Fourier-like bases and Integrable Probability

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Over the last two decades, a number of exactly solvable, integrable probabilistic systems have been analyzed (random matrices, random interface growth, lattice models, directed polymers in random media).

The solvability of many of them can be traced back to several remarkable Fourier-like multivariate bases.

Unraveling the basic structure that underlies the solvability leads to more powerful systems that yield new tractable physical models, new spectral theories, and new phenomena.

Fourier transform on $\mathbb{Z}$

$\{z^n\}$ is an orthogonal basis in $L^2(\mathbb{T}, \text{Lesbegue})$, and an orthogonal basis in $\ell^2(\mathbb{Z})$

$$\frac{1}{2\pi i} \oint_{|z|=1} z^m \cdot z^{-n} \frac{dz}{z} = \mathbb{1}_{m=n}, \qquad \sum_{n \in \mathbb{Z}} z^n w^{-n-1} = \delta(w-z)$$

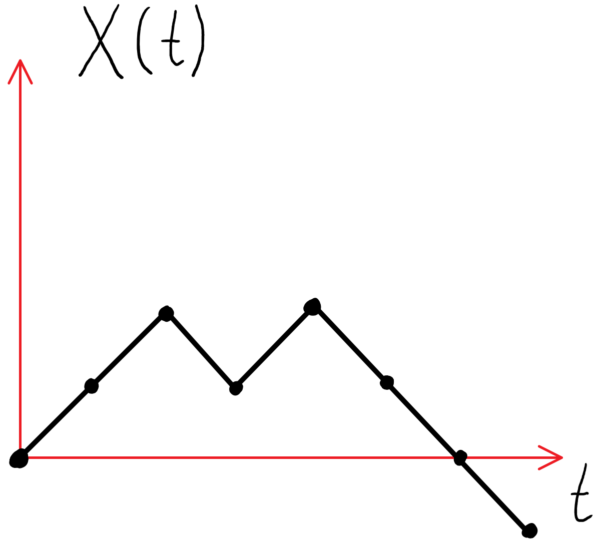
or, equivalently,

$$\frac{1}{2\pi i} \oint_{|z|=1} \underbrace{\sum_{m \in \mathbb{Z}} a_m z^m}_{\text{Direct Fourier transform}} \cdot z^{-n} \frac{dz}{z} = a_n, \qquad \sum_{n \in \mathbb{Z}} z^n \cdot \underbrace{\frac{1}{2\pi i} \oint_{|z|=1} f(w) w^{-n} \frac{dw}{w}}_{\text{Inverse Fourier transform}} = f(z)$$

Since $z^{n+1} = z \cdot z^n$, this basis diagonalizes any linear difference operator with constant coefficients.

Random walk on $\mathbb{Z}$

$$X(t+1)=\begin{cases} X(t)+1, & \text{prob. } p \\ X(t)-1, & \text{prob. } (1-p)\end{cases}$$

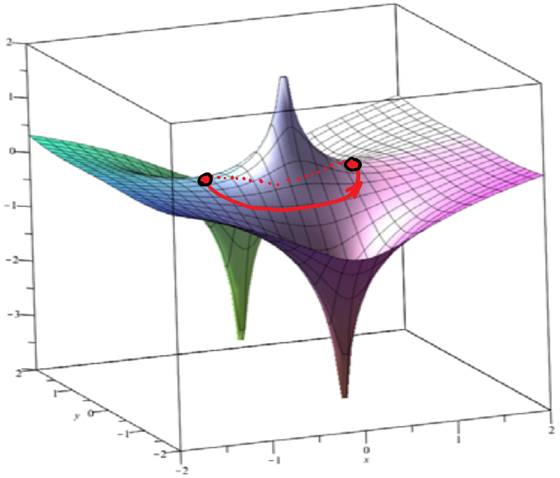


$$\mathrm{Prob}\{X(t+1)=m\} = p\cdot \mathrm{Prob}\{X(t)=m-1\} + (1-p)\cdot \mathrm{Prob}\{X(t)=m+1\}$$

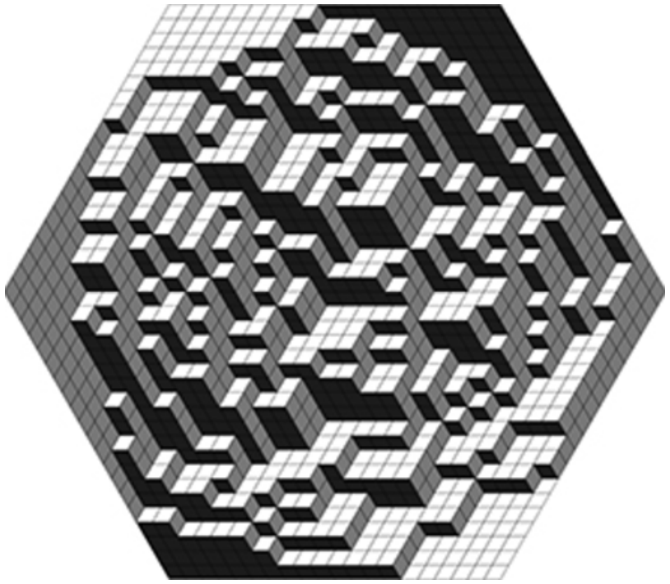
$$\mathrm{Prob}\{X(t)=m\} = \frac{1}{2\pi i}\oint_{|z|=1} \Big(\underbrace{pz+(1-p)z^{-1}}_{\text{Eigenvalue}}\Big)^{t} \cdot \underbrace{\sum_{n\in\mathbb{Z}} \mathrm{Prob}\{X(0)=n\}\, z^{n}}_{\text{Direct Fourier transform}} \cdot \underbrace{z^{-m}\,\frac{dz}{z}}_{\text{Inverse Fourier transform}}$$

Corollary (Central Limit Theorem)

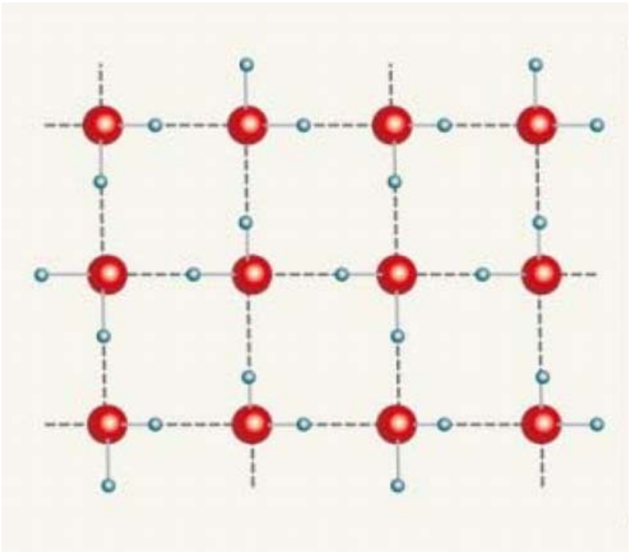
$$\mathrm{Prob}\left\{\frac{X(t)-X(0)-(2p-1)t}{2\sqrt{p(1-p)}\cdot\sqrt{t}} \le x\right\} \xrightarrow[t\to\infty]{} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$



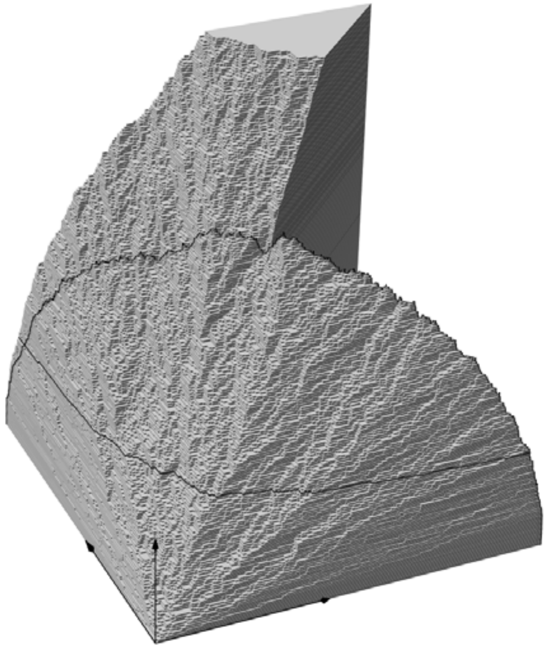
I would like to argue similar properties of other Fourier-like bases can be used to study 2d random interfaces, as opposed to 1d as in the random walk.



Random Plane Partitions



Ice-type models



Polynuclear growth (directed polymers)

However, there is a caveat - our Fourier-like bases will be unusual.

Rational Schur symmetric functions

$$s_\lambda(z_1,\dots,z_N) = \frac{\det\left[z_i^{\lambda_j+N-j}\right]_{i,j=1}^{N}}{\det\left[z_i^{N-j}\right]_{i,j=1}^{N}} \in \mathbb{C}\left[z_1^{\pm1},\dots,z_N^{\pm1}\right]^{\text{Symm}}, \qquad \lambda=(\lambda_1\geq\dots\geq\lambda_N)\in\mathbb{Z}^N.$$

Two orthogonality relations :

$$\frac{1}{N!}\frac{1}{(2\pi i)^N}\oint\cdots\oint_{|z_j|=1} s_\lambda(z)\,s_\mu(z^{-1})\prod_{i<j}|z_i-z_j|^2\,\frac{dz_1\cdots dz_N}{z_1\cdots z_N} = \mathbb{1}_{\lambda=\mu}$$

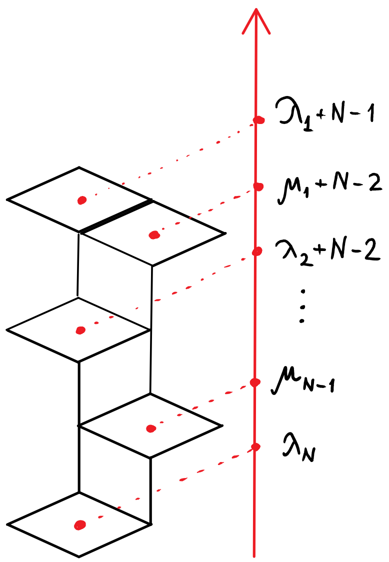
$$\sum_{-\infty<\lambda_N\leq\dots\leq\lambda_1<+\infty} s_\lambda(z)\,s_\lambda(w^{-1})\cdot\prod_{i<j}(z_i-z_j)(w_i^{-1}-w_j^{-1})\cdot\frac{1}{w_1\cdots w_N} = \det\left[\delta(w_i-z_j)\right]_{i,j=1}^{N}$$

The Schur functions are characters of the (complex) irreducible representations of $GL(N,\mathbb{C})$ (or $U(N)$).

