(p, k)-Dirac structures for higher analogues of Courant algebroids

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October 26, 2021

Outline

1. Background and Motivation
2. (p, k)-maximal isotropic subspaces
3. (p, k)-Dirac structures

Courant algebroid

There is a standard Courant algebroid structure on the direct sum bundle $TM \oplus T^*M$. The standard Courant bracket is given by

$$[\![X+\xi, Y+\eta]\!] = [X, Y] + L_X\eta - L_Y\xi + \frac{1}{2}(\mathrm{d}i_Y\xi - \mathrm{d}i_X\eta).$$

The nondegenerate symmetric pairing $(\cdot,\cdot)_+$ is given by

$$(X+\xi, Y+\eta)_+ = \frac{1}{2}(i_X\eta + i_Y\xi), \quad \forall\ X, Y \in \mathfrak{X}(M),\ \xi,\eta \in \Omega(M).$$

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Alternatively, one can also use the standard Dorfman bracket:

$$\{X + \xi, Y + \eta\} = [X, Y] + L_X\eta - L_Y\xi + \mathrm{d}i_Y\xi.$$

Dirac structures

A Dirac structure is a maximal isotropic subbundle of $TM \oplus T^*M$, which is also involutive under the standard Courant bracket operation $[\![\cdot,\cdot]\!]$. Due to the relation

$$\{X+\xi, Y+\eta\} = [\![X+\xi, Y+\eta]\!] + d\,(X+\xi, Y+\eta)_+\,,$$

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Examples:

- the graph $G_{\pi^\sharp}$ of a Poisson structure $\pi \in \mathfrak{X}^2(M)$ is a Dirac structure;
- the graph $G_{\omega^\natural}$ of a presymplectic structure $\omega \in \Omega^2(M)$ is a Dirac structure.

Higher Courant algebroids

Consider the direct sum bundle $\mathbb{T}^p \triangleq TM \oplus \Lambda^p T^*M$.

- $\Lambda^{p-1}T^*M$-valued pairing $(\cdot,\cdot)_+$:

$$(X+\xi, Y+\eta)_+ = \frac{1}{2}(i_X\eta + i_Y\xi), \quad \forall\, X, Y \in \mathfrak{X}(M),\ \xi, \eta \in \Omega^p(M).$$

- The higher Dorfman bracket $\{\cdot,\cdot\}^p$:

$$\{X+\xi, Y+\eta\}^p = [X,Y] + L_X\eta - L_Y\xi + \mathrm{d}i_Y\xi.$$

- Anchor ρ is the projection:

$$\rho(X+\xi) = X, \quad \forall\, X+\xi \in \Gamma(\mathbb{T}^p).$$

$(\mathbb{T}^p, (\cdot,\cdot)_+, \{\cdot,\cdot\}^p, \rho)$ is called higher analogues of the standard Courant algebroid.

Theorem

(i) *For any* $e_1, e_2 \in \Gamma(\mathbb{T}^p)$, $f \in C^\infty(M)$, *we have*

$$\begin{aligned} \{e_1, fe_2\}^p &= f\{e_1, e_2\}^p + \rho(e_1)(f)e_2, \\ \{fe_1, e_2\}^p &= f\{e_1, e_2\}^p - \rho(e_2)(f)e_1 + \mathrm{d}f \wedge 2(e_1, e_2)_+. \end{aligned}$$

(ii) *The higher Dorfman bracket* $\{\cdot,\cdot\}^p$ *is a Leibniz bracket:*

$$\{e_1, \{e_2, e_3\}^p\}^p = \{\{e_1, e_2\}^p, e_3\}^p + \{e_2, \{e_1, e_3\}^p\}^p.$$

Consequently, $(\mathbb{T}^p, \{\cdot,\cdot\}^p, \rho)$ *is a Leibniz algebroid.*

(iii) *The pairing and the higher Dorfman bracket are compatible:*

$$L_{\rho(e_1)}(e_2, e_3)_+ = (\{e_1, e_2\}^p, e_3)_+ + (e_2, \{e_1, e_3\}^p)_+ .$$

Higher Dirac (in the sense of Zambon)

Definition

A higher Dirac structure *for the higher analogues of the standard Courant algebroid* $(\mathbb{T}^p, (\cdot,\cdot)_+, \{\cdot,\cdot\}^p, \rho)$ *is a maximal isotropic subbundle with respect to the pairing* $(\cdot,\cdot)_+$*, which is involutive under the higher Dorfman bracket* $\{\cdot,\cdot\}^p$*.*

- For any closed $(p+1)$-form ω, the graph of the induced bundle map $\omega^\sharp : TM \longrightarrow \Lambda^p T^*M$, which we denote by G_ω, is a higher Dirac structure.

