3.5 First Mathematical Approaches to Elections

3.5.1 Ramon Llull's Election Method Reinvented 500 Years Later by Condorcet

The first medieval scholar known to consider elections from the mathematical standpoint is Ramon Llull (1232–1315); for his electoral texts and comments see [Hägele and Pukelsheim 2001] and [McLean and Urken 1995]. Having been brought up at the royal court of Majorca, at the frontier of Christian and Islamic worlds, he learned Catalan, Arabic and Latin, and was much inspired by Arabian mathematics, particularly combinatorics. In 1265 he had visions of Christ on the Cross who called him to write 'the best book in the world, against the errors of unbelievers' [Llull 1311, p. 15]. After this experience Llull turned to missionary work in North Africa and Asia Minor, having continued his intensive studies, and according to his principle: 'Without producing, no man can love, nor can he understand or remember, nor have the power of feeling and being.'

Llull hoped to develop a universal logical tool to persuade unbelievers through combinatorial (pairwise) operations on common statements from different religions.[5] His lifework, *Ars magna (Great Art)*, appeared in 1305–1308 when he was already over 70. It comprises numerous treatises as contributions to the *Ars generalis (General Skills)*, a code of thinking rules involving symbolic notation and combinatorial diagrams to relate all forms of knowledge, a kind of universal computer[6] [Bonner 1997]. In total, Llull left about 300 texts, including *Blanquerna*, the first novel in Catalan — indeed, in any European language at all. In 1376, some 60 years after Llull's death, Pope Gregory XI charged him with confusing faith with reason and condemned his teachings; there are however suspicions that the Pope's bull was faked by Llull's enemies [Llull 2019 (RUS)]. Llull was beatified in 1857 by Pope Pius IX [Llull 2019].

[5] Four centuries later Leibniz (1646–1716) in his *Mathesis universalis (Art of Discovery)*, 1685, expressed similar ideas for creating a scientific language, which would permit any two disputants to settle their differences by taking pencil and paper and saying, 'Let us calculate' [Leibniz 2019].

[6] Similar attempts were known throughout the Middle Ages, including a 'music computer', Musical-Rhythmical Ark, designed by Athanasius Kircher (1601/2–1680) in *Musurgia universalis (Universal Music)*, see [Kircher 1650, Kaul 1932]; but none of them was as extreme and ambitious as Llull's endeavor.

Llull's approach to elections is presented in at least three of his works, including the novel [Llull 1283, *Blanquerna*]. In short, the subject of the novel is as follows. Blanquerna is a young man of 18 who wants to be a hermit. His mother sets her friend's beautiful daughter Natana on him in the hope of persuading him to stay. Instead, he influences Natana; she enters a nunnery and rises up to the position of abbess. Blanquerna becomes successively a monk, an abbot, a bishop, and finally, the Pope. Then he renounces everything and becomes a hermit [Llull 1283].

In the course of the novel, Llull shows scenes of Christian life and inserts practical passages. As Blanquerna matures, he listens to the advice of a 'wise fool' named Ramon and learns from him 'the Art' (of Llull himself; Llull thereby promotes his own ideas). In particular, as Natana and Blanquerna are elected to abbess and bishop, respectively, a voting procedure according to 'the Art' is described in detail.

The procedure is two-stage. At first, an electoral board is determined. Here Llull compromises between democracy and competent choice by restricting the electors to the most qualified members of the collective. Next, candidates are voted on in pairs in all combinations, with the results recorded into 'cells'. For instance, while considering nine candidates, Llull mentions $\binom{9}{2} = 36$ pairs. For each candidate, the pairs with the given candidate are considered, and the one 'who has the most votes in the most cells' is elected:

> Let us suppose that this number [of candidates] consists of 9 persons from whom our pastor is to be elected. Firstly, it is necessary that the 7 [electors; all electors are among the candidates] are divided into two groups: 2 on the one side and 5 on the other; and then it is necessary that the 5 decide who of the 2 should be elected; and in secrecy let it be written down who has the most votes. Afterwards, the 1 who has the most votes should be compared to another one of the 5, and that this sister be set in the place of the one who is defeated by fewer votes; and the one who is defeated is put in the place of the sister who is now compared with the first or with the second. And let this be done, in order, with all the others; and to this number are added the 8th and 9th persons who are not among the electors. Following this numbering, 36 cells are generated in which appear the votes of each; and let her be elected who has the most votes in the most cells.
> [Llull 1283, *Blanquerna*, Chapter 24]

[McLean and Urken 1995, p. 18] are not sure of how to interpret the phrase 'who has the most votes in the most cells'. We believe it means: 'who has a majority in the most cells'. Indeed, in the middle of the quote one

finds 'who has the most votes' in application to a single pairwise contest where it clearly means 'a majority of votes'. Moreover, in the earlier article [Llull 1274–1283, *Ars notandi (Notorious Art)*] the same method is described, and 'the most votes' is used there to mean 'a majority' as well.

Thus, Llull's rule prescribes selecting the one who has the most victories in pairwise contests. This is equivalent to the Condorcet method (see Section 4.8) in the modification by [Copeland 1951]; see [Fishburn 1973] and for a simplified description [Tangian 1991, pp. 18–19]. We remind that the Copeland method is based on indexing the candidates by the difference between victories and losses in pairwise contests. Since the sum of victories and losses is always the same (the number of candidates minus one), these indices can be replaced by the number of victories.