Rational Schur symmetric functions

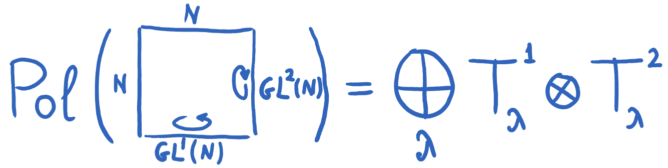
Branching rule (restriction from $U(N)$ to $U(N-1)$)

$$S_\lambda(z_1,\dots,z_{N-1},c) = \sum_{\lambda_1\geq\mu_1\geq\lambda_2\geq\dots\geq\mu_{N-1}\geq\mu_N} c^{\sum\lambda_i-\sum\mu_i}\cdot S_\mu(z_1,\dots,z_{N-1})$$



Cauchy identity (reproducing kernel)

$$\sum_{\lambda_1\geq\dots\geq\lambda_N\geq 0} S_\lambda(z_1,\dots,z_N)\,S_\lambda(w_1,\dots,w_N) = \prod_{i,j=1}^{N}\frac{1}{1-z_iw_j}$$



$$\mathrm{Pol}\Big({}_N\underset{GL^1(N)}{\overset{N}{\boxed{\quad\circlearrowleft\quad}}}\circlearrowleft GL^2(N)\Big) = \bigoplus_\lambda T^1_\lambda\otimes T^2_\lambda$$

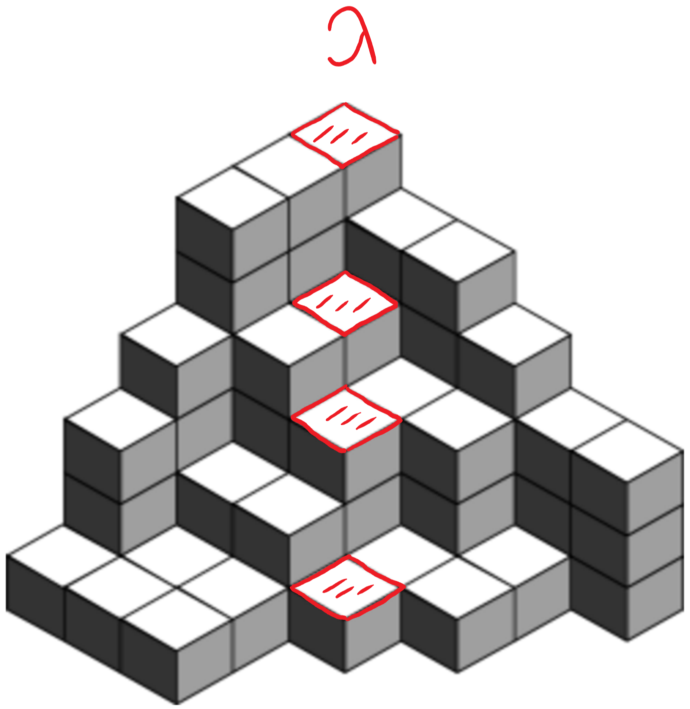
Difference operators

$$(z_1+\dots+z_N)\,S_\lambda(z) = \sum_{\mu=\lambda+\underset{(0,\dots,0,1,0,\dots,0)}{\underset{''}{\vec{e}_r}}} S_\mu(z)$$

$$\sum_{i=1}^{N}\prod_{j\neq i}\frac{z_j-qz_i}{z_j-z_i}\,S_\lambda(z_1,\dots,qz_i,\dots z_N) = \Big(\sum_{i=1}^{N} q^{\lambda_i+N-i}\Big)\cdot S_\lambda(z)$$

Eigenvalues

Random plane partitions



$$\text{weight(plane partition)} = Q^{\text{volume}}$$
$$= \left(s_\lambda(Q^{1/2}, Q^{3/2}, Q^{5/2}, \dots)\right)^2$$

$$\sum_{\text{plane partitions}} Q^{\text{volume}} = \prod_{n\geq 1} \frac{1}{(1-Q^n)^n}$$

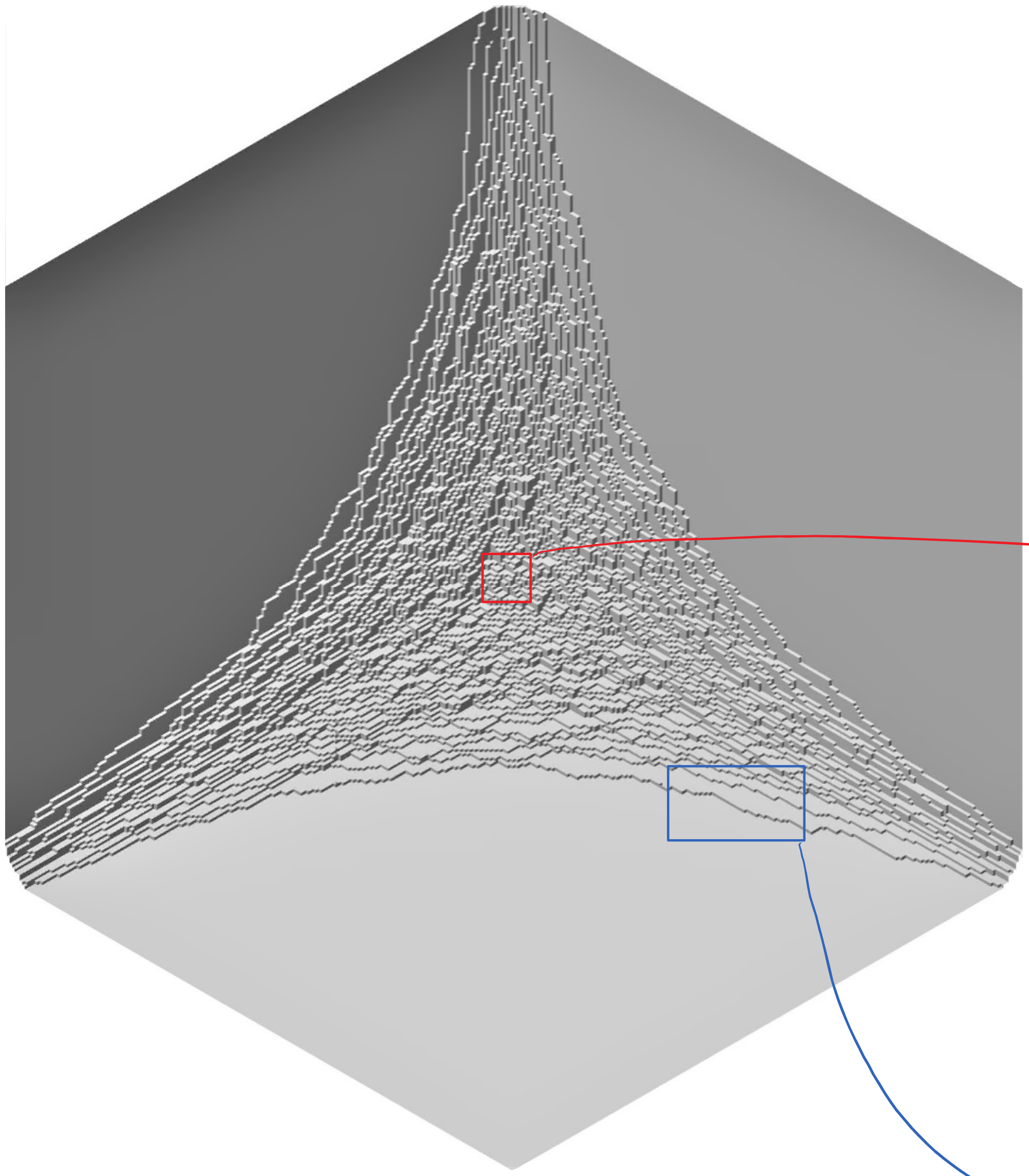
Cauchy/MacMahon identity

$$E\left(\sum_i q_1^{\lambda_i+N-i} \cdots \sum_i q_m^{\lambda_i+N-i}\right) = \frac{\mathcal{D}_1^{(q_1)} \cdots \mathcal{D}_1^{(q_m)} \sum_\lambda s_\lambda(z)\, s_\lambda(w)}{\sum_\lambda s_\lambda(z)\, s_\lambda(w)}\Bigg|_{z=w=(Q^{1/2}, Q^{3/2}, \dots)}$$

$$\mathcal{D}_1^{(q)} s_\lambda(z) := \sum_{i=1}^{N} \prod_{j\neq i} \frac{z_j - q z_i}{z_j - z_i}\, s_\lambda(z_1, \dots, q z_i, \dots z_N) = \left(\sum_{i=1}^{N} q^{\lambda_i+N-i}\right) \cdot s_\lambda(z)$$

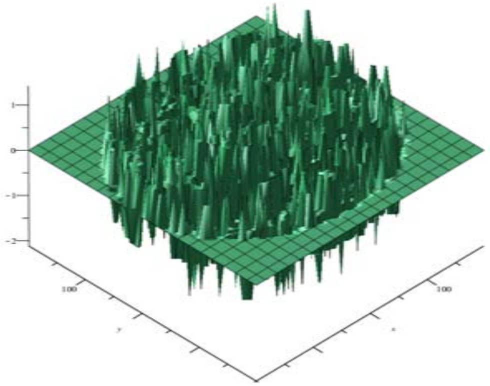
$$\sum_{\lambda_1\geq\dots\geq\lambda_N\geq 0} s_\lambda(z_1, \dots, z_N)\, s_\lambda(w_1, \dots, w_N) = \prod_{i,j=1}^{N} \frac{1}{1 - z_i w_j}$$

Random plane partitions

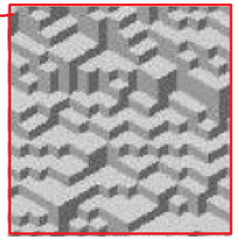


Global limit shape (Wulff droplet or 'crystal', Ronkin function of a complex line)

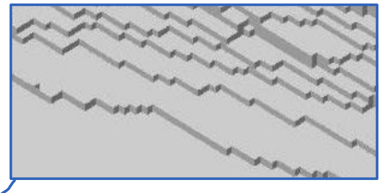
Global fluctuations (Gaussian Free Field)

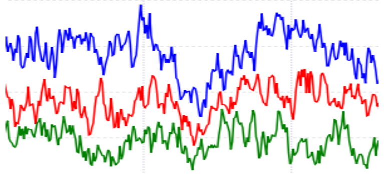


Local correlations (translation invariant Gibbs measures)



Edge fluctuations (Airy processes)





The Schur symmetric functions (or representation theory of the unitary groups) is a major source of integrable probabilistic systems that include

- Totally Asymmetric Simple Exclusion Process (TASEP) and its relatives
- Last Passage percolation in the plane with special weights
- Uniformly random and q-weighted plane partitions with various boundary conditions
- (2+1)-Dimensional models of random interface growth
- Unitarily invariant models of random matrices

Spin Hall-Littlewood symmetric rational functions

$$F_\lambda(u_1,\dots,u_N) = \frac{(1-q)^N}{\prod_{i=1}^{N}(1-su_i)} \cdot \sum_{\sigma\in S_N} \sigma\left(\prod_{i<j} \frac{u_i-qu_j}{u_i-u_j} \cdot \prod_{i=1}^{N} \left(\frac{u_i-s}{1-su_i}\right)^{\lambda_i} \right)$$

$$\lambda=(\lambda_1\geq\dots\geq\lambda_N)\in\mathbb{Z}^N$$

Specializing $s=q=0$ brings us back to the Schur, while setting $s=0$ yields the Hall-Littlewood polynomials that arise in connection with finite p-groups and representation theory of groups of p-adic type.

