Generalized connections, spinors and T-duality

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For details on the results which I will present, see:

V. Cortés, L. David: T-duality for transitive Courant algebroids, arxiv:2101.07184 (2021) (68 pages).

V. Cortés, L. David: Generalized connections, spinors and integrability of generalized structures on Courant algebroids, Moscow Math J, 21 (4), (2021), pag. 695 – 736.

T-dual exact Courant algebroids

Let $\pi : M \to B$ and $\tilde{\pi} : \tilde{M} \to B$ be principal T^k-bundles, $H \in \Omega^3(M)$ and $\tilde{H} \in \Omega^3(\tilde{M})$ closed T^k-invariant 3-forms. We denote by $(\mathbb{T}M, H)$ and $(\mathbb{T}\tilde{M}, \tilde{H})$ the corresponding exact Courant algebroids.

Definition

The exact Courant algebroids $(\mathbb{T}M, H)$ *and* $(\mathbb{T}\tilde{M}, \tilde{H})$ *are called* T*-dual if there is a* T^{2k}*-invariant 2-form* F *on*

$$N = M \times_B \tilde{M} = \{(m, \tilde{m}) \in M \times \tilde{M},\ \pi(m) = \tilde{\pi}(\tilde{m})\}$$

such that:

i) $\pi_N^* H - \tilde{\pi}_N^* \tilde{H} = dF$ *where* $\pi_N : N \to M$ *and* $\tilde{\pi}_N : N \to \tilde{M}$ *are natural projections;*

ii) $F|_{\mathrm{Ker}\,(d\tilde{\pi}_N) \times \mathrm{Ker}\,(d\pi_N)}$ *is non-degenerate at any point.*

Theorem (Bouwknegt, Hannabuss, Mathai, 2004)

Let $\pi : M \to B$ be a principal T^k-bundle and $H \in \Omega^3(M)$ a closed T^k-invariant 3-form which represents an integral cohomology class and satisfies $H|_{\Lambda^2 \mathrm{Ker}\, d\pi} = 0$. Then $(\mathbb{T}M, H)$ admits a T-dual.

Theorem (Bouwknegt, Evslin, Mathai, 2004)

If $(\mathbb{T}M, H)$ and $(\mathbb{T}\tilde{M}, \tilde{H})$ are T-dual then there is an isomorphism

$$\tau : \Omega_{T^k}(M) \to \Omega_{T^k}(\tilde{M})$$

which satisfies $\tau \circ d_H = d_{\tilde{H}} \circ \tau$, where $d_H\omega := d\omega + H \wedge \omega$ for any $\omega \in \Omega(M)$. It is given by

$$\tau(\omega) = \int_{T^k} e^F \wedge \pi^*\omega.$$

Theorem (Cavalcani, Gualtieri, 2010)

If $(\mathbb{T}M, H)$ *and* $(\mathbb{T}\tilde{M}, \tilde{H})$ *are* T*-dual then there is a canonical isomorphism of Courant algebroids*

$$\rho : (\mathbb{T}M/T^k, [\cdot,\cdot]_H) \to (\mathbb{T}\tilde{M}/T^k, [\cdot,\cdot]_{\tilde{H}})$$

compatible with τ*:*

$$\tau \circ \gamma_u = \gamma_{\rho(u)} \circ \tau,\ \forall u \in \Gamma_{T^k}(\mathbb{T}M).$$

Standard Courant algebroids

Definition

i) A vector bundle $\mathcal{G} \to M$ *endowed with a tensor field* $[\cdot, \cdot] \in \Gamma(\wedge^2 \mathcal{G}^* \otimes \mathcal{G})$ *satisfying the Jacobi identity is called a Lie algebra bundle if in a neighborhood of every point* $p \in M$ *the tensor field has constant coefficients with respect to some local frame.*
ii) A bundle of quadratic Lie algebras is a Lie algebra bundle $(\mathcal{G}, [\cdot, \cdot])$ *endowed with a metric* $\langle \cdot, \cdot \rangle \in \Gamma(\mathrm{Sym}^2 \mathcal{G}^*)$ *of neutral signature, which is* ad*-invariant:*

$$\langle [u, v], w \rangle + \langle v, [u, w] \rangle = 0, \ \forall u, v, w \in \mathcal{G}.$$

Theorem (Chen, Stiénon, Xu, 2013)

