# Where does the curvature of a Courant connection live?


Miquel Cueca Ten
(joint with Rajan Mehta)

Georg-August-Universität
Göttingen, Germany


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# Lie algebroid picture

Let $(A \to M, \rho, [\cdot, \cdot])$ be a Lie algebroid and $B \to M$ vector bundle.

$A$-connection: $\nabla : \Gamma A \times \Gamma B \to \Gamma B$ with

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Cartan calculus of $A$: $(C^\bullet(A), d, \mathcal{L}, \iota)$ with $C^k(A) = \Gamma \wedge^k A^*$ and

$$d\omega(a_0, \cdots, a_k) = \sum_{i=0}^{k} (-1)^i \rho(a_i) \omega(a_0, \cdots, \widehat{a}_i, \cdots, a_k) + \sum_{i<j} (-1)^{i+1} \omega(a_0, \cdots, \widehat{a}_i, \cdots, [a_i, a_j], \cdots, a_k)$$

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Curvature and Bianchi identity: $F_\nabla \in C^2(A; \mathrm{End}(B)) \quad D^{\widetilde{\nabla}} F_\nabla = 0.$

Representations of $A$: Flat $A$-connections.

# Definition of Courant algebroids

Vector Bundle $E \to M$ with the following structure:

- $\langle \cdot, \cdot \rangle$ nondegenerate symmetric pairing.
- $\rho : E \to TM$ a bundle map.
- $[\![\cdot, \cdot]\!]$ a bracket.

Satisfiyng:

1. $[\![e_1, [\![e_2, e_3]\!]]\!] = [\![[\![e_1, e_2]\!], e_3]\!] + [\![e_2, [\![e_1, e_3]\!]]\!]$,
2. $[\![e_1, fe_2]\!] = \rho(e_1)(f)e_2 + f[\![e_1, e_2]\!]$,
3. $\rho(e_1)\langle e_2, e_3 \rangle = \langle [\![e_1, e_2]\!], e_3 \rangle + \langle e_2, [\![e_1, e_3]\!] \rangle$,
4. $[\![e_1, e_2]\!] + [\![e_2, e_1]\!] = \mathcal{D}\langle e_1, e_2 \rangle$,

where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is given by $\langle \mathcal{D}f, e \rangle = \rho(e)(f)$.

This is due to Liu-Weinstein-Xu, 1997.

# Courant connections

## Alekseev-Xu

$(E \to M, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!], \rho)$ Courant algebroid, $B \to M$ vector bundle.

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The curvature is

$$F_\nabla(e_1, e_2)(b) = [\nabla_{e_1}, \nabla_{e_2}](b) - \nabla_{[\![e_1, e_2]\!]} b.$$

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**Questions:**

- Why $F_\nabla(fe_1, e_2) \neq f\ F_\nabla(e_1, e_2)$?
- Where does it lives?
- Cartan calculus for $E$?

**Theorem (Roytenberg, Severa)**

*Courant algebroids are in one-to-one correspondence with degree 2 symplectic Q-manifolds.*

$$(E, \langle \cdot \cdot \rangle, [\![\cdot, \cdot]\!], \rho) \Longleftrightarrow (\mathcal{M}, \{\cdot, \cdot\}, Q)$$

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Examples:

- $TM \oplus T^*M$ corresponds to $T^*[2]T[1]M$
- $\mathfrak{g}$ corresponds to $\mathfrak{g}[1]$

In general, the correspondence was only implicitly defined:
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**Questions:**

- How to find an easy-to-work description?
- Any Cartan-type formula for the differential?

# Keller-Waldmann Algebra

Given a vector bundle with nondegenerate pairing $(E \to M, \langle \cdot, \cdot \rangle)$ , define $k$-cochains as maps

$$\omega : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k} \to C^{\infty}(M)$$

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- $C^\infty(M)$-linear in the last entry
- There exists a map

  $$\sigma_\omega : \underbrace{\Gamma(E) \times \cdots \times \Gamma(E)}_{k-2} \to \mathfrak{X}(M)$$

  such that

  $$\omega(e_1, \ldots, e_i, e_{i+1}, \ldots, e_k) + \omega(e_1, \ldots, e_{i+1}, e_i, \ldots, e_k)$$
  $$= \sigma_\omega(e_1, .\hat{.}..\hat{.}., e_k)(\langle e_i, e_{i+1} \rangle)$$

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The space of cochains is denoted $C^\bullet(E)$. It is a graded commutative algebra with a degree $-2$ Poisson bracket.