M. Zambon, L_∞-algebras and higher analogues of Dirac structures and Courant algebroids, *J. Symplectic Geom.* 10 (2012), no. 4, 563-599.

Nambu-Dirac (in the sense of Hagiwara)

Definition

A subbundle $L \subset \mathbb{T}^p$ is said to be **an almost Nambu-Dirac structure of order p** *if $(i_X\eta + i_Y\xi)|_{\Lambda^{p-1}\rho(L)} = 0$, for any $(X,\xi),(Y,\eta) \in \Gamma(L)$ and $\Lambda^p\rho(L) = \rho(L^\top)$, where $L^\top$ denotes the annihilator of L with respect to the pairing $\langle\cdot,\cdot\rangle_+ : \mathbb{T}^p \times \mathbb{T}_p \longrightarrow C^\infty(M)$, in which $\mathbb{T}_p = \Lambda^p TM \oplus T^*M$ and*

$$\langle(X,\xi),(Y,\eta)\rangle_+ = \frac{1}{2}(\xi(Y)+\eta(X)), \quad \forall\, (X,\xi) \in \mathbb{T}^p,\ (Y,\eta) \in \mathbb{T}_p.$$

An almost Nambu-Dirac structure is **a Nambu-Dirac structure** if it is involutive under the higher Dorfman bracket.

Y. Hagiwara, Nambu-Dirac manifolds, *J. Phys. A*, 35 (5) (2002), 1263-1281.

Nambu-Dirac (in the sense of Hagiwara)

- For any Nambu-Poisson structure $\pi \in \mathfrak{X}^{p+1}(M)$, the graph $G_{\pi^\sharp}$ is a Nambu-Dirac structure.

Recall that a $(p+1)$-vector field $\pi \in \mathfrak{X}^{p+1}(M)$ is a Nambu-Poisson structure if and only if for $f_1, \cdots, f_p \in C^\infty(M)$, we have

$$L_{\pi^\sharp(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_p)}\pi = 0.$$

Motivation

Unify higher Dirac structures and Nambu-Dirac structures.

Outline

Let V be a vector space, and V^* its dual space. Denote by

$$\mathcal{V}^p \triangleq V \oplus \Lambda^p V^*.$$

Let W be a subspace of V and denote by W^0 its null space, i.e.

$$W^0 = \{\xi \in V^* | \langle \xi, u \rangle = 0, \ \forall \ u \in W\}.$$

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Definition (Bi-S.)

Let L be a subspace of $\mathcal{V}^p$ and $W = \rho(L)$. L is called a **(p, k)-isotropic** *subspace, where $0 \leq k \leq p-1$, if for any $l_1, l_2 \in L$ and $u_1, \cdots, u_k \in W$, we have $i_{u_k} \cdots i_{u_1}(l_1, l_2)_+ = 0$. L is called a* **linear (p, k)-Dirac structure** *if L is (p, k)-maximal isotropic, i.e.*

$$L = L_k^{\perp} \triangleq \{e \in \mathcal{V}^p | i_{u_k} \cdots i_{u_1}(e, L)_+ = 0, \quad \forall \ u_1, \cdots u_k \in W\}. \quad (1)$$

Proposition

The following two statements are equivalent:

- *L is a linear $(p, p-1)$-Dirac structure in $\mathcal{V}^p$;*
- *L is an almost Nambu-Dirac structure of order p in $\mathcal{V}^p$.*

Proof: It is obvious that the condition being isotropic are the same. So we only need to show

$$L = L_{p-1}^{\perp} \Longleftrightarrow \rho(L^{\top}) = \Lambda^p \rho(L).$$

If L is isotropic, we have

$$0 = i_{u^{p-1}}((u, \xi), (u, \xi))_+ = i_{u^{p-1}} i_u \xi, \quad \forall\, (u, \xi) \in L,\ u^{p-1} \in \Lambda^{p-1}\rho(L).$$