Any transitive Courant algebroid E is isomorphic to a Courant algebroid with underlying bundle $TM \oplus \mathcal{G} \oplus T^*M$, where $(\mathcal{G}, \langle \cdot, \cdot \rangle_{\mathcal{G}}, [\cdot, \cdot]_{\mathcal{G}})$ is a bundle of quadratic Lie algebras, with anchor the natural projection to TM scalar product

$$\langle \xi + r_1 + X, \eta + r_2 + Y \rangle = \frac{1}{2}(\eta(Y) + \xi(X)) + \langle r_1, r_2 \rangle_{\mathcal{G}},$$

and Dorfmann bracket defined by (∇, R, H), where ∇ is a connection on the vector bundle $\mathcal{G}$, $R \in \Omega^2(M, \mathcal{G})$ and $H \in \Omega^3(M)$, such that ∇ preserves $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ and $[\cdot, \cdot]_{\mathcal{G}}$, the following relations hold

$$d^{\nabla} R = 0, \quad (1)$$

$$dH = \langle R \wedge R \rangle_{\mathcal{G}} \quad (2)$$

and the curvature R^{∇} of ∇ is given by $R^{\nabla}(X, Y)r = [R(X, Y), r]_{\mathcal{G}}$, for any $X, Y \in \mathfrak{X}(M)$ and $r \in \Gamma(\mathcal{G})$.

A Courant algebroid of the form $E = TM \oplus \mathcal{G} \oplus T^*M$ as above will be called standard.

The Dirac generating operator for standard Courant algebroids

Let $E = TM \oplus \mathcal{G} \oplus T^*M$ be a standard Courant algebroid and $S_{\mathcal{G}}$ an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle. Consider the canonical weighted spinor bundle defined by

$$\mathbb{S} := \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}}, \tag{3}$$

where $\mathcal{S}_{\mathcal{G}} := S_{\mathcal{G}} \otimes |\det S^*_{\mathcal{G}}|^{1/r}$. It is an irreducible spinor bundle of E, with Clifford action

$$\gamma_{X+r+\xi}(\omega \otimes s) = (i_X\omega + \xi \wedge \omega) \otimes s + (-1)^{|\omega|}\omega \otimes (r \cdot s), \tag{4}$$

for any $X \in TM$, $r \in \mathcal{G}$, $\xi \in T^*M$, $\omega \in \Lambda(T^*M)$ and $s \in S_{\mathcal{G}}$.

Theorem

The canonical Dirac generating operator $\not{d} : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ *is given by*

$$\not{d}(\omega \otimes s) = (d\omega - H \wedge \omega) \otimes s + \nabla^{\mathcal{S}_{\mathcal{G}}}(s) \wedge \omega + \frac{1}{4}(-1)^{|\omega|+1}\omega \otimes (C_{\mathcal{G}} s) + (-1)^{|\omega|+1}\bar{R}^E(\omega \otimes s), \tag{5}$$

where $\omega \in \Omega(M)$ *and* $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$. *Above* $C_{\mathcal{G}} \in \Gamma(\Lambda^3 \mathcal{G}^*) \subset \Gamma(\mathrm{Cl}(\mathcal{G}))$ *is the Cartan form* $C_{\mathcal{G}}(u, v, w) := \langle [u, v]_{\mathcal{G}}, w\rangle_{\mathcal{G}}$ *which acts on* s *by Clifford multiplication,* $\nabla^{\mathcal{S}_{\mathcal{G}}}$ *is a connection on* $\mathcal{S}_{\mathcal{G}}$ *induced by a connection* $\nabla^{S_{\mathcal{G}}}$ *on* $S_{\mathcal{G}}$ *compatible with* ∇,

$$\nabla^{\mathcal{S}_{\mathcal{G}}}(s) \wedge \omega = \sum_i \alpha_i \wedge \omega \otimes (\nabla^{\mathcal{S}_{\mathcal{G}}}_{X_i} s)$$

and

$$\bar{R}^E(\omega \otimes s) = \frac{1}{2}\sum_{i,j,k} \langle R(X_i, X_j), r_k\rangle_{\mathcal{G}} (\alpha_i \wedge \alpha_j \wedge \omega) \otimes (\tilde{r}_k s),$$

where (r_k) *is a local frame of* $\mathcal{G}$ *and* $(\tilde{r}_k)$ *the metrically dual frame.*

