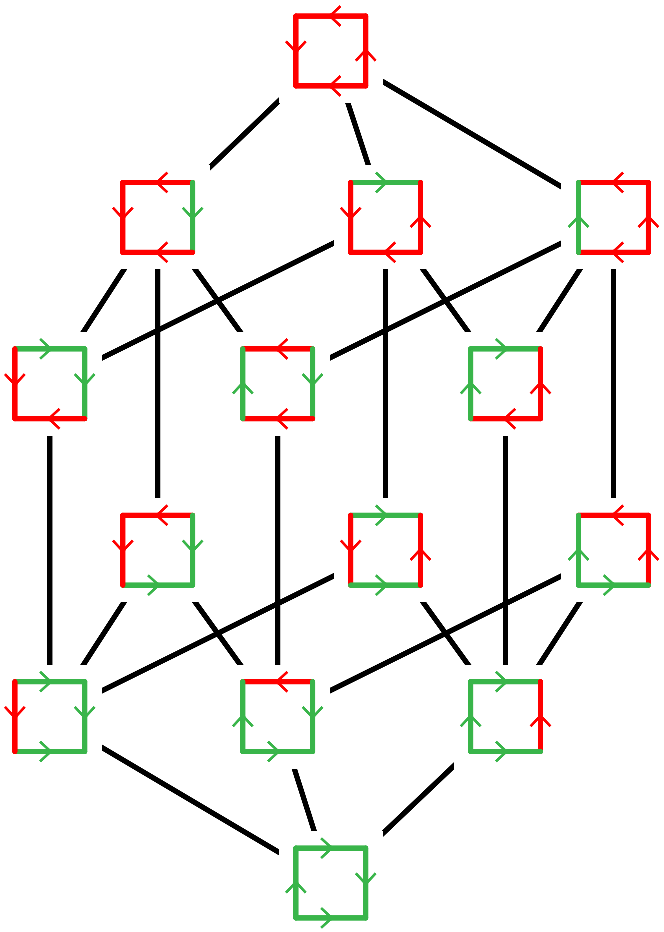
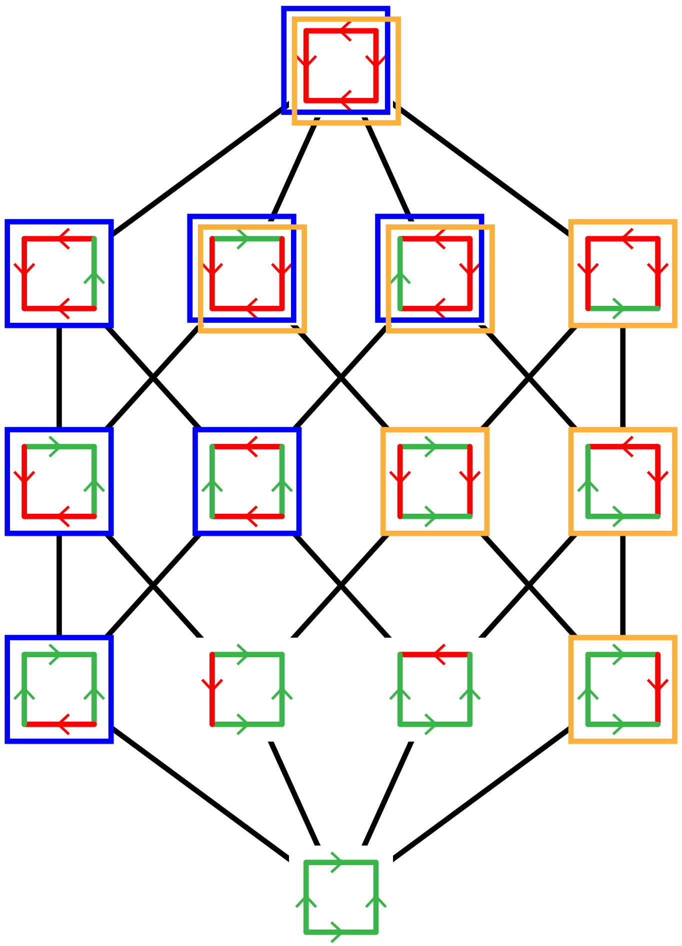


lattice



not lattice

ACYCLIC REORIENTATION LATTICES

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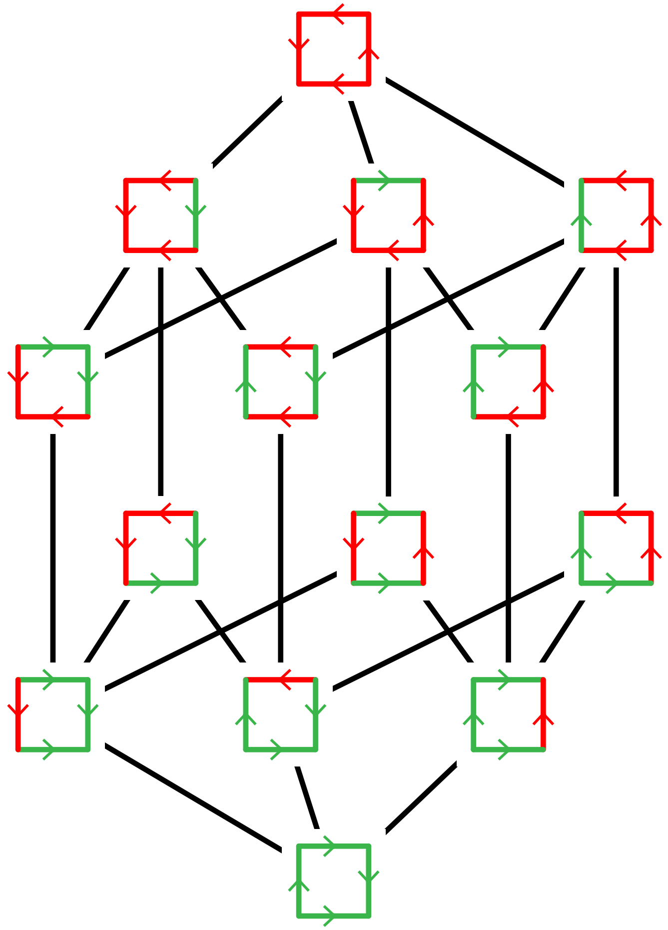


X subset of arcs of D is

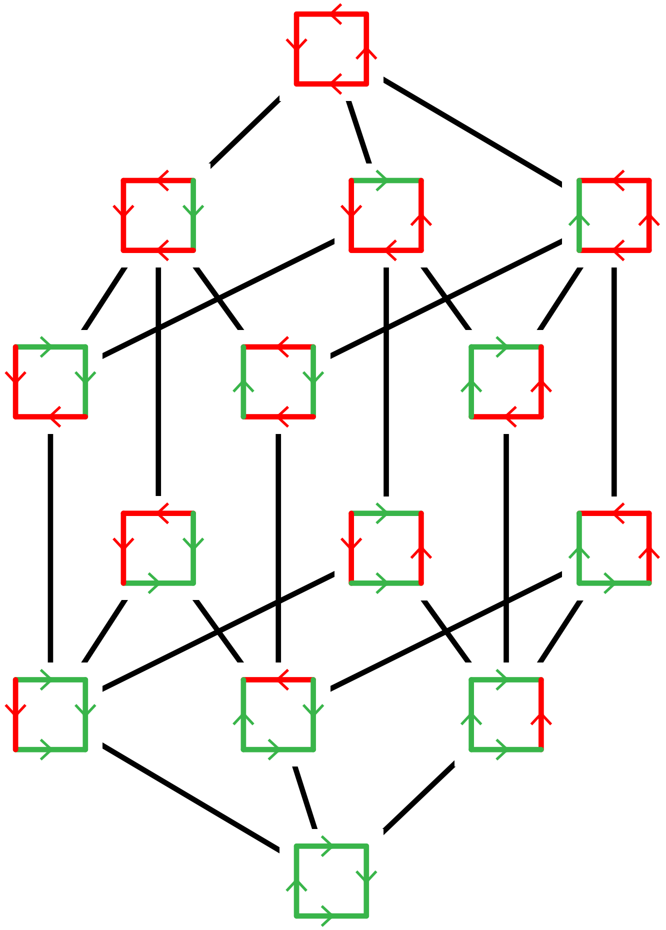
- closed if all arcs of D in the transitive closure of X also belong to X
- coclosed if its complement is closed
- biclosed if it is closed and coclosed

PROP. If D vertebrate, P. ('25)
X biclosed $\iff$ the reorientation of X is acyclic

ACYCLIC REORIENTATION LATTICES

D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest

THM. $\mathcal{AR}_D$ lattice $\iff$ D vertebrate P. ('25)



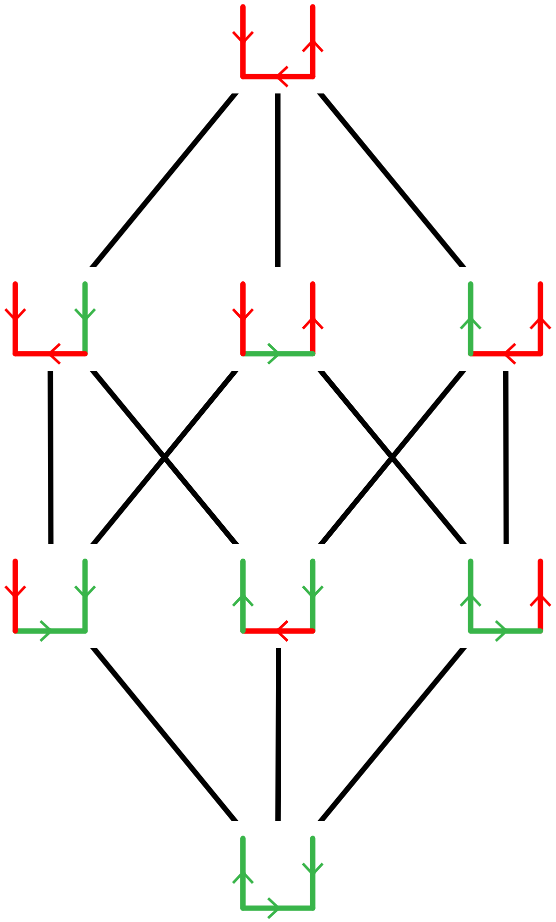
PROP. If D vertebrate, P. ('25)

$$\mathrm{bwd}(E_1 \vee \ldots \vee E_k) =$$
transitive closure of $\mathrm{bwd}(E_1) \cup \cdots \cup \mathrm{bwd}(E_k)$

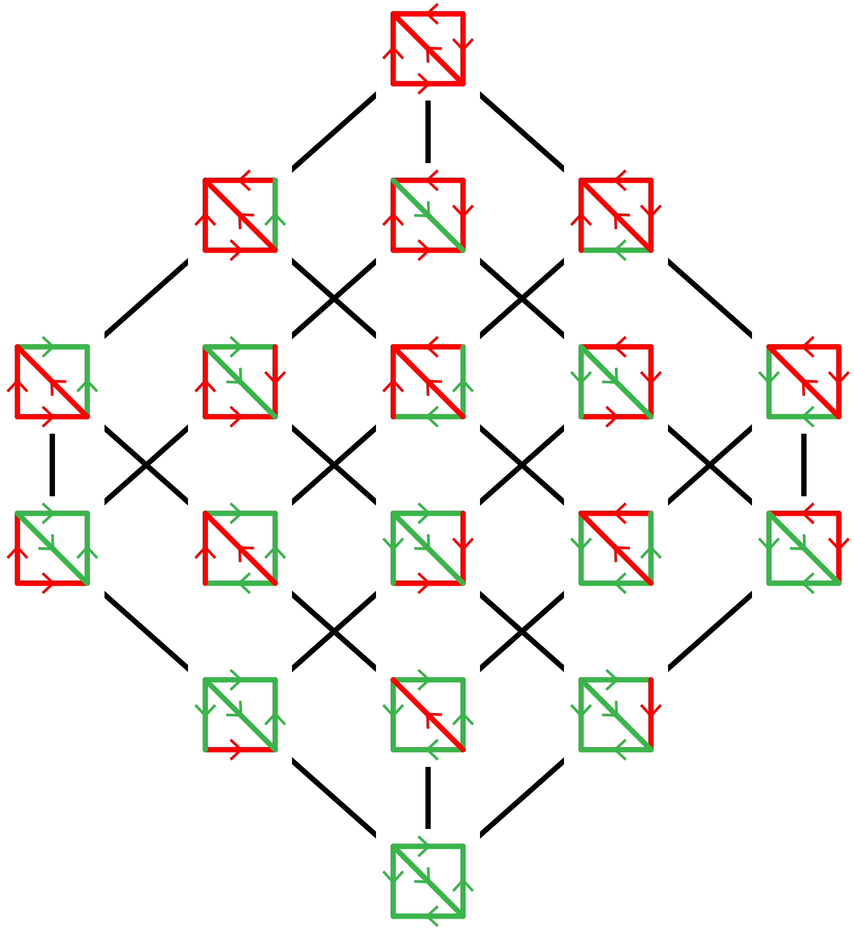
$$\mathrm{fwd}(E_1 \wedge \ldots \wedge E_k) =$$
transitive closure of $\mathrm{fwd}(E_1) \cup \cdots \cup \mathrm{fwd}(E_k)$

DISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

THM. $\mathcal{AR}_D$ distributive lattice $\iff$ D forest $\iff$ $\mathcal{AR}_D$ boolean lattice   P. ('25)



distributive



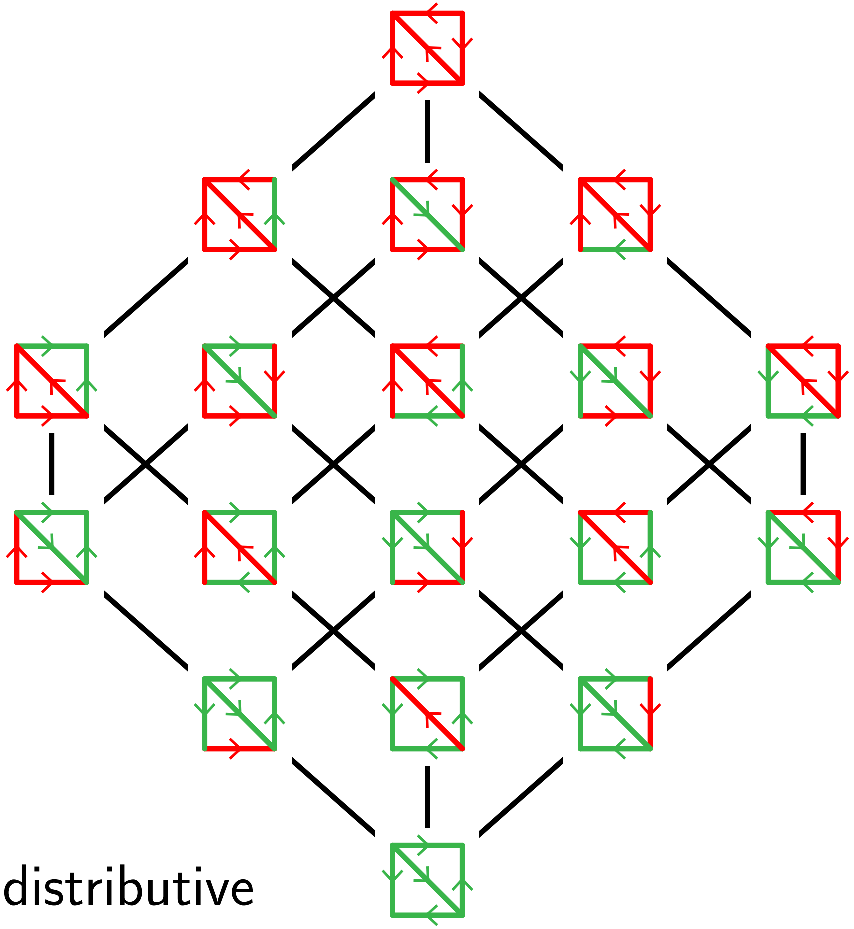
not distributive

SEMIDISTRIBUTIVE ACYCLIC REORIENTATION LATTICES

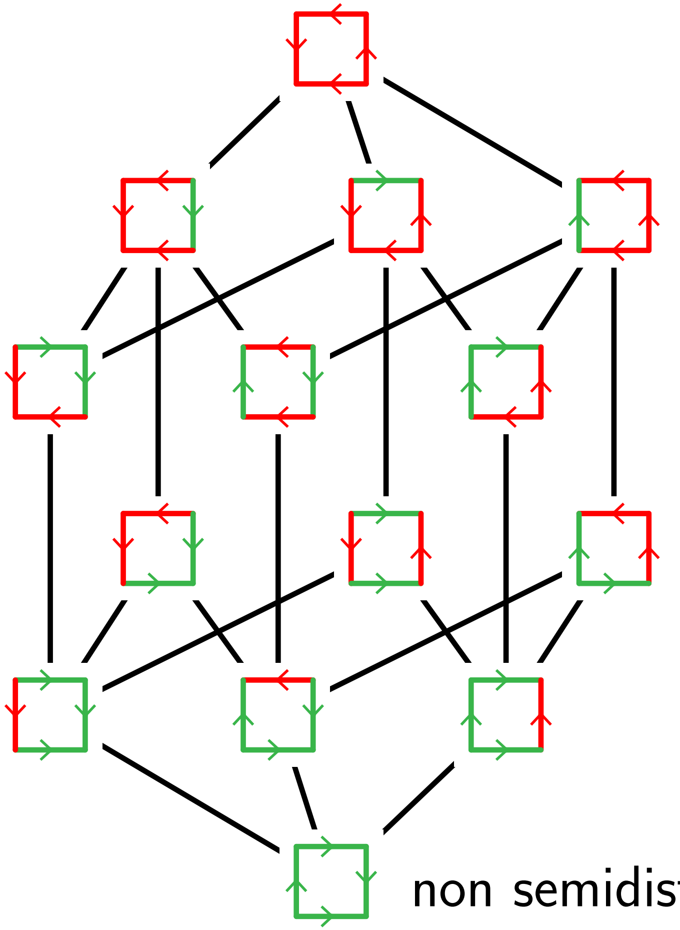
D skeletal $=$

- D vertebrate $=$ transitive reduction of any induced subgraph of D is a forest
- D filled $=$ any directed path joining the endpoints of an arc in D induces a tournament

THM. $\mathcal{AR}_D$ semidistributive lattice $\iff$ D is skeletal P. ('25)



semidistributive



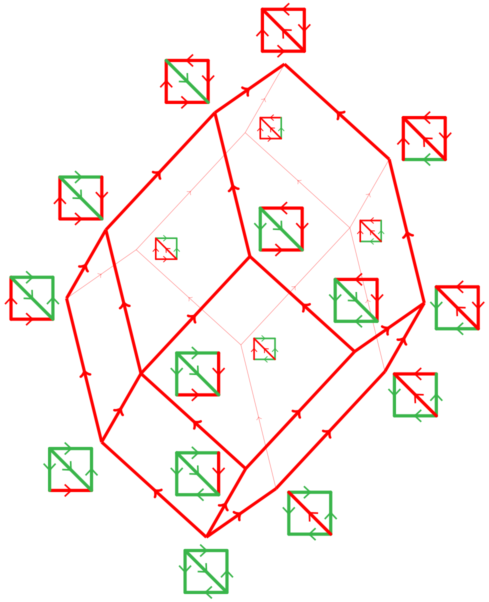
non semidistributive

LITTLE MORE IS KNOWN IN GENERAL

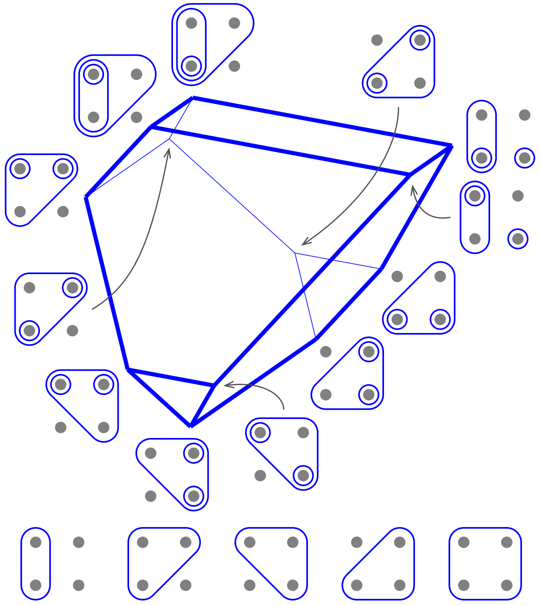
PROP. $\triangle_{\mathbb{H}}$-posets $=$ transitive closures of acyclic orientations of $\mathbb{H}$

PROB. Characterize hypergraphs $\mathbb{H}$ whose hypergraphic polytope $\triangle_{\mathbb{H}}$ is simple

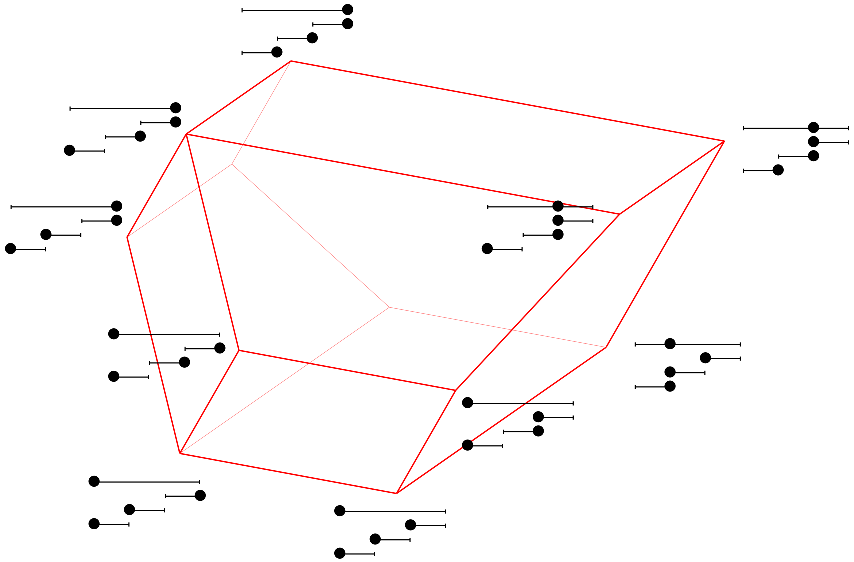
PROB. Characterize hypergraphs $\mathbb{H}$ whose oriented skeleton defines a lattice



graphical zonotopes



nestohedra



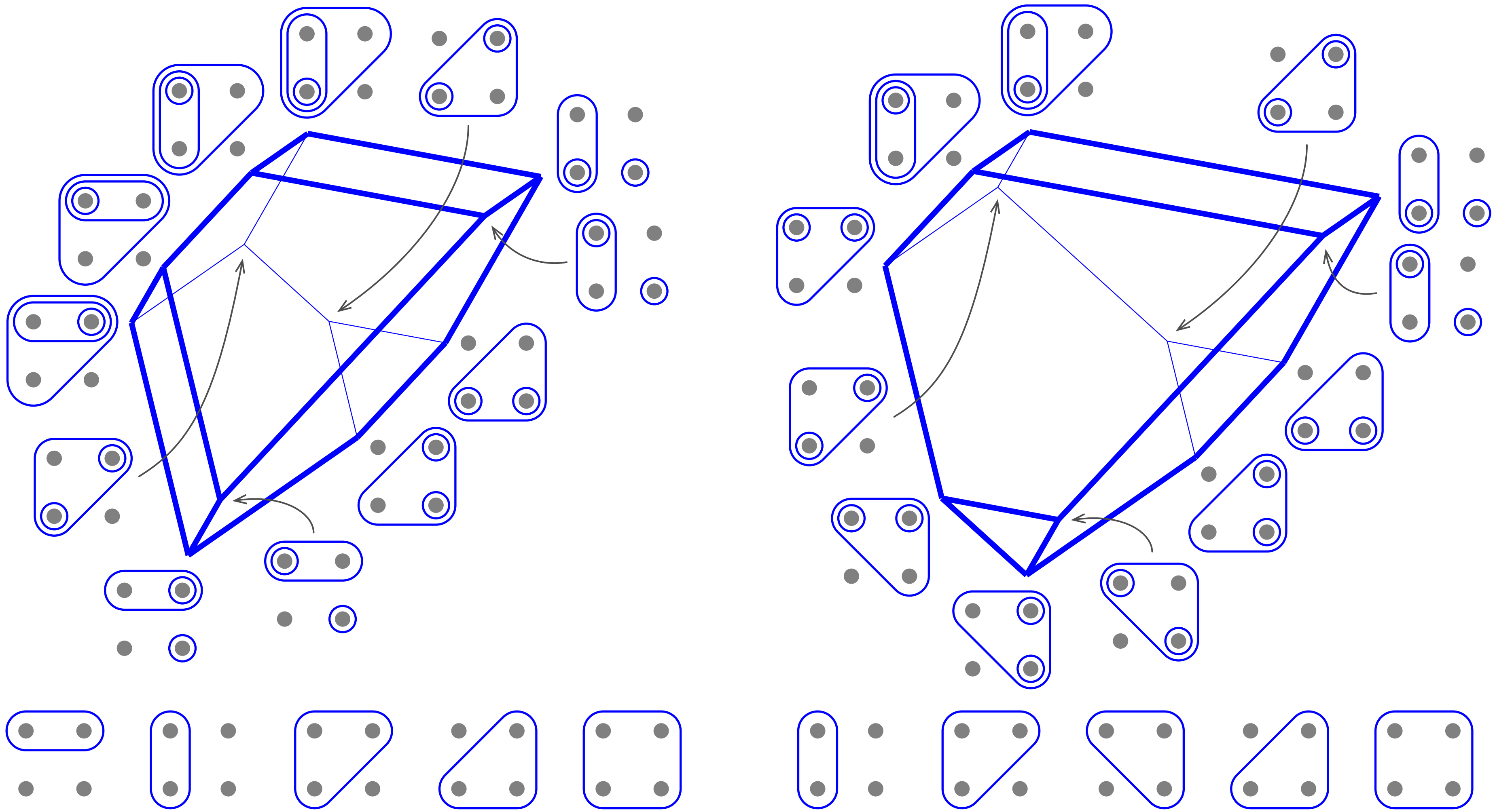
interval hypergraphic polytopes

NESTOHEDRA

building set on $[n]$ = collection $\mathcal{B}$ of non-empty subsets of $[n]$ such that

- $\mathcal{B}$ contains all singletons $\{s\}$ for $s \in [n]$
- if $B, B' \in \mathcal{B}$ and $B \cap B' \neq \varnothing$, then $B \cup B' \in \mathcal{B}$

nestohedron $\mathbb{N}\mathrm{est}(\mathcal{B}) = \sum_{B \in \mathcal{B}} \triangle_B$ where $\triangle_B := \mathrm{conv}\{\boldsymbol{e}_b \mid b \in B\}$