# Courant algebroid differential

If $E \to M$ is a Courant algebroid, then

$$T(e_1, e_2, e_3) = \langle [\![ e_1, e_2 ]\!], e_3 \rangle \in C^3(E).$$

Courant axioms $\iff \{T, T\} = 0$

$d_E = \{T, \cdot\}$ is a differential on $C^\bullet(E)$.

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**Result 1:** Let $E \to M$ a Courant algebroid.
The dg-algebras $(\mathcal{O}_\mathcal{M}, Q)$ and $(C^\bullet(E), d_E)$ are ismorphic via

$$\Upsilon(\psi)(e_1, \cdots, e_k) = \{e_k, \{e_{k-1}, \cdots \{e_1, \psi\}, \cdots\}.$$

Keller-Waldmann introduced $C^\bullet(E)$ in an algebraic setting where the correspondence with Q-manifolds doesn't apply.

# Cartan calculus

**Result 2:** The differential satisfies the Cartan formula

$$d_E\omega(e_0, \cdots, e_k) = \sum_{i=0}^{k} (-1)^i \rho(e_i)\omega(e_0, \cdots, \widehat{e_i}, \cdots, e_k)$$
$$+ \sum_{i<j} (-1)^{i+1} \omega(e_0, \cdots, \widehat{e_i}, \cdots, [\![e_i, e_j]\!], \cdots, e_k).$$

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If we introduce contractions and Lie derivatives

$$(\iota_e\omega)(e_1,\cdots,e_{k-1})=\omega(e,e_1,\cdots,e_{k-1}),\quad \mathcal{L}_e\omega=\{\{e,T\},\omega\}.$$

**Result 3:** The following Cartan relations hold

$$\begin{aligned} d_E^2 &= 0 & [\mathcal{L}_e,d_E] &= 0\\ [\iota_e,d_E] &= \mathcal{L}_e & [\mathcal{L}_e,\mathcal{L}_{e'}] &= \mathcal{L}_{[\![e,e']\!]}\\ [\mathcal{L}_e,\iota_{e'}] &= \iota_{[\![e,e']\!]} \end{aligned}$$

**Warning:** $[\iota_e,\iota_{e'}]\neq 0.$

# E-connections

For $\nabla$ an $E$-connection on $B$ define $D^{\nabla} : C^{\bullet}(E;B) \to C^{\bullet+1}(E;B)$

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\begin{aligned}
D^{\nabla}\omega(e_0,\cdots,e_k) &= \sum_{i=0}^{k}(-1)^i \nabla_{e_i}\omega(e_0,\cdots,\widehat{e_i},\cdots,e_k) \\
&\quad + \sum_{i<j}(-1)^{i+1}\omega(e_0,\cdots,\widehat{e_i},\cdots,[\![e_i,e_j]\!],\cdots,e_k)
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**Result 4:** There is a correspondence between:

- $\nabla$ $E$-connections on $B$.
- $D$ differentials on $C^{\bullet}(E;B)$ with $D(\omega \smile \tau) = d_E\omega \smile \tau + (-1)^k \omega \smile D\tau$.

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**Result 5:** For $\nabla$ an $E$-conenction on $B$ the curvature $F_{\nabla}$ satisfy

- $F_{\nabla} \in C^2(E;\mathrm{End}(B))$.
- The Bianchi identity $D^{\widetilde{\nabla}}F_{\nabla} = 0$.

## Applications

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- For $\hat{\nabla}$ a $TM$-connection on $E$ we define the adjoint connection

$$\nabla^E_{e_1} e_2 = [\![ e_1, e_2 ]\!] + \hat{\nabla}_{\rho(e_2)} e_1 - \rho^* \langle D^{\hat{\nabla}} e_1, e_2 \rangle.$$

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$\nabla^E$ induce secondary char. classes for Courant algebroids.

- For $TM \oplus_H T^*M$ and $\nabla$ torsion free $TM$-conn. on $TM$ then

$$\hat{\nabla}_X Y + \beta = \nabla_X Y + \nabla^\dagger_X \beta + \frac{1}{2} i_X i_Y H \quad \text{so} \quad \nabla^E_{X+\alpha} Y + \beta = \nabla_X Y + \nabla^\dagger_X \beta$$

and all the secondary characteristic classes vanish.

**Thanks !!**