Proposition

Let $I_E : E_1 \to E_2$ *be an isomorphism of standard Courant algebroids,* S_i *irreducible* $\mathrm{Cl}(E_i)$*-bundles and* $\mathbb{S}_i = \mathcal{S}_i \otimes |\det T^*M|^{1/2}$ *their canonical weighted spinor bundles. Then for any* $U \subset M$ *open and sufficiently small, there is a unique (up to multiplication by* ± 1*) isomorphism*

$$I_{\mathbb{S}|_U} : \mathbb{S}_1|_U \to \mathbb{S}_2|_U$$

such that

$$I_{\mathbb{S}|_U} \circ \gamma_u = \gamma_{I_E(u)} \circ I_{\mathbb{S}|_U}.$$

Moreover, $\not{d}_{E_2} \circ I_{\mathbb{S}|_U} = I_{\mathbb{S}|_U} \circ \not{d}_{E_1}$.

Pullback and pushforward on spinors

Let $f : M \to N$ be a submersion and $E_N = TN \oplus \mathcal{G} \oplus T^*N$ a standard Courant algebroid, defined by a bundle of quadratic Lie algebras $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H).

Lemma

The data

$$(f^*\mathcal{G},\ [\cdot,\cdot]_{f^*\mathcal{G}} := f^*[\cdot,\cdot]_{\mathcal{G}},\ \langle\cdot,\cdot\rangle_{f^*\mathcal{G}} := f^*\langle\cdot,\cdot\rangle_{\mathcal{G}})$$

together with $(f^*\nabla, f^*R, f^*H)$ defines a standard Courant algebroid on M, denoted by $f^!E_N = E_M$. It is called the pullback Courant algebroid.

We fix an irreducible $\mathrm{Cl}(\mathcal{G})$-bundle $S_{\mathcal{G}}$. Then $S_{f^*\mathcal{G}} := f^* S_{\mathcal{G}}$ is an irreducible $\mathrm{Cl}(f^*\mathcal{G})$-bundle. We consider the canonical weighted spinor bundles

$$\mathbb{S}_N = \Lambda(T^*N)\hat{\otimes}\mathcal{S}_{\mathcal{G}},\ \mathbb{S}_M = \Lambda(T^*M)\hat{\otimes} f^*\mathcal{S}_{\mathcal{G}}.$$

Definition

The natural map

$$\begin{aligned} f^* : \Gamma(\mathbb{S}_N) =& \Omega(N, \mathcal{S}_{\mathcal{G}}) \to \Gamma(\mathbb{S}_M) = \Omega(M, f^*\mathcal{S}_{\mathcal{G}}), \\ & \omega \otimes s \to f^*(\omega) \otimes f^*(s) \end{aligned} \tag{6}$$

is called the pullback on spinors.

Definition

In the above setting, assume that f has compact oriented fibers. The pushforward on spinors is the map

$$f_* : \Gamma(\mathbb{S}_M) \to \Gamma(\mathbb{S}_N)$$

by

$$f_*(\omega \otimes f^* s) = (-1)^{r|s| + nr + \frac{r(r-1)}{2}} (f_* \omega) \otimes s, \tag{7}$$

where n and r are the dimensions of N and the fibers of f, $\omega \in \Omega(M)$ and $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$ is homogeneous of degree s.

Theorem

The canonical Dirac generating operators are compatible with the pullback and pushforward on spinors, i.e.

$$\not{d}_M \circ \pi^* = \pi^* \circ \not{d}_N, \ \not{d}_N \circ \pi_* = \pi_* \circ \not{d}_M.$$

Actions on Courant algebroids

Assume that

$$E = TM \oplus \mathcal{G} \oplus T^*M \tag{8}$$

is a standard Courant algebroid, defined by a quadratic Lie algebra bundle $(\mathcal{G}, [\cdot,\cdot]_{\mathcal{G}}, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ and data (∇, R, H). Let $\mathfrak{g}$ be a Lie algebra acting on M by an infinitesimal action

$$\psi : \mathfrak{g} \to \mathfrak{X}(M),\ a \mapsto \psi(a) = X_a,$$

assumed to be free. This means that the fundamental vector fields X_a are non-vanishing, for all $a \in \mathfrak{g} \setminus \{0\}$. We define

$$\nabla^{\Psi}_{X_a(p)} r := (\Psi(a)(r))\,(p),\ \forall a \in \mathfrak{g},\ r \in \Gamma(\mathcal{G}),\ p \in M,$$

which is a partial connection on $\mathcal{G}$.