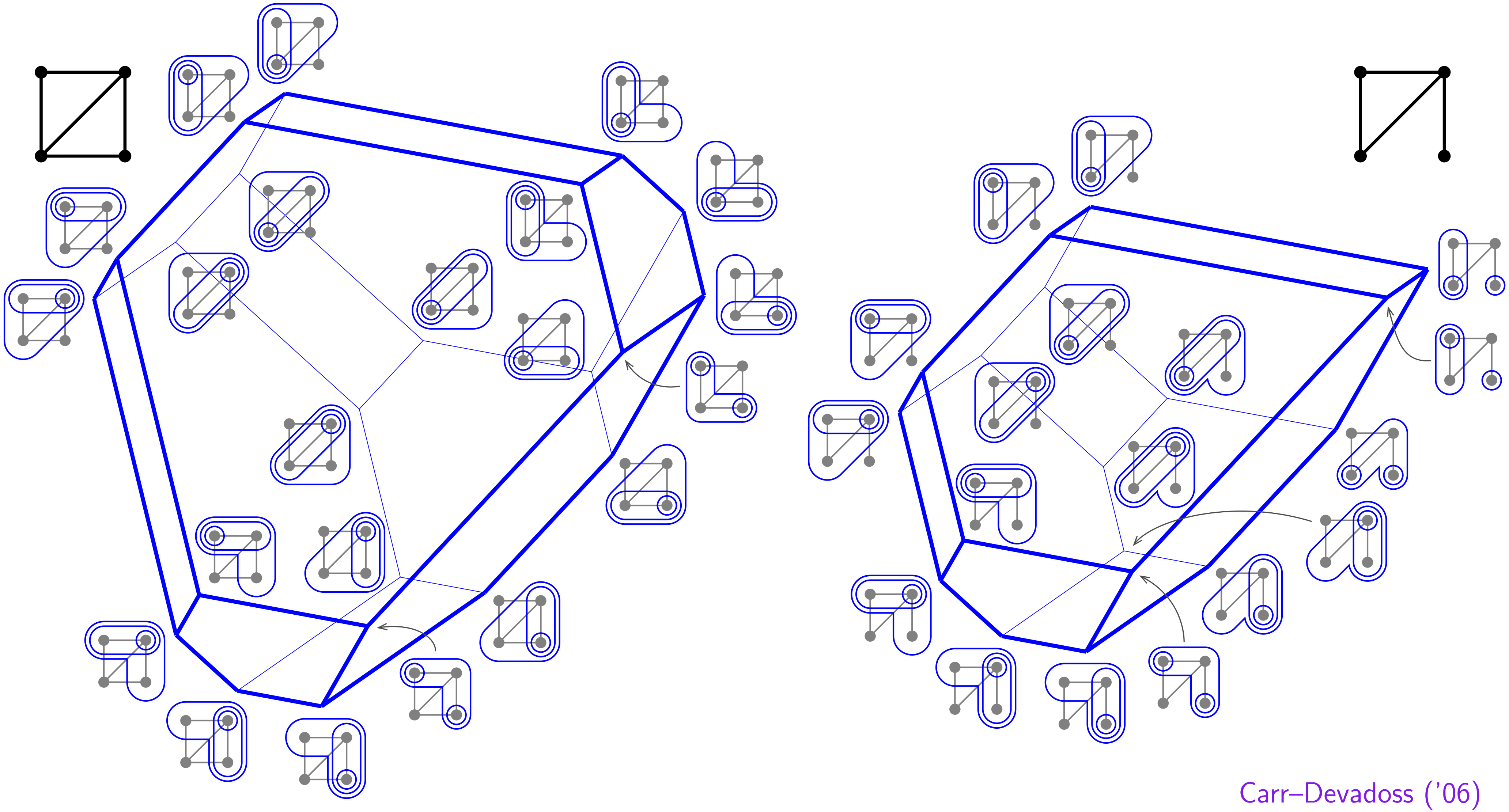


Feichtner–Koslov ('04), Feichtner–Sturmfels ('05), Postnikov ('09), Zelevinski ('06)

GRAPHICAL NESTOHEDRA

<u>tube</u> of G = connected induced subgraph of G

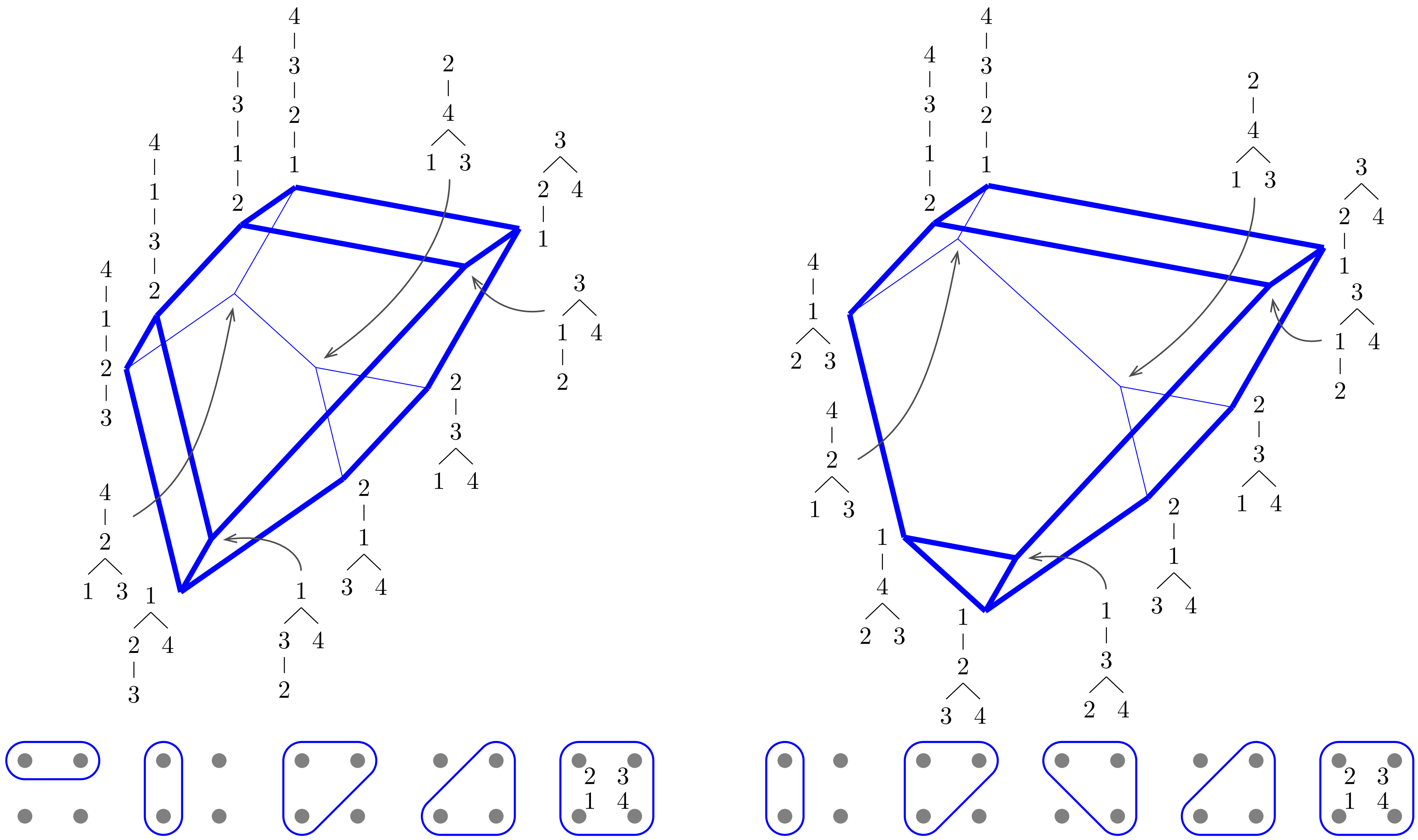
<u>graphical building set</u> of G = collection of all tubes of G



Carr–Devadoss ('06)

PROP. $\mathbb{N}$est$(\mathcal{B})$-posets $=$ rooted forests on $[n]$ such that $\mathcal{B}$ contains the descendant set of each node, but no union of the descendant sets of $k \geq 2$ incomparable nodes

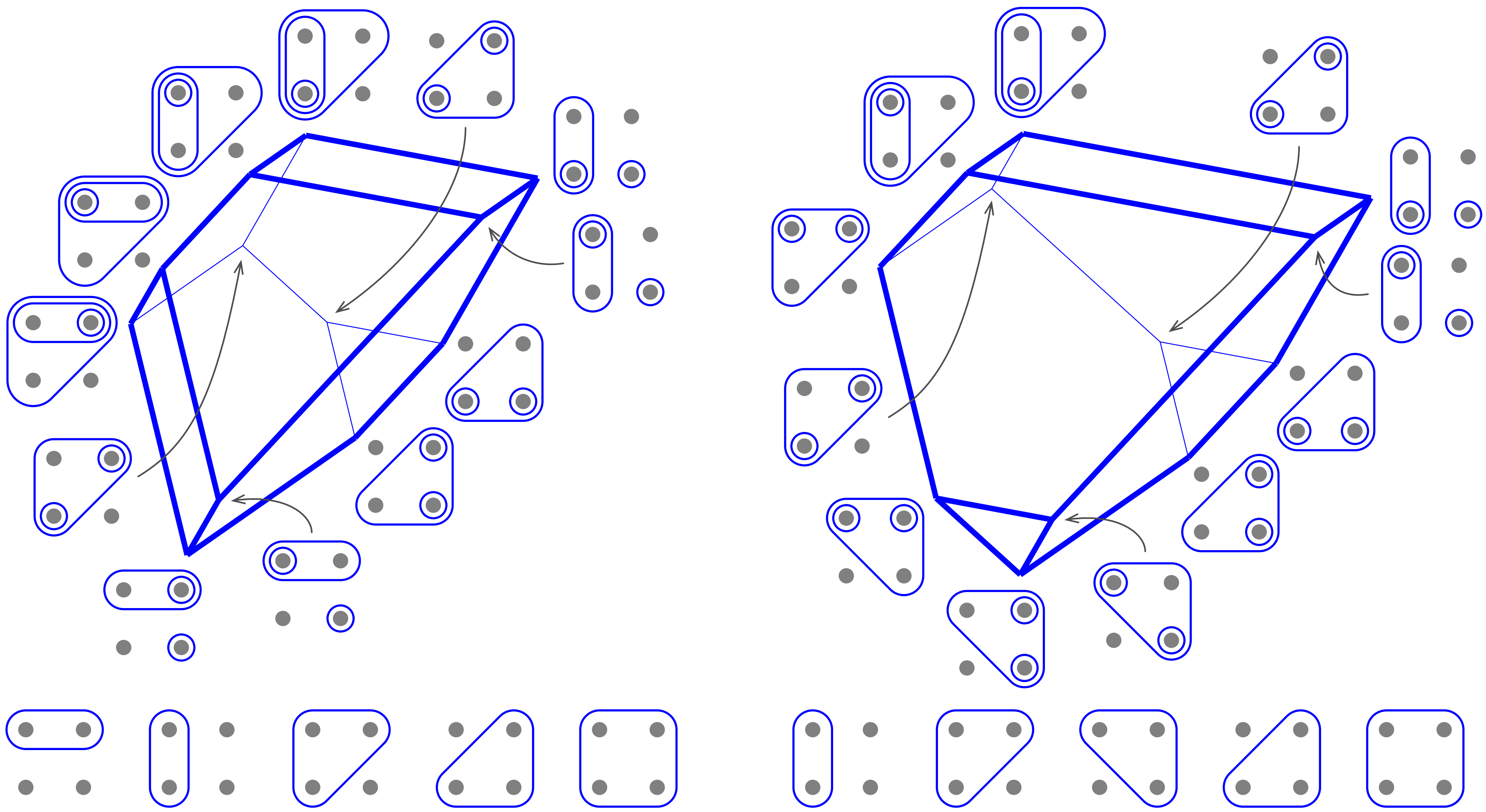
Postnikov ('09)



NESTOHEDRA ARE SIMPLE

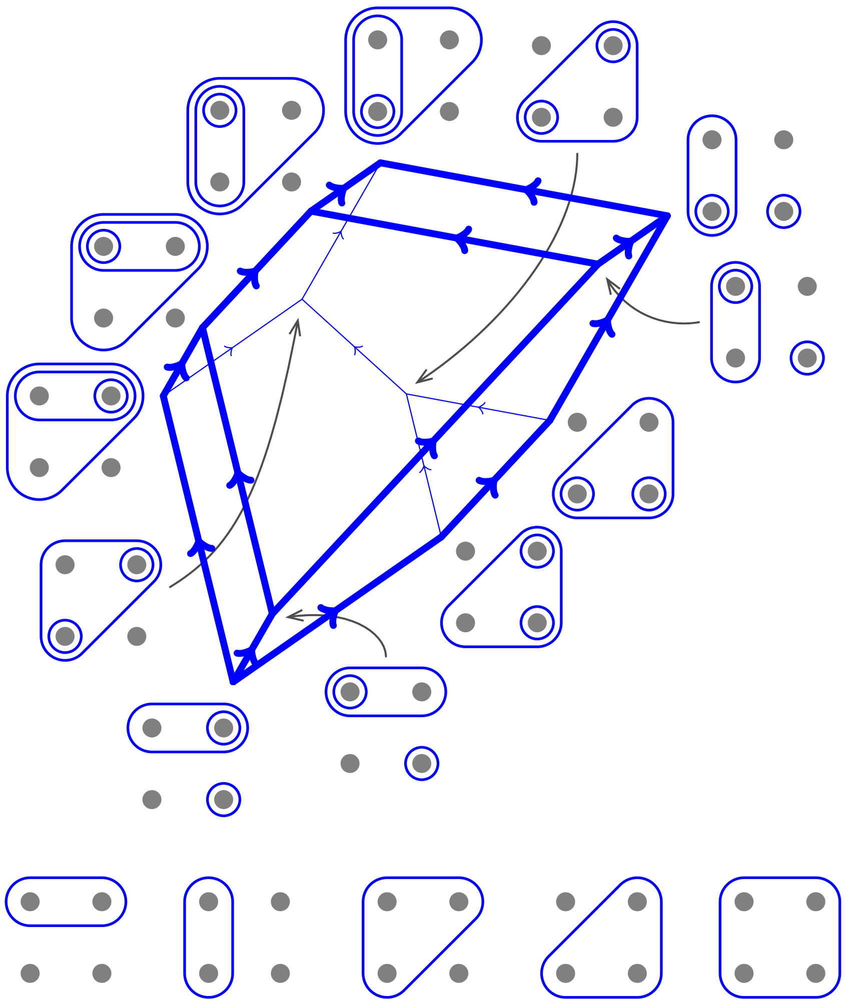
PROP. $\mathbb{N}\mathrm{est}(\mathcal{B})$ is a simple polytope — Feichtner–Sturmfels ('05) — Postnikov ('09)

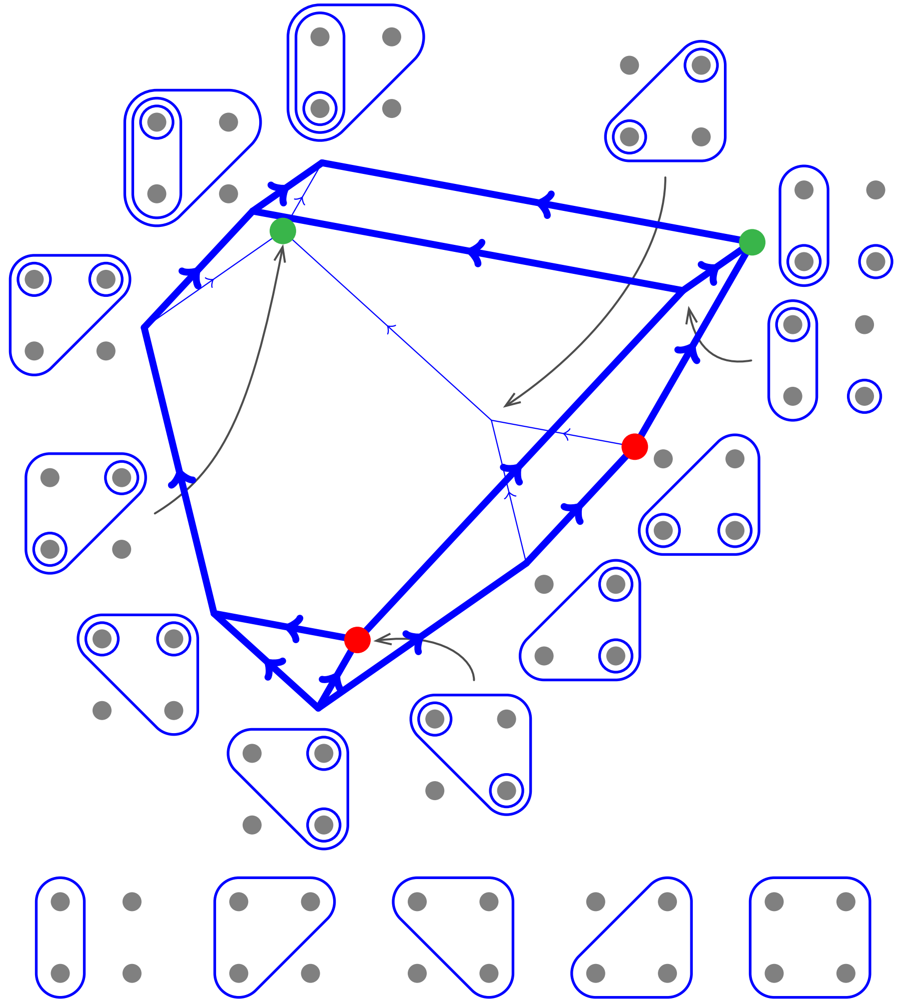
(the boundary complex of the polar of $\mathbb{N}\mathrm{est}(\mathcal{B})$ is the nested complex of $\mathcal{B}$)



ORIENTED SKELETA OF NESTOHEDRA

REM. No known characterization of the building sets B such that $\mathcal{P}_{\mathbb{N}\mathrm{est}(B)}$ is a lattice (even for graph associahedra) Barnard–McConville ('21)





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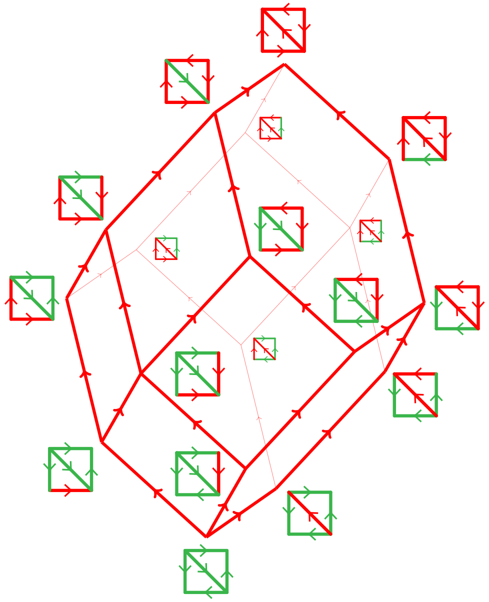
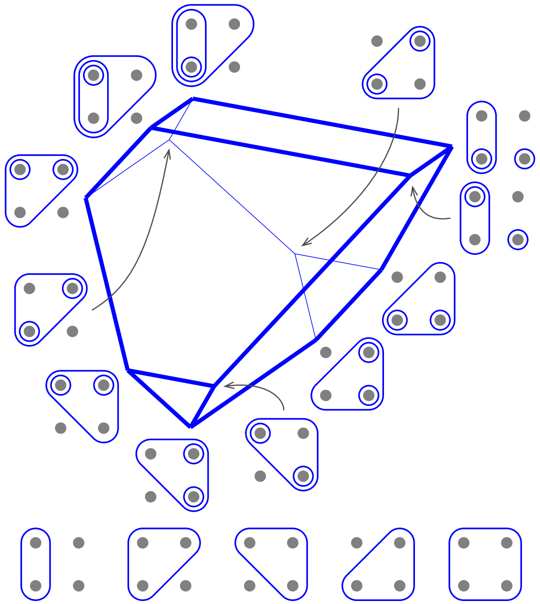
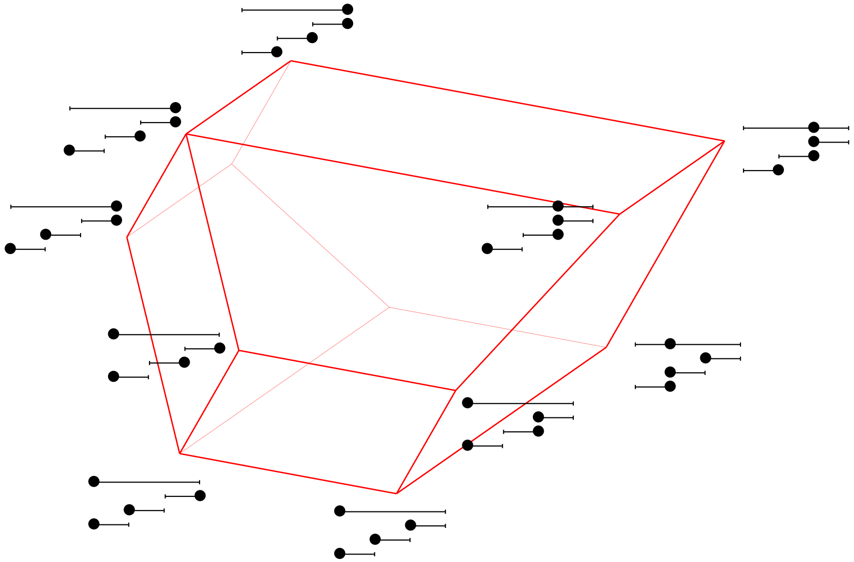


graphical zonotopes



nestohedra

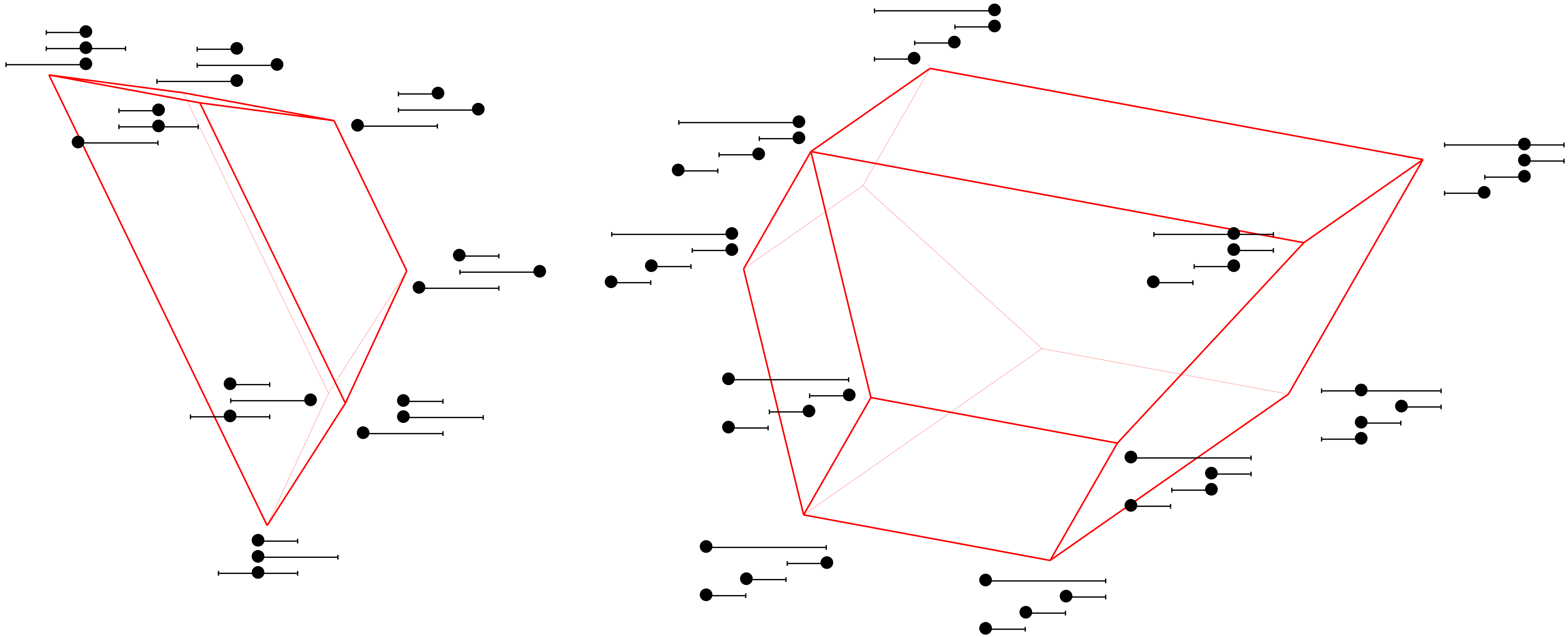


interval hypergraphic polytopes

INTERVAL HYPERGRAPHIC POLYTOPES

interval hypergraph $\mathbb{I}$ = subset of the set of intervals $[i,j]$ of $[n]$

interval hypergraphic polytope = hypergraphic polytope of an interval hypergraph

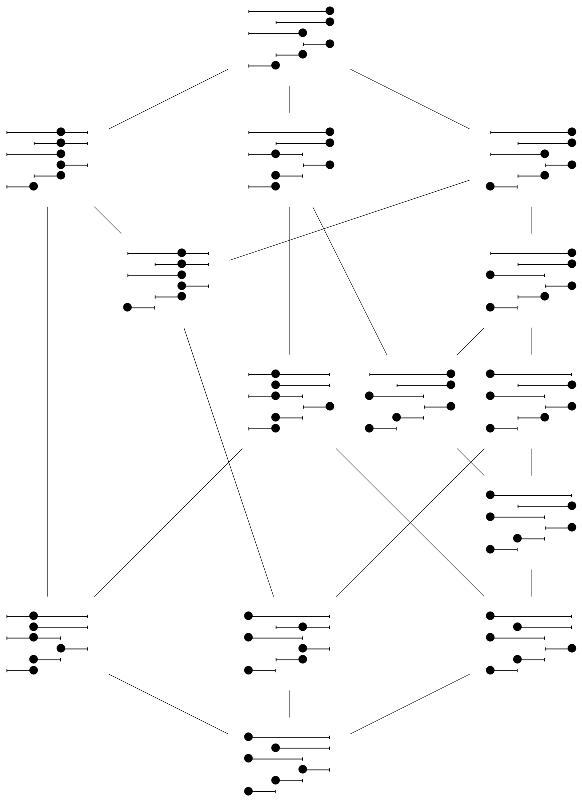


THM. interval hypergraphic polytopes = deformed associahedra

Bazier-Matte–Chapelier-Laguet–Douville–Mousavand–Thomas–Yıldırım ('23) Padrol–Palu–P.–Plamondon ('23)

SOME INTERVAL HYPERGRAPHIC POLYTOPES

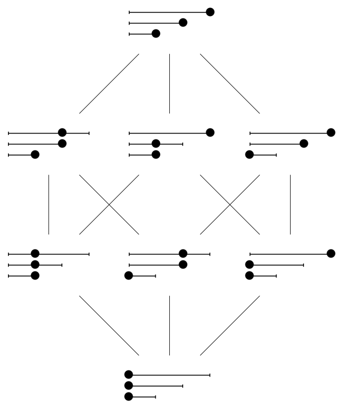
all intervals



associahedron

Tamari ('51), Loday ('04)

