From mechanics to weakly-Lagrangian condition of higher Dirac structures

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ONEW on Higher Dirac structures

time-depending Hamiltonian...a geometrical point of view

$$
\begin{array}{ccc}
T^*Q \times \mathbb{R}^2 & \xrightarrow[H]{} & \mathbb{R} \\
& \searrow & \\
& & Q \times \mathbb{R} \\
& & \Big\downarrow \pi \\
& & \mathbb{R}
\end{array}
\tag{1}
$$

A suitable Hamiltonian in this setting is H so that $\dfrac{\partial H}{\partial s_2} = 0$ Solution of equation of motion are $\sigma : \mathbb{R} \to T^*Q \times \mathbb{R}^2$ that is secction of the canonical projection

two issues...

This is a suitable setting for **classical field theories** in the dual of a dual jet bundle	condition $\dfrac{\partial H}{\partial s_2} = 0$ says that the dinamic lives on $T^*Q \times R$ as Poisson manifold

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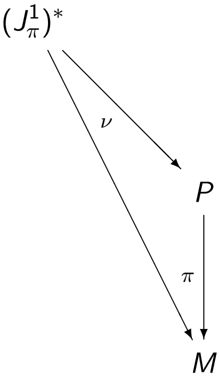
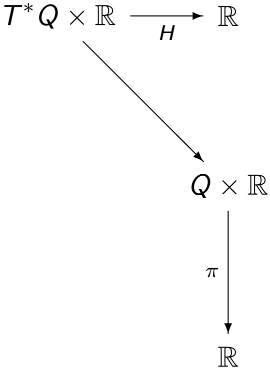
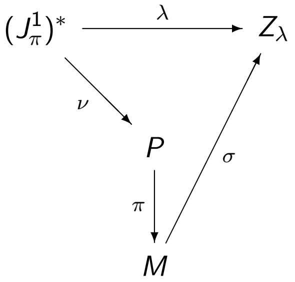




how could we include both in just one geometrical setting?

A solution



(2)

Hamiltonian:	sections of λ
Solution of eq. of motion:	sections σ **but** in $(J^1_\pi)^*$
geometry:	–?

geometry of Z_λ

$\mathbb{R} \curvearrowright T^*(Q \times \mathbb{R})$ preserving the symplectic form $\Rightarrow T^*Q \times \mathbb{R}$ is **Poisson**

$\mathbb{R} \curvearrowright (J^1_\pi)^*$ preserving the multisymplectic form $\Rightarrow Z_\lambda$ is **multi-Poisson**

Finally, this geometry recovers the solution of the equation of motion

[M. Thm 5.3.1]

$\sigma : M \to Z_\lambda$ is solution of equation of motion in CFT if and only if for all $\alpha \in \Gamma(S_\lambda)$ the following equation holds

$$trace(i_{\bar{\sigma}(\cdot)}\alpha) = -(d\mathcal{H}_h)(P_\lambda(\alpha)) \circ s\sigma \qquad (3)$$

where $\bar{\sigma}$ and $\mathcal{H}_h$ comes from horizontal lifting of the section σ and the hamiltonian $h : Z_\lambda \to (J^1_\pi)^*$.

What about the higher-Dirac setting?

1. multi-Poisson should fit on higher-Dirac structures
2. Annihilator condition is a key ingrdiente....**but** is not compatible with the lagrangian condition

weakly-Lagrangian condition:

A subbundle L of $TM \oplus \wedge^k T^*M$ is called weakly-Lagrangian if

$$L^{\perp} \cap (L + TM)$$

Higher-Dirac structures:

A subbundle L of $TM \oplus \wedge^k T^*M$ is called Higher-Dirac structure if (i) is weakly-Lagrangian (ii)involutive under the bracket

$$[X \oplus \alpha, Y \oplus \beta] = [X, Y] \oplus L_X\beta - i_Y d\alpha$$

Some facts for HD:

1. Integration by **pre-multi-symplectic** groupoids

[Bursztyn,M.,Rubio, Thm 5.3]

integrable Higher-Dirac structures over M are in one-to-one correspondence with Lie groupoids $\mathcal{G} \rightrightarrows M$ with $\omega \in \Omega^{k+1}(\mathcal{G})$ closed, multiplicative so that

(a) $(\ker(\omega) \cap \ker(dt) \cap \ker(ds))|_g = \{0\}$,

(b) $d_g t(\ker(\omega) \cap \ker(ds)) = (\ker(\omega) \cap TM)|_{t(g)}$.

2. Foliation

Some facts for HD:

1. Integration by **pre-multi-symplectic** groupoids
2. Foliation

[Bursztyn,M.,Rubio,Thm 4.2 and Thm 4.10]

Higher-Dirac structures over M are in one-to-one correspondence with a triple (E, A, ε) so that on each leave $\mathcal{O}$ of the foliation pf the respective Lie algebroid we have

(a)$E \leq TM$	(b) $A \leq$Ann(E)
(c)$\varepsilon : E \to \wedge^k T^*M/A$	(d)$L_{\Gamma(E)}\Gamma(A\vert_{\mathcal{O}}) \subset \Gamma(A\vert_{\mathcal{O}})$
(e)$\varepsilon_{\mathcal{O}}(\mathrm{pr}_2(L)^o) = \{0\}$	(f)$\varepsilon_{\mathcal{O}}$ is δ-closed

where $(\Omega^{\cdot}_{sk}(\mathcal{O}, \wedge^k T^*M/A), \delta)$ is differential complex.

Some facts for HD(cont.):

Note that this works also for higher-Dirac $L \leq TM \oplus T^*M \otimes E$ taking values on a vector bundle E

1. Vertical foliation, $\{\mathcal{O}\}$ and $\{\mathcal{O}_\lambda\}$, of the geometric setting of CFT are endowed with E-valued Dirac structures

[M.Prop 5.2.6]

The restriction $\lambda : \mathcal{O} \to \mathcal{O}_\lambda$ is forward-higher-dirac map

2. poly-Poisson-sigma-model needs the condition $S^0 = \{0\}$ to obtain poly-symplectic groupoid that integrates the poly-Poisson structure (via reduction), indeed a key part of the construction is

[Contreras,M.,Prop 4.6]

For poly-Poisson (M, S, P) we get $path(S)$ is poly-symplectic submanifold of $\oplus T^*path(M)$