Lemma

There is a one to one correspondence between actions $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ *which lift* ψ *and preserve the factors* TM, $\mathcal{G}$, T^*M *of* E *and partial connections* ∇^{Ψ} *on* $\mathcal{G}$ *such that the following conditions are satisfied:*
i) ∇^{Ψ} *is flat and preserves* $[\cdot,\cdot]_{\mathcal{G}}$ *and* $\langle\cdot,\cdot\rangle_{\mathcal{G}}$*;*
ii) H *and* R *are invariant, i.e. for any* $a \in \mathfrak{g}$*,*

$$\mathcal{L}_{X_a} H = 0,\ \mathcal{L}_{\Psi(a)} R = 0 \tag{9}$$

where

$$(\mathcal{L}_{\Psi(a)} R)(X, Y) := \nabla^{\Psi}_{X_a}(R(X, Y)) - R(\mathcal{L}_{X_a} X, Y) - R(X, \mathcal{L}_{X_a} Y) \tag{10}$$

for any $X, Y \in \mathfrak{X}(M)$*;*
iii) for any $a \in \mathfrak{g}$*, the endomorphism* $A_a := \nabla^{\Psi}_{X_a} - \nabla_{X_a}$ *of* $\mathcal{G}$ *satisfies*

$$(\nabla_X A_a)(r) = [R(X_a, X), r]_{\mathcal{G}},\ \forall X \in \mathfrak{X}(M),\ r \in \Gamma(\mathcal{G}). \tag{11}$$

If the above conditions are satisfied, the corresponding action Ψ acts naturally (by Lie derivative) on the subbundle $TM \oplus T^*M$ of E, i.e.

$$\Psi(a)(\xi + X) = \mathcal{L}_{X_a}(\xi + X),\ X \in \mathfrak{X}(M),\ \xi \in \Omega^1(M), \tag{12}$$

and on $\mathcal{G}$ by

$$\Psi(a)(r) = \nabla^{\Psi}_{X_a} r,\ \ r \in \Gamma(\mathcal{G}). \tag{13}$$

A class of T^k-actions

Assume that M is the total space of a principal T^k-bundle $\pi : M \to B$ and let $\psi : \mathfrak{t}^k \to \mathfrak{X}(M)$ be the vertical paralellism. Let $\mathcal{H}$ be a connection on π, with connection form $\theta = \sum_{i=1}^{k} \theta_i e_i$, where (e_i) is a basis of $\mathfrak{t}^k$.

We obtain a Courant algebroid $E = TM \oplus T^*M \oplus \mathbb{R}$ with an action of $\mathfrak{t}^k$ from the following data:

• a quadratic Lie algebra bundle $(\mathcal{G}_B, \langle\cdot,\cdot\rangle_{\mathcal{G}_B}, [\cdot,\cdot]_{\mathcal{G}_B})$, whose adjoint action is an isomorphism;

• a connection ∇^B on $\mathcal{G}_B$, which preserves $\langle\cdot,\cdot\rangle_{\mathcal{G}_B}$ and $[\cdot,\cdot]_{\mathcal{G}_B}$;

• sections $r_i^B \in \Gamma(\mathcal{G}_B)$, 2-forms $H_{(2)}^{i,B} \in \Omega^2(B)$ $(1 \leq i \leq k)$ and a 3-form $H_{(3)}^B \in \Omega^3(B)$ such that

$$\mathcal{K}_i := H_{(2)}^{i,B} + 2\langle \mathfrak{r}^B, r_i^B\rangle_{\mathcal{G}_B} - \langle r_i^B, r_j^B\rangle_{\mathcal{G}_B}(d\theta_j)^B \quad (14)$$

is closed and

$$dH_{(3)}^B = \langle \mathfrak{r}^B \wedge \mathfrak{r}^B\rangle_{\mathcal{G}_B} - \mathcal{K}_i \wedge d\theta_i. \quad (15)$$

Above

$$R^{\nabla^B}(X, Y) = \mathrm{ad}_{\mathfrak{r}^B(X,Y)}$$

for $\mathfrak{r}^B \in \Omega^2(B, \mathcal{G}_B)$.

Action on spinors

Let $E = TM \oplus \mathcal{G} \oplus T^*M$ be a standard Courant algebroid over a manifold M and $\Psi : \mathfrak{g} \to \mathrm{Der}(E)$ an action on E, which lifts an action $\psi : \mathfrak{g} \mapsto \mathfrak{X}(M)$, $a \mapsto X_a$ of $\mathfrak{g}$ on M and leaves the factors TM, $\mathcal{G}$ and T^*M invariant.