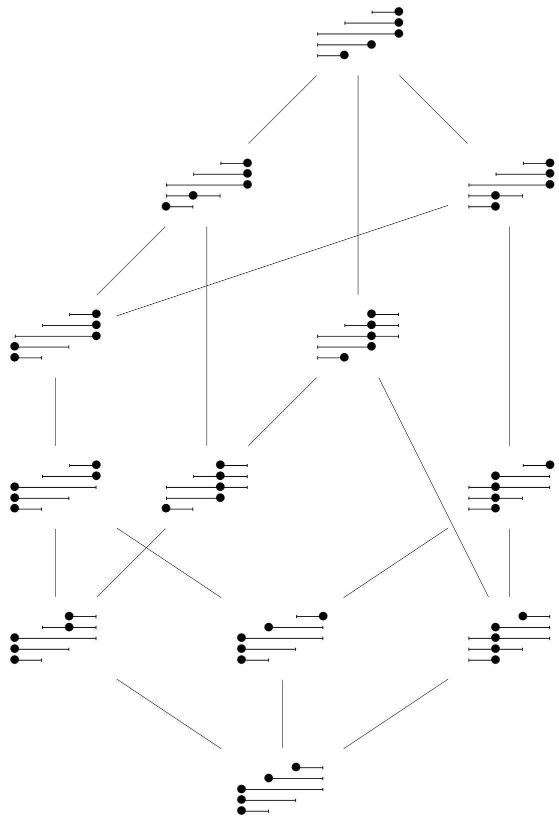
initial intervals



cube

Pitman–Stanley ('02)

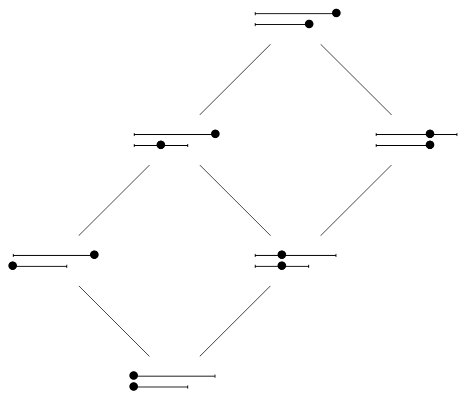
initial and final intervals



freehedron

Saneblidze ('09)

nested or disjoint



fertilitope

Defant ('23)

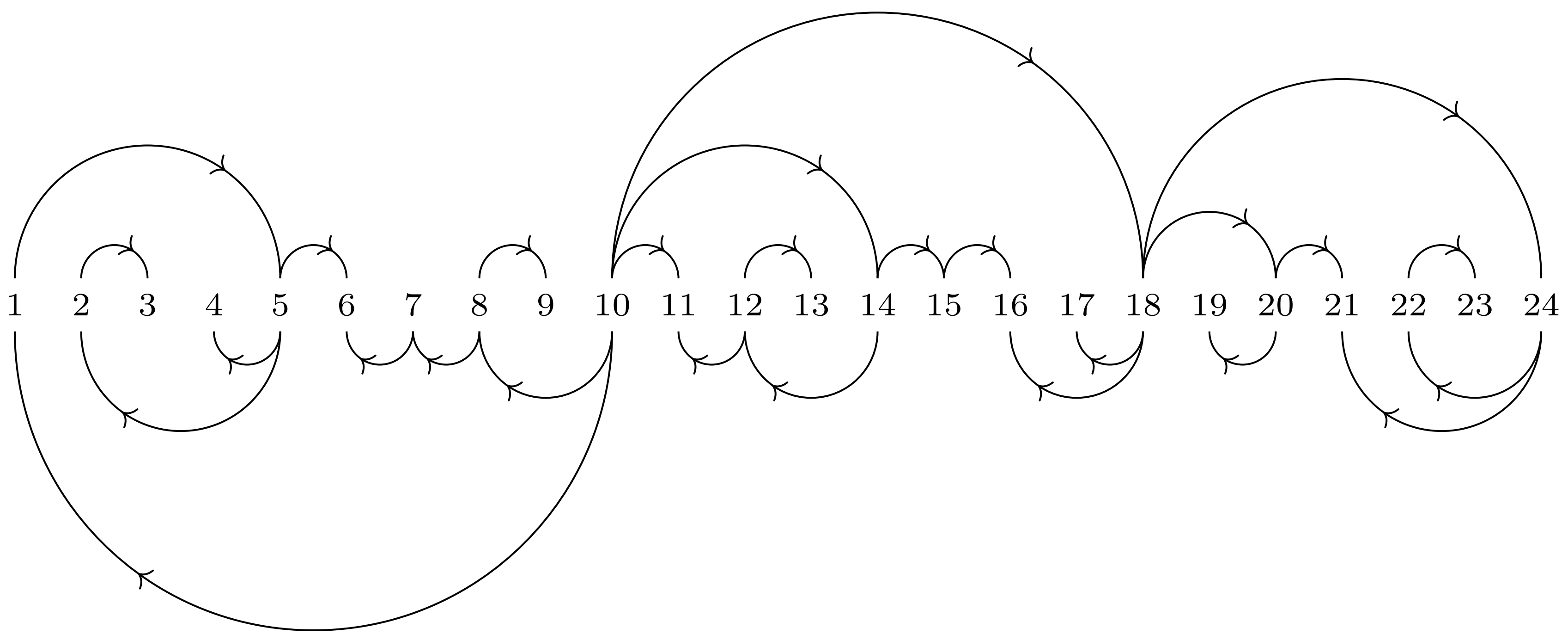
TAMARI INTERVAL POSETS & INTERVAL HYPERGRAPHIC POLYTOPES

weak order interval poset $=$ poset $\triangleleft$ on $[n]$ such that — Björner–Wachs ('91)

- the linear extensions of $\triangleleft$ form an interval of the weak order
- $a \triangleleft c$ implies $a \triangleleft b$ or $b \triangleleft c$ and $a \triangleright c$ implies $a \triangleright b$ or $b \triangleright c$ for all $a < b < c$

Tamari interval poset $=$ poset $\triangleleft$ on $[n]$ such that — Chatel–Pons ('15)

- the linear extensions of $\triangleleft$ is the union of the sets of linear extensions of the binary trees in an interval of the Tamari lattice
- $a \triangleleft c$ implies $a \triangleleft b$ and $a \triangleright c$ implies $b \triangleright c$ for all $a < b < c$
- for any $a \in [n]$, the upper set $a^{\triangleleft} := \{b \in [n] \mid a \triangleleft b\}$ of a is an interval of $[n]$



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REM. $\#$ Tamari interval posets $= \dfrac{2}{(3n+1)(3n+2)}\dbinom{4n+1}{n+1}$ Chapoton ('07)

also counting rooted 3-connected planar triangulations with $2n+2$ faces... Tutte ('63)

... and there are nice bijections! Bernardi–Bonichon ('09) Fang–Fusy–Nadeau ('25)

(in contrast, I don't know how to count Tamari interval preposets...)

TAMARI INTERVAL POSETS & INTERVAL HYPERGRAPHIC POLYTOPES

weak order interval poset $=$ poset $\lhd$ on $[n]$ such that Björner–Wachs ('91)

- the linear extensions of $\lhd$ form an interval of the weak order
- $a \lhd c$ implies $a \lhd b$ or $b \lhd c$ and $a \rhd c$ implies $a \rhd b$ or $b \rhd c$ for all $a < b < c$

Tamari interval poset $=$ poset $\lhd$ on $[n]$ such that Chatel–Pons ('15)

- the linear extensions of $\lhd$ is the union of the sets of linear extensions of the binary trees in an interval of the Tamari lattice
- $a \lhd c$ implies $a \lhd b$ and $a \rhd c$ implies $b \rhd c$ for all $a < b < c$
- for any $a \in [n]$, the upper set $a^{\lhd} \coloneqq \{b \in [n] \mid a \lhd b\}$ of a is an interval of $[n]$

PROP. Vertex posets of interval hypergraphic polytopes $=$ Tamari interval posets
Face preposets of interval hypergraphic polytopes $=$ Tamari interval preposets

Bergeron–P. ('25) Bastidas–Gélinas–P.–Poullot–Sack–Tzanaki ('26^+)

PROP. $\lhd$ is a vertex poset of $\triangle_{\mathbb{I}}$ if and only if

- for any $I \in \mathbb{I}$, there is $a \in [n]$ such that $a \in I \subseteq a^{\lhd}$
- for any cover relation $a \lessdot b$, there is $I \in \mathbb{I}$ such that $\{a, b\} \subseteq I \subseteq a^{\lhd}$

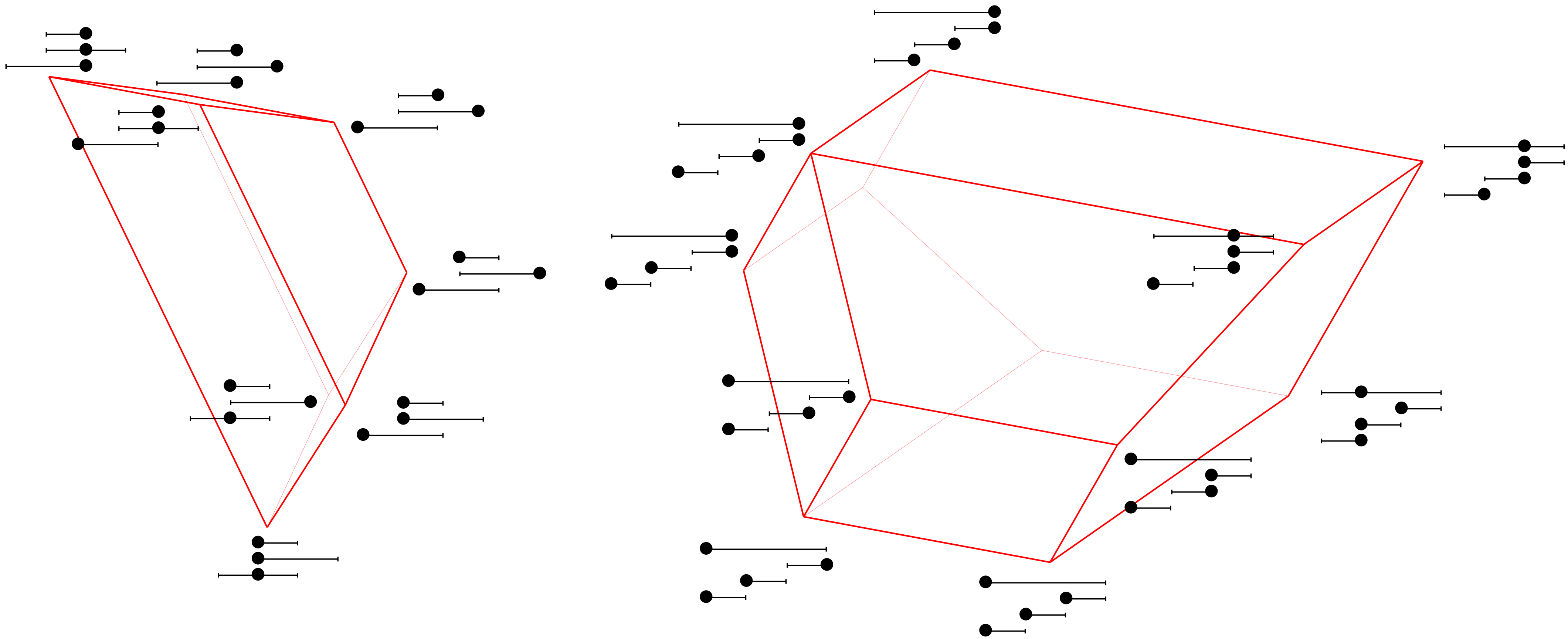
Bastidas–Gélinas–P.–Poullot–Sack–Tzanaki ('26^+)

SIMPLE INTERVAL HYPERGRAPHIC POLYTOPES

THM. $\triangle_{\mathbb{I}}$ is simple $\iff$ for all $I, J \in \mathbb{I}$:

- if $I \cap J \neq \varnothing$ and $\exists\, K \in \mathbb{I}$ with $I \cup J \subseteq K$, then $I \cup J \in \mathbb{I}$
- if $|I \cap J| \geq 2$, then $I \cup J \in \mathbb{I}$ or $\exists\, K \in \mathbb{I}$ with $I \cap J \subseteq K$ and $I \not\subseteq K$ and $J \not\subseteq K$

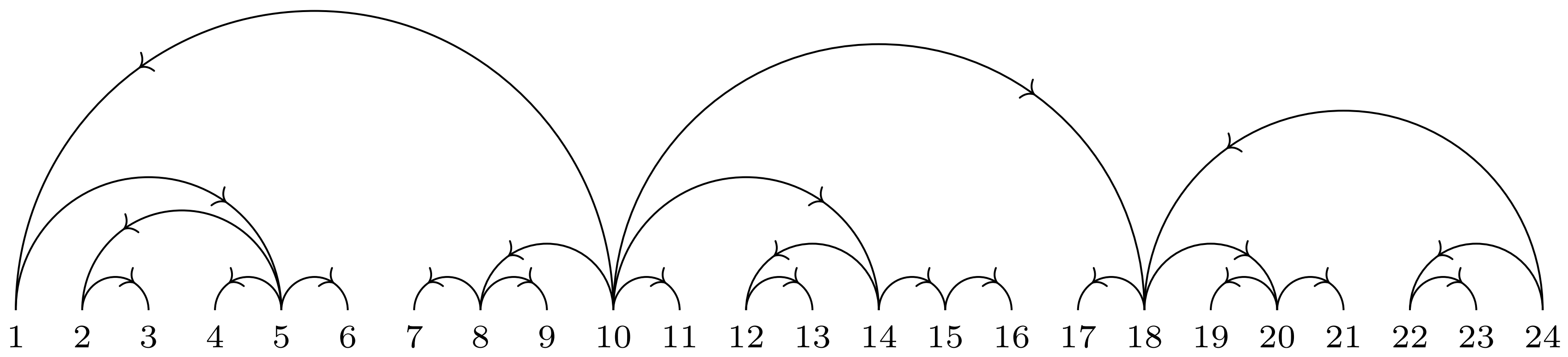
Bastidas–Gélinas–P.–Poullot–Sack–Tzanaki ('26^{+})



exm: union closed ($I \cap J \neq \varnothing \;\Rightarrow\; I \cup J \in \mathbb{I}$) or subinterval closed ($I \subseteq J \in \mathbb{I} \;\Rightarrow\; I \in \mathbb{I}$)

WEEPING WILLOWS

weeping willow = Tamari interval poset whose Hasse diagram is a tree
= non-crossing tree such that for all $a < b < c$,
$a \to c$ in T implies $a \rightsquigarrow b$ in T and $a \leftarrow c$ in T implies $b \leftsquigarrow c$ in T



THM. Vertex posets of simple interval hypergraphic polytopes = weeping willows
Vertex posets of interval nestohedra = rooted weeping willows
(extends to face preposets of simple interval hypergraphic polytopes...)

Bastidas–Gélinas–P.–Poullot–Sack–Tzanaki ('26$^+$)

WEEPING WILLOWS

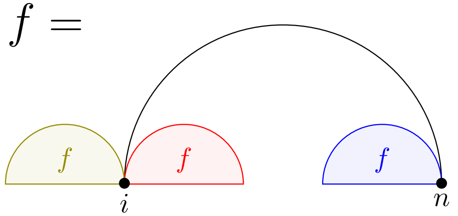
weeping willow $=$ Tamari interval poset whose Hasse diagram is a tree
$=$ non-crossing tree such that for all $a < b < c$,
$a \to c$ in T implies $a \rightsquigarrow b$ in T and $a \leftarrow c$ in T implies $b \leftsquigarrow c$ in T

REM. Counting weeping willows:

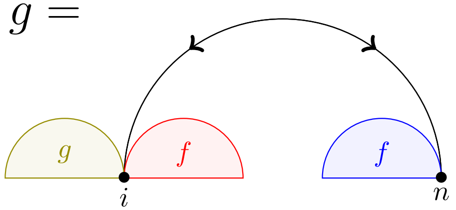
unoriented non-crossing trees	weeping willows	rooted weeping willows
$f = 1 + xf^3$	$g = 1 + 2xgf^2$	$h = 1 + xf^2(h + f)$
$xf^3 - f + 1 = 0$	$(8x - 1)g^3 - g^2 + g + 1 = 0$	$x^2h^3 - 2xh^2 + h - 1 = 0$
$\dfrac{1}{2n+1}\dbinom{3n}{n}$	$2\dbinom{3n}{n} - \displaystyle\sum_{k=0}^{n}\dbinom{3n}{k}$	$\dfrac{1}{n+1}\dbinom{3n+1}{n}$

Bastidas–Gélinas–P.–Poullot–Sack–Tzanaki ('26$^+$)

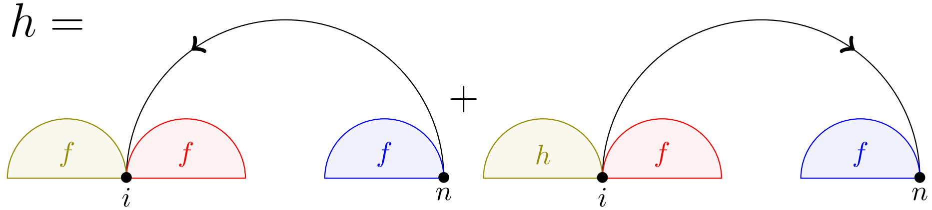
$f =$



$g =$



$h =$

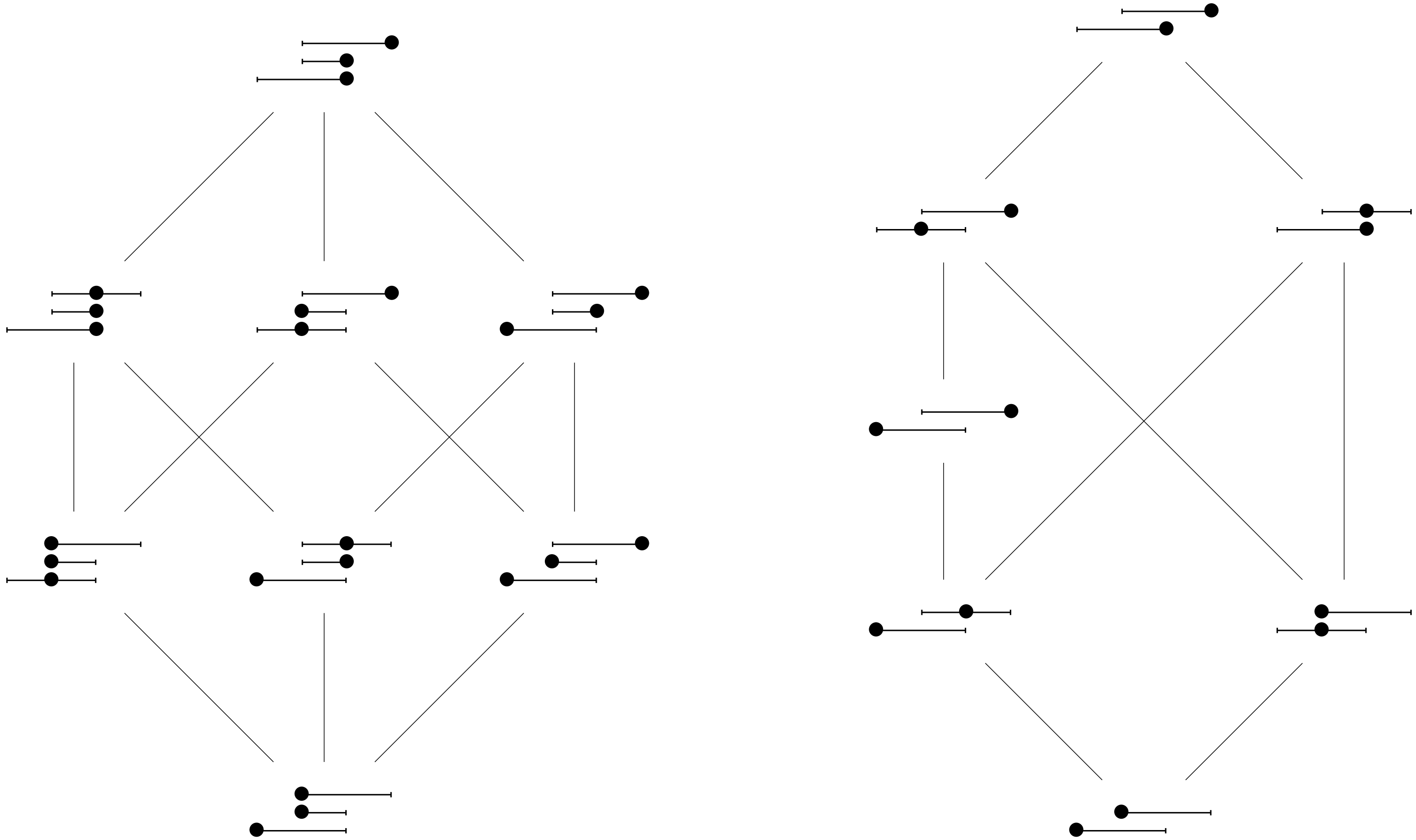


(counting formulas extend to face preposets of simple interval hypergraphic polytopes...)

INTERVAL HYPERGRAPHIC LATTICES

THM. $P_{\mathbb{I}}$ lattice $\iff$ $\mathbb{I}$ is closed under intersection — Bergeron–P. ('24$^+$)

moreover, all these lattices are quasi-lattice quotients of the weak order

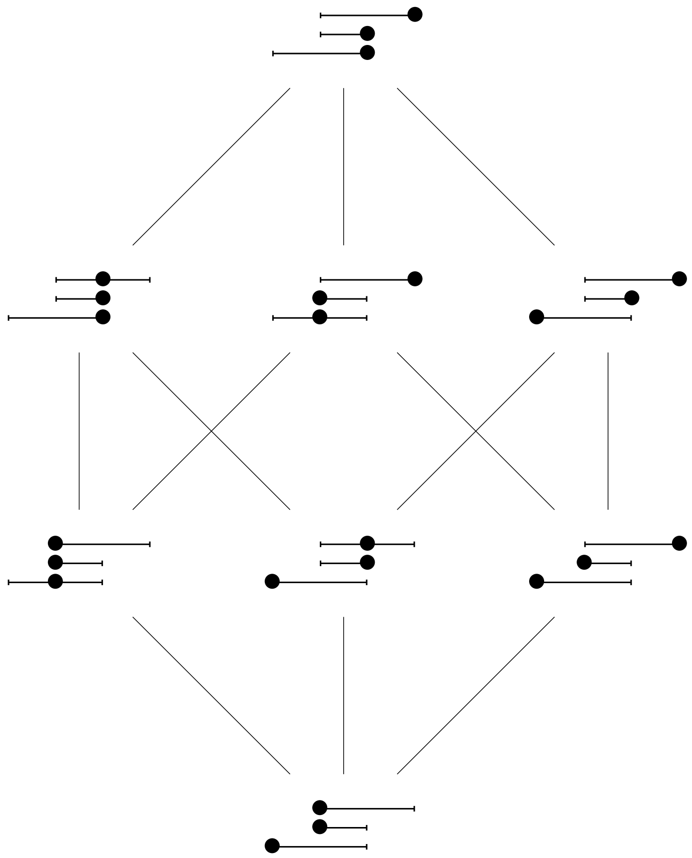


lattice — not lattice

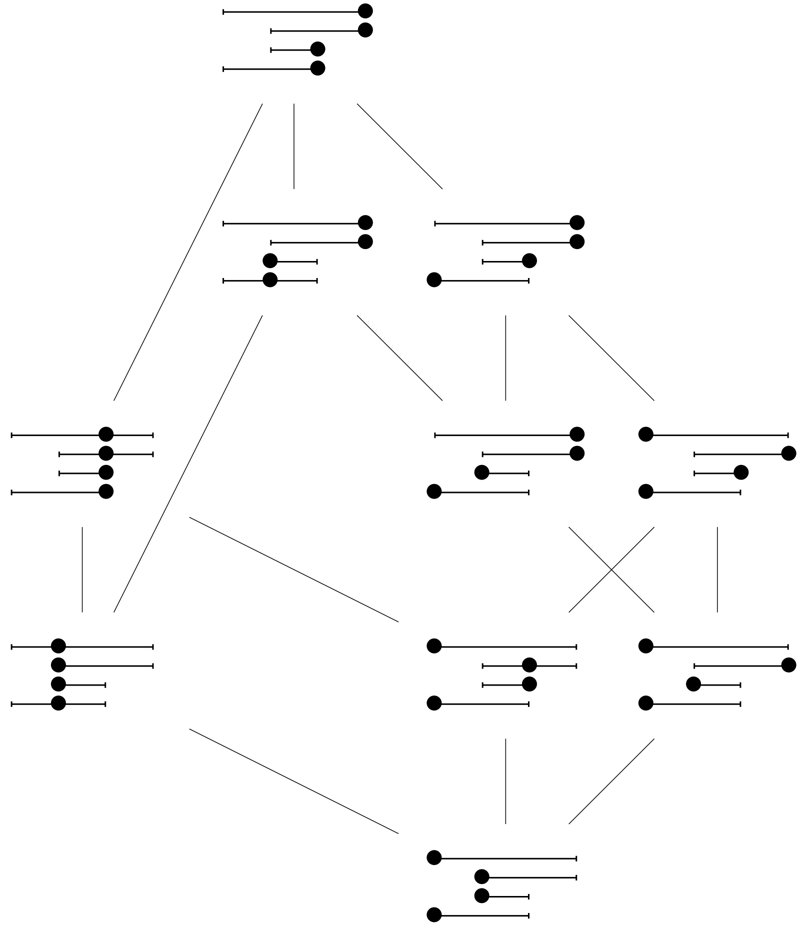
DISTRIBUTIVE INTERVAL HYPERGRAPHIC LATTICES

THM. $P_{\mathbb{I}}$ distributive lattice $\iff$ for all $I, J \in \mathbb{I}$ such that $I \not\subseteq J$, $I \not\supseteq J$ and $I \cap J \neq \varnothing$, the intersection $I \cap J$ is in $\mathbb{I}$ and is initial or final in any $K \in \mathbb{I}$ with $I \cap J \subseteq K$

Bergeron–P. ('24$^+$)