In case of ties Llull proposes to consider the tied candidates only and to apply the procedure anew, which is however not a universal way to break ties:

> When Natana had explained the electoral system, one of the sisters asked her: 'If it turns out that in the cells there are some who have equal votes, in which manner does the system proceed?' Natana replied: 'The system demands that among these 2 or 3 or more one decides solely by means of the system; and that [in this way] one decides who of them combines best the 4 conditions mentioned above. And she who combines the conditions best is worthy to be elected.'
> [Llull 1283, *Blanquerna*, Chapter 24]

In the article [Llull 1274–1283, *Ars notandi (Notorious Art)*] the problem of ties is first treated in this way but afterwards resolved by drawing lots:

> If, however, it happens that two persons or more have an equal number of votes, it is necessary that they leave the hall and that the other remaining persons, no matter how many votes they have, again take an oath to select, while observing the above-mentioned three things, the one who is more worthy and suitable for this dignity, and that the one for whom more votes are counted will be elected. If now it happens [yet again] that as many votes are counted for one [person] as for another, lots are thrown over those who had an equal number of votes in the last election, and the one whose lot wins is elected.
> [Llull 1274–1283, *Ars notandi (Notorious Art)*]

Llull's last work on elections is the two-page article *Ars eleccionis*. It was drafted on July 1, 1299, in Paris where Llull tried to teach his General Art to the students of the University of Paris. In his autobiography Llull complains that nobody understood him as he lectured because of his 'Arabic way of speaking' [McLean and Urken 1995, p. 17].

Table 3.1 Election matrix from [Llull 1299, *Ars eleccionis (Electoral Art)*]

bc	*cd*	*de*	*ef*	*fg*	*gh*	*hi*	*ik*
bd	*ce*	*df*	*eg*	*fh*	*gi*	*hk*	
be	*cf*	*dg*	*eh*	*fi*	*gk*		
bf	*cg*	*dh*	*el*	*fk*			
bg	*ch*	*di*	*ek*				
bh	*ci*	*dk*					
bi	*ck*						
bk							

Source: [Hägele and Pukelsheim 2001]

The article presents a method of election which differs from the one in *Blanquerna*. It is again based on Condorcet pairwise comparisons of candidates $b, c, d, \ldots$ which are illustrated by 'cells' of a triangular tournament matrix labeled by candidate pairs $bc, bd, \ldots$ as shown in Table 3.1. However, this method, unlike the one from *Blanquerna*, runs in elimination rounds (knock-out rounds): the first candidate b is compared with others as long as he gains victories, while the losers are all eliminated; as b is himself defeated he is eliminated and replaced by the victor, for instance, d who is contested with the remaining candidates as long he is not defeated, and so on. Llull explains that, in this way, one finds the true winner.

The method itself is again based on voting on all candidates in pairs and counting the victories of each candidate, as in the Condorcet method. For accounting of votes and the pairwise victories of the candidates, Llull uses score tables similar to tournament tables, which are in fact matrices or their triangular halves. He explains the operations necessary to find the winner. Here, however, neither cycles nor tied outcomes are considered. Llull might think that his assumption of odd numbers of electors excludes such cases, or, perhaps he just simplified his earlier method for the Parisian students, mainly promoting his key idea — deriving knowledge from pairwise combinations.

[Hägele and Pukelsheim 2001] bridge the gaps and ambiguities in Llull's methods, referring to Llull's religious position:

> As a devoted Christian in his time, Llull's concern was not the aggregation of many individual truths, but the discovery of the one and only truth existing, the truth of God. We understand Llull's electoral systems as manifestations of his Ars Generalis, and as such they are means to set mankind on a trail leading to the unique, divine truth: 'By this method,' said Natana, 'is found the truth; by this truth we will be able to find the sister who is most suitable and best to be our abbess'.

> ...
> Formally, the exhaustive comparisons of [all Llull's election methods] are different systems, possibly leading to different results. Under mild hypotheses, however, the differences vanish and the final conclusions coincide. For an illustration, let us agree on the following two hypotheses,
>
> (i) that the truth set by God is unique, and
> (ii) that, in any decision instance, it discloses itself to at least half of the electorate.
>
> Llull would have emphatically supported our assumptions, or so we believe. Under hypotheses (i) and (ii), the electoral systems of [all the methods] lead to the same result. [Formal proof follows]
> [Hägele and Pukelsheim 2001, *Llull's Writings on Electoral Systems*]

These arguments suggest that all Llull's methods are versions of the Condorcet count in a broad sense. The account of non-victorious votes does not meet the spirit of a single divine truth which manifests itself in the number of victories of some candidates over others:

> The truth emerges on the side of the victor no matter how easy or how hard he had to fight. This way of arriving at a decision is concordant with Llull's belief in the uniqueness of the divine truth: among two candidates, one must be more worthy than the other.
> [Hägele and Pukelsheim 2001, *Llull's Writings on Electoral Systems*]

The remarkable peculiarities of Llull's electoral works are an advanced use of pairwise comparisons, reinvented by Condorcet in the 1785, and matrix notation, believed to be introduced in this context six centuries later by Dodgson in 1873. Another interesting feature is a warning against strategic voting (declaring false preferences to eliminate the strongest competitors to favorite candidates): 'First, let all voters take an oath that they will elect the better and more suitable candidate' [Llull 1299, p. 73].

Llull's studies exhibit his admiration for combinatorial formulas which he learned from Arabian mathematics; it is also possible that he got his ideas on elections from the Arab world. He believed that the *Ars generalis* could produce all the truth of the holy Catholic faith by combining virtues in pairs. The modern state of knowledge shows how far these hopes were from reality, justifying Llull's reputation as 'one of the most inspired madmen who ever lived' [McLean and Urken 1995, p. 19]. Nevertheless Llull's insightful application of combinatorics to elections was absolutely relevant.