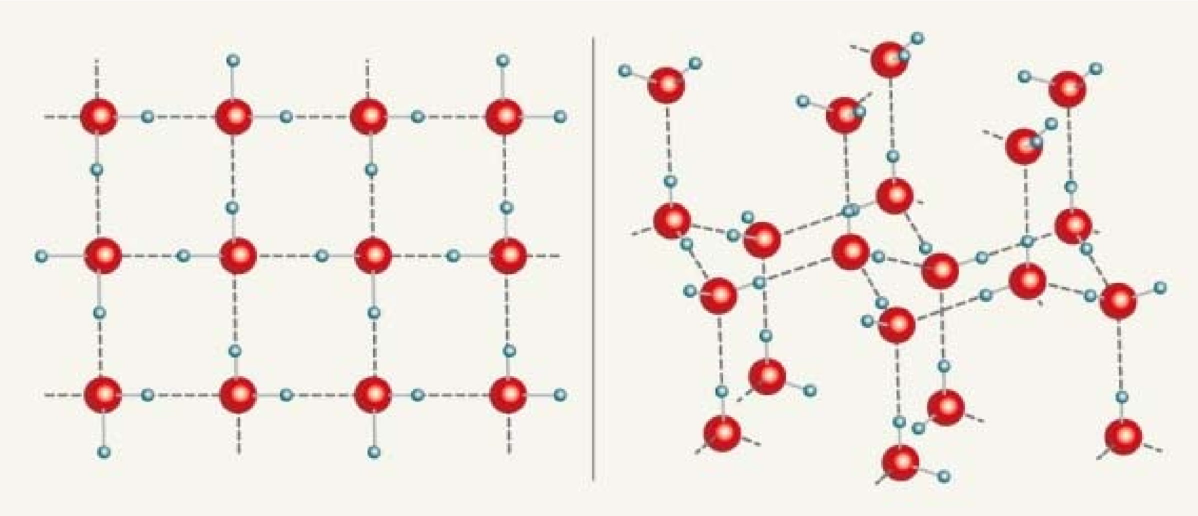
More generally, one can add parameters $\{s_i,\xi_i\}_{i\in\mathbb{Z}}$ and define

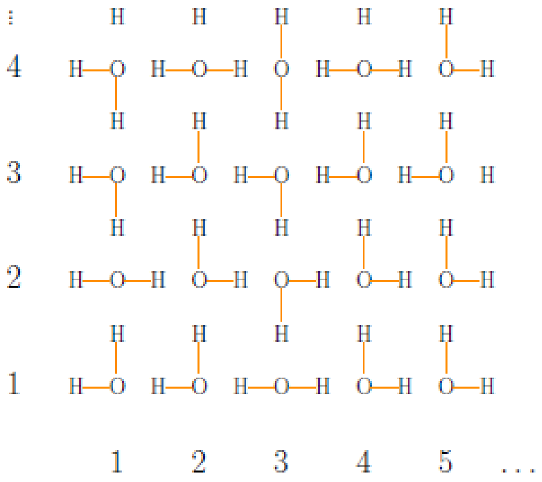
$$F_\lambda(u_1,\dots,u_N) = \sum_{\sigma\in S_N} \sigma\left(\prod_{i<j} \frac{u_i-qu_j}{u_i-u_j} \cdot \prod_{i=1}^{N} \varphi_{\lambda_i}(u_i)\right)$$

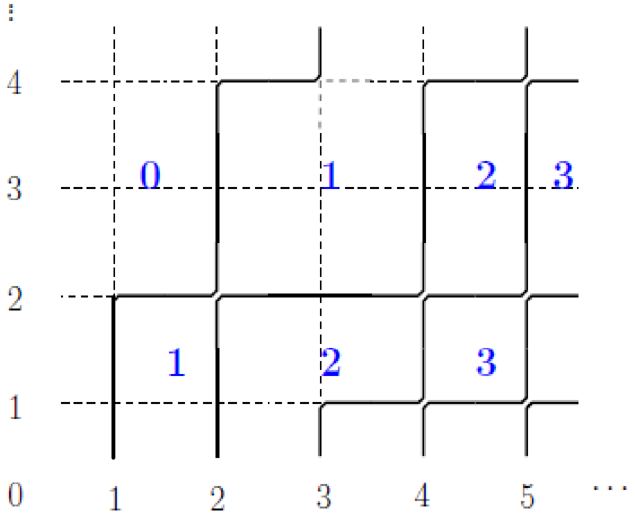
$$\varphi_k(u) = \frac{1-q}{1-s_k\xi_k u} \cdot \prod_{i=0}^{k-1} \frac{\xi_i u - s_i}{1-s_i\xi_i}, \qquad k\geq 0.$$

These functions originate from the six vertex model.

The six vertex model (Pauling, 1935)

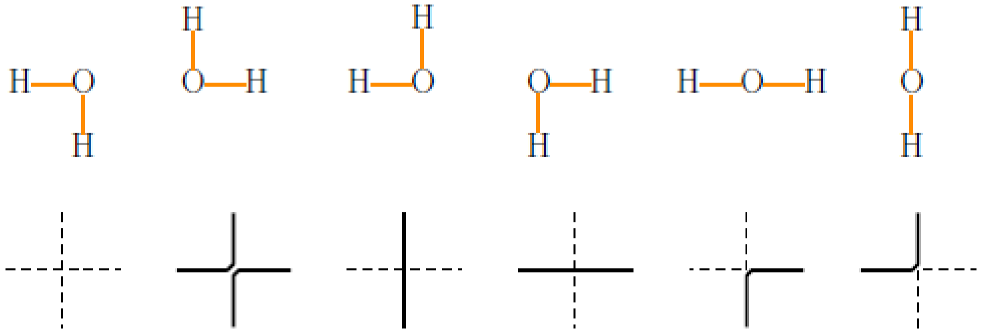






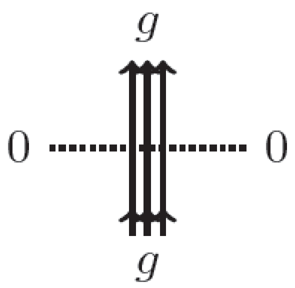
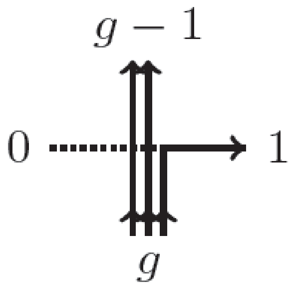
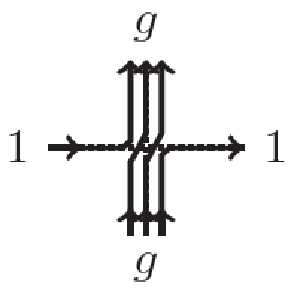
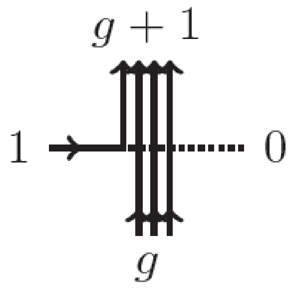
In 'square ice', which has been seen between graphene sheets, water molecules lock flat in a right-angled formation. The structure is strikingly different from familiar hexagonal ice (right).

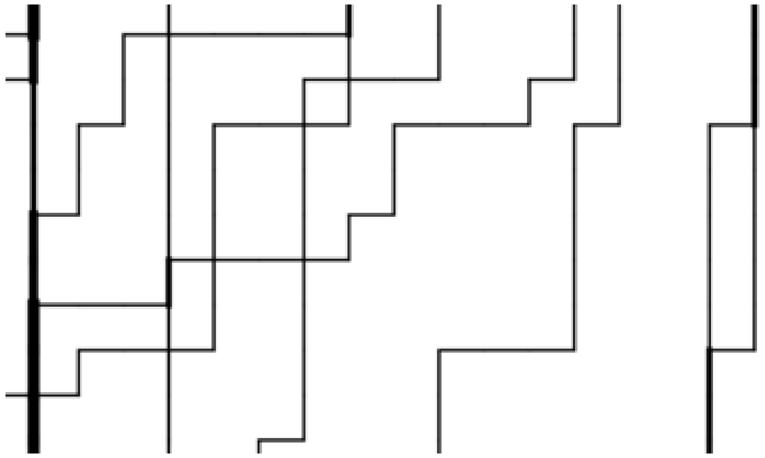
From <http://www.nature.com/news/graphene-sandwich-makes-new-form-of-ice-1.17175>



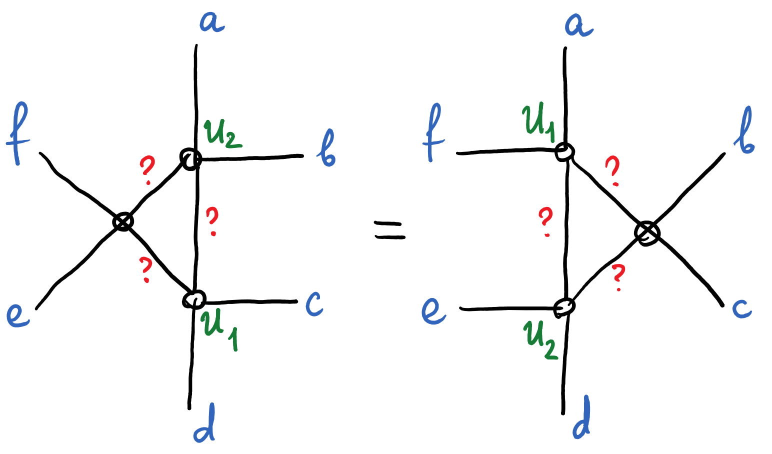
Lieb in 1967 computed the partition function of the square ice on a large torus – an estimate for the residual entropy of real ice.