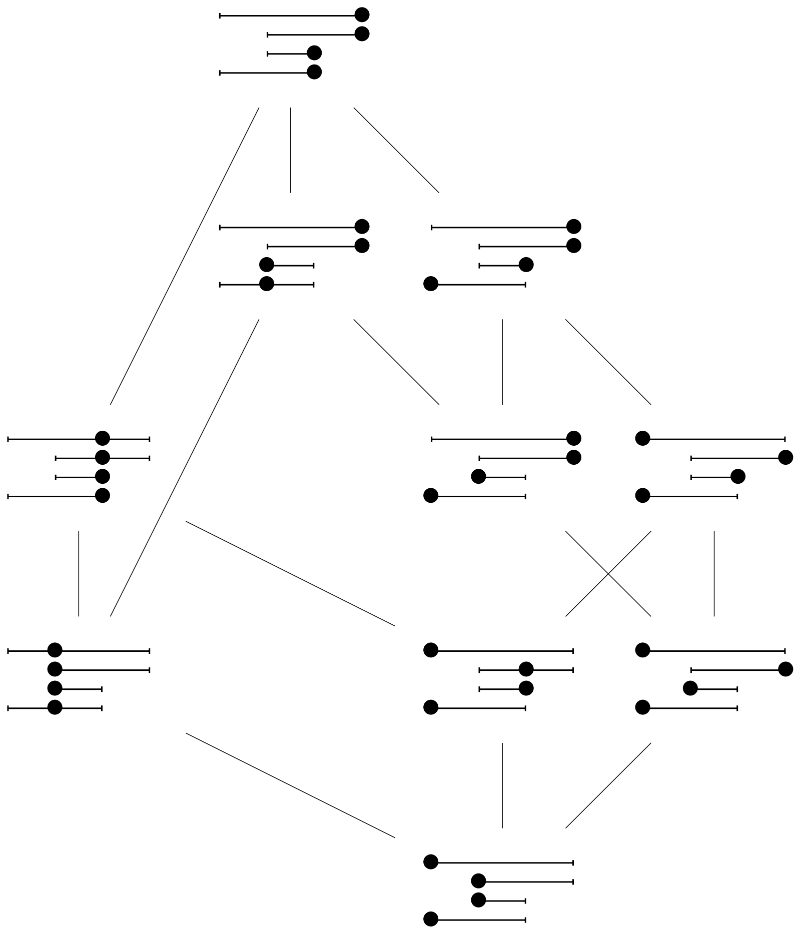
distributive



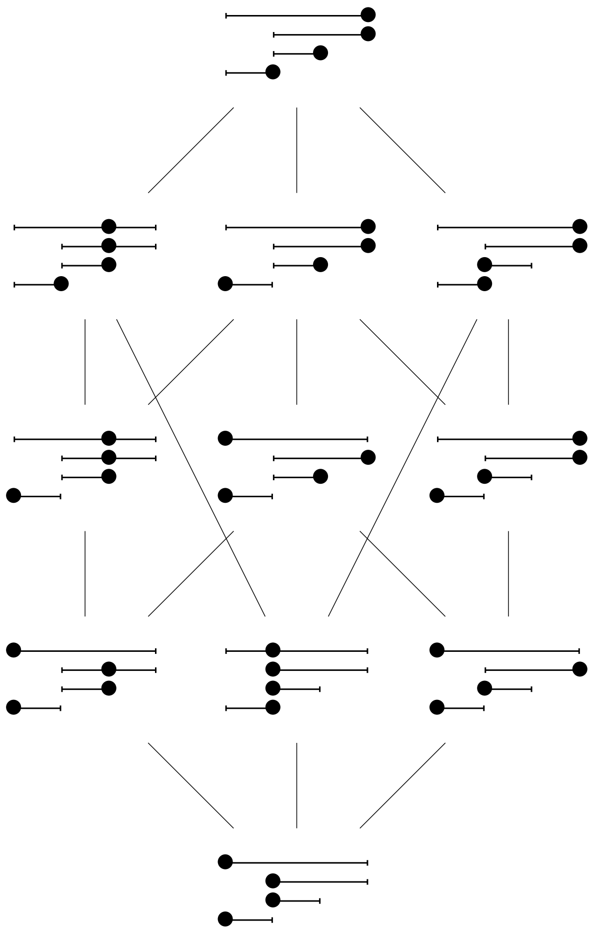
not distributive

SEMIDISTRIBUTIVE INTERVAL HYPERGRAPHIC LATTICES

THM. $P_{\mathbb{I}}$ join semidistributive lattice $\iff$ $\mathbb{I}$ is closed under intersection and for all $[r,r'],[s,s'],[t,t'],[u,u'] \in \mathbb{I}$ with $r<s\le r'<s'$, $r<t\le s'<t'$, $u<\min(s,t)$, $s'<u'$, there is $[v,v'] \in \mathbb{I}$ such that $v<s$ and $s'<v'<t'$ Bergeron–P. ('24$^+$)



semidistributive



not join semidistributive

INTERLUDE: PLUMBING BIJECTION

PROP. The following families are in bijection

- interval hypergraphs closed under intersection: $I, J \in \mathbb{I} \;\Rightarrow\; I \cap J \in \mathbb{I}$
- interval hypergraphs closed under union: $I, J \in \mathbb{I}$ and $I \cap J \neq \varnothing \;\Rightarrow\; I \cup J \in \mathbb{I}$
- graphs $([n+1], E)$ where $\{a, c\}, \{b, d\} \in E \;\Rightarrow\; \{b, c\} \in E$ for all $a < b < c < d$
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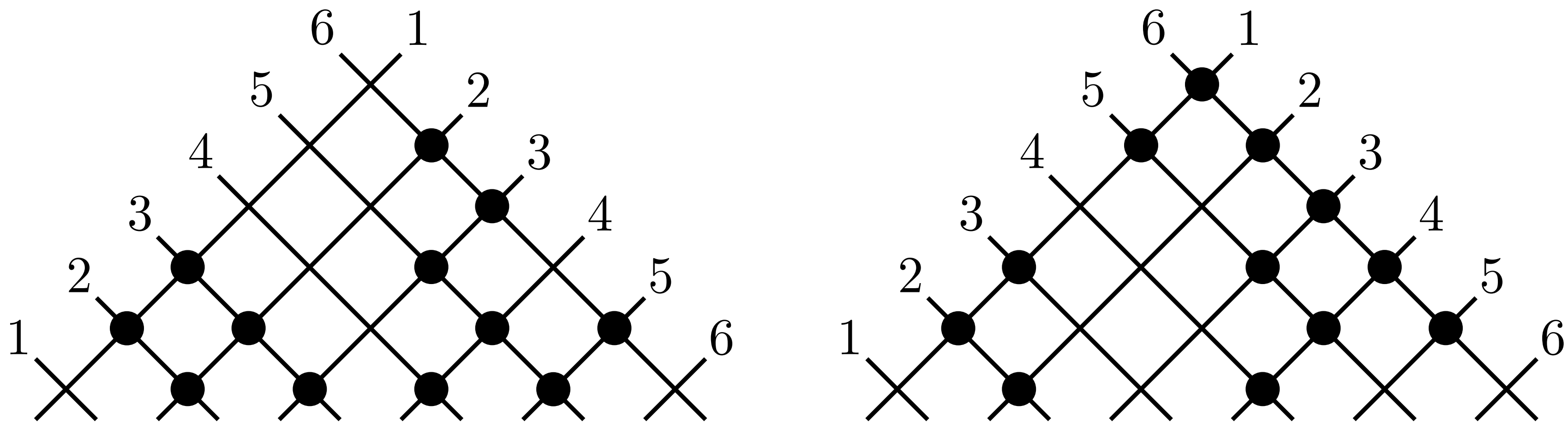

INTERLUDE: PLUMBING BIJECTION

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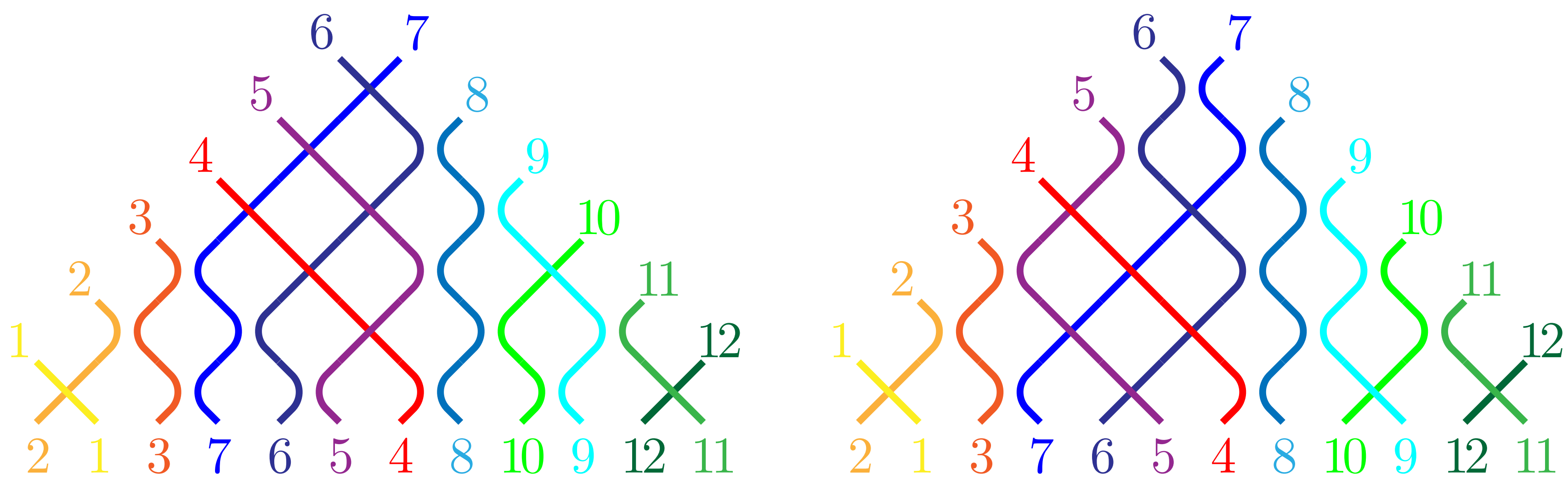

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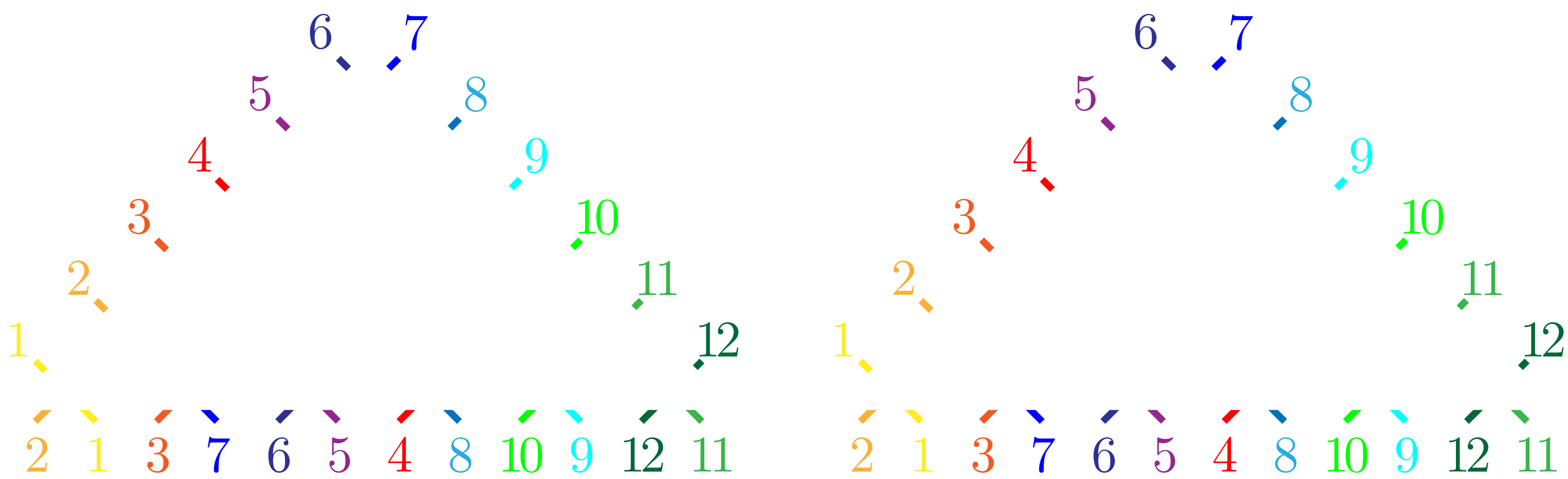

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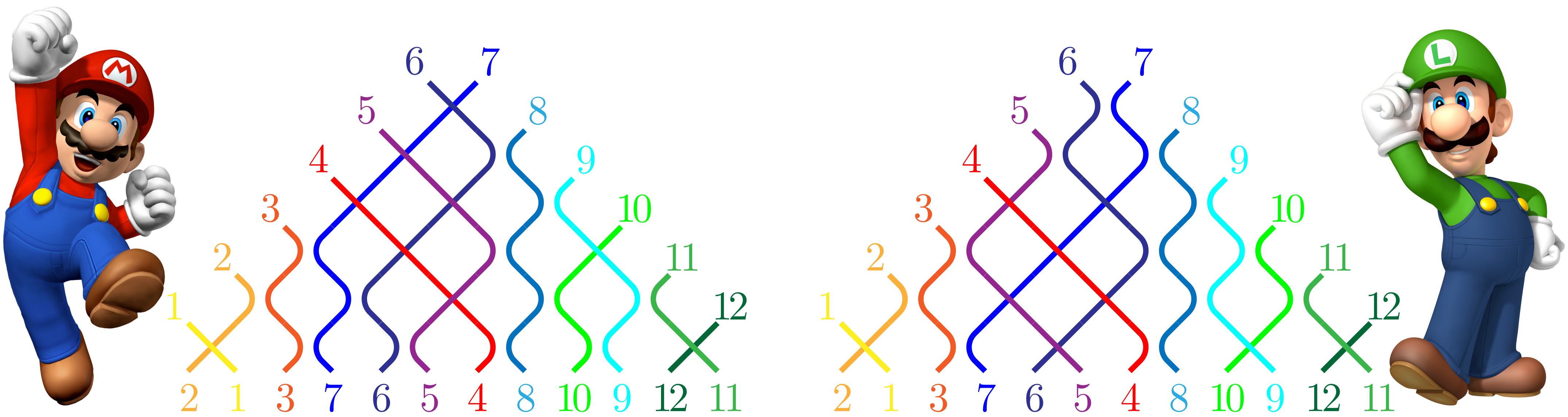
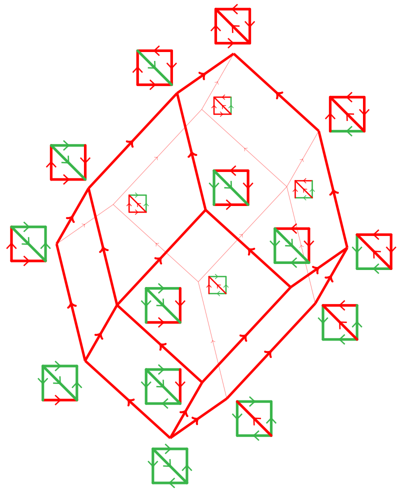
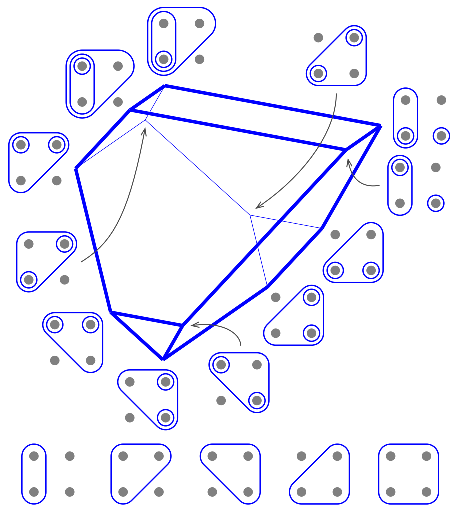
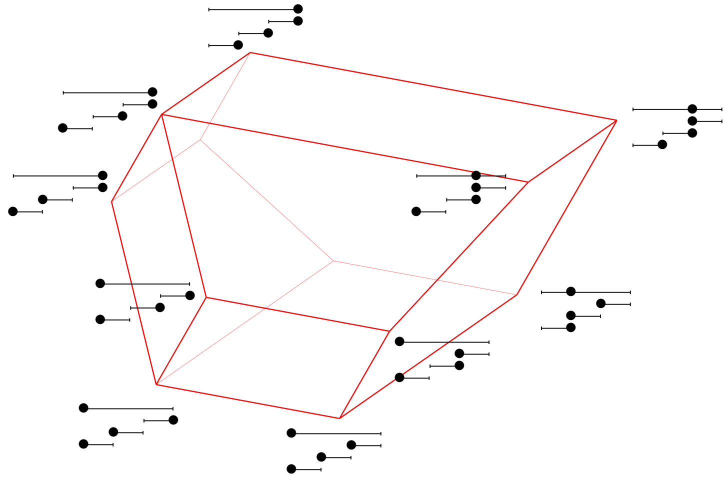


SUMMARY...

hypergraphic polytope $\triangle(\mathbb{H}) = \sum_{H \in \mathbb{H}} \triangle_H$ where $\triangle_H = \{\boldsymbol{e}_h \mid h \in H\}$

$\triangle_{\mathbb{H}}$-posets $=$ transitive closures of acyclic orientations of $\mathbb{H}$

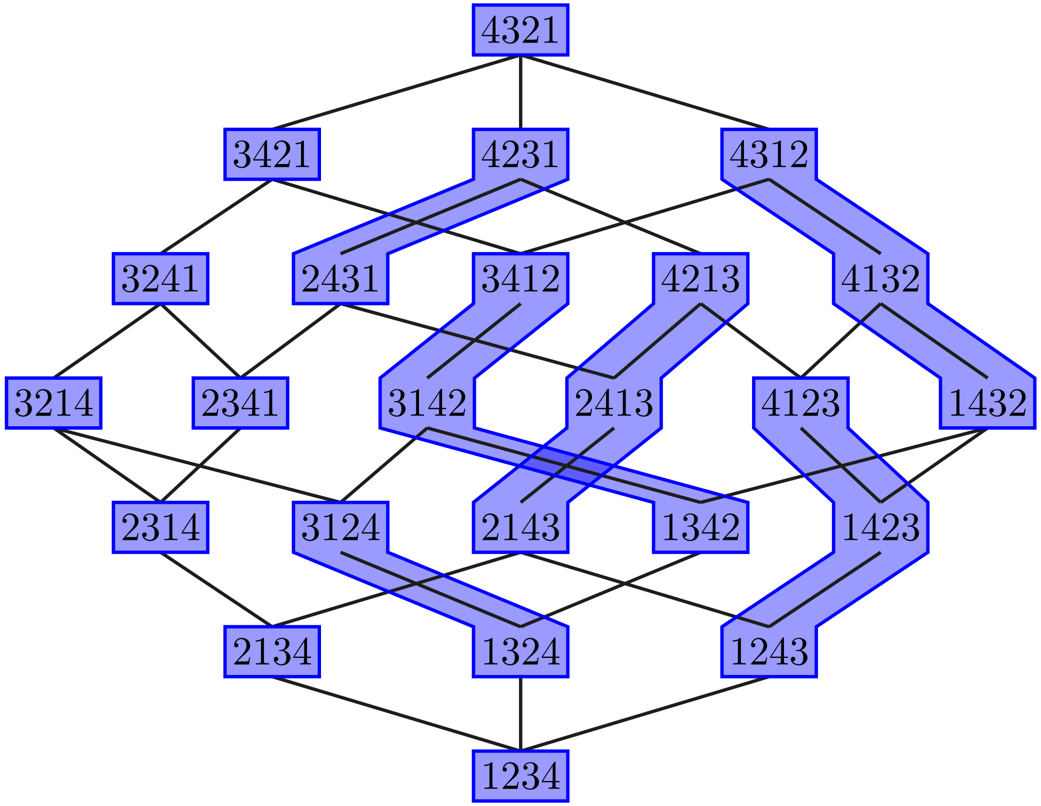
families	graphical zonotopes	nestohedra	interval hypergraphic polytopes
			
$\mathbb{P}$-posets	acyclic reorientations	$\mathcal{B}$-forests	Tamari interval posets
simple?	chordful	always	2 closedness conditions
lattice?	skeletal	??	intersection closed

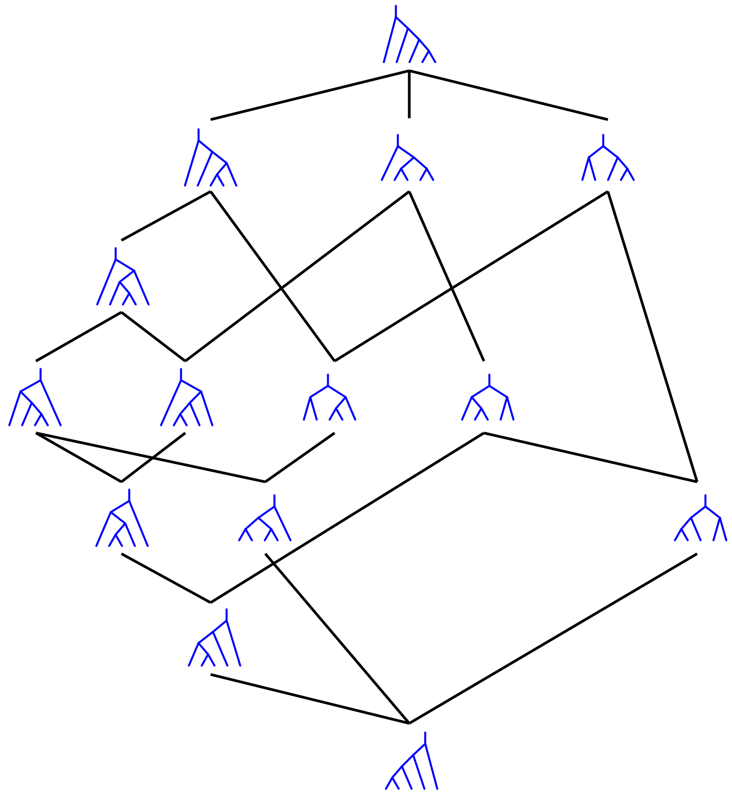
QUOTIENTOPES

Reading, *Lattice congruences, fans and Hopf algebras* ('05)
P.–Santos, *Quotientopes* ('19)
Padrol–P.–Ritter, *Shard polytopes* ('22)
Barnard–Novelli–P., *Separating trees and simple congruences of the weak order* ('25)

LATTICES: WEAK ORDER AND TAMARI LATTICE

lattice = partially ordered set L where any $X \subseteq L$ admits a meet $\bigwedge X$ and a join $\bigvee X$





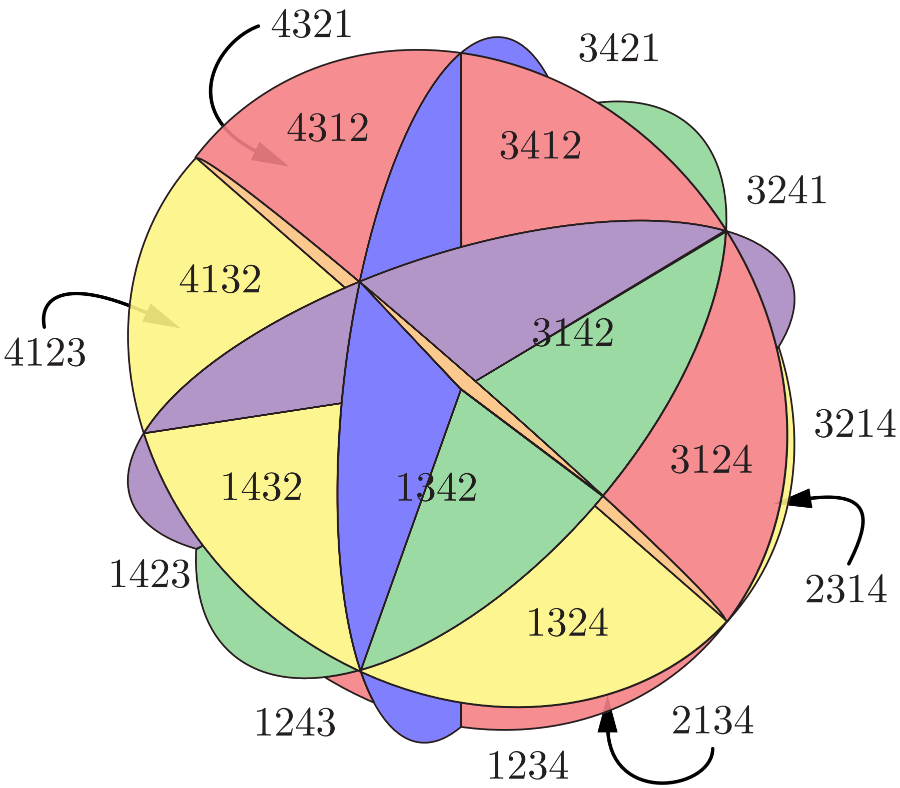
weak order = permutations of $[n]$
ordered by paths of simple transpositions

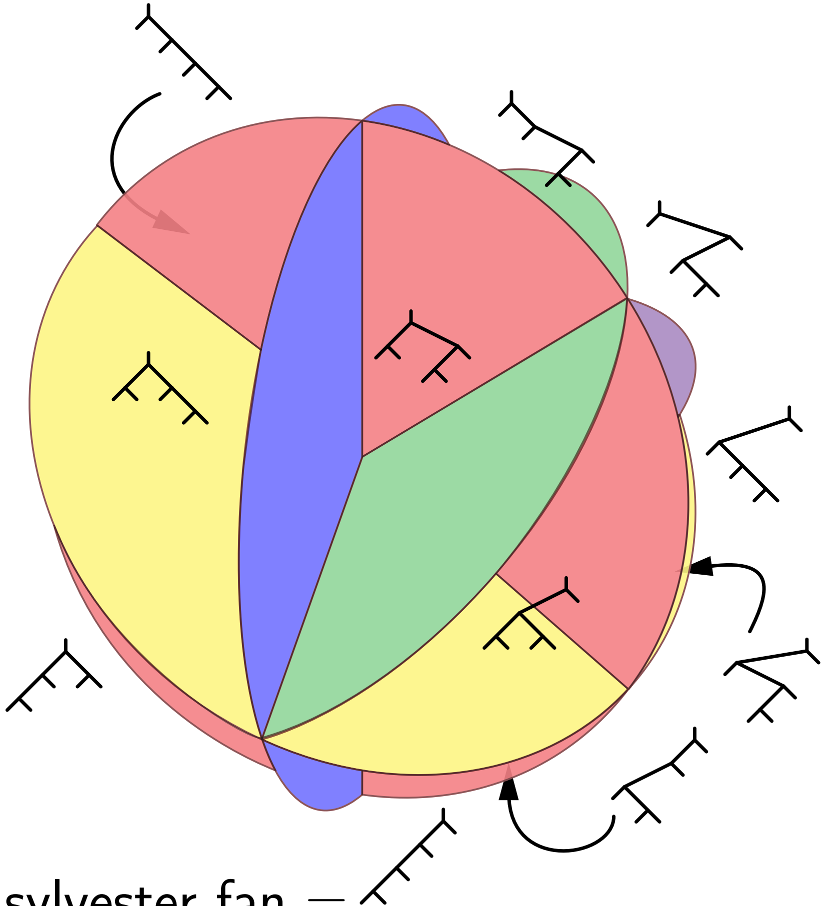
Tamari lattice = binary trees on $[n]$
ordered by paths of right rotations

sylvester congruence = equivalence classes are sets of linear extensions of binary trees
= equivalence classes are fibers of BST insertion
= rewriting rule $UacVbW \equiv_{\text{sylv}} UcaVbW$ with $a < b < c$