Let $\mathbb{S} := \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}}$ be a canonical weighted spinor bundle of E.

Definition

The map $\Psi^{\mathbb{S}} : \mathfrak{g} \to \mathrm{End}\,\Gamma(\mathbb{S})$ *defined by*

$$\Psi^{\mathbb{S}}(a)(\omega \otimes s) = (\mathcal{L}_{X_a}\omega) \otimes s + \omega \otimes \nabla^{\Psi, \mathcal{S}_{\mathcal{G}}}_{X_a} s, \qquad (16)$$

for any $a \in \mathfrak{g}$, $\omega \in \Omega(M)$ *and* $s \in \Gamma(\mathcal{S}_{\mathcal{G}})$ *is called the action on spinors defined by* Ψ.

Theorem

The Dirac generating operator maps invariant spinors to invariant spinors.

Pullback actions

Let $f : M \to N$ be a submersion and

$$\psi^M : \mathfrak{g} \to \mathfrak{X}(M),\ a \mapsto X_a^M$$
$$\psi^N : \mathfrak{g} \to \mathfrak{X}(N),\ a \to X_a^N$$

be f-related infinitesimal actions, i.e. $X_a^N \circ f = df X_a^M$ for all $a \in \mathfrak{g}$.

Definition

Let $E_N := T^*N \oplus \mathcal{G} \oplus TN$ be a standard Courant algebroid and $\Psi^N : \mathfrak{g} \to \mathrm{Der}(E_N)$ be an action given by

$$\Psi^N(a)(\xi + r + X) := \mathcal{L}_{X_a^N}\xi + \nabla^{\Psi}_{X_a^N} r + \mathcal{L}_{X_a^N} X, \tag{17}$$

where $\xi \in \Omega^1(N)$, $r \in \Gamma(\mathcal{G})$ and $X \in \mathfrak{X}(N)$. Then

$$\Psi^M(\xi + r + X) := \mathcal{L}_{X_a^M}\xi + (f^*\nabla^{\Psi})_{X_a^M} r + \mathcal{L}_{X_a^M} X, \tag{18}$$

where $\xi \in \Omega^1(M)$, $r \in \Gamma(f^*\mathcal{G})$ and $X \in \mathfrak{X}(M)$ is an action of $f^! E_N$, called pullback action.

T-duality for transitive Courant algebroids

Let $\pi : M \to B$ and $\tilde{\pi} : \tilde{M} \to B$ be principal bundles over the same manifold B with structure group the k-dimensional torus T^k. We denote the structure group of $\tilde{\pi}$ by $\tilde{T}^k$ and its Lie algebra by $\tilde{t}^k$. We assume that M, $\tilde{M}$ and B are oriented. Let

$$\mathrm{Lie}\,(T^k) = \mathfrak{t}^k \ni a \mapsto \psi^M(a) := X_a^M,\ \tilde{\mathfrak{t}}^k \ni \tilde{a} \mapsto \psi^{\tilde{M}}(\tilde{a}) := X_{\tilde{a}}^{\tilde{M}},$$

be the vertical paralellism of π and $\tilde{\pi}$. We denote by

$$N := M \times_B \tilde{M} := \{(m, \tilde{m}) \in M \times \tilde{M} \mid \pi(m) = \tilde{\pi}(\tilde{m})\}$$

the fiber product of M and $\tilde{M}$ and by $\pi_N : N \to M$ and $\tilde{\pi}_N : N \to \tilde{M}$ the natural projections. The actions of T^k on M and $\tilde{T}^k$ on $\tilde{M}$ induce naturally an action of $T^{2k} = T^k \times \tilde{T}^k$ on N. We denote by X_a^N, $X_{\tilde{a}}^N$ the fundamental vector fields, $a \in \mathrm{Lie}\,(T^k)$, $\tilde{a} \in \mathrm{Lie}\,(\tilde{T}^k)$.

Let E and $\tilde{E}$ be standard Courant algebroids over M and $\tilde{M}$, and assume they come with actions

$$\Psi : \mathfrak{t}^k \to \mathrm{Der}(E),\ \tilde{\Psi} : \tilde{\mathfrak{t}}^k \to \mathrm{Der}(\tilde{E}),$$

which lift ψ^M and $\psi^{\tilde{M}}$ and preserve the decompositions $E = TM \oplus \mathcal{G} \oplus T^*N$ and $\tilde{E} = T\tilde{M} \oplus \tilde{\mathcal{G}} \oplus T^*\tilde{M}$.

