Holomorphic Dirac-Jacobi structures

AISSA WADE

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Dirac geometry: a brief history

- Dirac structures were introduced towards the end of 1980s by, (independently,
- (1) Ted Courant and Weinstein in " *Beyond Poisson structures*" a paper that appeared in the *Proceedings of the "Journées Lyonnaises de la Société Mathématique de France"* in 1988.
- (2) Dorfman in " *Dirac structures of integrable evolution equations*" (appeared in *Physics Letters A*, 1887). Dorfman applied Dirac structures to integrable equations.

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- (2) Dorfman in " *Dirac structures of integrable evolution equations*" (appeared in *Physics Letters A*, 1887). Dorfman applied Dirac structures to integrable equations.
- The idea of Weinstein and Courant was to extend Dirac's constrained Poisson brackets (see Dirac's book *Lectures on quantum mechanics*) to functions that are constant on *presymplectic* leaves of a foliated manifold.

Dirac structures: brief history

- In his thesis, Courant developed the Dirac structures bringing a unified approach to presymplectic and Poisson structures.
- One of the novelties in Courant's thesis was what is nowadays called the Courant algebroid of a smooth manifold M, namely the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$.

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- One of the novelties in Courant's thesis was what is nowadays called the Courant algebroid of a smooth manifold M, namely the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$.
- In his thesis, Courant defined a Dirac structure as a vector subbundle $L \subseteq \mathbb{T}M$ which in maximally isotropic with respect to the canonical symmetric fiberwise bilinear form on $\mathbb{T}M$, and satisfies the integrability condition $[\![\Gamma(L),\ \Gamma(L)]\!] \subseteq \Gamma(L)$, where the bracket $[\![\ ,\ ]\!]$ is given by:

$$[\![(X,\alpha),\ (Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X)).$$

Dirac structures: brief history

- In 1997, Liu, Weinstein and Xu introduced more general Courant algebroids extending the case of $\mathbb{T}M$ (see their paper on "*Manin triples for bialgebroids*").
- In 1998, Kosmann-Schwarzbach pointed out that the skew-symmetric Courant bracket is equivalent to Dorfman bracket: $[\![(X,\alpha),\ (Y,\beta)]\!] = ([X,Y],\ \mathcal{L}_X\beta - i_Y d\alpha)$.

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- In his thesis, Roytenberg developed Courant algebroids (1999).
- I became interested in Dirac geometry in 2000 when I first attempted to extend the framework of Dirac structures to contact and Jacobi structures introducing $\mathcal{E}^1(M)$-Dirac structures which were later called Dirac-Jacobi structures by Grabowski.

Dirac structures: brief history

Since then, several developments and applications of Courant algebroids and Dirac structures have emerged, among others:

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- E-Courant algebroids (Chen, Liu and Sheng);
- contact -Courant algebroids (Grabowski).
- In addition, Courant algebroids play a fundamental role in the study of generalized complex geometry.
- Similarly, they are also essential tools in the study of odd-dimensional analogues of generalized complex manifolds, these are **generalized contact manifolds**.

Motivations

- In this talk, we will focus on holomorphic Dirac-Jacobi structures.
- The study of these objects is motivated by the paper "Variations of Prequantization" by Weinstein and Zambon and Gualtieri's paper on "Generalized Kähler Geometry" where he considered the prequantization of generalized Kähler manifolds, we think that there is a need to better understand holomorphic Dirac-Jacobi structures.

Motivations

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- The study of these objects is motivated by the paper "Variations of Prequantization" by Weinstein and Zambon and Gualtieri's paper on "Generalized Kähler Geometry" where he considered the prequantization of generalized Kähler manifolds, we think that there is a need to better understand holomorphic Dirac-Jacobi structures.
- According to Weinstein and Zambon, the prequantization of a Dirac manifold is a $U(1)$-bundle with a compatible Dirac-Jacobi structure. So, it is natural to ask if the prequantization of generalized complex manifolds can be better understood through the framework of holomorphic Dirac-Jacobi structures.

Real Dirac-Jacobi structures

Before introducing holomorphic Dirac-Jacobi structures, let us first recall the line bundle approach to (real) Dirac geometry.

- Let $L \to M$ be a vector bundle over a smooth manifold M.
- A **derivation of** L is an $\mathbb{R}$-linear map $\delta : \Gamma(L) \to \Gamma(L)$ such that there exists a unique vector field $X = \sigma(\delta) \in \mathfrak{X}(M)$, called the **symbol of** δ and satisfying:

$$\delta(fs) = f\delta(s) + X(f)s,$$

  for all $f \in C^\infty(M)$ and $s \in \Gamma(L)$. In fact, every first order differential operator $\Gamma(L) \to \Gamma(L)$ is a derivation.
- For instance, if $L = M \times \mathbb{R}$ is the trivial line bundle then derivations of L can be identified with sections of $TM \times \mathbb{R}$.