FANS: BRAID FAN AND SYLVESTER FAN

fan = collection of polyhedral cones closed by faces and intersecting along faces





braid fan =

$$\mathbb{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \right\}$$

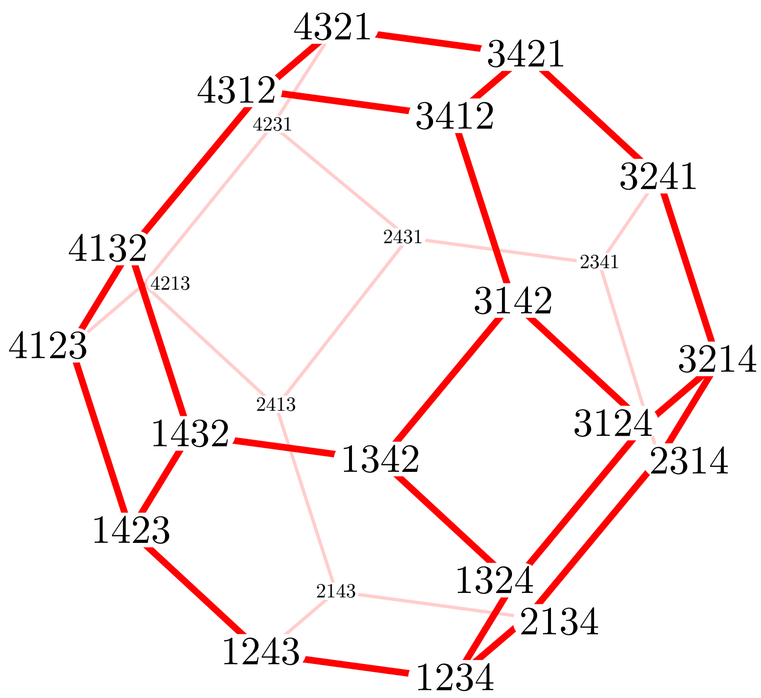
sylvester fan =

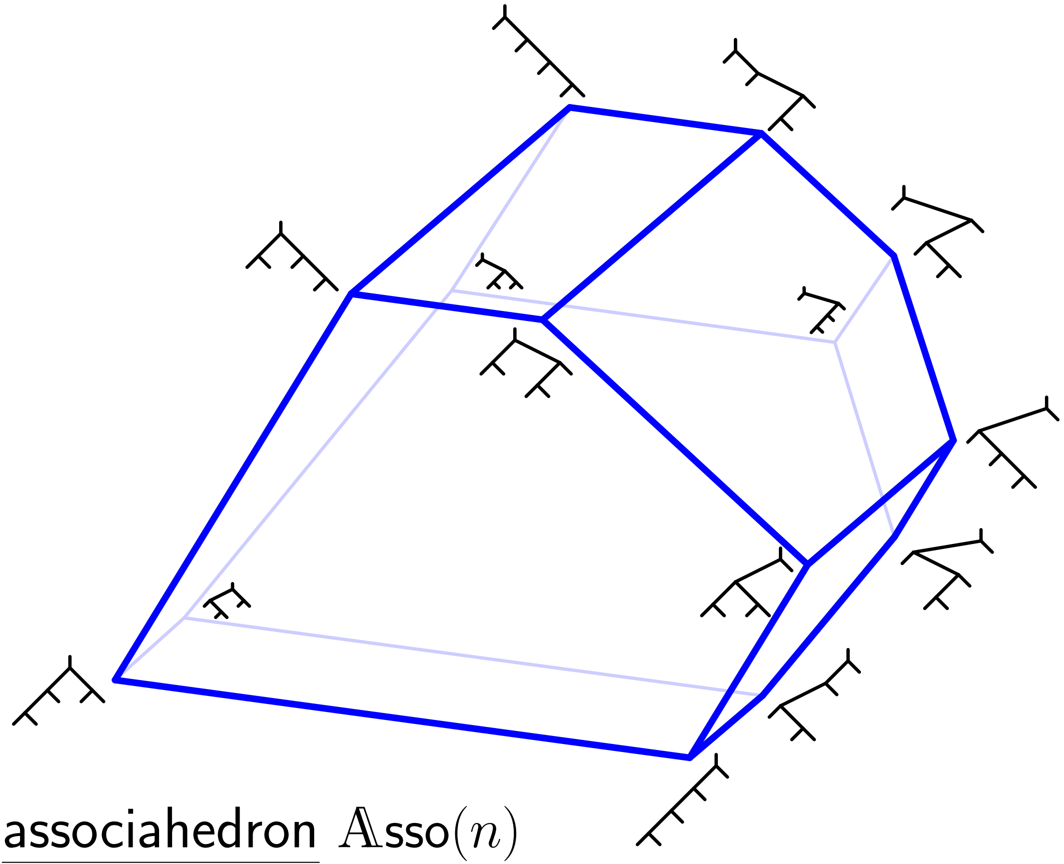
$$\mathbb{C}(T) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid x_i \le x_j \text{ if } i \to j \text{ in } T \}$$

quotient fan = $\mathbb{C}(T)$ is obtained by glueing $\mathbb{C}(\sigma)$ for all linear extensions σ of T

POLYTOPES: PERMUTAHEDRON AND ASSOCIAHEDRON

polytope $=$ convex hull of a finite set of points
$=$ bounded intersection of a finite set of affine half-spaces





permutahedron $\mathbb{P}\mathrm{erm}(n)$

$$= \mathrm{conv}\left\{ [\sigma^{-1}(i)]_{i\in[n]} \;\middle|\; \sigma \in \mathfrak{S}_n \right\}$$

$$= \mathbb{H} \cap \bigcap_{\varnothing \neq J \subsetneq [n]} \mathbb{H}_J$$

$$= \sum_{1\le i<j\le n} \triangle_{\{i,j\}}$$

associahedron $\mathbb{A}\mathrm{sso}(n)$

$$= \mathrm{conv}\left\{ [\ell(T,i)\cdot r(T,i)]_{i\in[n]} \;\middle|\; T \text{ binary tree} \right\}$$

$$= \mathbb{H} \cap \bigcap_{1\le i<j\le n} \mathbb{H}_{[i,j]}$$

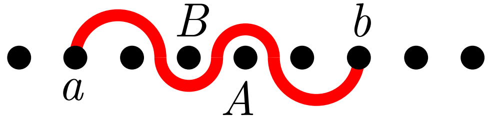
$$= \sum_{1\le i\le j\le n} \triangle_{[i,j]}$$

Stasheff ('63)
Shnider–Sternberg ('93)
Loday ('04)
Postnikov ('09)

FROM PERMUTATIONS TO NONCROSSING ARC DIAGRAMS

draw all points (σ_i, i) and all segments
from (σ_i, i) to $(\sigma_{i+1}, i+1)$ with $\sigma_i > \sigma_{i+1}$
and project down to an horizontal line allowing
arcs to bend but not to cross or pass points

arc $=$
(a, b, A, B) with $1 \le a < b \le n$ and $A \sqcup B =]a, b[$

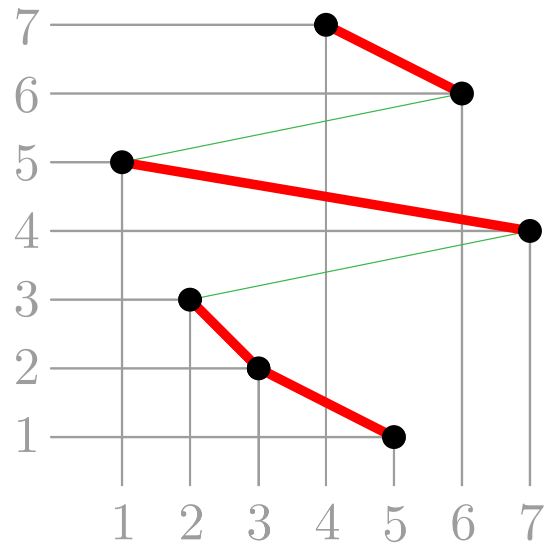


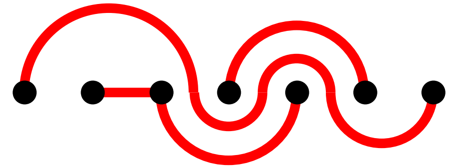
crossing arcs $=$
(a, b, A, B) and (a', b', A', B') such that there is $x \ne x'$ with

$$x \in (A \cup \{a, b\}) \cap (B' \cup \{a', b'\})$$
$$\text{and} \quad x' \in (B \cup \{a, b\}) \cap (A' \cup \{a', b'\})$$

noncrossing arc diagrams $=$
set of pairwise non-crossing arcs

permutation $\sigma = 5327164$

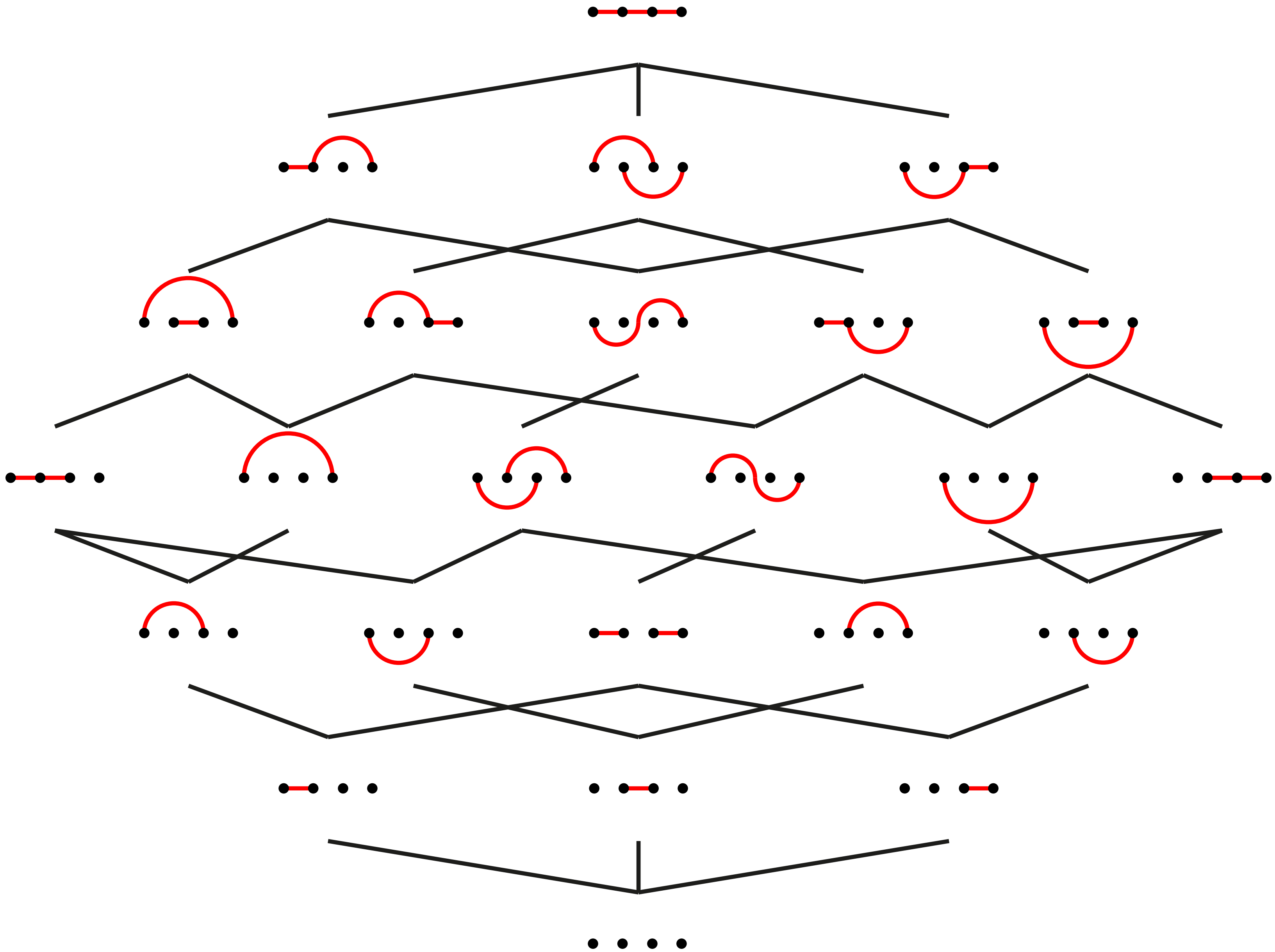




noncrossing arc diagram $\delta(\sigma)$

THM. δ is a bijection from permutations to noncrossing arc diagrams — Reading ('15)

CANONICAL JOIN REPRESENTATIONS & NONCROSSING ARC DIAGRAMS



THM. $\sigma = \bigvee_{\alpha \in \delta(\sigma)} \delta^{-1}(\{\alpha\})$ is the canonical join representation Reading ('15)

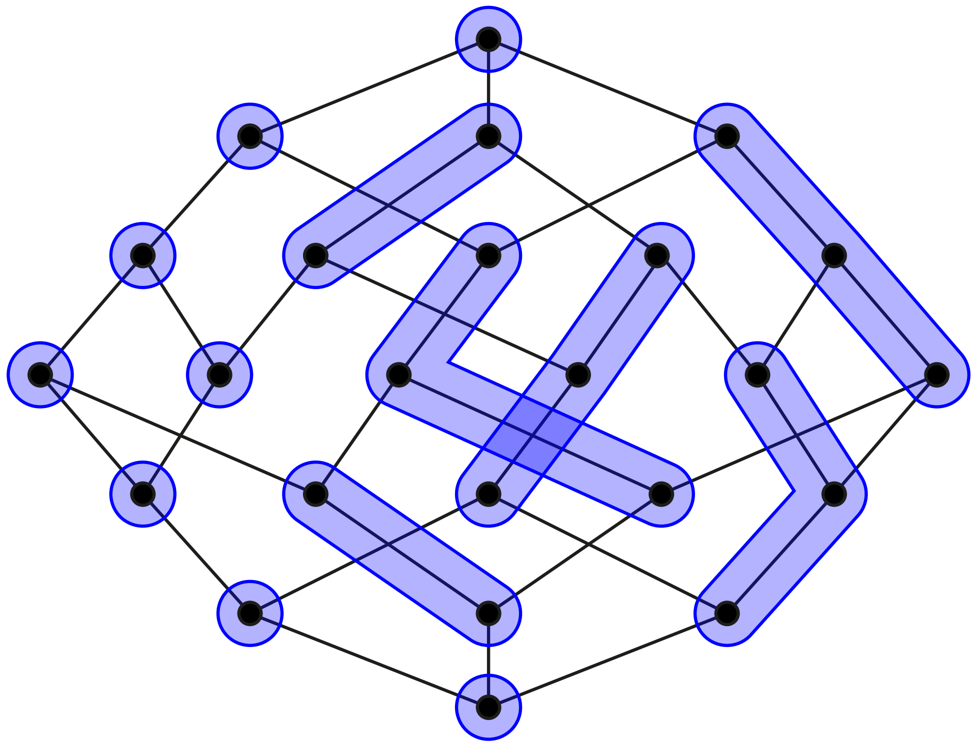
LATTICE CONGRUENCES & LATTICE QUOTIENTS

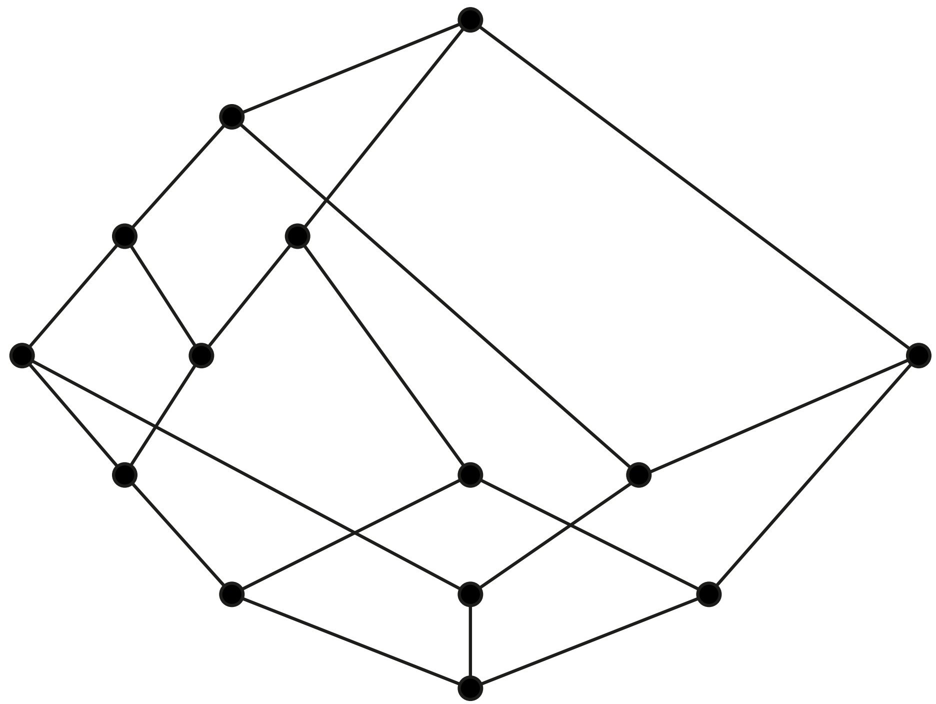
lattice congruence of L = equivalence relation $\equiv$ which respects meets and joins

$$x \equiv x' \text{ and } y \equiv y' \implies x \wedge y \equiv x' \wedge y' \text{ and } x \vee y \equiv x' \vee y'$$

lattice quotient of $L/{\equiv}$ = lattice on equivalence classes of L under $\equiv$ where

- $X \le Y \iff \exists\, x \in X,\ y \in Y, \quad x \le y$
- $X \wedge Y$ = equiv. class of $x \wedge y$ for any $x \in X$ and $y \in Y$
- $X \vee Y$ = equiv. class of $x \vee y$ for any $x \in X$ and $y \in Y$

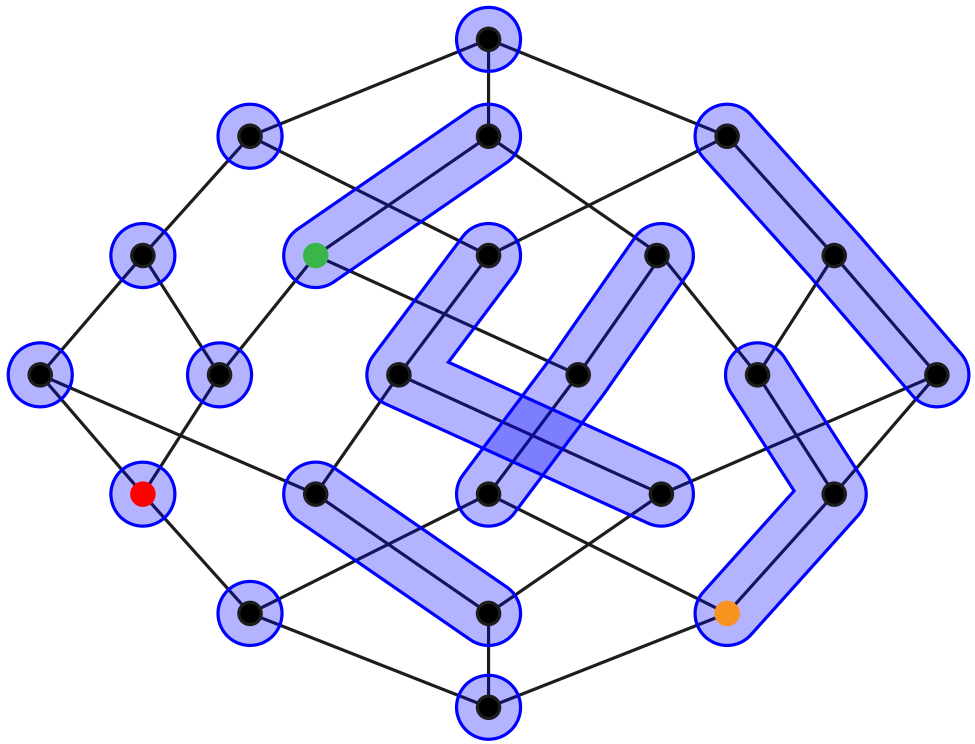


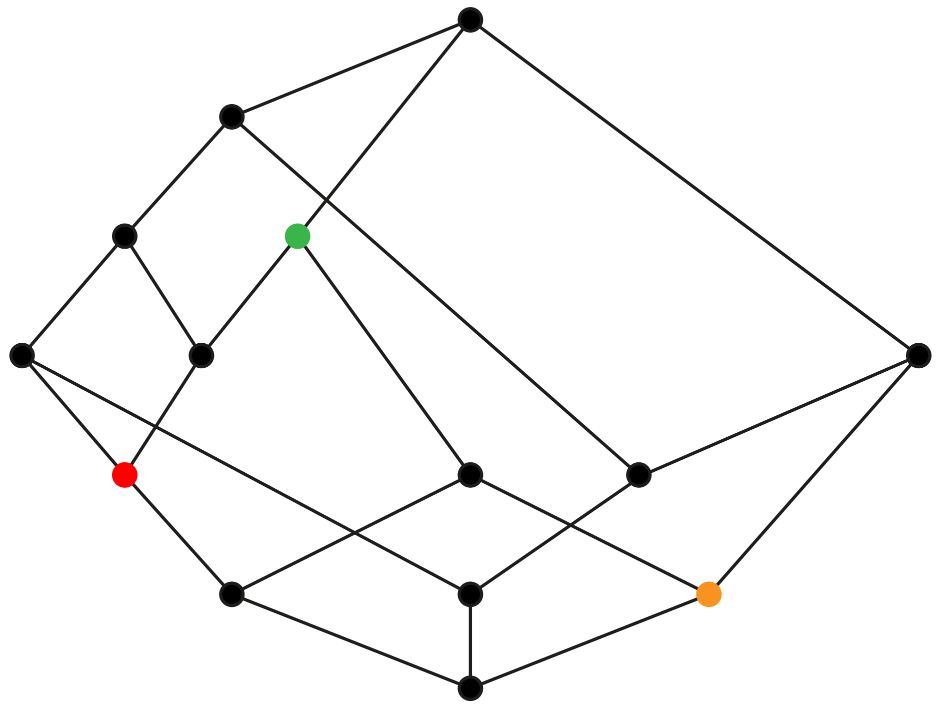


LATTICE QUOTIENTS & CANONICAL JOIN REPRESENTATIONS

$\equiv$ lattice congruence on L, then

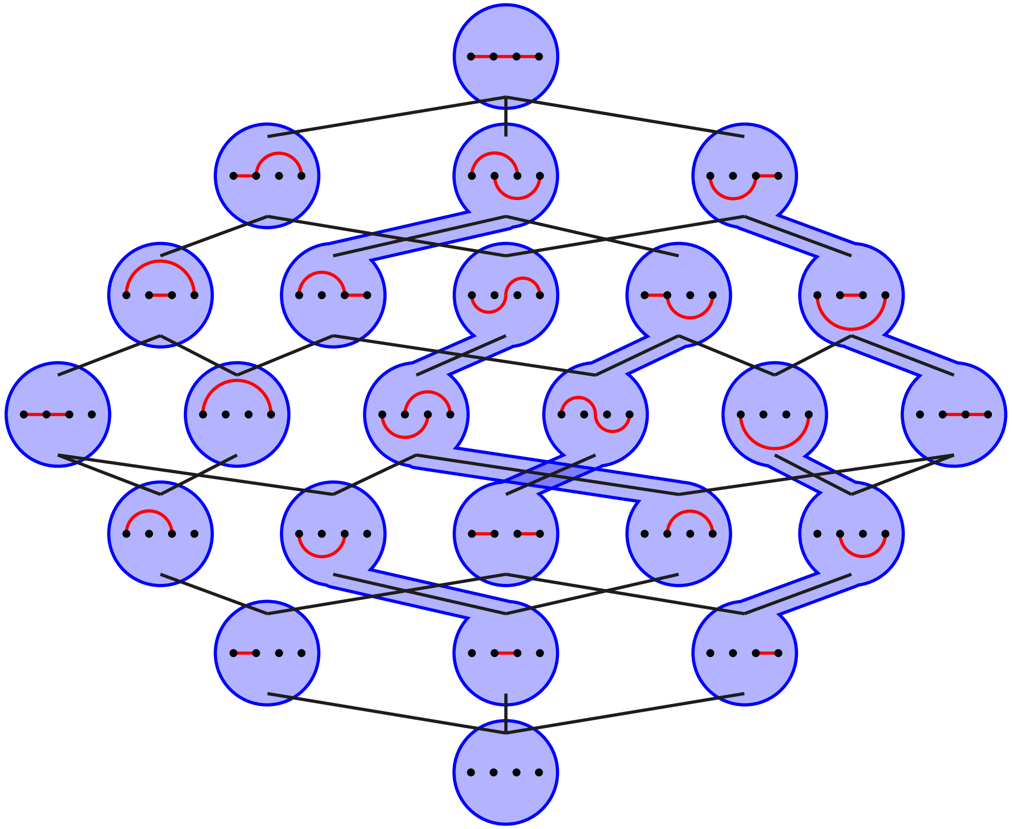
- each class X is an interval $[\pi_\downarrow(X), \pi^\uparrow(X)]$
- $L/{\equiv}$ is isomorphic (as poset) to the restriction of L to the elements x with $\pi_\downarrow(x) = x$
- $\pi_\downarrow(x) = x$ if and only if $\pi_\downarrow(j) = j$ for all canonical joinands j of x
- canonical join representations in $L/{\equiv}$ are canonical join representations in L that only involve join irreducibles j with $\pi_\downarrow(j) = j$