Thus, we have

$$\langle (u,\xi),(u\wedge u^{p-1}, i_{u^{p-1}}\xi)\rangle_+ = \frac{1}{2}\Big(\xi(u\wedge u^{p-1}) + i_u i_{u^{p-1}}\xi\Big) = 0,$$

which implies that $(u\wedge u^{p-1}, i_{u^{p-1}}\xi)\in L^\top$. Therefore, we have

$$\Lambda^p\rho(L)\subset\rho(L^\top).$$

If L is a linear $(p, p-1)$-Dirac structure, we have

$$L\cap\Lambda^p V^* = L^\perp_{p-1}\cap\Lambda^p V^* = \rho(L)^0\wedge\Lambda^{p-1}V^*.$$

For any $(U,\mu)\in L^\top\subset\Lambda^p V\oplus V^*$, we have

$$\langle (U,\mu),(0,\alpha)\rangle_+ = 0,\quad \forall\,\alpha\in\rho(L)^0\wedge\Lambda^{p-1}V^*,$$

which implies that $U\in\Lambda^p\rho(L)$. Thus, we have

$$\rho(L^\top)\subset\Lambda^p\rho(L).$$

Therefore, we have $\rho(L^\top)=\Lambda^p\rho(L)$.

We describe linear (p, k)-Dirac structures by characteristic pairs. Let W be a subspace of V. Since $W^* \cong V^*/W^0$, we have the natural projection $\pi : V^* \longrightarrow W^*$, and

$$0 \longrightarrow W^0 \xrightarrow{i} V^* \xrightarrow{\pi} W^* \longrightarrow 0. \tag{2}$$

For any $0 \leq m \leq p$, π also induces a projection $\pi^p_m : \Lambda^p V^* \longrightarrow \Lambda^m W^* \wedge \Lambda^{p-m} V^*$ given by

$$\pi^p_m(\xi_1 \wedge \cdots \wedge \xi_p) = \sum_{1 \leq i_1 < \cdots < i_m \leq p} \xi_1 \wedge \cdots \wedge \pi(\xi_{i_1}) \wedge \cdots \wedge \pi(\xi_{i_m}) \wedge \cdots \wedge \xi_p.$$

Denote by $\iota_u : \Lambda^m W^* \wedge \Lambda^{p-m} V^* \longrightarrow \Lambda^{m-1} W^* \wedge \Lambda^{p-m} V^*$ the operator given for all $w_i \in W^*$, $\xi \in \Lambda^{p-m} V^*$ by

$$\iota_u(w_1 \wedge \cdots \wedge w_m \wedge \xi) = \sum_{i=1}^{m} (-1)^{i+1} \langle u, w_i \rangle w_1 \wedge \cdots \hat{w}_i \wedge \cdots \wedge w_m \wedge \xi.$$

For any $\Omega \in \Lambda^{p+1}V^*$, set $\Omega^{p+1}_{k+2} = \pi^{p+1}_{k+2}(\Omega)$.

Theorem

For any subspace $W \subset V$ satisfying $\dim(W) \leq \dim(V) - (p - k)$, or $W = V$, and $\Omega \in \Lambda^{p+1}V^*$, the subspace $L(W, \Omega^{p+1}_{k+2}) \subset \mathcal{V}^p$:

$$L(W, \Omega^{p+1}_{k+2}) \triangleq \{(u, \xi) | u \in W,\ \xi \in \Lambda^p V^*,\ \iota_u \Omega^{p+1}_{k+2} = \pi^p_{k+1}(\xi)\}$$

is a linear (p, k)-Dirac structure.
Conversely, for any linear (p, k)-Dirac structure $L \subset \mathcal{V}^p$, denote by $W = \rho(L)$, and define $\omega \in W^* \otimes (\Lambda^{k+1}W^* \wedge \Lambda^{p-k-1}V^*)$ by

$$\iota_u \omega = \pi^p_{k+1}(\xi), \quad \forall\ (u, \xi) \in L.$$

Then $\omega \in \Lambda^{k+2}W^* \wedge \Lambda^{p-k-1}V^*$. Choose $\Omega \in \Lambda^{p+1}V^*$ satisfying $\pi^{p+1}_{k+2}(\Omega) = \omega$, then $L = L(W, \Omega^{p+1}_{k+2})$.

We call the pair (W,Ω^{p+1}_{k+2}) a **characteristic pair** associated to the linear (p,k)-Dirac structure L.