Real Dirac-Jacobi structures

- Derivations of L are sections of a Lie algebroid $DL \to M$, called the **gauge algebroid of** L.
- Geometrically, sections of DL can be considered as infinitesimal vector bundle automorphisms of L. Denote by $J^1L \to M$ the first jet bundle of L then $(DL)^* = J^1L \otimes L^*$. Now consider

$$\mathbb{D}L = DL \oplus J^1L.$$

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  The vector bundle $\mathbb{D}L$ was first introduced by Chen and Liu (in 2010) who called it an **omni-Lie algebroid**.
- This vector bundle is a special **contact-Courant algebroid** in the sense of Grabowski.

Real Dirac-Jacobi structures

There is a canonical non-degenerate, symmetric L-valued pairing on $\mathbb{D}L$ defined by:

$$\langle\langle(\delta_1, \varphi_1), (\delta_2, \varphi_2)\rangle\rangle := \langle \delta_1, \varphi_2 \rangle + \langle \delta_2, \varphi_1 \rangle,$$

but one can also consider an analogue of the Dorfman bracket:

$$[\![(\delta_1, \varphi_1), (\delta_2, \varphi_2)]\!] := ([\delta_1, \delta_2],\ \mathcal{L}_{\delta_1}\varphi_2 - d_{DL}\varphi_1(\delta_2, \cdot)) \qquad (1)$$

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where d_{DL} is the de Rham differential of the gauge algebroid DL.

Definition: A Dirac-Jacobi bundle over a manifold M is a line bundle L over M endowed with a subbundle $\mathfrak{L} \subseteq \mathbb{D}L$ which is is *maximal isotropic* with respect to $\langle\langle -,-\rangle\rangle$ and satisfies the integrability condition: $[\![\Gamma(\mathfrak{L}),\Gamma(\mathfrak{L})]\!] \subseteq \Gamma(\mathfrak{L})$.

In particular Jacobi structures on M are exactly to Dirac-Jacobi structures $\mathfrak{L}$ such that $\mathfrak{L} \cap J^1L = 0$.

Properties of real Dirac-Jacobi structures

- Dirac-Jacobi structures on (M, L) can be viewed as foliations on M whose leaves are submanifolds equipped with either a pre-contact form or a locally conformal presymplectic structure.
- In other words, Dirac-Jacobi structures are pre-Jacobi structures.
- Jacobi structures can be described as homogeneous Poisson structures.
- In a similar way Dirac-Jacobi structures can be considered as homogeneous Dirac structures via a procedure called the homogenization scheme.

Holomorphic case

- Let M be a (real) smooth manifold equipped with a complex structure $j : TM \to TM$. Denote $X = (M, j)$.
- Recall a *holomorphic vector bundle* $E \to X$ can be seen as a complex vector bundle equipped with a flat $T^{0,1}X$-connection. In particular there is an operator $\overline{\partial} : \Gamma(E) \to \Omega^{0,1}(X, E)$ whose kernel consists of *holomorphic sections of* E.

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- Now consider a holomorphic line bundle $L \to X$. It can also be viewed as a (real) vector bundle over M of rank 2.
- Let $D_{\mathbb{R}} L$ be the real gauge algebroid, i.e. the vector bundle whose sections are $\mathbb{R}$-linear operators $\delta : \Gamma(L) \to \Gamma(L)$ satisfying the derivation property (also called Liebniz rule).
- The holomorphic gauge algebroid $DL \to X$ is the subalgebroid of $D_{\mathbb{R}} L$ whose sections are $\mathbb{C}$-linear derivations of L.

Holomorphic case

- Notice that $D_{\mathbb{C}}L$ is a complex (**but not holomorphic**) Lie algebroid. We have $D_{\mathbb{C}}L = D^{1,0}L \oplus T^{0,1}X$.
- The holomorphic gauge algebroid $DL \to X$ is isomorphic to the complex subbundle $D^{1,0}L \subseteq D_{\mathbb{C}}L$ whose sections are derivations δ having symbols in $\mathfrak{X}^{1,0}(X) = \Gamma(T^{1,0}(TX))$.
- Identify $DL \simeq D^{1,0}L$. Then the holomorphic first jet bundle of $L \to X$ is $\mathfrak{J}^1 L = \mathrm{Hom}_{\mathbb{C}}(D^{1,0}L, L)$.
- We thus obtain a **holomorphic contact-Courant algebroid**

$$\mathbb{D}^{1,0}L = DL \oplus \mathfrak{J}^1 L.$$

Holomorphic case

These vector bundles fit in the following triangle of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{End}_{\mathbb{C}} L & \longrightarrow & D^{1,0}L & \longrightarrow & T^{1,0}X & \longrightarrow & 0 \\
 & & \| & \nearrow & \Big\downarrow \iota & \nearrow & \Big\downarrow 2\,\mathrm{Re} & & \\
0 \longrightarrow & \mathrm{End}_{\mathbb{C}} L & \longrightarrow & D_{\mathbb{C}} L & \longrightarrow & (TM)^{\mathbb{C}} & \longrightarrow & 0 & \\
 & & \| & \nwarrow & \Big\downarrow & \nwarrow & \Big\downarrow & & \\
0 & \longrightarrow & \mathrm{End}_{\mathbb{C}} L & \longrightarrow & DL & \longrightarrow & TX & \longrightarrow & 0
\end{array}
\quad .
$$