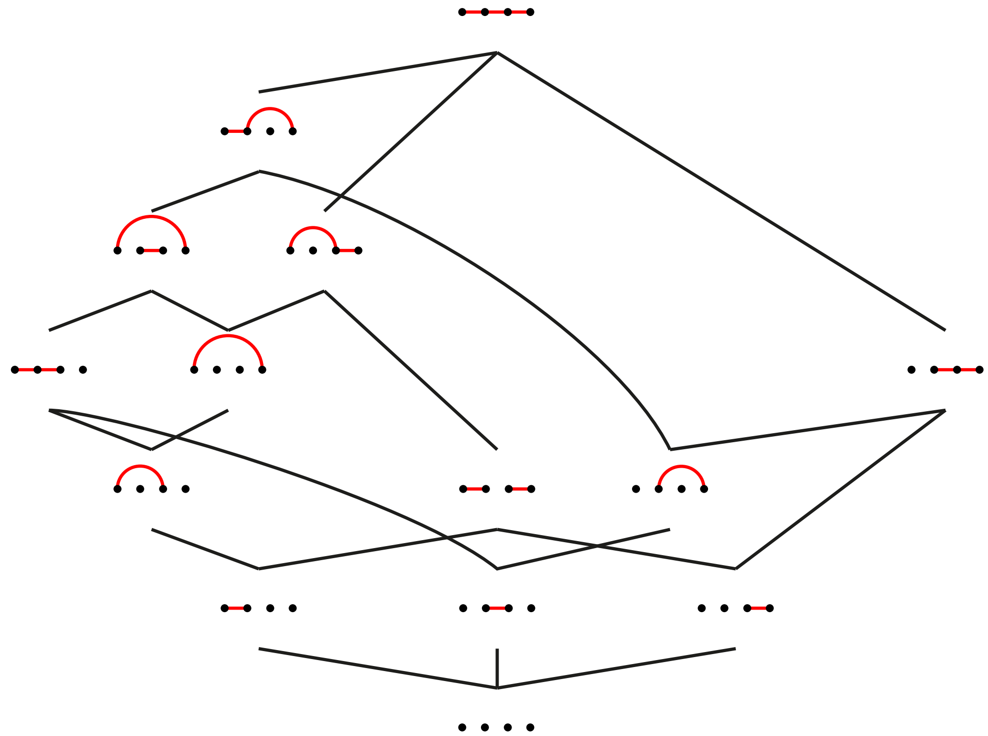




LATTICE QUOTIENTS OF THE WEAK ORDER

THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
$\mathcal{A}_\equiv$ = arcs corresponding to join irreducibles σ with $\pi_\downarrow(\sigma) = \sigma$
$\mathfrak{S}_n/\!\equiv\ \simeq$ subposet induced by noncrossing arc diagrams with all arcs in $\mathcal{A}_\equiv$



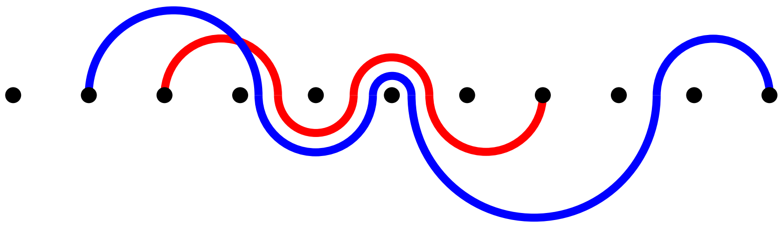


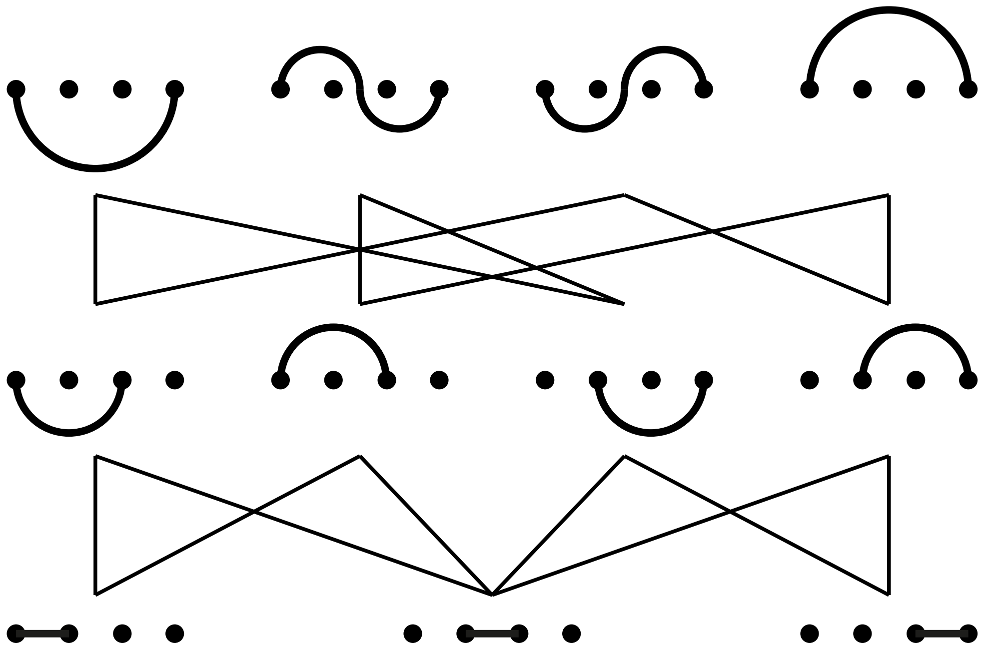
Reading ('15)

SUBARC ORDER

THM. $\equiv$ lattice congruence of the weak order on $\mathfrak{S}_n$
$\mathcal{A}_\equiv$ = arcs corresponding to join irreducibles j with $\pi_\downarrow(j) = j$
$\mathfrak{S}_n/\equiv \;\simeq$ subposet induced by noncrossing arc diagrams with all arcs in $\mathcal{A}_\equiv$

THM. The following are equivalent for a set of arcs $\mathcal{A}$:
- there exists a lattice congruence $\equiv$ on $\mathfrak{S}_n$ with $\mathcal{A} = \mathcal{A}_\equiv$
- $\mathcal{A}$ is a lower ideal of the subarc order

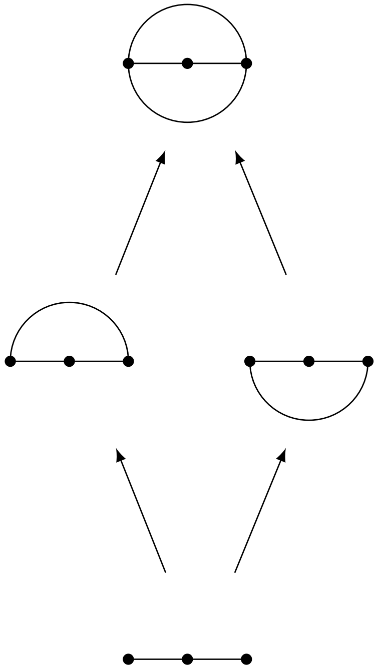




Reading ('15)

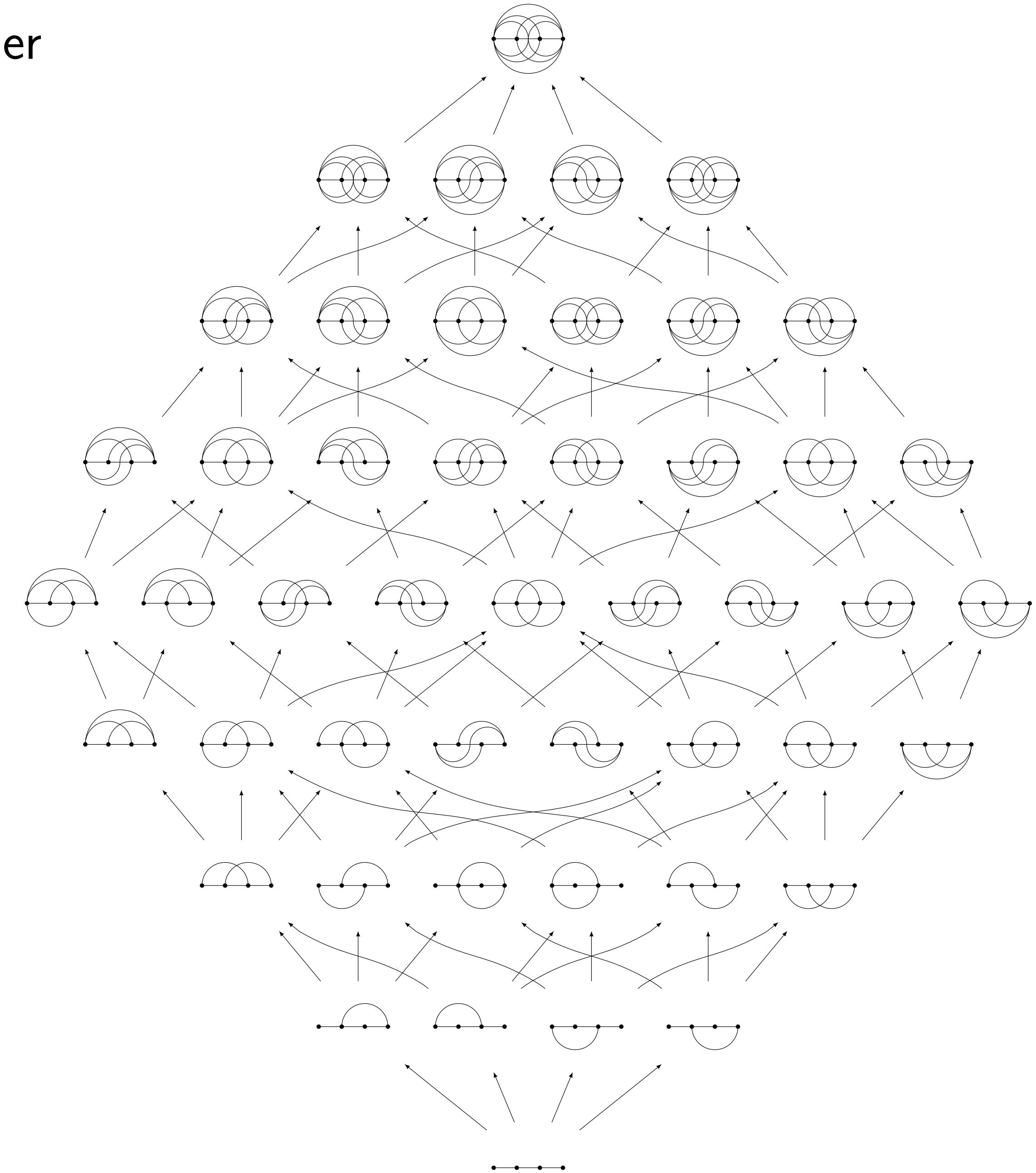
ARC IDEALS

arc ideal = lower ideal of the subarc order



essential congruences:
1, 1, 4, 47, 3322, ...
OEIS A330039

all congruences
1, 2, 7, 60, 3444, ...
OEIS A091687



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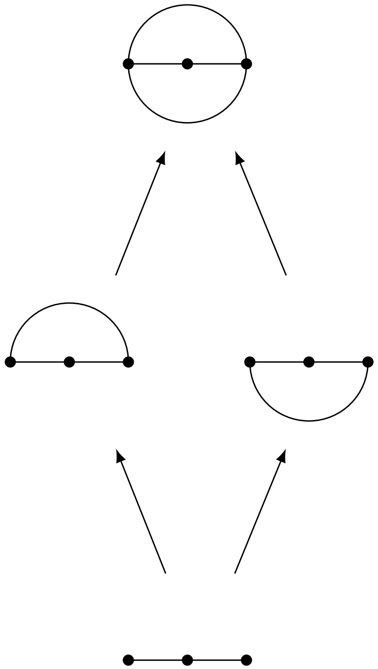

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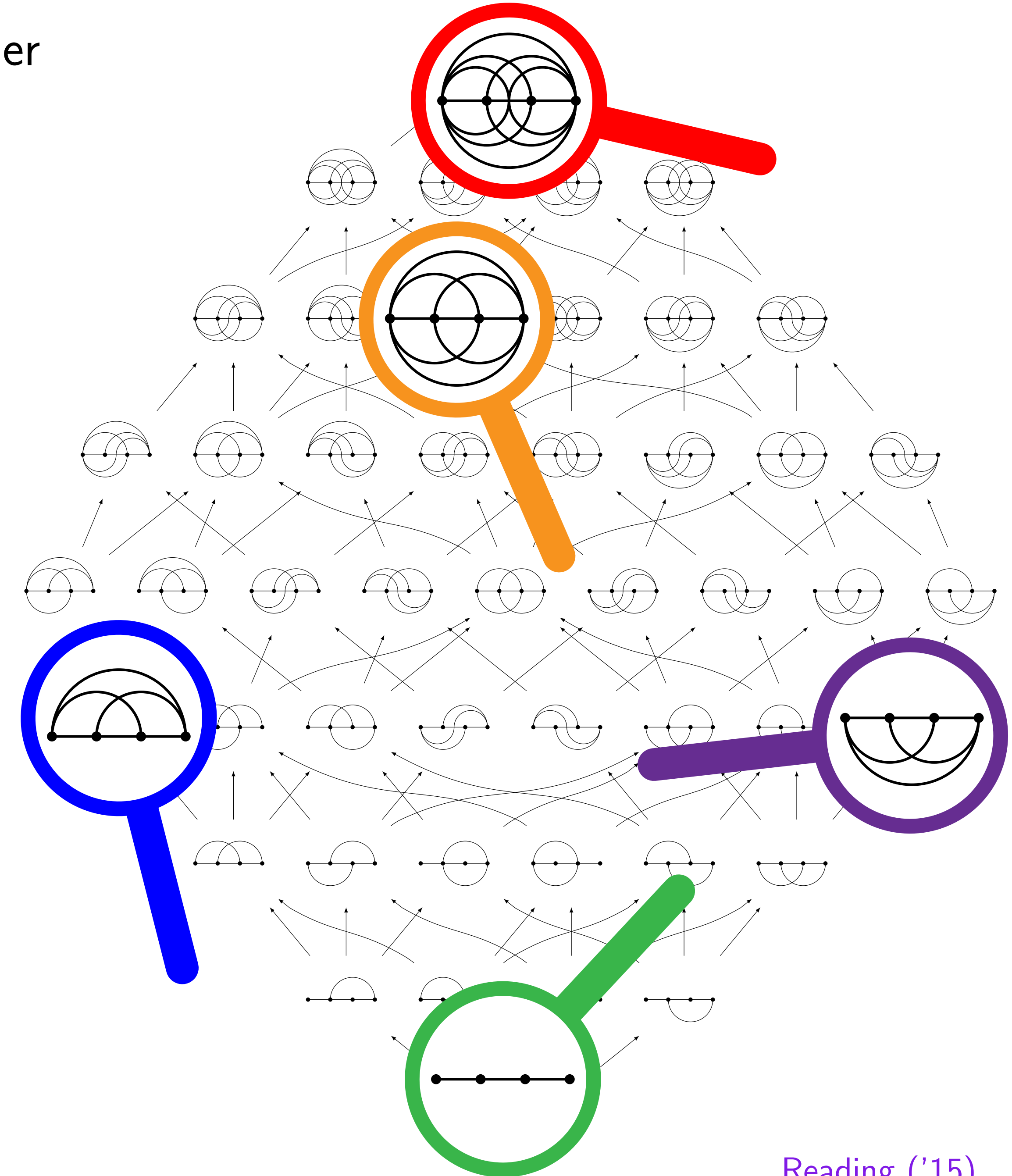
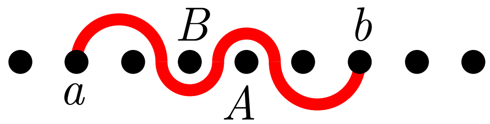


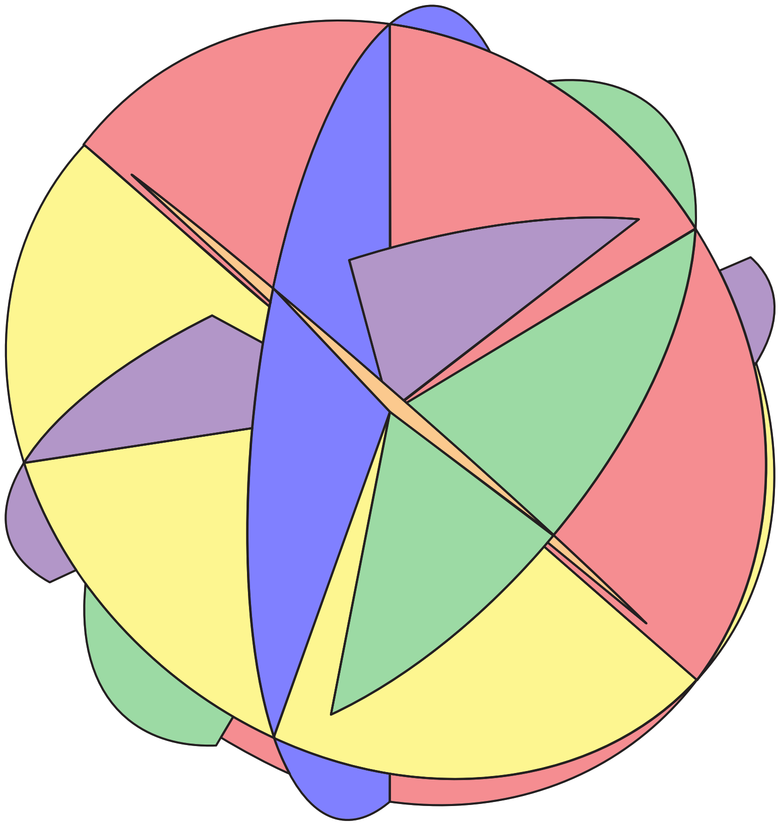
Reading ('15)

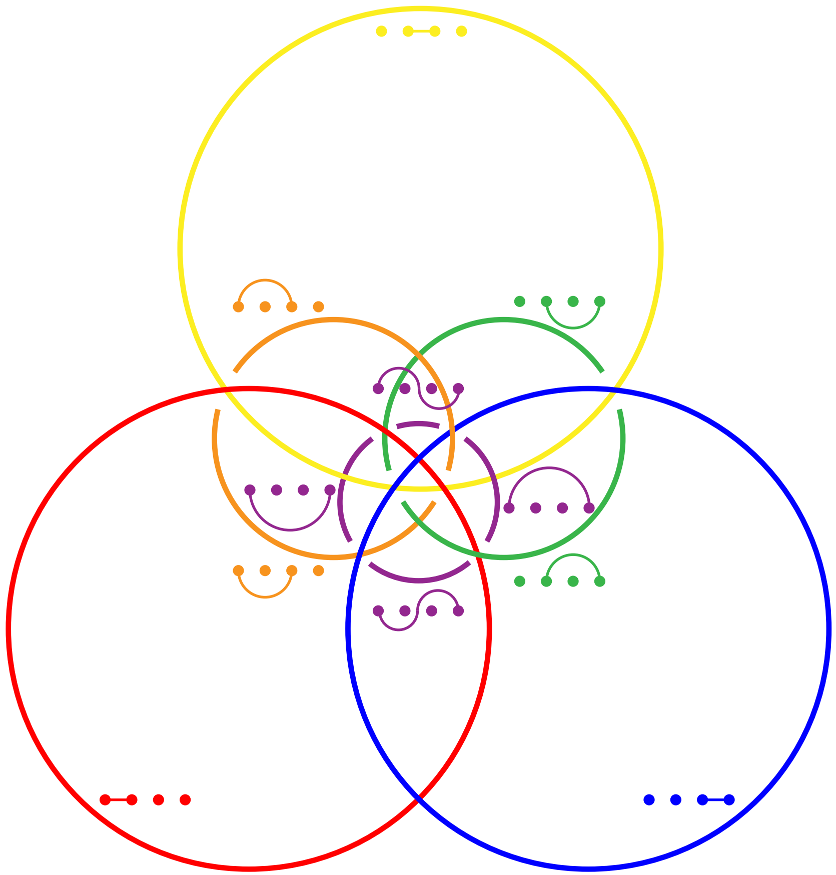
ARCS AND SHARDS

<u>arc</u> (a, b, A, B) with $1 \le a < b \le n$ and $A \sqcup B = \,]a, b[$



<u>shard</u> $\Sigma(a, b, A, B) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; x_{a'} \le x_a = x_b \le x_{b'} \text{ for all } a' \in A \text{ and } b' \in B \right\}$





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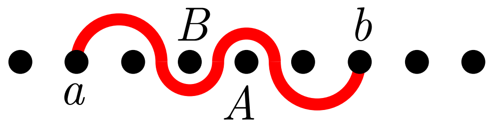

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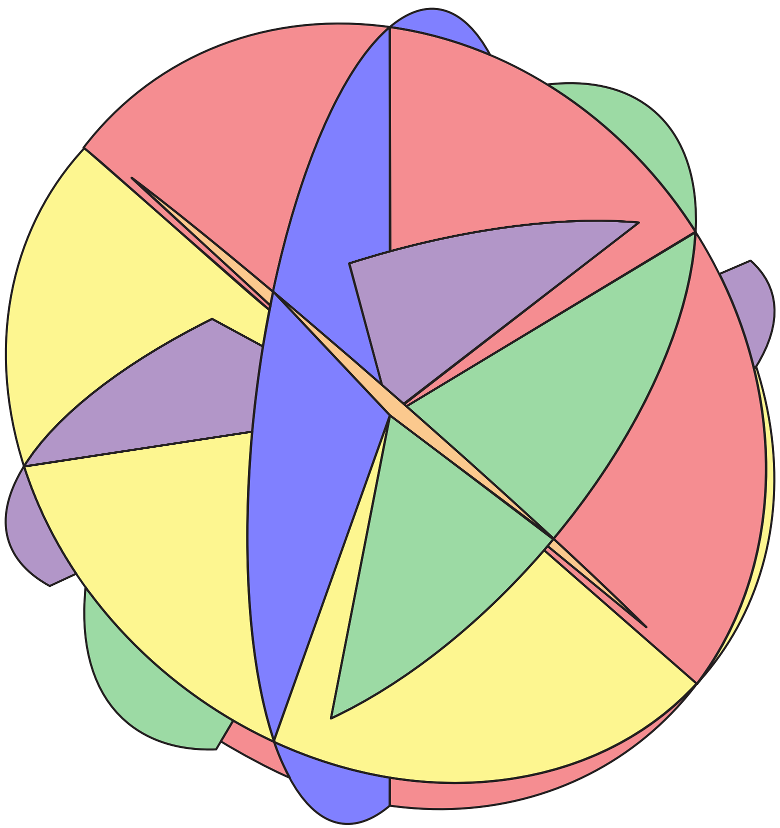
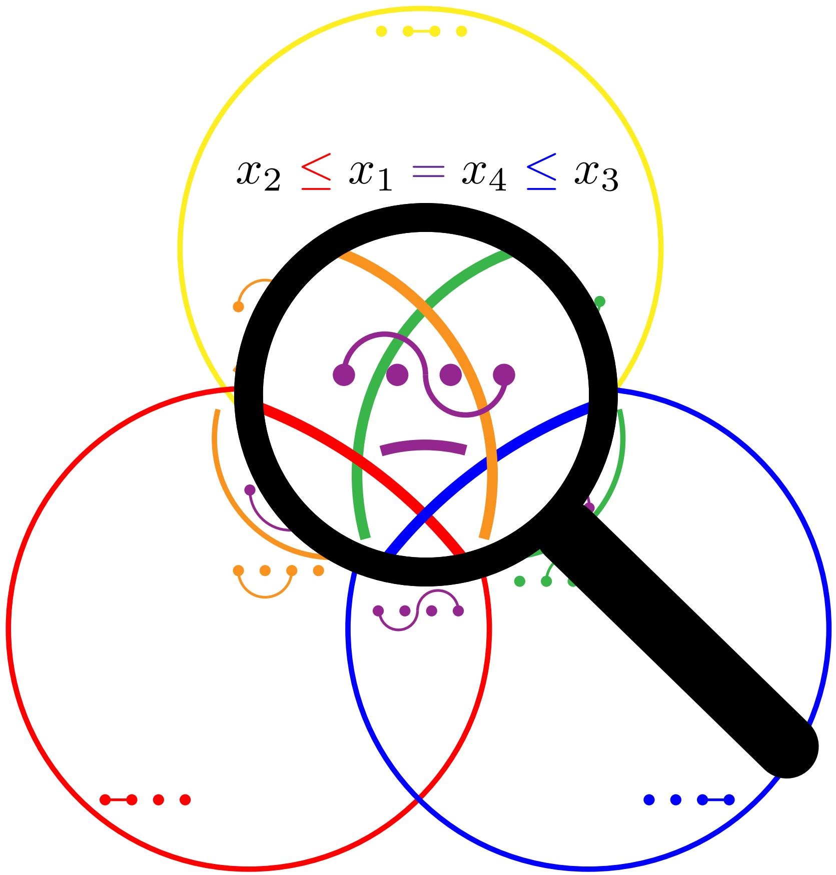


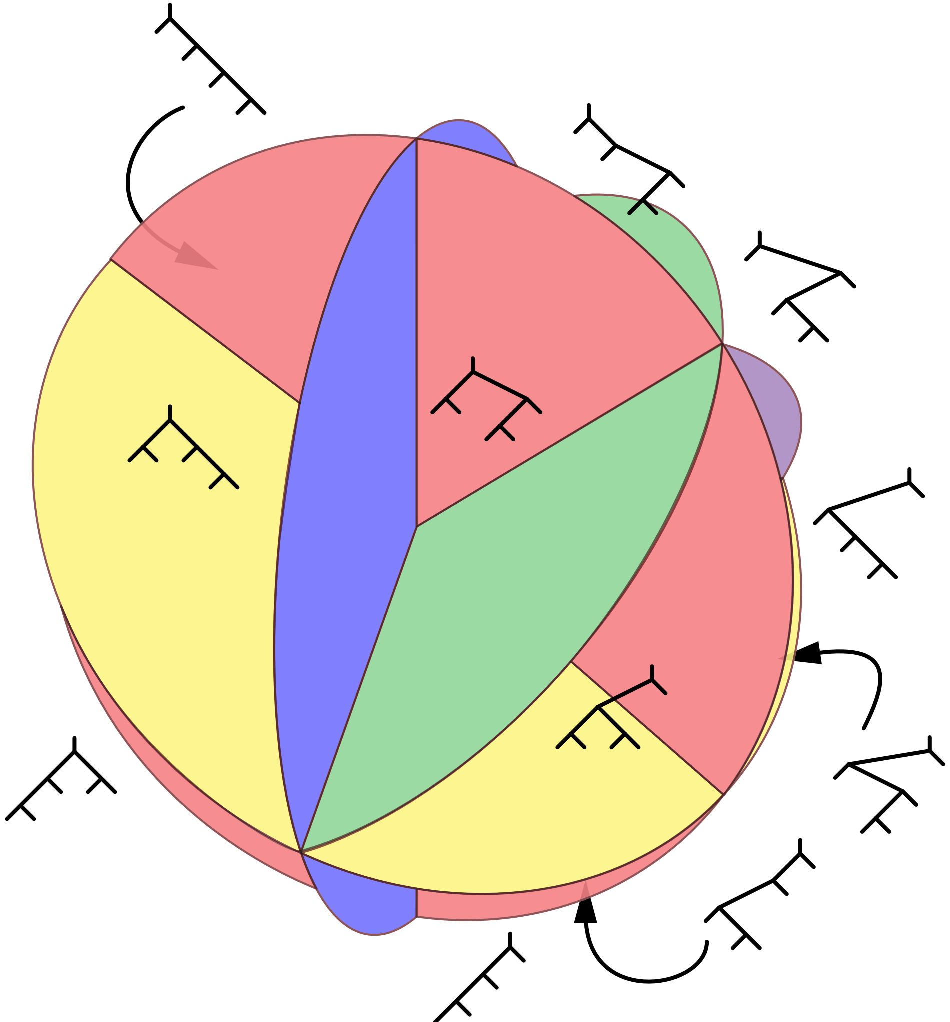


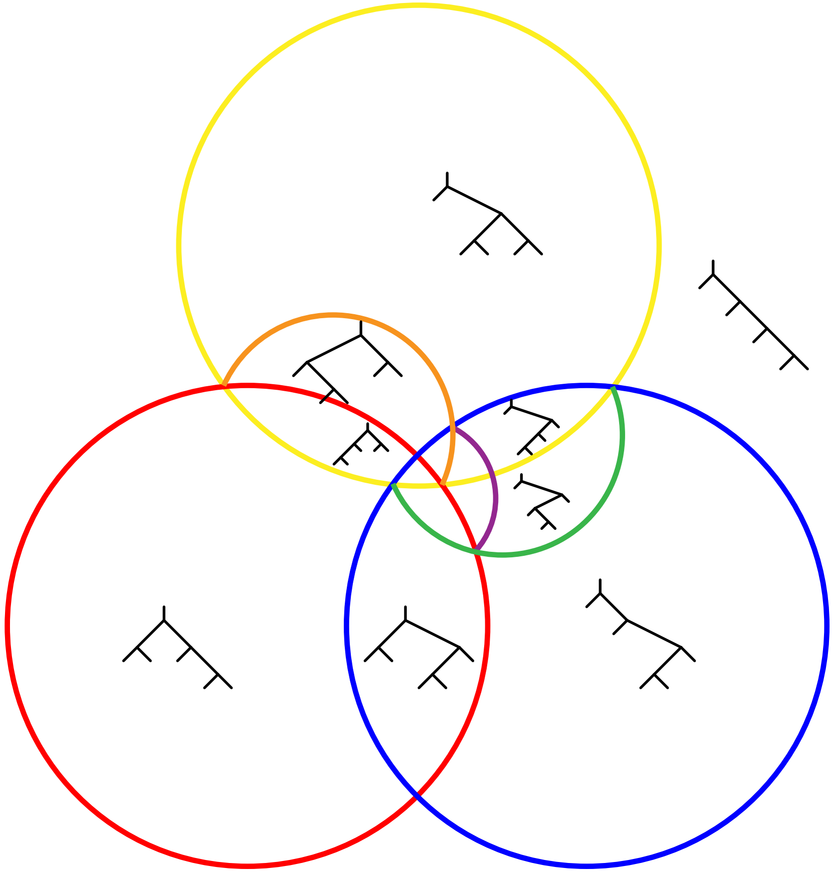
QUOTIENT FANS

THM. quotient fan $\mathcal{F}_{\equiv}$ =

• the chambers are obtained by glueing the chambers $\mathbb{C}(\sigma)$ of the permutations σ in the same congruence class of $\equiv$

