Higher Dirac as Lagrangian Q-submanifolds

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Motivation

Let M be a manifold, $H \in \Omega^{k+1}_{cl}(M)$ define

$$\begin{aligned}
\mathbb{T}^{k-1}_H M \equiv & \quad (TM \oplus \wedge^{k-1} T^*M, \langle \cdot, \cdot \rangle, p_1, [\![\cdot, \cdot]\!]_H) \\
\langle X + \alpha, Y + \beta \rangle = & \quad i_X \beta + i_Y \alpha \\
p_1(X + \alpha) = & \quad X \\
[\![X + \alpha, Y + \beta]\!]_H = & \quad [X, Y] + \mathcal{L}_X \beta - i_Y d\alpha + i_X i_Y H
\end{aligned}$$

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Higher Courant algebroid.

Q1: Which is the definition of a Higher Courant algebroid?

Q2: Are those the only examples? $TM \leftrightarrow A$ Lie algebroid.

Q3: What are Higher Dirac structures?

Severa-Roytenberg correspondence

Classical object		Graded geometry
Courant algebroids	$\rightleftharpoons$	Degree 2 symplectic Q-manifolds.
Dirac structures	$\rightleftharpoons$	Lagrangian Q-submanifolds.

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In particular:

$\mathbb{T}^1_H M$	$\rightleftharpoons$	$(T^*[2]T[1]M,\ \omega_{can},\ Q)$
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Use the RHS to define Higher Courant and Higher Dirac

Idea

There is a correspondence between:

Higher Courant algebroids $\rightleftharpoons$ Graded cotangent bundles

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Let $A \to M$ be a vector bundle.

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Recall:

Degree k Symplectic Q-manifold $\leftrightarrow (\mathcal{M}, \omega, Q)$ then $Q = X_\theta$ with

$$\theta \in \mathcal{O}_{\mathcal{M}}^{k+1} \quad \text{s.t.} \quad \{\theta, \theta\} = 0.$$

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Problem: It was difficult to describe $T^*[k]A[1]$ in classical terms.

Geometrization functor

Theorem [Bursztyn, C, Mehta]

There is an equivalence of categories:

- Coalgebra bundles.
- Graded manifolds.

Claim: To give a geometric description of a k-manifold is enough to identify $\mathcal{O}^i = \Gamma E_i$ for $i = 1, \cdots k$ and know the maps $\mathcal{O}^i \cdot \mathcal{O}^j \subseteq \mathcal{O}^{i+j}$ for $i + j < k$.

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The k-manifold $T^*[k]A[1]$ is equivalent to the algebra bundle (E, m) where for $k > 2$,

- $E_i = \wedge^i A^*$ if $1 \leq i \leq k - 2$.
- $E_{k-1} = A \oplus \wedge^{k-1} A^*$
- $E_k = Der(A) \oplus \wedge^k A^*$

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The symplectic form is codified in $\langle \cdot, \cdot \rangle : E_{k-1} \oplus E_{k-1} \to E_{k-2}$ and the Atiyah alg structure of $Der(A)$.

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So $A \to M$ is a Lie algebroid and $d_{CE} H = 0$

Prop: The Q's are equivalent iff $\tau \cong \tau'$ and $H = H' + d_{TM}\beta$.

Semi-Direct products

Sheng-Zhu

Let $(A \to M, [\cdot,\cdot], \rho)$ be a Lie algebroid, $H \in \wedge^{k+1} A^*$ with $d_A H = 0$ and $(T^*M \xrightarrow{\rho} A^*, \nabla^{T^*M} \nabla^{A^*}, K)$ the coadjoint representation up to homotopy for some ∇ TM-connection on A.

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Proposition

For $k > 2$, $A \ltimes_H (T^*M \to A^*)[k-1]$ is a L_k-algebroid, with $L_{-k+1} = T^*M,\ L_{-k+2} = A^*$ and $L_0 = A$ and brackets

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$$\rho = \rho \qquad \ell_1 = \rho^*$$

$$\ell_2(a, a') = [a, a'] \qquad \ell_2(a, \alpha) = \nabla_a^{A^*} \alpha \qquad \ell_2(a, \omega) = \nabla_a^{T^*M} \omega$$

$$\ell_3(a, a', \alpha) = K(a, a')(\alpha) \qquad \ell_k(a_1, \cdots, a_k) = i_{a_k} \cdots i_{a_1} H$$

$$\ell_{k+1}(a_1, \cdots, a_{k+1}) = d(i_{a_{k+1}} \cdots i_{a_1} H) - \sum \langle D(a_i), i_{a_{K+1}} \cdots i_{a_1} H \rangle$$

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Proposition [-]

∇ induces an isomorphism between the Q-manifolds

$$T^*[k+1]A[1],\ \{\theta + H, \cdot\} \cong \Big(A \ltimes_H (T^*M \to A^*)[k]\Big)[1],\ d_{CE}$$

