KU LEUVEN

Higher analogues of Dirac structures and embeddings

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Outline

Terminology

Fix $p \geq 1$. Consider the "higher Courant algebroid"

$$(E^p, \langle \cdot, \cdot \rangle, pr_{TM}, [\![\cdot, \cdot]\!])$$

with

$$E^p := TM \oplus \wedge^p T^*M$$
$$\langle \cdot, \cdot \rangle \colon E^p \times E^p \to \wedge^{p-1} T^*M$$
$$[\![X + \alpha, Y + \beta]\!] = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha.$$

Definition

1) $L \subset E^p$ is isotropic if

$$\langle L, L \rangle = 0.$$

2) $L \subset E^p$ is Lagrangian if

$$L = L^{\perp} := \{e \in E^p : \langle e, L \rangle = 0\}.$$

Examples: forms and distributions

Proposition

Let $\omega \in \Omega^{p+1}(M)$ and S an integrable distribution on M, such that

$$d\omega|_{\wedge^3 S \otimes \wedge^{p-1} TM} = 0.$$

Then

$$L := \{X - \iota_X \omega + \alpha : X \in S, \alpha \in \wedge^p S^\circ\}$$

is an isotropic, involutive subbundle of E^p.

When L as above is Lagrangian, the converse holds.

Example:
For $\omega \in \Omega^{p+1}(M)$: $graph(\omega)$ is involutive iff $d\omega = 0$.

Remark:
When $p \geq 2$, Lagrangian subbundles of E^p are quite rigid (get restrictions on the pointwise rank of $S := pr_{TM}(L)$.)

Examples: multivector fields

Proposition

Let $\pi \in \Gamma(\wedge^{p+1}TM)$ *be either*

- *a Poisson bivector field,*
- *a* $dim(M)$*-multivector field, or*
- $\pi = 0$.

Then

$$graph(\pi) := \{\iota_\alpha \pi + \alpha : \alpha \in \wedge^p T^* M\}$$

is an isotropic involutive subbundle of E^p.

All isotropic involutive subbundles that project isomorphically onto $\wedge^p T^* M$ are of the above form, and are Lagrangian.

Remark:
Graphs to Nambu-Poisson multivector fields are not isotropic in general.

L_∞-algebras

Definition

An L_∞-algebra is a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ endowed with a sequence of multi-brackets ($n \geq 1$)

$$l_n \colon \ \wedge^n V \to V$$

of degree $2 - n$, satisfying higher Jacobi identities (quadratic relations).

- $l_1 =: d$ makes V into a cochain complex
- l_2 does not satisfy the (graded) Jacobi identity

The binary bracket of hamiltonian forms

Let $p \geq 1$. Let L be a isotropic, involutive subbundle of $E^p = TM \oplus \wedge^p T^*M$.

Definition

1) $\alpha \in \Omega^{p-1}(M)$ is Hamiltonian if there exists a smooth vector field X_α such that

$$X_\alpha + d\alpha \in \Gamma(L).$$

2) We define a bracket $\{\cdot, \cdot\}$ on $\Omega^{p-1}_{ham}(M, L)$ by

$$\{\alpha, \beta\} := \iota_{X_\alpha} d\beta,$$

where X_α is any Hamiltonian vector field for α.

An L_∞-algebra for involutive isotropic subbundles

Theorem

There is an L_∞-algebra structure on the complex

$$C^\infty(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{p-2}(M) \xrightarrow{d} \Omega^{p-1}_{ham}(M, L).$$

The only non-vanishing multibrackets are:

- *de Rham differential* d
- $$l_k(\alpha_1, \dots, \alpha_k) = \quad \pm \iota_{X_{\alpha_k}} \dots \iota_{X_{\alpha_3}} \{\alpha_1, \alpha_2\}$$

for $\alpha_1, \dots, \alpha_k \in \Omega^{p-1}_{ham}(M, L)$ *and* $k = 2, \dots, p+1$.