• the walls are given by the union of shards $\Sigma(\alpha)$ for $\alpha \in \mathcal{A}_{\equiv}$ Reading ('05)





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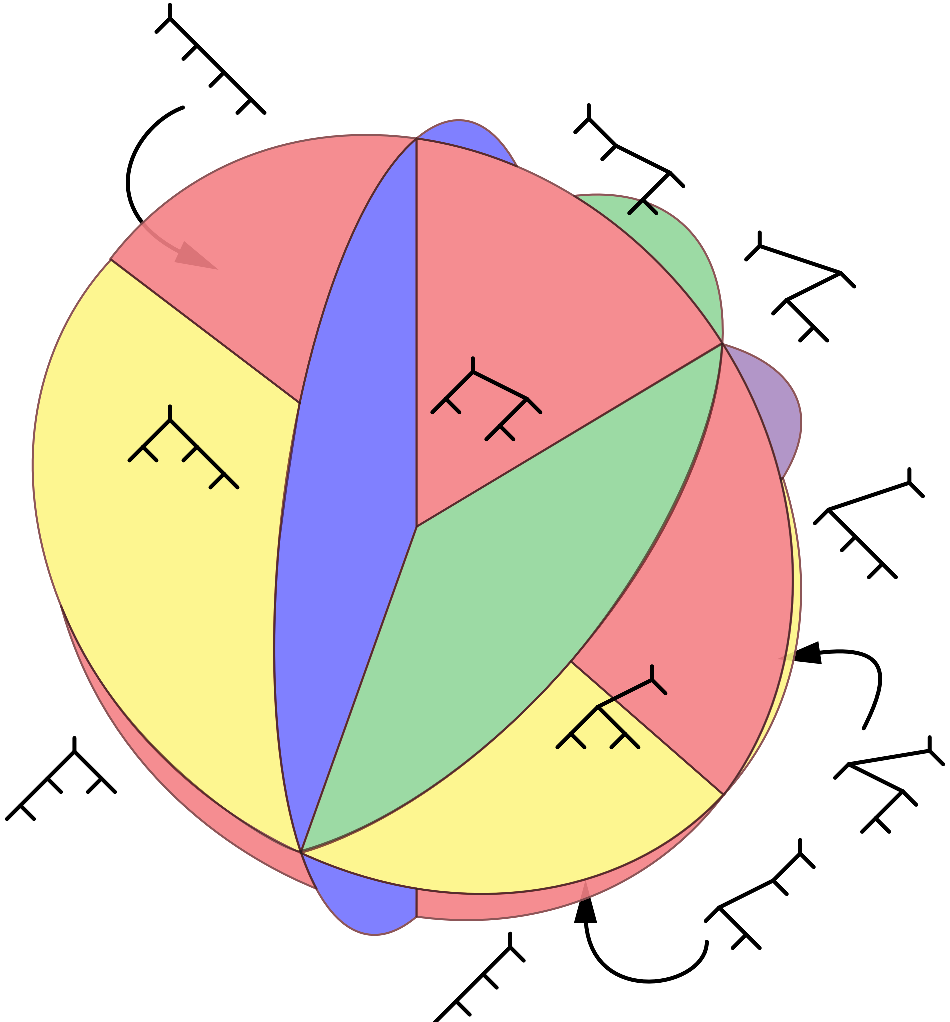
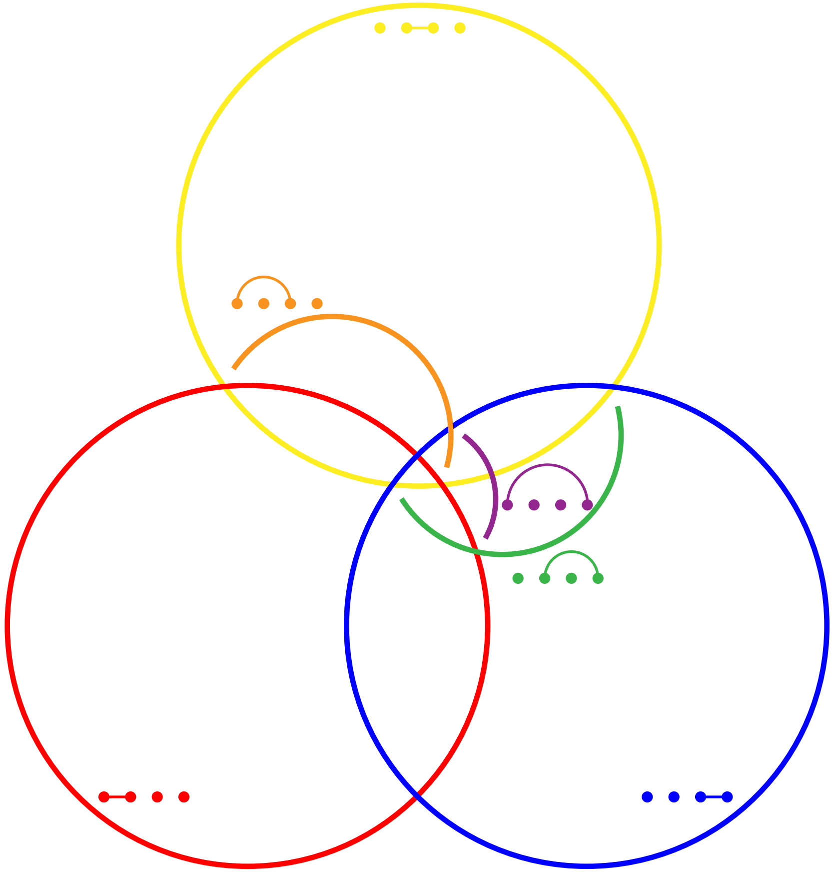
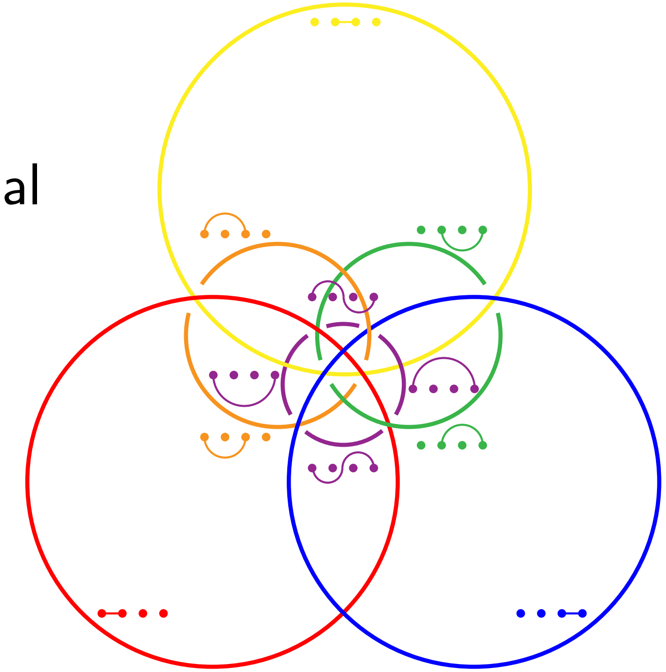


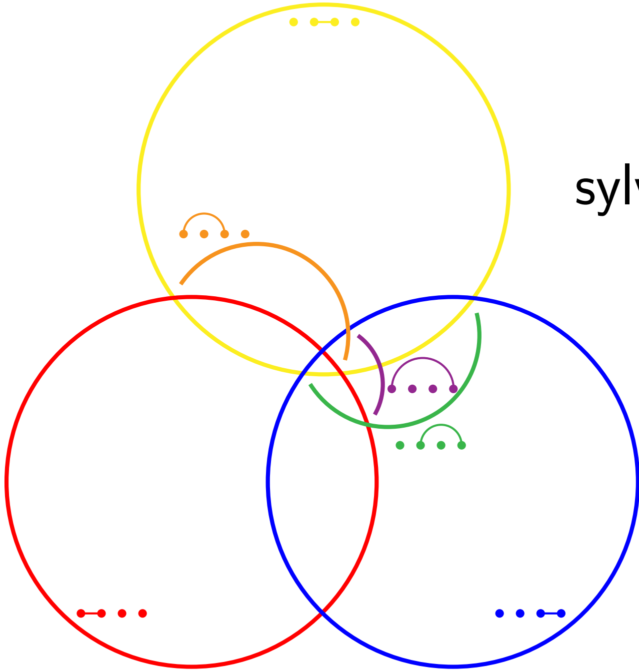


QUOTIENT FANS

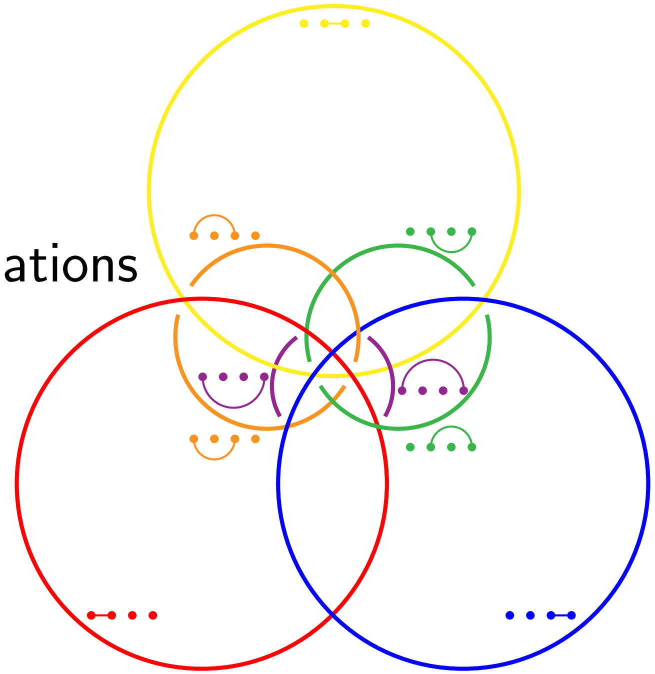
trivial



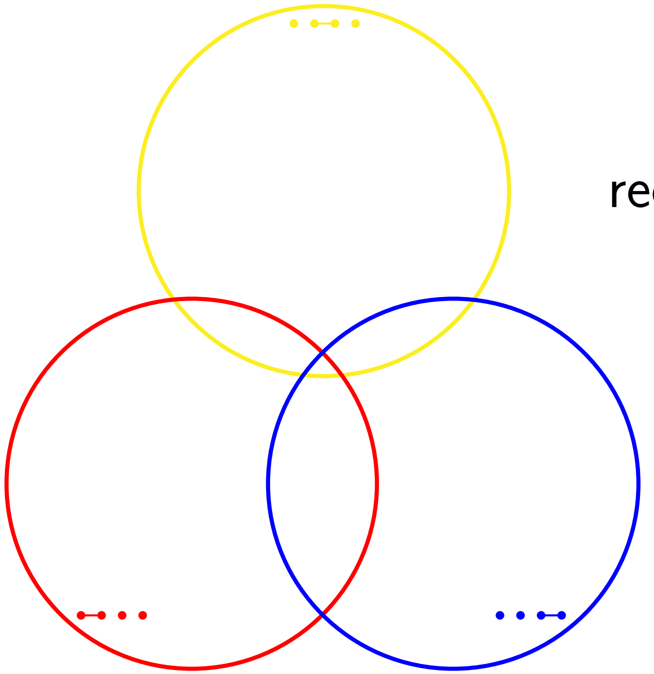
sylvester



diagonal
rectangulations



recoils



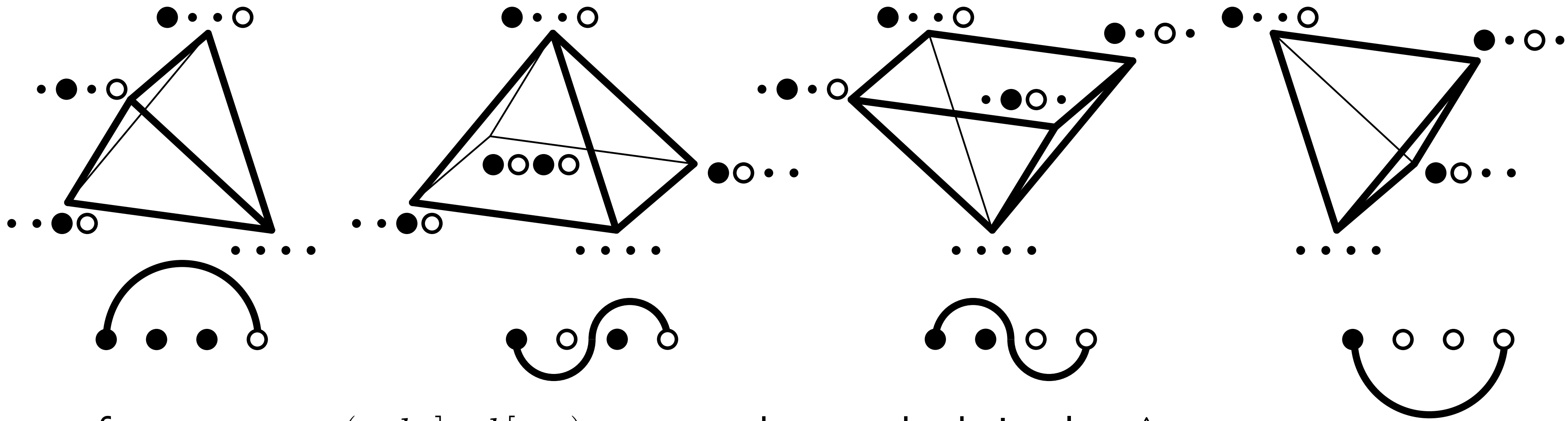
SHARD POLYTOPES

for an arc $\alpha = (a, b, A, B)$, define

- <u>α-matching</u> $=$ sequence $a \le a_1 < b_1 < \cdots < a_k < b_k \le b$ where $\begin{cases} a_i \in \{a\} \cup A \\ b_i \in B \cup \{b\} \end{cases}$
- <u>characteristic vector</u> $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$

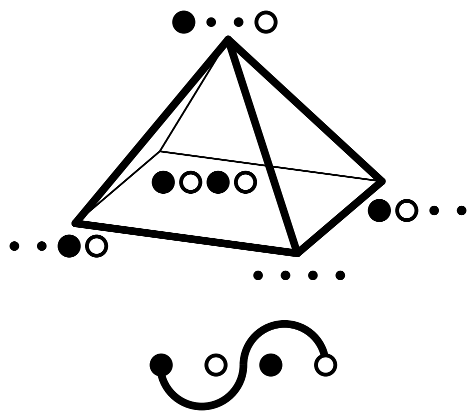
DEF. <u>shard polytope</u> $\mathbb{SP}(\alpha)$ $= \operatorname{conv} \big\{ \chi(M) \;\big|\; M \text{ } \alpha\text{-matching} \big\}$

$$= \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{cl} x_j = 0 & \text{for all } j \in [n] \smallsetminus [a, b] \\ 0 \le x_{a'} \le 1 & \text{for all } a' \in \{a\} \cup A \\ -1 \le x_{b'} \le 0 & \text{for all } b' \in B \cup \{b\} \\ 0 \le \sum_{i \le j} x_i \le 1 & \text{for all } j \in [n] \end{array} \right\}$$

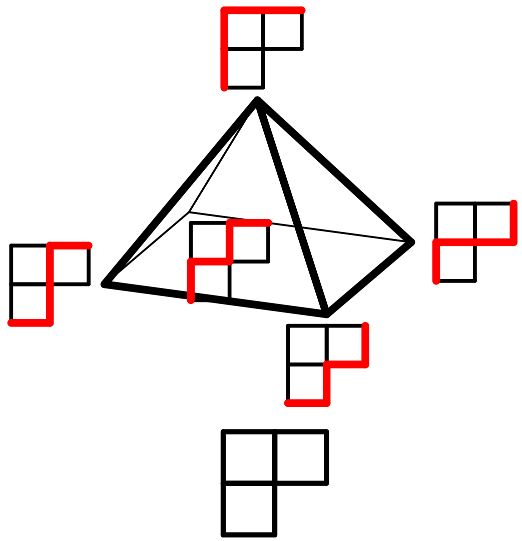


exm: for an up arc $(a, b,]a, b[, \varnothing)$, we get the standard simplex $\triangle_{[a,b]} - \boldsymbol{e}_b$

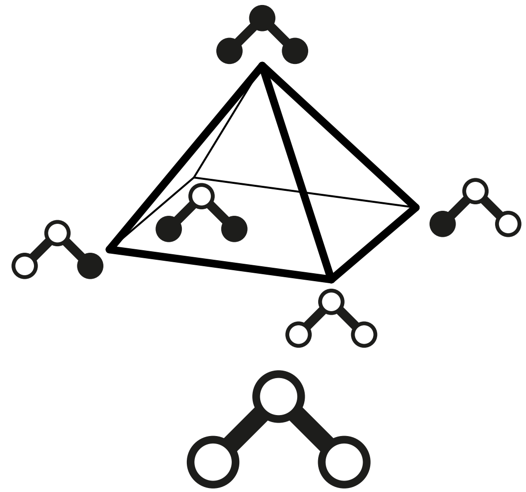
MANY PERSPECTIVES ON SHARD POLYTOPES



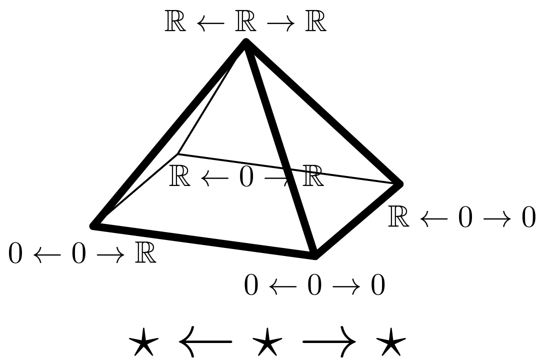
Shard polytopes of arcs



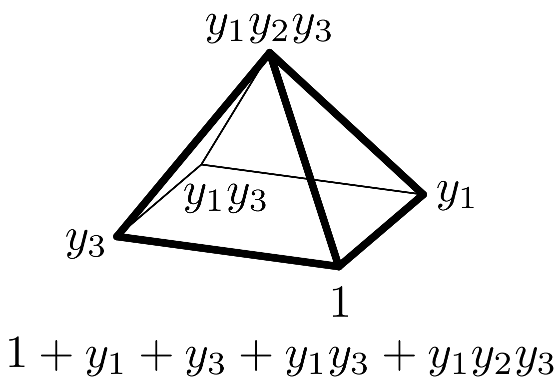
Base polytopes of
lattice path matroids
of snakes



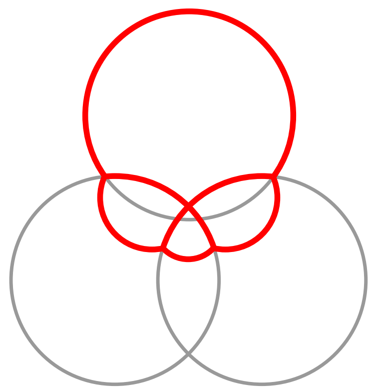
Order polytopes
of fence posets



Harder–Narasimhan
polytopes of
oriented paths



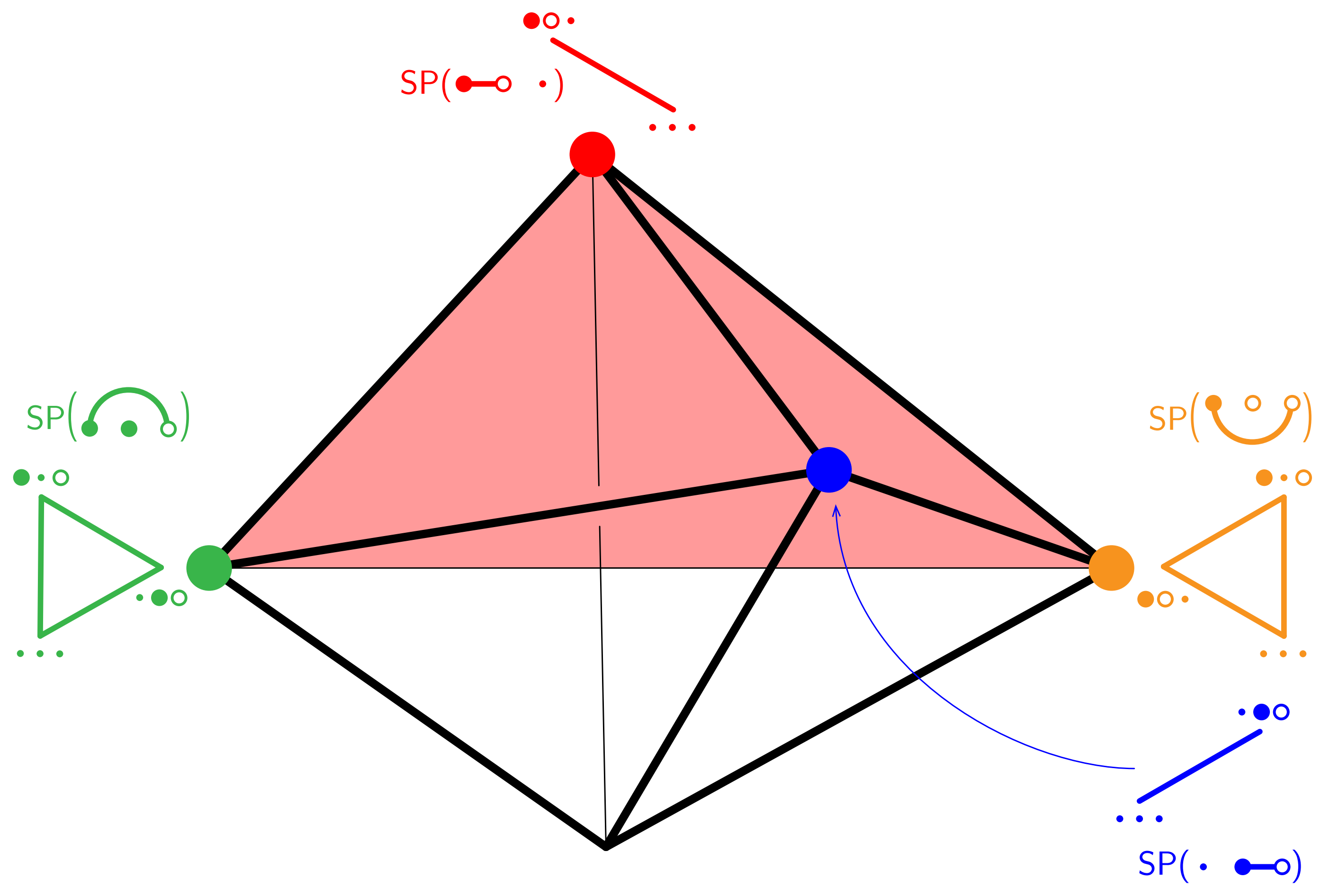
Newton polytopes
of F-polynomials
of cluster variables



Minkowski indecomposable
deformations of
Cambrian associahedra

SHARD POLYTOPES FORM A BASIS

PROP. $\{\mathbb{SP}(i,j,A,B) \mid 1 \le i < j \le n,\ A \sqcup B = [n]\}$ linear basis of submodular cone