Definition

The Courant algebroids E and $\tilde{E}$ are T-dual if there is an invariant fiber preserving Courant algebroid isomorphism $F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$ such that the following non-degeneracy condition is satisfied. If F is defined by (β, Φ, K), where $\beta \in \Omega^2(N)$, $\Phi \in \Omega^1(N, \tilde{\pi}_N^* \tilde{\mathcal{G}})$ and $K \in \mathrm{Isom}(\pi_N^* \mathcal{G}, \tilde{\pi}_N^* \tilde{\mathcal{G}})$, then

$$\beta - \Phi^* \Phi : \mathrm{Ker}\ (d\pi_N) \times \mathrm{Ker}\ (d\tilde{\pi}_N) \to \mathbb{R} \tag{19}$$

is non-degenerate.

Assume that E and $\tilde{E}$ are T-dual standard Courant algebroids and let

$$F : \pi_N^! E \to \tilde{\pi}_N^! \tilde{E}$$

be an invariant isomorphism as above. Let

$$\mathbb{S}_E = \Lambda(T^*M)\hat{\otimes}\mathcal{S}_{\mathcal{G}},\ \mathbb{S}_{\tilde{E}} = \Lambda(T^*\tilde{M})\hat{\otimes}\mathcal{S}_{\tilde{\mathcal{G}}},$$
$$\mathbb{S}_{\pi_N^! E} = \Lambda(T^*N)\hat{\otimes}\pi_N^*(\mathcal{S}_{\mathcal{G}}),\ \mathbb{S}_{\tilde{\pi}_N \tilde{E}} = \Lambda(T^*N)\hat{\otimes}\tilde{\pi}_N^*(\mathcal{S}_{\tilde{\mathcal{G}}}),$$

be canonical weighted spinor bundles of E, $\tilde{E}$, $\pi_N^! E$ and $\tilde{\pi}_N^! \tilde{E}$ respectively. Assume that $F_{\mathbb{S}} : \Gamma(\mathbb{S}_{\pi_N^! E}) \to \Gamma(\mathbb{S}_{\tilde{\pi}_N^! \tilde{E}})$ is globally defined.

Theorem

i) The map

$$\tau := (\tilde{\pi}_N)_! \circ F_{\mathbb{S}} \circ \pi_N^! : \Gamma(\mathbb{S}_E) \to \Gamma(\mathbb{S}_{\tilde{E}}) \quad (20)$$

intertwines the canonical Dirac generating operators of E and $\tilde{E}$ and maps invariant spinors to invariant spinors.
ii) There is an isomorphism $\rho : \Gamma_{\mathfrak{k}^k}(E) \to \Gamma_{\tilde{\mathfrak{k}}^k}(\tilde{E})$ of $C^\infty(B)$-modules which preserves Courant brackets, scalar products and is compatible with τ, i.e.

$$\tau(\gamma_u s) = \gamma_{\rho(u)}\tau(s),\ [\rho(u), \rho(v)]_{\tilde{E}} = \rho[u, v]_E,\ \langle \rho(u), \rho(v)\rangle_{\tilde{E}} = \langle u, v\rangle_E, \quad (21)$$

for any $u, v \in \Gamma_{\mathfrak{k}^k}(E)$ and $s \in \Gamma_{\mathfrak{k}^k}(\mathbb{S}_E)$.

Let (E, Ψ) be a standard Courant algebroid over the total space of a principal T^k-bundle $\pi : M \to B$, with action $\Psi : \mathfrak{t}k \to \mathrm{Der}(E)$, which belongs to the class constructed before. Let (e^i) be the dual basis of (e_i) and $\tilde{T}^k$ the dual torus of T^k.

Theorem

Assume that the closed forms $\mathcal{K}_i$ represent integral cohomology classes in $H^2(B, \mathbb{R})$ and let $\tilde{\pi} : \tilde{M} \to B$ be a principal $\tilde{T}^k$-bundle with connection form $\tilde{\theta} = \sum_{i=1}^{k} \tilde{\theta}_i e^i$, such that $d\tilde{\theta}_i = \mathcal{K}_i$ for any i. Then E admits a T-dual $\tilde{E}$ over $\tilde{M}$.

Thank you for your attention!