Notation: $L_\infty(M, L)$, the "observables".

Variation: Replace $\Omega^{p-1}_{ham}(M, L)$ by

$$\{(X, \alpha) : X + d\alpha \in \Gamma(L)\} \subset \mathfrak{X}(M) \oplus \Omega^{p-1}_{ham}(M, L).$$

Notation: $Ham_\infty(M, L)$, the "observables with choice of hamiltonian v.f."

An L_∞-algebra for "higher Courant algebroids"

Theorem

For any manifold M, integer $p \geq 1$ and $H \in \Omega^{p+1}_{closed}(M)$, there is a canonical L_∞-algebra structure on the complex

$$C^\infty(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-2}(M) \xrightarrow{0 \oplus d} \underbrace{\chi(M) \oplus \Omega^{p-1}(M)}_{=\Gamma(E_H^{p-1})},$$

Notation: $L_\infty(M, E_H^{p-1})$.

Remark

The theorem follows from [Getzler].

The n-th multibracket involves the Bernoulli number B_{n-1}. Hence it i zero for $n = 4, 6, 8, ...$

The case of Courant algebroids

Example:
For $p = 2$ due to [Roytenberg-Weinstein].
Explicitly, the L_∞-algebra is on the complex

$$C^\infty(M) \xrightarrow{d} \Gamma(TM \oplus T^*M)$$

with:

- The binary bracket is the H-twisted Courant bracket $[\![\cdot,\cdot]\!]_{Cou,H}$ on $\Gamma(E^1)$ and
$$[e, f] := \frac{1}{2}\langle e, df\rangle$$
- The trinary bracket is
$$[e_1, e_2, e_3] := -\frac{1}{6}\left(\langle [\![e_1, e_2]\!]_{Cou,H}, e_3\rangle + c.p.\right).$$

An embedding of L_∞-algebras

Let $\omega \in \Omega^{p+1}(M)$ be closed.

Expected result

There is a canonical L_∞-embedding

$$Ham_\infty(M,\omega) \to L_\infty(E^{p-1},\omega),$$

whose first component is the inclusion.

Remark:

For ω non-degenerate this was proven

- $p = 2$: by [Rogers]
- $p \leq 5$: by [Miti-Zambon]
- for all p: unpublished preprint [Ritter-Sämann] (no explicit formulae)

The symplectic case ($p = 1$)

Example:

Let ω be a symplectic form on M.
Then $Ham_\infty(M,\omega) = L_\infty(M,\omega) = C^\infty(M,\omega)$.
Have Lie algebra embedding

$$C^\infty(M,\omega) \to \Gamma(TM \oplus \mathbb{R})_\omega, \quad f \mapsto (X_f, f). \tag{1}$$

Geometric interpretation: prequantization

Assume $\frac{1}{2\pi}[\omega] \in H^2(M,\mathbb{Z})$ is integral.
Have S^1-bundle $\pi\colon P \to M$. Using a connection θ s.t. $d\theta = \pi^*\omega$, get

- Lie algebra embedding

$$C^\infty(M,\omega) \to \mathfrak{X}(P)^{S^1}, \quad f \mapsto X_f^{\text{lift}} + \pi^* f \cdot E.$$

- Isomorphism between sections of Lie algebroids

$$\mathfrak{X}(P)^{S^1} = \Gamma(TP/S^1) \underset{\theta}{\cong} \Gamma(TM \oplus \mathbb{R})_\omega.$$

The composition is (1), and is independent of the choice of θ.

References

A. Miti and M. Zambon
Observables on multsymplectic manifolds and higher Courant algebroids
In progress

C.L. Rogers
L_∞-algebras from multisymplectic geometry
Lett. Math. Phys. **100** (2012), 29–50

M. Zambon
L_∞-algebras and higher analogues of Dirac structures and Courant algebroids
J. Symplectic Geom. **10** (2012), 563–599

Thank you for your attention