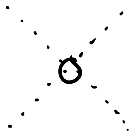
The higher spin six vertex model [Kulish-Reshetikhin-Sklyanin '81]

	 	 	 	
$w_{u,\mathrm{s}}$	$\frac{1-\mathrm{s}q^{g}u}{1-\mathrm{s}u}$	$\frac{(1-\mathrm{s}^{2}q^{g-1})u}{1-\mathrm{s}u}$	$\frac{u-\mathrm{s}q^{g}}{1-\mathrm{s}u}$	$\frac{1-q^{g+1}}{1-\mathrm{s}u}$

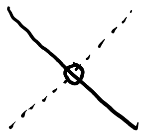


The Yang-Baxter (star-triangle) equation:

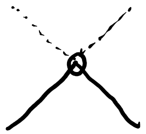




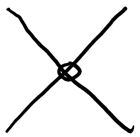
$u_1 - q u_2$



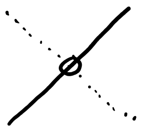
$u_1 - u_2$



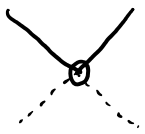
$(1-q)u_1$



$u_1 - q u_2$



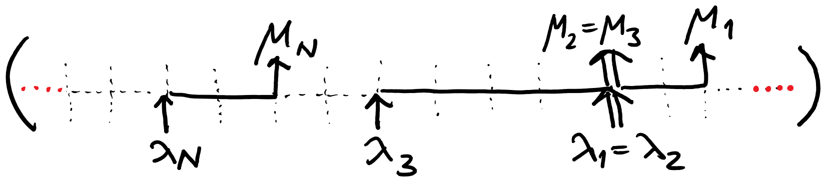
$q(u_1 - u_2)$

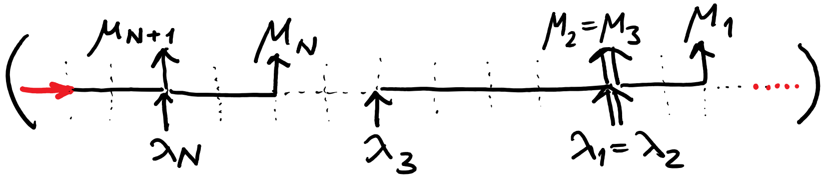


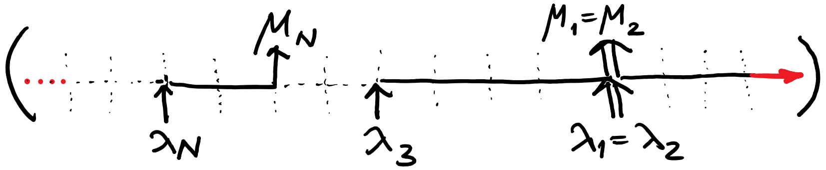
$(1-q)u_2$

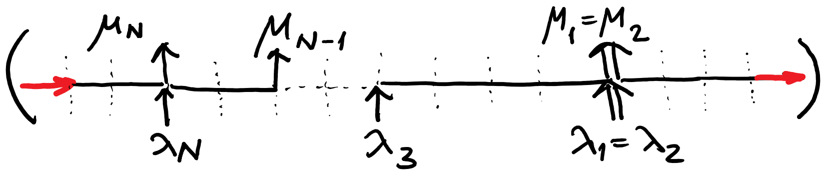
Operators of the Algebraic Bethe Ansatz

In $\mathrm{Span}\{e_\lambda : \lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)\}$ define

$A(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u($  $)\, e_\mu$

$B(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u($  $)\, e_\mu$

$C(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u($  $)\, e_\mu$

$\mathcal{D}(u)\, e_\lambda = \sum_\mu \mathrm{weight}_u($  $)\, e_\mu$

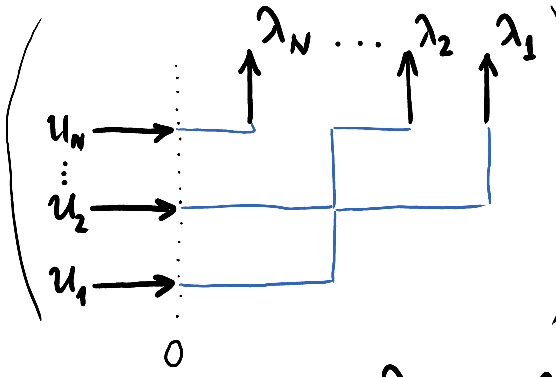
In infinite volume, C and $\mathcal{D}$ need to be normalized:

$$\overline{\mathcal{D}}(u) := \lim_{L\to\infty} \frac{\mathcal{D}^{(L)}(u)}{\left(\mathrm{weight}_u(\rightarrow\bullet\rightarrow)\right)^L},$$

L is the length of the strip

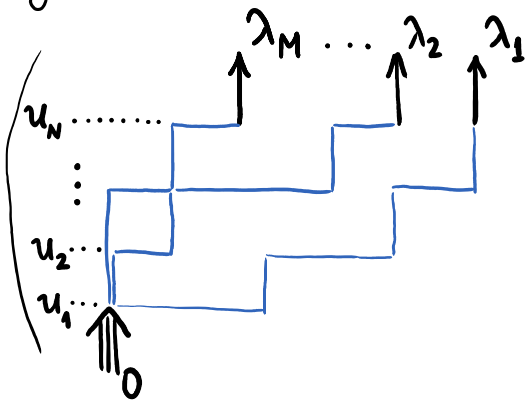
Spin Hall-Littlewood symmetric rational functions

$$F_\lambda(u_1,\dots,u_N) = \left\langle B(u_1)\cdots B(u_N)e_\varnothing, e_\lambda\right\rangle = \text{weight}\left(\quad\right)$$



$$= \frac{(1-q)^N}{\prod_{i=1}^N (1-su_i)} \cdot \sum_{\sigma\in S_N} \left(\prod \frac{u_i - qu_j}{u_i-u_j}\cdot \prod_{i=1}^N \left(\frac{u_i-s}{1-su_i}\right)^{\lambda_i}\right)$$

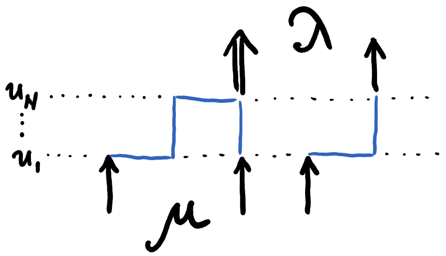
$$G_\lambda(u_1,\dots,u_N) = \left\langle A(u_1)\cdots A(u_N)e_{(0^M)}, e_\lambda\right\rangle = \text{weight}\left(\quad\right)$$

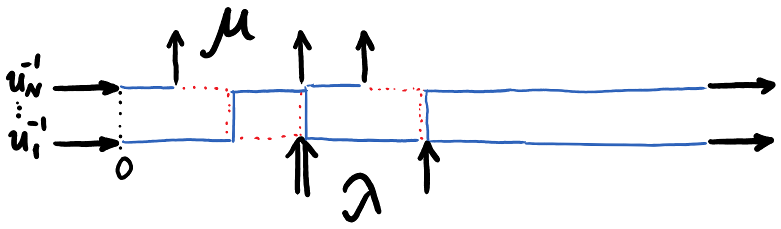


More generally,

$$c(0^{m_0}1^{m_1}2^{m_2}\dots) = \prod_{k\geq 0}\frac{(s^2;q)_{m_k}}{(q;q)_{m_k}}$$

$$G_{\lambda/\mu}(u_1,\dots,u_N) = \left\langle A(u_1)\cdots A(u_N)e_\mu, e_\lambda\right\rangle = \frac{c(\mu)}{c(\lambda)}\left\langle \bar{\mathcal{D}}(u_1^{-1})\cdots\bar{\mathcal{D}}(u_N^{-1})e_\lambda, e_\mu\right\rangle$$





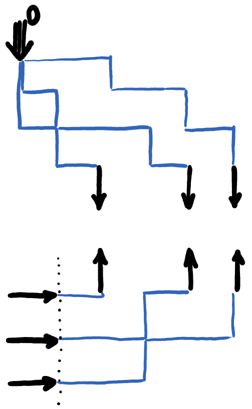
Spin Hall-Littlewood symmetric rational functions

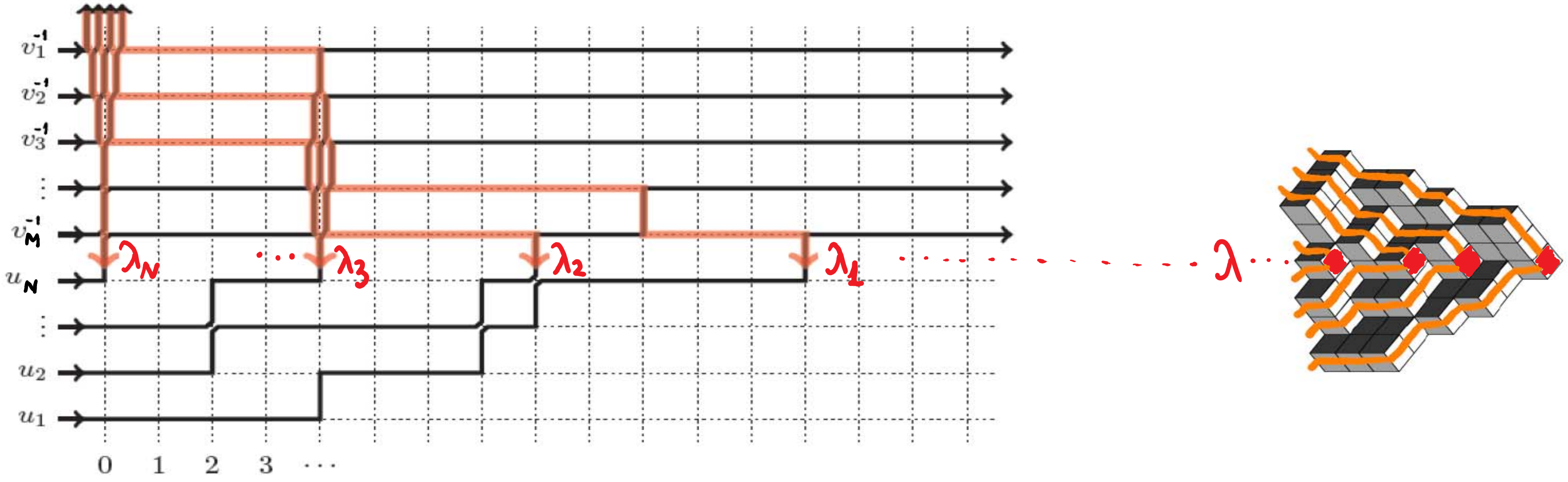
Difference operator (transfer-matrix)

$$\sum_{\mu} \frac{c(\mu)}{c(\lambda)} G_{\mu/\lambda}(v)\cdot F_{\mu}(u_1,\dots,u_N) = \prod_{i=1}^{N} \frac{1-qu_iv}{1-u_iv}\cdot F_{\lambda}(u_1,\dots,u_N)$$