Corollary

Let $L\subset\mathcal{V}^p$ be a linear (p,k)-Dirac structure, and (W,Ω^{p+1}_{k+2}) its characteristic pair, i.e. $L=L(W,\Omega^{p+1}_{k+2})$. Then L can be described by W and Ω as follows:

$$L=L(W,\Omega^{p+1}_{k+2})=\{(u,i_u\Omega+\alpha)|u\in W,\alpha\in\Lambda^{p-k}W^0\wedge\Lambda^kV^*\}.$$

Proof. Any $(u,\xi)\in L$ satisfies

$$\pi^p_{k+1}(\xi)=\iota_u\Omega^{p+1}_{k+2}=\iota_u\pi^{p+1}_{k+2}(\Omega)=\pi^p_{k+1}(i_u\Omega).$$

Thus, we can assume that

$$\xi=i_u\Omega+\alpha,\ \ \alpha\in\mathrm{Ker}(\pi^p_{k+1})=\Lambda^{p-k}W^0\wedge\Lambda^kV^*.$$

Therefore, L can be described as above. ∎

Outline

(p, k)-Dirac structures

An almost (p, k)-Dirac structure in $\mathbb{T}^p$ is a subbundle $L \subset \mathbb{T}^p$, which is pointwise a linear (p, k)-Dirac structure.

Definition

A (p, k)-Dirac structure *in $\mathbb{T}^p$ is an almost (p, k)-Dirac structure, which is involutive under the higher Dorfman bracket $\{\cdot, \cdot\}^p$.*

Main Theorem

For an almost (p, k)-Dirac structure L of $\mathbb{T}^p$, assume that $W = \rho(L) \subset TM$ is a regular distribution. L can be described by W and some $\Omega \in \Omega^{p+1}(M)$ as follows:

$$L = \{(X, i_X\Omega + \alpha) | X \in \Gamma(W),\ \alpha \in \Gamma(\Lambda^{p-k} W^0 \wedge \Lambda^k T^* M)\}.$$

Theorem (Bi-S.)

With the above notations, an almost (p, k)-Dirac structure L in $\mathbb{T}^p$ is a (p, k)-Dirac structure if and only if

(a) *W is an involutive distribution;*

(b) $\pi^{p+2}_{k+3}(d\Omega) = 0$.

Proof

For any $(X, i_X\Omega),\ (Y, i_Y\Omega) \in \Gamma(L), \alpha \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M)$,

$$\begin{aligned}
\{(X, i_X\Omega),\ (Y, i_Y\Omega)\}^p &= ([X, Y], L_X i_Y\Omega - i_Y d i_X\Omega) \\
&= ([X, Y], L_X i_Y\Omega - i_Y L_X\Omega + i_Y i_X d\Omega) \\
&= ([X, Y], i_{[X,Y]}\Omega + i_Y i_X(d\Omega)), \\
\{(X, i_X\Omega), \alpha\}^p &= L_X\alpha, \\
\{\alpha, (X, i_X\Omega)\}^p &= -i_X d\alpha.
\end{aligned}$$

Thus, $\Gamma(L)$ is involutive if and only if

$$[X, Y] \in \Gamma(W), \quad i_Y i_X(d\Omega) \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M),$$
$$L_X\alpha \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M), \quad i_X d\alpha \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M).$$

Obviously, for any $X, Y \in \Gamma(W)$, $[X, Y] \in \Gamma(W)$ means that W is an involutive distribution.

Proof

Under the condition that W is an involutive distribution, it is not hard to deduce that

$$L_X\alpha \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M), \quad i_X d\alpha \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M).$$

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The condition

$$i_Y i_X(d\Omega) \in \Gamma(\Lambda^{p-k}W^0 \wedge \Lambda^k T^*M)$$

is equivalent to that

$$\pi^p_{k+1}(i_Y i_X(d\Omega)) = 0, \quad \forall X, Y \in \Gamma(W).$$

Since $\iota_Y\iota_X\pi^{p+2}_{k+3} = \pi^p_{k+1} i_Y i_X$, this is equivalent to

$$\iota_Y\iota_X\pi^{p+2}_{k+3}(d\Omega) = 0, \quad \forall X, Y \in \Gamma(W),$$

and hence to $\pi^{p+2}_{k+3}(d\Omega) = 0$.

References

Y. Bi and Y. Sheng, Dirac structures for higher analogues of Courant algebroids, *Int. J. Geom. Methods Mod. Phys.* 12 (2015), 1550010.

Thanks for your attention!