Holomorphic case

Definition: A holomorphic Dirac-Jacobi bundle is a holomorphic subbundle $\mathfrak{L}$ of $\mathbb{D}^{1,0}L \to X$ which is maximally isotropic with respect to the fiberwise L-valued inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{D}^{1,0}L$ and whose sheaf of holomorphic sections is closed under the Dorfman bracket.

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- Now, we want to look at Dirac-Jacobi bundles from the real differential geometric view point.
- Obviously any holomorphic Dirac-Jacobi bundle is a holomorphic Lie algebroid. We know that any holomorphic Lie algebroid, one can define its underlying real and imaginary Lie algebroid. We obtain:

Holomorphic case

Proposition: Given a holomorphic Dirac-Jacobi bundle $\mathfrak{L}$ over $X = (M, j)$, there are two underlying real Dirac structures $\mathfrak{L}_{\mathrm{Re}}$ and $\mathfrak{L}_{\mathrm{Im}}$ that coincide with the real part and the imaginary part of $\mathfrak{L}$ viewed as a Lie algebroid.

Sketch of the proof: We denote by $\mathfrak{J}^1_{\mathbb{R}} L$ the first (real) jet bundle of $L \to M$, and by $\mathrm{j}^1_{\mathbb{R}} : \Gamma(L) \to \Gamma(\mathfrak{J}^1_{\mathbb{R}} L)$, the first (real) jet prolongation of sections of L. In fact, $\mathfrak{J}^1_{\mathbb{R}} L$ fits in the short exact sequence of vector bundles over M:

$$0 \longrightarrow T^*M \otimes_{\mathbb{R}} L \xrightarrow{\gamma} \mathfrak{J}^1_{\mathbb{R}} L \longrightarrow L \longrightarrow 0,$$

where the second arrow is the embedding $\gamma : T^*M \otimes_{\mathbb{R}} L \to \mathfrak{J}^1_{\mathbb{R}} L$, $df \otimes s \mapsto \mathrm{j}^1_{\mathbb{R}}(fs) - f\mathrm{j}^1_{\mathbb{R}} s$, for all $f \in C^\infty(M)$, and $s \in \Gamma(L)$.We have:$\mathfrak{J}^1_{\mathbb{R}} L = \mathfrak{J}^1 L \oplus (T^{0,1}X)^* \otimes L$. The real part $\mathfrak{L}_{\mathrm{Re}}$ can be considered as a subbundle of $D_{\mathbb{R}} L \oplus \mathfrak{J}^1_{\mathbb{R}} L$.

From holomorphic Dirac-Jacobi manifolds to holomorphic Dirac manifolds

- Let $L \to X$ be a holomorphic line bundle over a complex manifold $X = (M, j)$ and let L^* be its dual.
- Denote $\widetilde{M} := L^* \setminus 0$ The real manifold $\widetilde{M}$ is equipped with a complex structure $\widetilde{j}$ induced by the complex structure on L.
- Moreover $\widetilde{X} = (\widetilde{M}, \widetilde{j})$ is a holomorphic principal bundle over X with structure group the multiplicative $\mathbb{C}^\times := \mathbb{C} \setminus 0$.
- Denote by $H \in \Gamma(T^{1,0}\widetilde{X})$ the restriction to $\widetilde{X}$ of the holomorphic Euler vector field on L^*.
- Then H is the fundamental vector field corresponding to the canonical generator 1 in $\mathbb{C}$. We have: $H = \frac{1}{2}(\eta - i\widetilde{j}\eta)$, where $\eta \in \mathfrak{X}(\widetilde{M})$ is the restriction of the real Euler vector field of the rank 2 (real)vector bundle $L^* \to M$.

From holomorphic Dirac-Jacobi manifolds to holomorphic Dirac manifolds

Proposition: A holomorphic Dirac-Jacobi structure $\mathfrak{L}\ (X, L)$ induces a canonical holomorphic Dirac structure on $\widetilde{X}$ which is homogeneous with respect to H.

Remark: There are subtile differences between holomorphic Dirac-Jacobi structures and holomorphic Dirac structures. One of the key facets is that holomorphic line bundles correspond to rank 2 (but not rank 1) real vector bundle. Thus, a direct homogenization method won't work. To deal with with this difficulty, the key step is construct an intermediary real line bundle over a manifold $\widehat{M}$ starting from a holomorphic line bundle showing that. in some sense, holomorphic Dirac-Jacobi structures offer a richer framework.

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— Thank You —