Cauchy identity [B.'14, B.-Petrov '16]

$$\sum_{\lambda=(\lambda_1\geq\dots\geq\lambda_N\geq0)} F_{\lambda}(u_1,\dots u_N)\cdot \frac{c(\lambda)}{c(0^N)} G_{\lambda}(v_1,\dots,v_M) = \prod_{i=1}^{N} \frac{1-q^i}{1-su_i}\cdot \prod_{i,j} \frac{1-qu_iv_j}{1-u_iv_j}$$

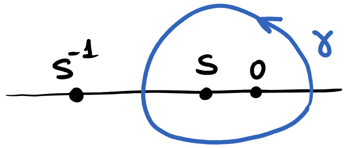




Spin Hall-Littlewood symmetric rational functions

Orthogonality relations [Tarasov-Varchenko '97, Povolotsky '13, B.-Corwin-Petrov-Sasamoto '14-15, B.-Petrov '16]

$$\frac{c(\lambda)}{(2\pi i)^N (1-q)^N N!} \oint_\gamma \cdots \oint_\gamma \prod_{1 \le A \ne B \le N} \frac{u_A - u_B}{u_A - q u_B} F_\lambda(u_1, \ldots, u_N) F_\mu(u_1^{-1}, \ldots, u_N^{-1}) \prod_{i=1}^N \frac{du_i}{u_i} = \mathbb{1}_{\lambda=\mu}$$



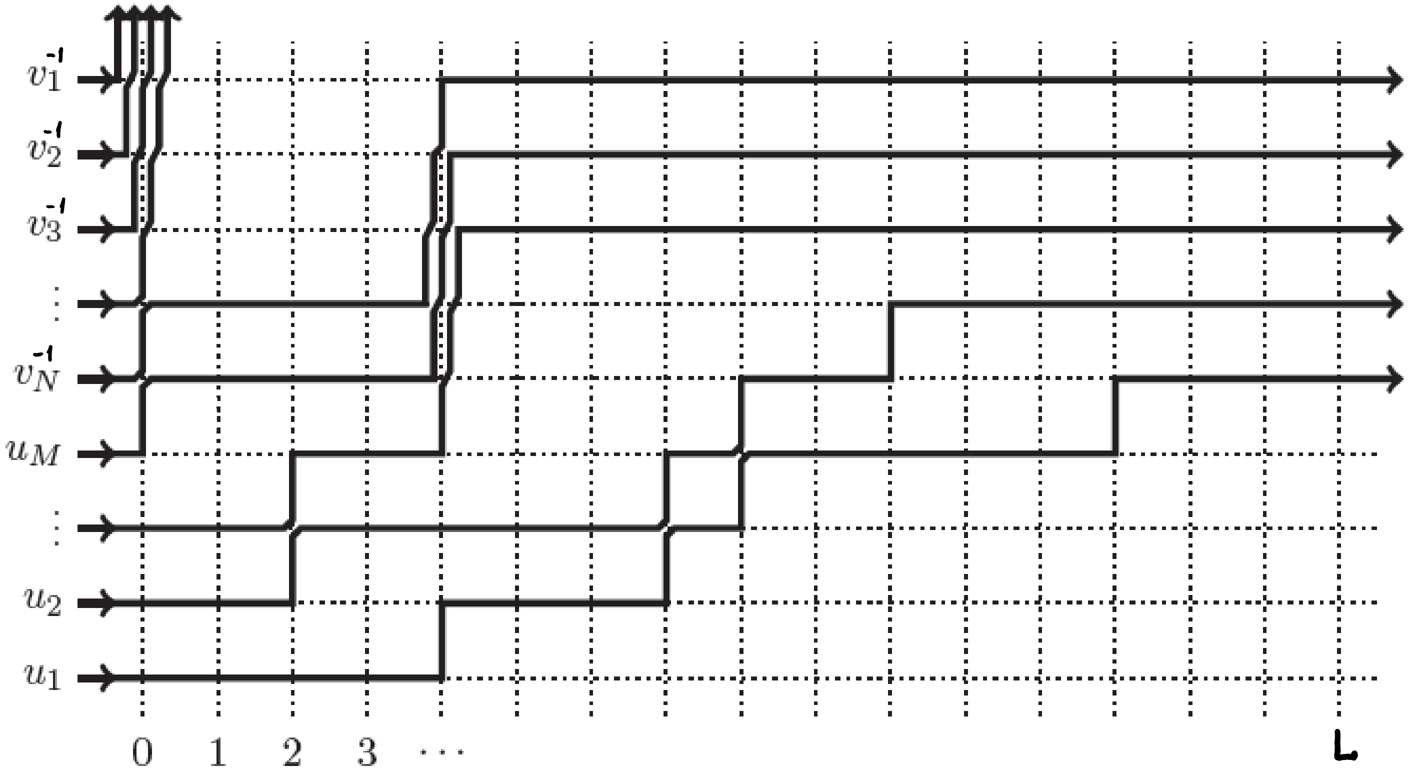
$$\sum_{-\infty < \lambda_N \le \ldots \le \lambda_1 < +\infty} \prod_{i<j} (u_i - u_j)(v_i - v_j) \cdot c(\lambda) F_\lambda(u_1, \ldots, u_N) F_\lambda(v_1^{-1}, \ldots, v_N^{-1})$$

$$= (-1)^{\frac{N(N-1)}{2}} \prod_{i,j=1}^N (u_i - q u_j) \det\left[\delta(v_i - u_j)\right]_{i,j=1}^N$$

$$c(0^{m_0} 1^{m_1} 2^{m_2} \ldots) = \prod_{k \ge 0} \frac{(s^2; q)_{m_k}}{(q; q)_{m_k}}$$

$$F_\lambda(u_1, \ldots, u_N) = \frac{(1-q)^N}{\prod_{i=1}^N (1 - s u_i)} \cdot \sum_{\sigma \in S_N} \left(\prod_{i<j} \frac{u_i - q u_j}{u_i - u_j} \cdot \prod_{i=1}^N \left(\frac{u_i - s}{1 - s u_i} \right)^{\lambda_i} \right)$$

The Cauchy identity



Theorem [B.'14] *The partition function normalized by* $\left(\prod_{j=1}^{N} \mathrm{weight}_{v_j^{-1}}(\to\cdot\to)\right)^{L}$ *equals*

$$\prod_{i=1}^{M}\frac{1-q^{i}}{1-su_i}\cdot\prod_{\substack{i=1,\dots,M\\ j=1,\dots,N}}\frac{1-qu_iv_j}{1-u_iv_j}$$

Convergence: $\left|\frac{u_i-s}{1-su_i}\cdot\frac{v_j-s}{1-sv_j}\right|<1 \quad \forall\, i,j$

Proof - operator approach

The Yang-Baxter equation is equivalent to certain quadratic commutation relations between these operators. For example,

$$B(u_1)\,\mathcal{D}(u_2) = \frac{u_1-u_2}{qu_1-u_2}\,\mathcal{D}(u_2)\,B(u_1) + \frac{(1-q)u_2}{u_2-qu_1}\,B(u_2)\,\mathcal{D}(u_1).$$

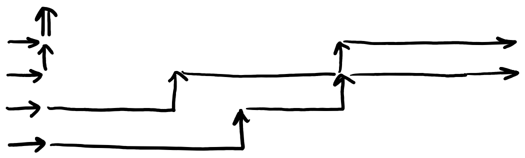
Assuming $\left|\text{weight}_{u_1}(\to\bullet\to)\right| < \left|\text{weight}_{u_2}(\to\bullet\to)\right|$, in infinite volume one gets

$$B(u_1)\,\bar{\mathcal{D}}(u_2) = \frac{u_1-u_2}{qu_1-u_2}\,\bar{\mathcal{D}}(u_2)\,B(u_1)$$

The result now follows from

$$\left(\bar{\mathcal{D}}(v_N^{-1})\cdots\bar{\mathcal{D}}(v_1^{-1})\,B(u_M)\cdots B(u_1)\,e_{\emptyset},\; e_{0^M}\right) =$$

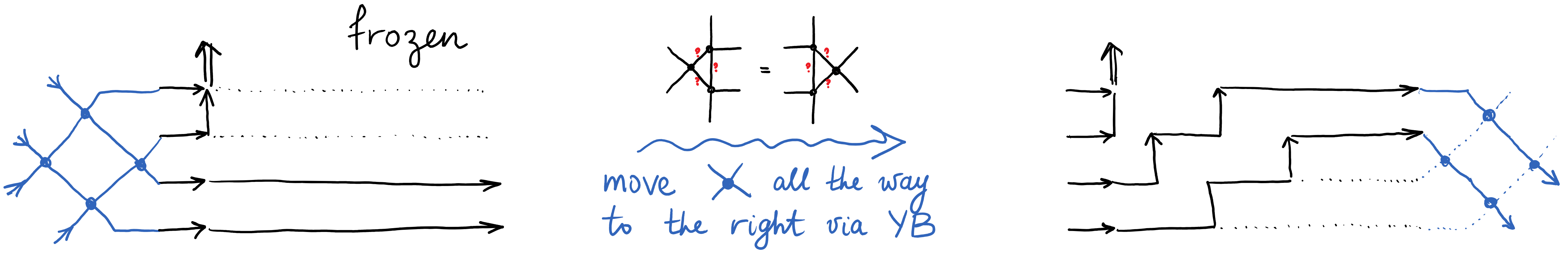
$$= \prod_{\substack{i=1,\dots,M\\ j=1,\dots,N}} \frac{1-q\,u_i v_j}{1-u_i v_j}\cdot\left(B(u_M)\cdots B(u_1)\,\bar{\mathcal{D}}(v_N^{-1})\cdots\bar{\mathcal{D}}(v_1^{-1})\,e_{\emptyset}, e_{0^M}\right)$$