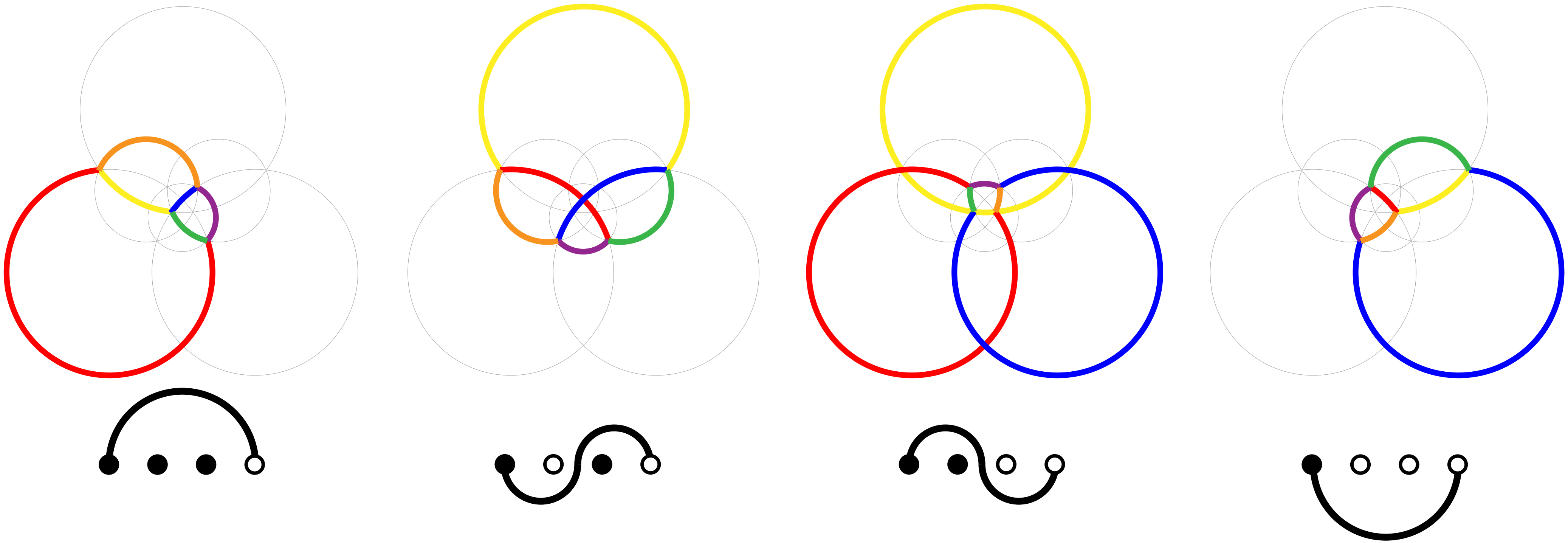


SHARD POLYTOPES

DEF. shard polytope $\mathbb{SP}(\alpha) = \operatorname{conv}\left\{\chi(M) \;\middle|\; M \; \alpha\text{-matching}\right\}$

PROP. The union of the walls of the normal fan of the shard polytope $\mathbb{SP}(\alpha)$

- contains the shard $\Sigma(\alpha)$,
- is contained in the union of the shards $\Sigma(\alpha')$ for α' subarc of α



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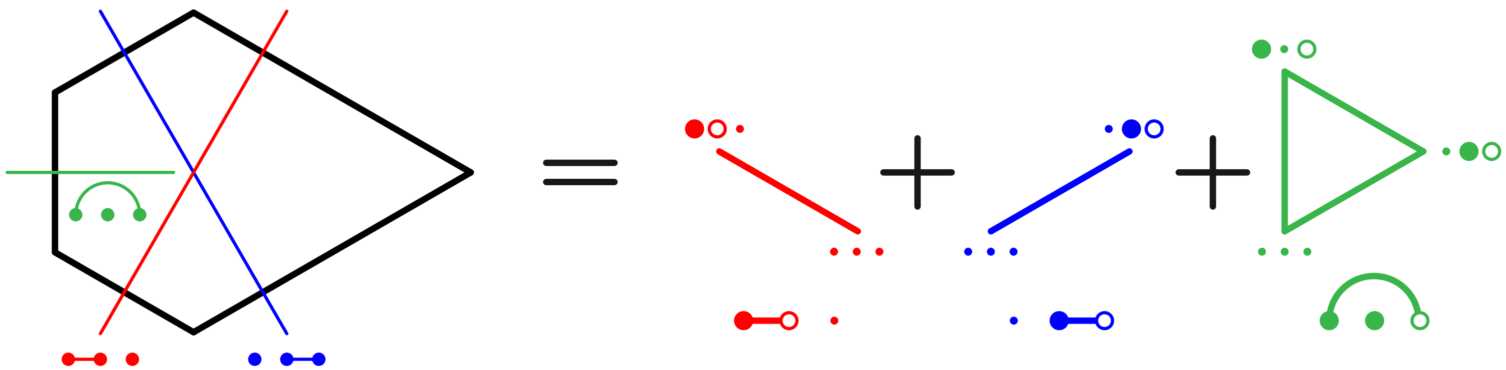

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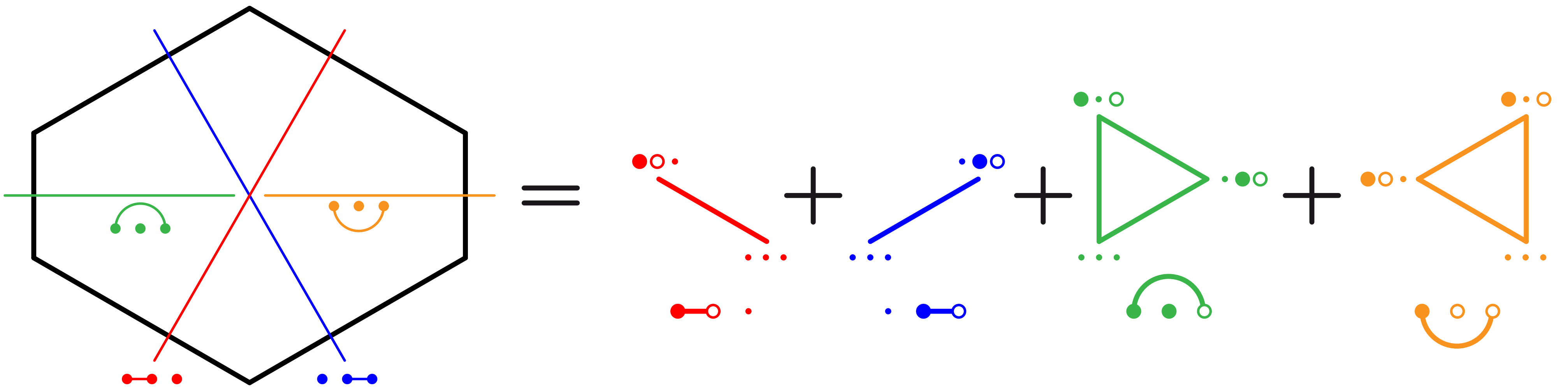
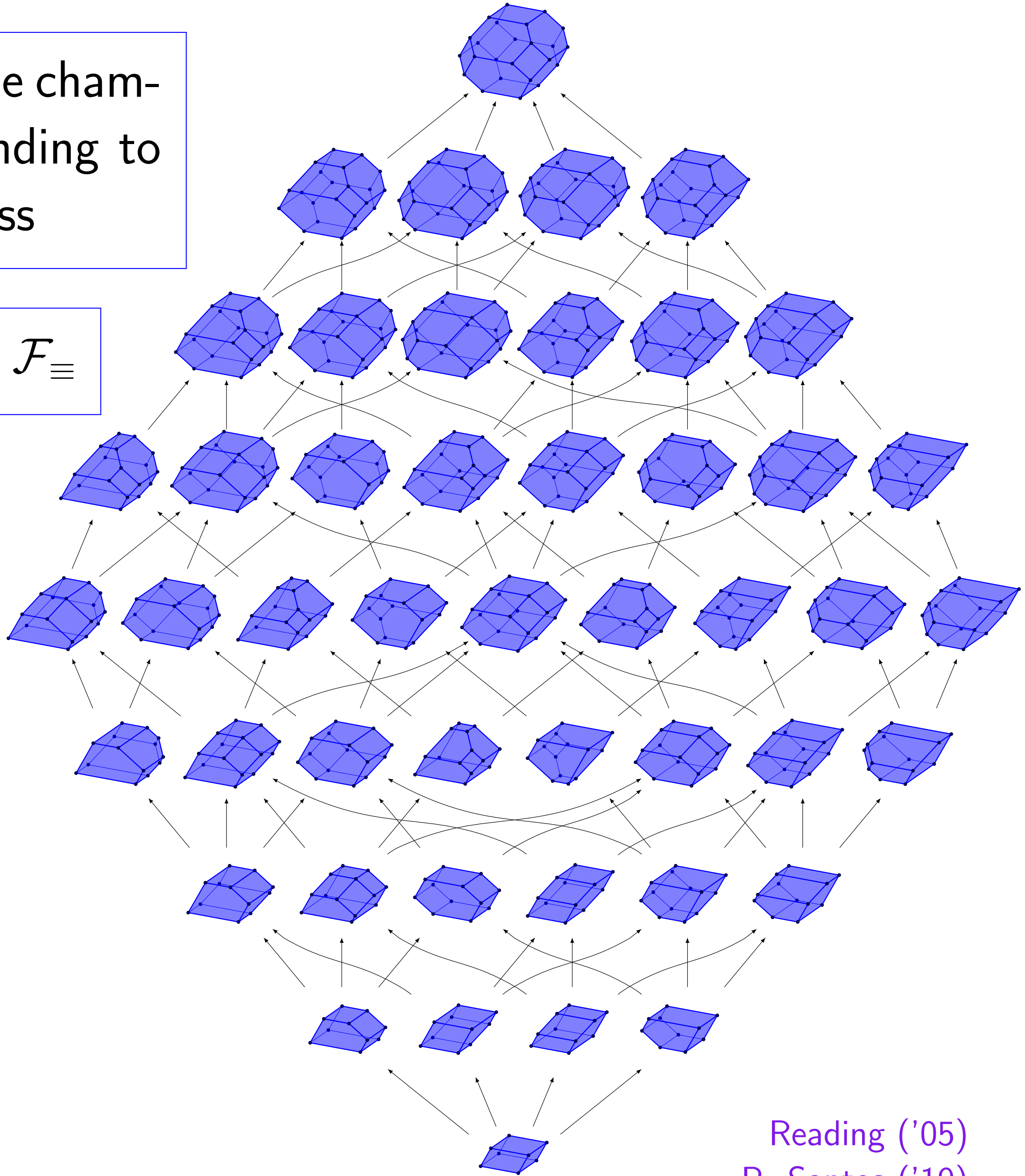


QUOTIENTOPES

quotient fan $\mathcal{F}_{\equiv}$ = obtained by glueing the chambers of the braid arrangement corresponding to chambers in the same $\equiv$-congruence class

quotientope = polytope with normal fan $\mathcal{F}_{\equiv}$



Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('23)

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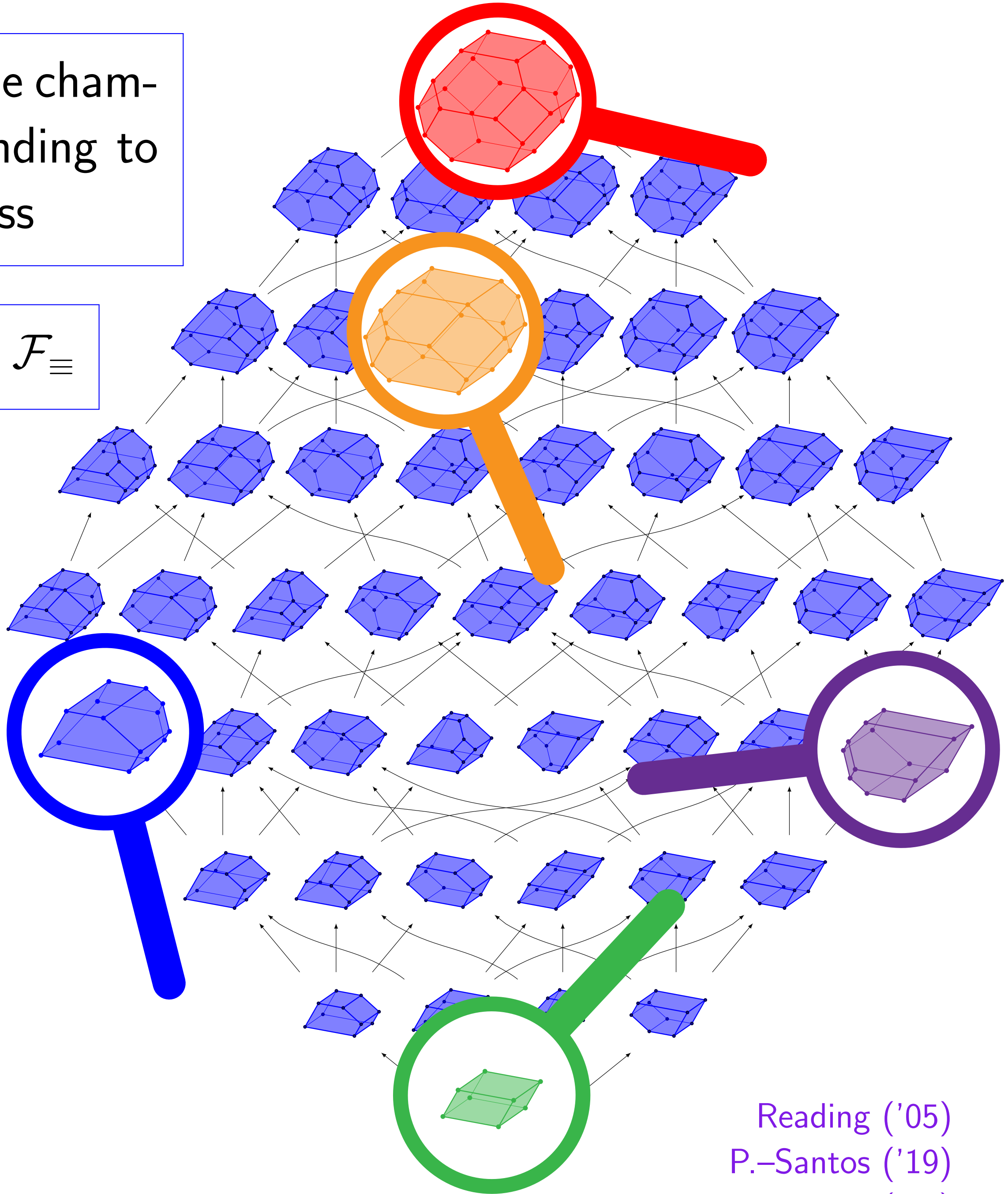

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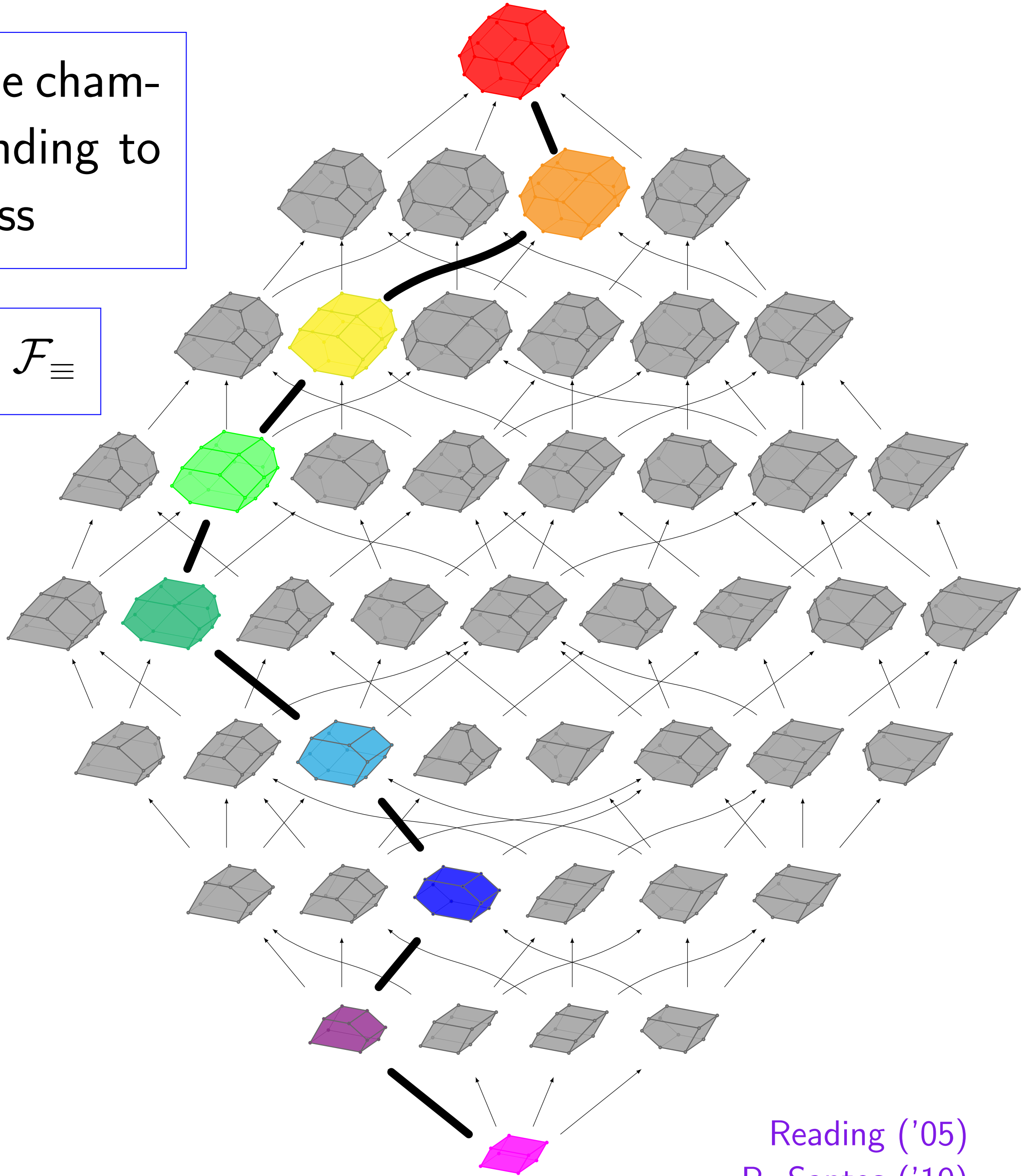
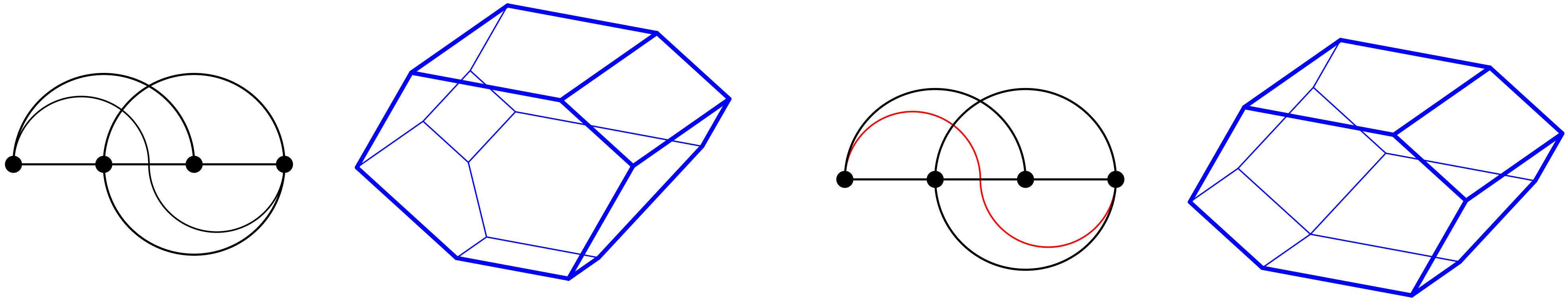


Reading ('05)
P.–Santos ('19)
Padrol–P.–Ritter ('23)

SIMPLE QUOTIENTOPES

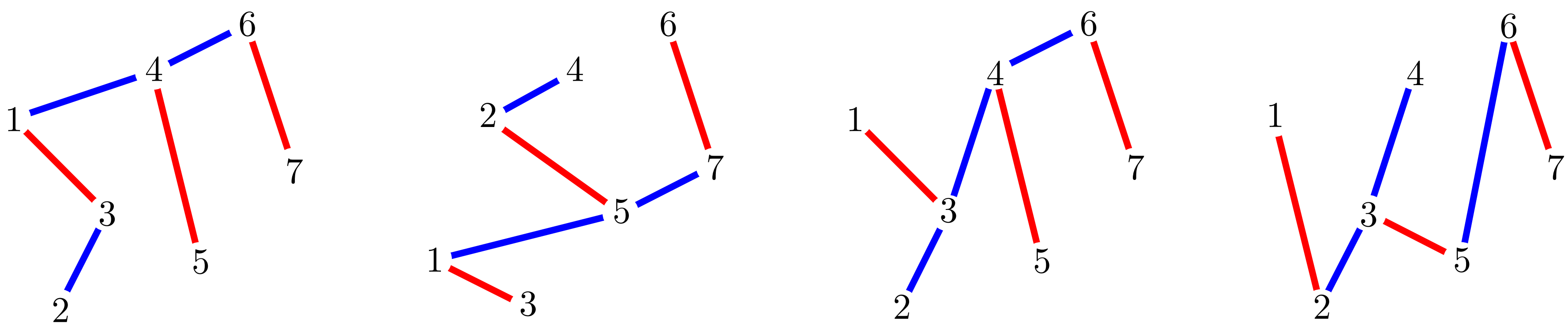
PROP. $\mathbb{Q}\mathrm{uot}_{\equiv}$ is simple $\iff$ no minimal arc $\mathcal{A}_n \smallsetminus \mathcal{A}_{\equiv}$ crosses the horizontal axis

Hoang–Mütze ('21) Demonet–Iyama–Reading–Reiten–Thomas ('23) Barnard–Novelli–P. ('25)



PROP. All $\mathbb{Q}\mathrm{uot}_{\equiv}$-posets are separating trees

Barnard–Novelli–P. ('25)



each node has ≤ 2 parents (resp. children) & separates its descendants (resp. ancestors)

SUMMARY

bijection $\{$permutations$\}$ $\longleftrightarrow$ $\{$non-crossing arc diagrams$\}$
encoding canonical join representations in the weak order

any lattice congruence $\equiv$ of the weak order defines a quotient fan $\mathcal{F}_\equiv$ obtained
- either by coarsening the braid fan according to the classes of $\equiv$,
- or by keeping the shards of arcs of $\mathcal{A}_\equiv$

the quotient fan is the normal fan of the Minkowski sum of the shard polytopes

shard polytopes = snake polytope = fence order polytopes
= HN-polytopes of paths = Newton polytopes of F-polynomials
= rays of the deformation cone of Cambrian associahedra

simple quotientopes = when minimal arcs of $\mathcal{A}_n \smallsetminus \mathcal{A}_\equiv$ do not cross the axis
their vertex posets are separating trees

DEFORMED CAMBRIAN ASSOCIAHEDRA

Giardino–Gützeit–Hoang–Leivaditis–P.,
On Minkowski summands of Cambrian associahedra ('26^{+++})

DEFORMED CAMBRIAN ASSOCIAHEDRA

for a fixed arc $\alpha = (i, j, \alpha^+, \alpha^-)$, the <u>$\alpha$-associahedron</u> is $\displaystyle\sum_{\beta<\alpha} \mathbb{SP}(\alpha)$

PROP. deformed α-associahedra $= \displaystyle\sum_{\beta\in\mathbb{B}} \mathbb{SP}(\alpha)$ where $\mathbb{B} \subseteq \{\text{subarcs of } \alpha\}$

PROP. Vertex posets of deformed α-associahedra $=$ α-Cambrian interval posets
ie. posets $\lhd$ on $[n]$ such that for all $a < b < c$,

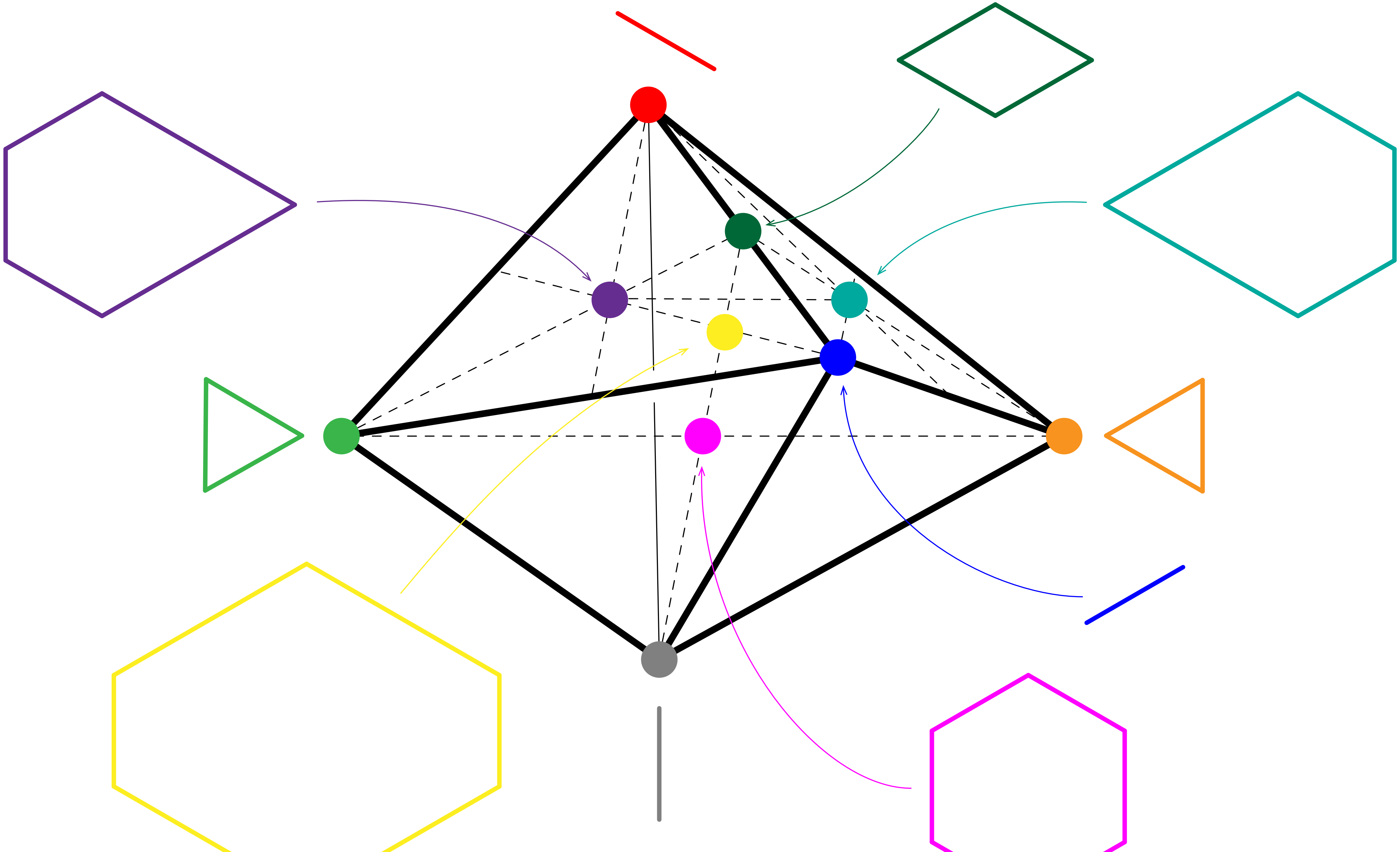
- $a \lhd c \;\Rightarrow\; a \lhd b$ if $b \in \alpha^+$ and $b \lhd c$ if $b \in \alpha^-$
- $a \rhd c \;\Rightarrow\; a \rhd b$ if $b \in \alpha^+$ and $b \rhd c$ if $b \in \alpha^-$

THM. The oriented squeleton of $\displaystyle\sum_{\beta\in\mathbb{B}} \mathbb{SP}(\alpha)$ is a lattice

$\iff$ any problematic pair in $\mathbb{B}$ has a fix in $\mathbb{B}$

Giardino–Gützeit–Hoang–Leivaditis–P. ('26^{+++})

PROB. Characterize simple deformed α-associahedra



THANK YOU