Lagrangian Q-submanifolds

Theorem [-] Definition of higher Dirac structure

For $k > 2$ Lagrangian Q-submanifolds of $(T^*[k]A[1], \{\cdot,\cdot\}, Q)$ are the same as a $L \to N$ subvector bundle of $A \oplus \wedge^{k-1} A^* \to M$ satisfying:

- $p_1(L) \subseteq A$ is a subbundle.
- $\langle L, L \rangle \subseteq Ann(p_1(L)) \wedge \bigwedge^{k-3} A^*$
- $L \cap \bigwedge^{k-1} A^* = Ann(p_1(L)) \wedge \bigwedge^{k-2} A^*$
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In the case when $N = M$ and without the first condition this was defined by Hagiwara under the name of Nambu-Dirac structures. There are other definitions for Higher Dirac Structures: Wade, Zabzine, Zambon, Bi-Sheng, Bursztyn-Martinez-Rubio.

Applications

Idea

Use the theory of symplectic Q-manifolds to study higher Courant and higher Dirac.

I Lie n-groupoids

Degree n Q-manifolds $\rightleftharpoons$ Lie n-algebroids (Bonavolonta-Poncin)

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Let $(A \to M, \rho, [\cdot,\cdot])$ be a Lie algebroid and $[H] \in H^{k+1}(A)$.

Assume A integrates to $G \rightrightarrows M$ and $VE([\mathcal{H}]) = [H]$.

Integration

The higher Courant algebroid $(A \oplus \wedge^{k-1} A^*, \langle\cdot,\cdot\rangle, \rho, [\![\cdot,\cdot]\!]_H)$ integrates to $G \ltimes_{\mathcal{H}} (T^*M \to A^*)[k]$

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Canonical integration?
Shifted sympelctic form?

II Actions

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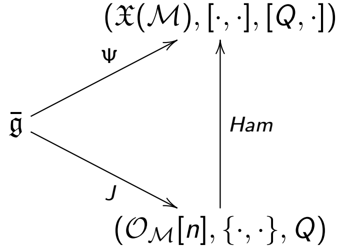
A Hamiltonian action $\bar{\mathfrak{g}} \curvearrowright (\mathcal{M}, \{\cdot,\cdot\}, Q)$ is a pair of L_∞-morphism, the action Ψ and the comoment map J s.t.



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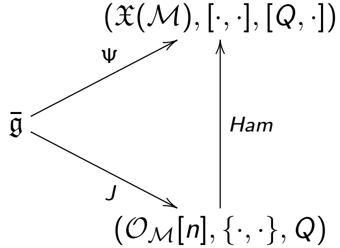

Define:

$$C^i_{BRST} = \bigoplus_{j+k=i} \mathcal{O}^j_{T^*\bar{\mathfrak{g}}[1]} \otimes \mathcal{O}^{k+n}_{\mathcal{M}}$$

with bracket $\{\cdot,\cdot\}_{can} + \{\cdot,\cdot\}$

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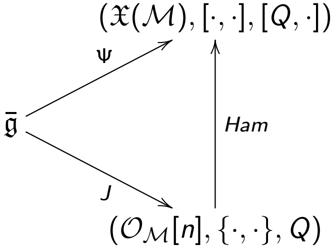

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Then $\theta_{\bar{\mathfrak{g}}}, \theta \in C^1_{BRST}$. A comoment map $J : \bar{\mathfrak{g}} \to \mathcal{O}_{\mathcal{M}}[n]$ defines also a degree 1 element θ_J. The element $\Theta = \theta + \theta_{\bar{\mathfrak{g}}} + \theta_J$ satisfy

$$\{\Theta, \Theta\} = 0.$$

Under favourable hypothesis $H^0(C_{BRST}, \{\Theta, \cdot\}) = \mathcal{O}_{\mathcal{M}_{red}}$

III Prequantum bundle

$(\mathcal{M}, \omega, Q)$ deg k symplectic Q-manifold.

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Prop. If $k \geq 1$ then a preguantum bundle is given by:

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- For a Courant algebroid ? Higher Courant ?

IV AKSZ Topological σ-models

Alexandrov, Kontsevich, Schwarz, Zaboronsky

$$\text{Geometric data} = \begin{cases} \Sigma \quad d\text{-dimensional manifold}, \\ (\mathcal{M}, \omega_{\mathcal{M}} = d\lambda_{\mathcal{M}}, \theta) \; d-1 \text{ symplectic } Q\text{-manifold.} \end{cases}$$

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- Gauge fixing: $\mathcal{L}_{BV}$ Lagrangian submanifold of $(\mathcal{F}_{BV}, \omega_{BV})$.

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When $= T^*[k]\mathfrak{g}[1]$ we obtain BF theory!!!

Thanks !!