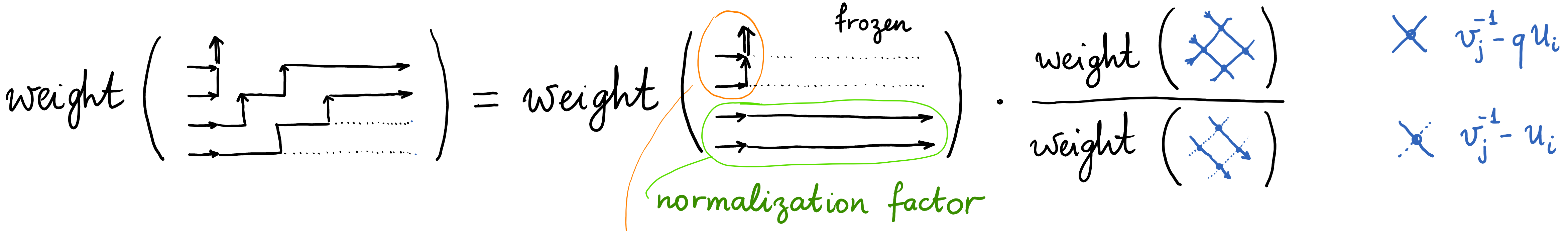


$\mathcal{D}\mathcal{D}BBe_{\emptyset} \rightsquigarrow e_{(0^2)}$

$BB\mathcal{D}\mathcal{D}e_{\emptyset} \rightsquigarrow e_{(0^2)}$

Proof - pictorial approach





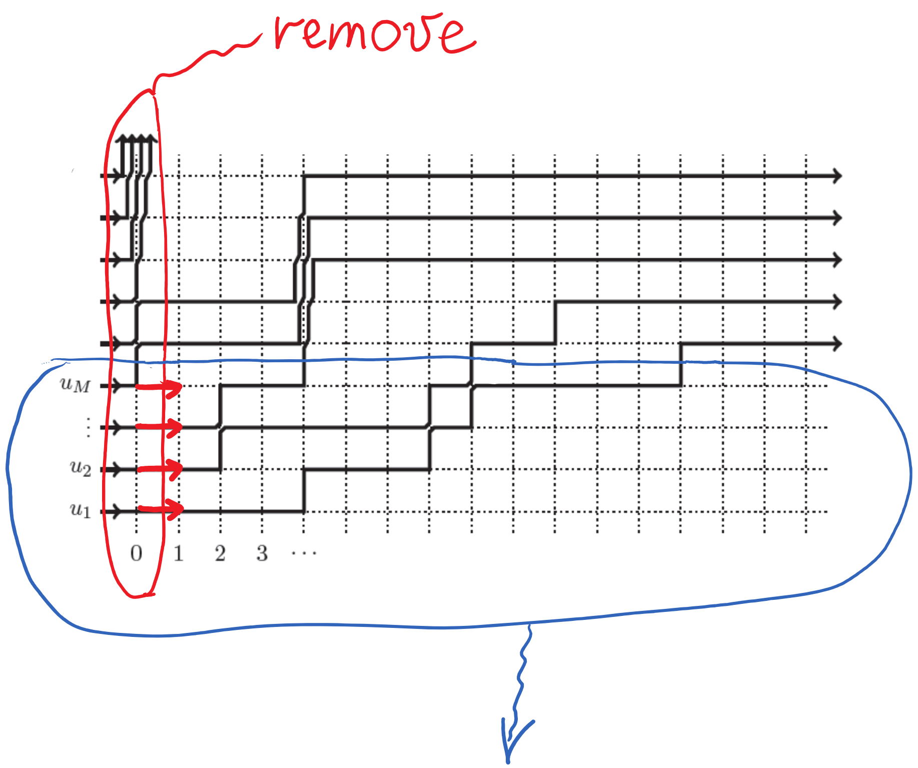
$$\prod_i \frac{1-q^i}{1-su_i} \prod_{i,j} \frac{1-qu_iv_j}{1-u_iv_j}$$

Markovian specialization

Set

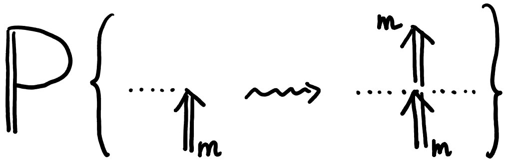
$$\prod_j \frac{1-quv_j}{1-uv_j}\Bigg|_{\substack{(v_1,\dots,v_N)=\\=(v,qv,\dots,q^{N-1}v)}} = \frac{1-q^N uv}{1-uv}\Bigg|_{q^N=(sv)^{-1}}$$

$$= \frac{1-u/s}{1-uv}\Bigg|_{v=0} = 1-\frac{u}{s}$$

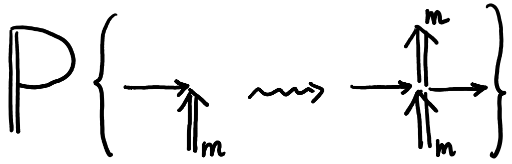


Theorem [B. '14, Corwin-Petrov '15]

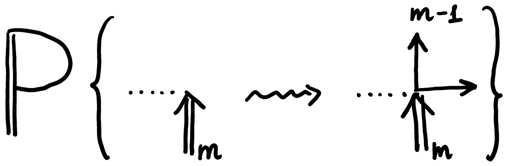
The resulting random paths in the bottom part of the picture can be constructed recursively via



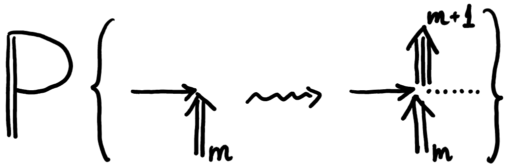
$$= \frac{1-sq^m u_y}{1-su_y}$$



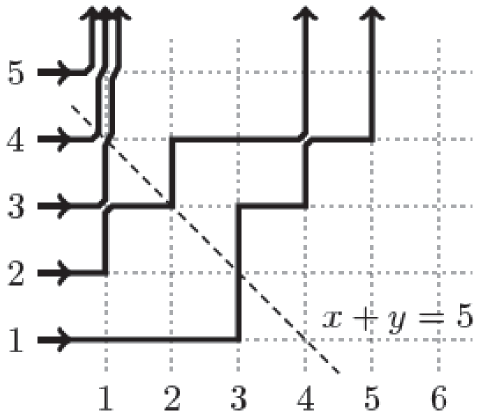
$$= \frac{s^2q^m - su_y}{1-su_y}$$



$$= \frac{(q^m-1)su_y}{1-su_y}$$



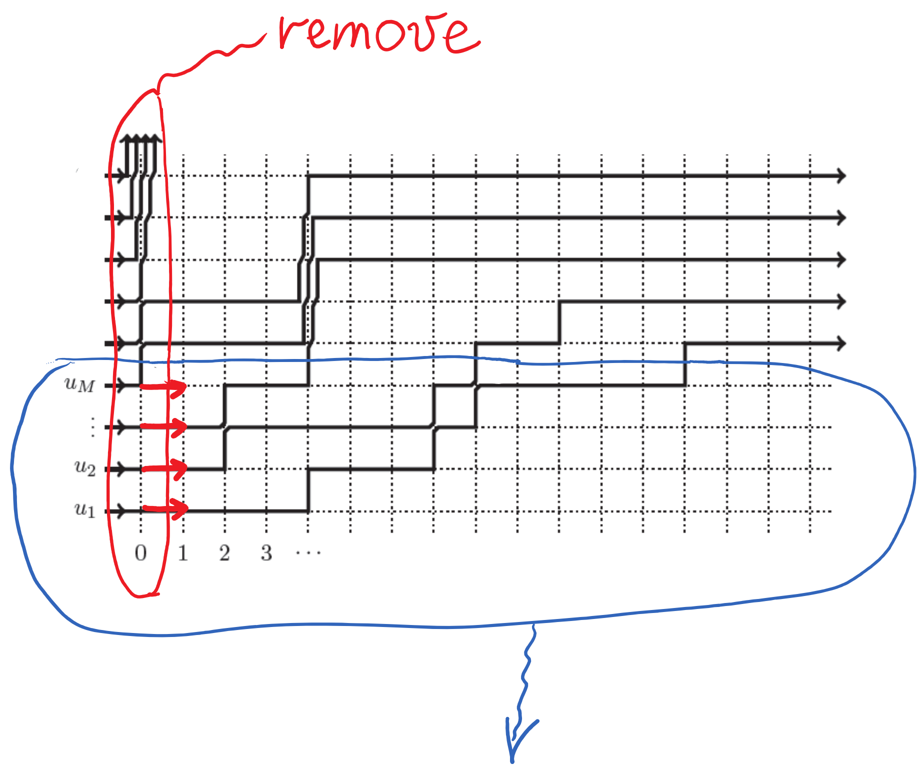
$$= \frac{-s^2q^m}{1-su_y}$$



Markovian specialization

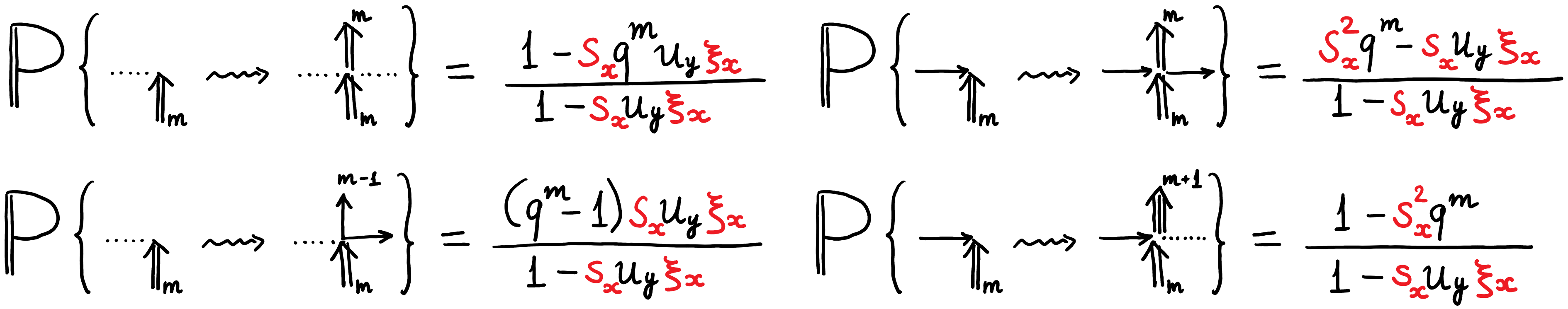
$$\prod_j \frac{1-quv_j}{1-uv_j}\Bigg|_{\substack{(v_1,\dots,v_N)=\\=(v,qv,\dots,q^{N-1}v)}} = \frac{1-q^N uv}{1-uv}\Bigg|_{q^N=(sv)^{-1}}$$

$$= \frac{1-u/s}{1-uv}\Bigg|_{v=0} = 1-\frac{u}{s}$$



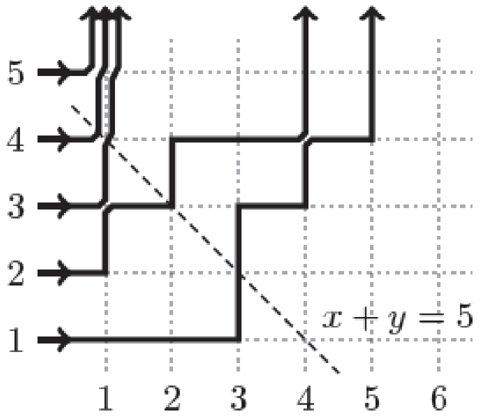
Theorem [B. '14, Corwin-Petrov '15]

The resulting random paths in the bottom part of the picture can be constructed recursively via



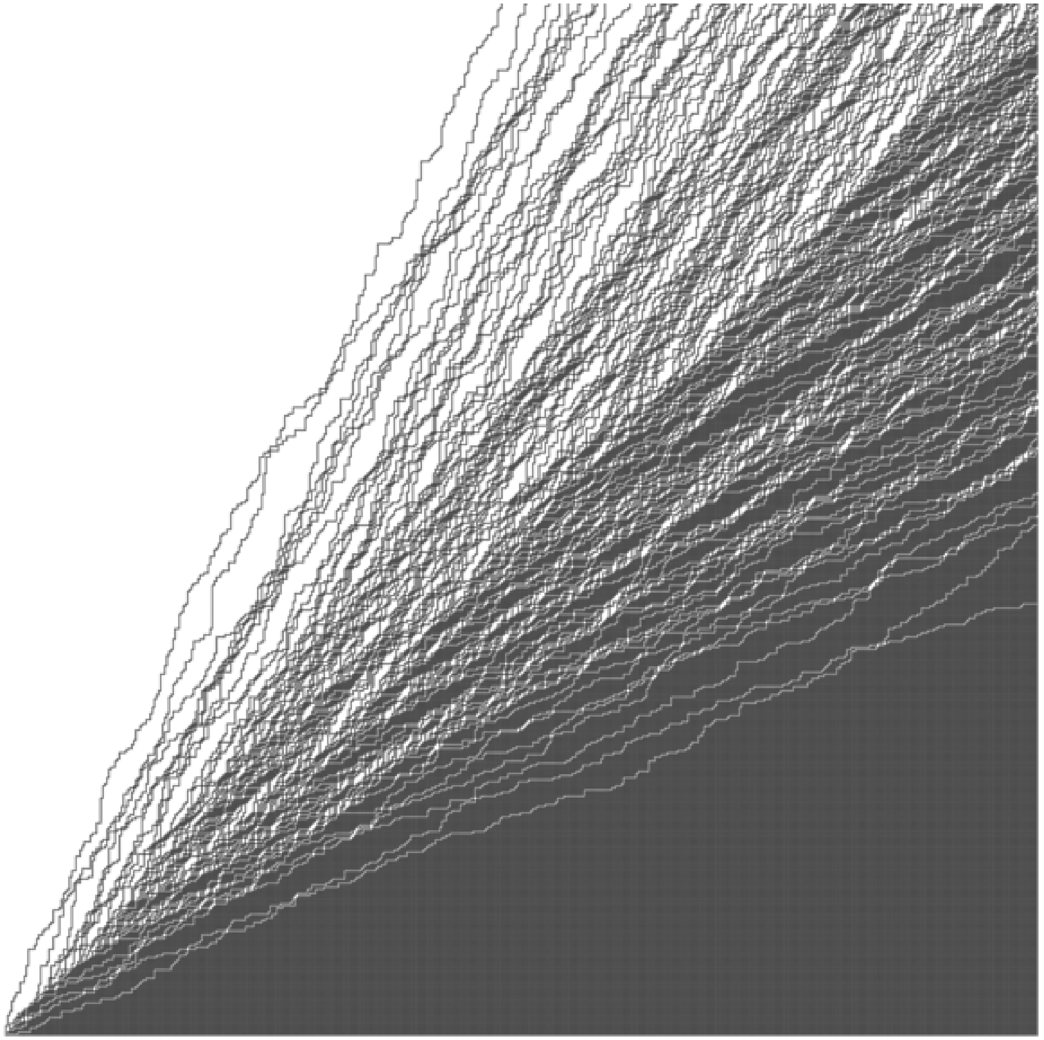
$$= \frac{1-s_x q^m u_y \xi_x}{1-s_x u_y \xi_x} \qquad = \frac{s_x^2 q^m - s_x u_y \xi_x}{1-s_x u_y \xi_x}$$

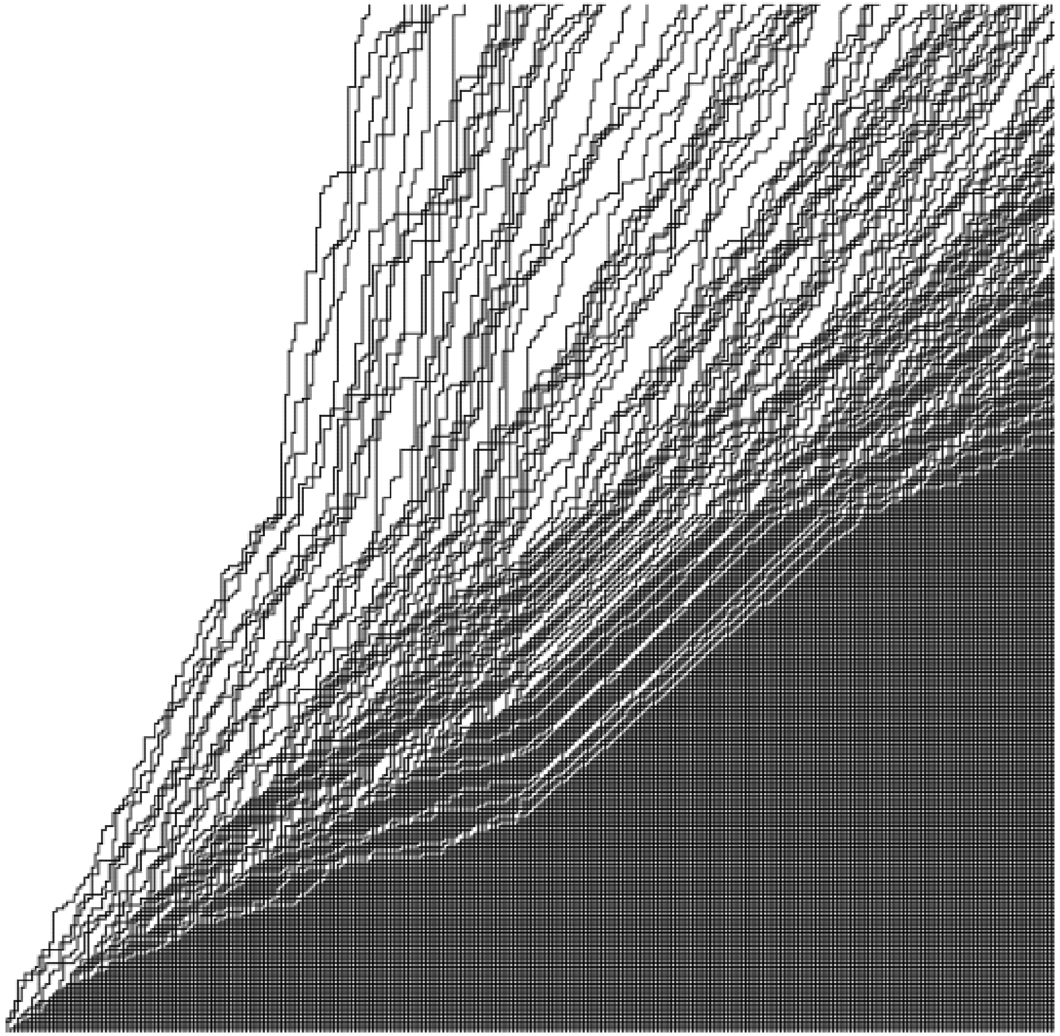
$$= \frac{(q^m-1)s_x u_y \xi_x}{1-s_x u_y \xi_x} \qquad = \frac{1-s_x^2 q^m}{1-s_x u_y \xi_x}$$



with additional sets of parameters $\{s_x\}_{x\ge1}, \{\xi_x\}_{x\ge1}$.

Sampling (the six vertex case)



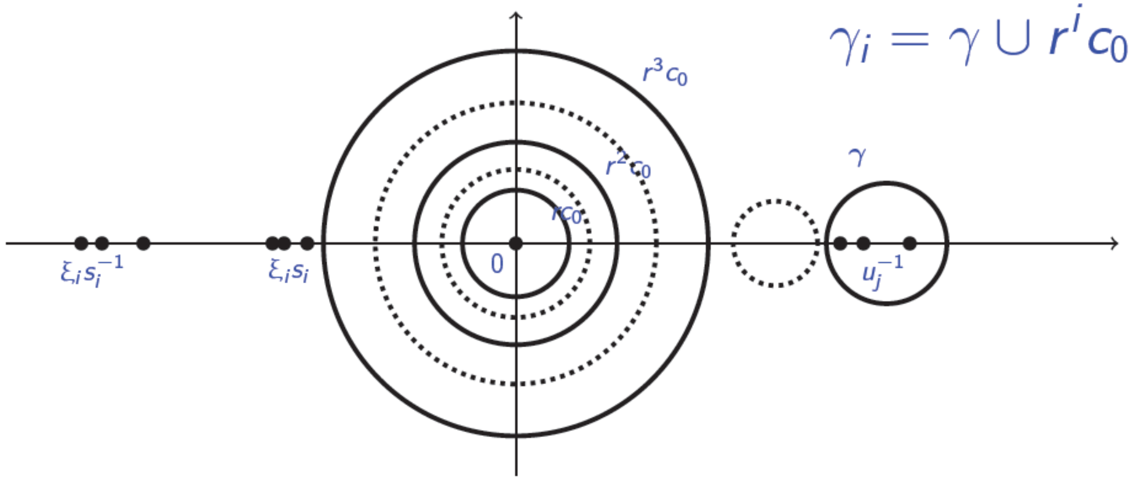


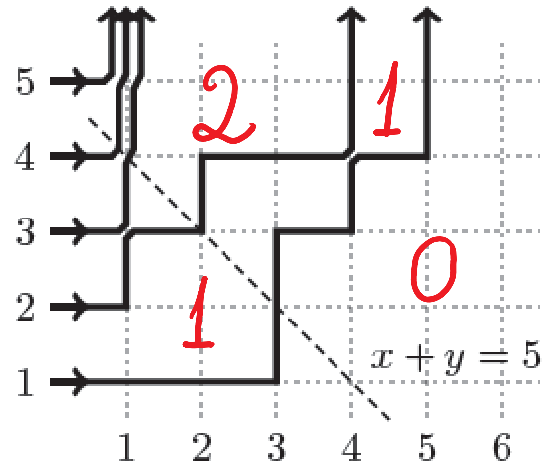
Courtesy of Leo Petrov

q-Moments of the height function

Theorem [B.-Petrov '16] *For any* $x_1 \geq \dots \geq x_\ell$, $y \geq 1$,

$$\mathbb{E}\, q^{h(x_1,y)+\dots+h(x_\ell,y)} = \frac{q^{\ell(\ell-1)/2}}{(2\pi i)^\ell} \oint_{\gamma_1}\cdots\oint_{\gamma_\ell} \prod_{1\leq A<B\leq \ell} \frac{z_A-z_B}{z_A-qz_B} \cdot \prod_{i=1}^{\ell}\left(\prod_{j=1}^{x_i-1}\frac{\xi_j-s_jz_i}{\xi_j-s_j^{-1}z_i}\cdot\prod_{k=1}^{y}\frac{1-qu_kz_i}{1-u_kz_i}\cdot\frac{dz_i}{z_i}\right)$$





$h(x,y)$ = number of paths to the right or through (x,y).

The six vertex case - asymptotics

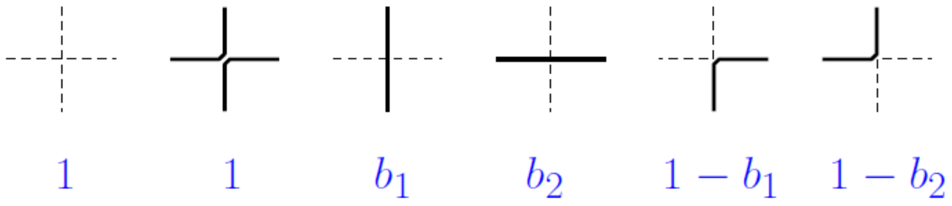
Theorem [B.-Corwin-Gorin '14]

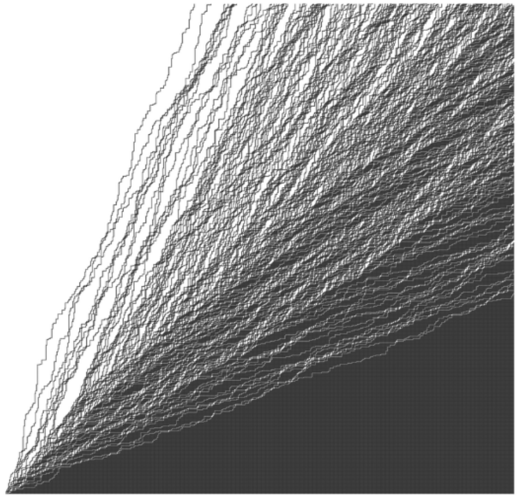
Assume $b_1 > b_2$. *Then for* $\frac{1-b_1}{1-b_2} < \frac{x}{y} < \frac{1-b_2}{1-b_1}$

$$\lim_{L\to\infty} \mathbb{P}\left\{ \frac{h(Lx,Ly) - L\cdot H(x,y)}{L^{1/3}\cdot \sigma(x,y)} \le -s \right\} = F_{GUE}(s)$$

where $H(x,y) = \frac{\left(\sqrt{x(1-b_2)} - \sqrt{y(1-b_1)}\right)^2}{x-y}$, $\sigma(x,y)$ *is explicit,*

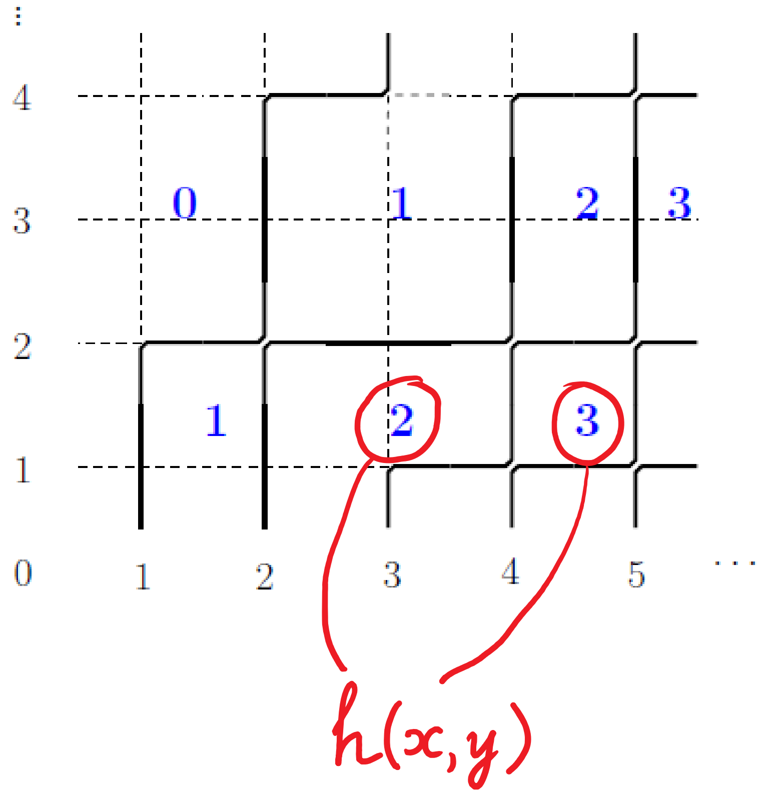
$F_{GUE}(s)$ *is the GUE Tracy-Widom distribution.*





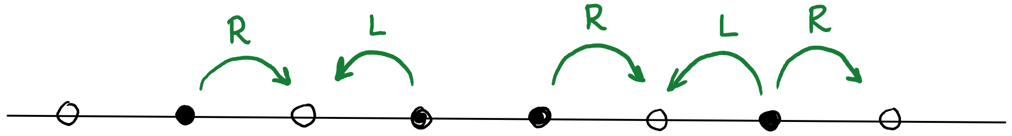
Gwa-Spohn (1992):

This is a member of the KPZ universality class. This class was related to TW in late 1990's.

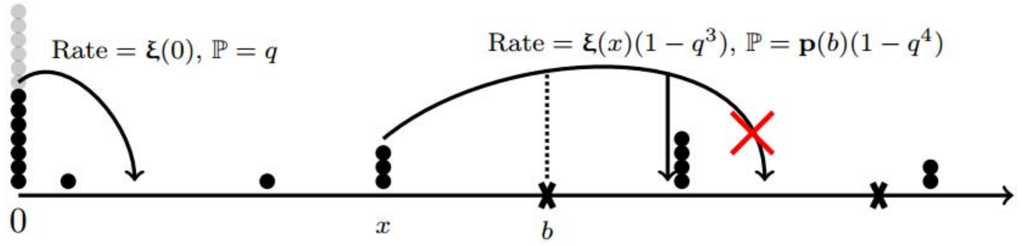


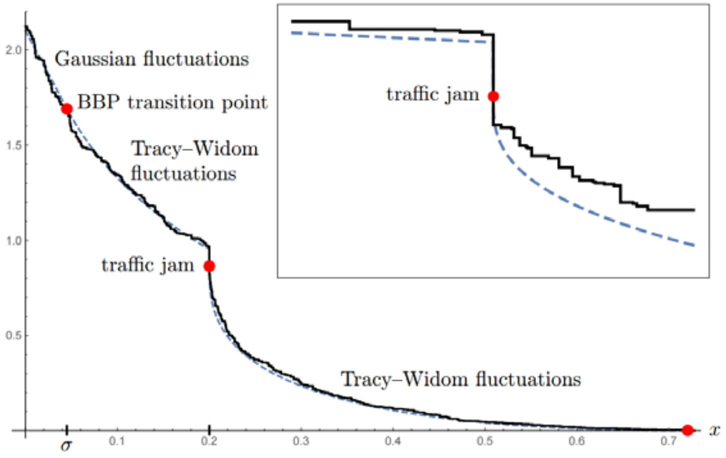
Other models that benefit from s-HL functions

- Asymmetric Simple Exclusion Process (ASEP)
  [Tracy-Widom '08-09, B.-Corwin-Petrov-Sasamoto '14-15, B.-Petrov '16, B.-Aggarwal '16, Aggarwal '16]



- Inhomogeneous Exponential Jump Model [B.-Petrov '17]



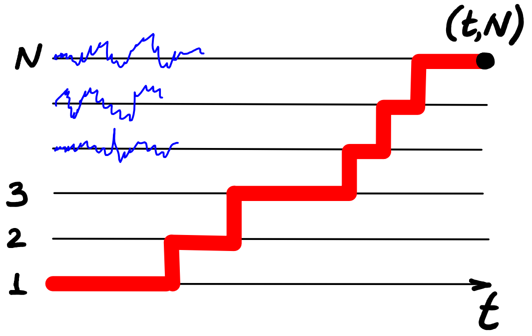


Other models that benefit from s-HL functions

- Directed Polymers in Random Media, KPZ equation, q-versions of TASEP, q-Boson models, Random Walks in Random Environment

  [Dotsenko '10+, Le Doussal et al. '10, Sasamoto-Spohn '10+, B.-Corwin '11+, Povolotsky '13, B.-Corwin-Petrov-Sasamoto '14-15, Corwin-Petrov '15, Barraquand-Corwin '15, B.-Petrov '16]

$$F_t^N = \log \int_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)} ds_1 \dots ds_{N-1}$$



- Quantum non-linear Schroedinger equation (Lieb-Liniger model), quantum Heisenberg ferromagnet (XXZ model)

  [Babbitt-Thomas '77, Babbitt-Gutkin '90, Andrei et al. '12+]

$$\sum_{j=1}^{N} J_x \sigma_j^x \sigma_{j+1}^x + J_x \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z, \qquad \sigma_j^{\left\{\substack{x\\y\\z}\right\}} = \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes \overset{(j)}{\left\{\begin{matrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{matrix}\right\}} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}.$$

Other Fourier-like bases with probabilistic applications

- Elliptic versions of the s-HL functions (a.k.a. weight functions)

  Probability: Markovian IRF (Interaction-Round-a-Face) Models, Exclusion Processes with a dynamic parameter [Felder-Varchenko-Tarasov '96+, B.'17, Aggarwal '17]

- (q-) Whittaker functions (eigenfunctions of (q-) Toda lattice)

  Probability: Directed polymers in random media, q-deformations of TASEP and of (2+1)-dimensional interface growth [O'Connell '09+, B.-Corwin '11, B.-Corwin-Ferrari '16]

- Jack polynomials and Heckman-Opdam hypergeometric functions

  Probability: General beta random matrix ensembles, log-gases at arbitrary temperature [Forrester et al. '95+, B.-Gorin '13, Moll '15, Fyodorov-Le Doussal '15, Gorin-Zhang '17]

Summary

Fourier-like bases are a key tool in studying integrable probabilistic systems.

Their properties are directly linked with probabilistic properties of these systems on one hand, and are deeply rooted in their representation theoretic origins on the other one.

Building bridges between the two sides has been very beneficial to both.