Simplicity simplified*

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Abstract

We present the fundamentals of simplicity theory starting from general model theory and finishing in the proof of the Independence Theorem.

1 Introduction

Simple theories are a natural and fruitful generalization of stable theories. They were introduced by Shelah in [12] and rediscovered and fully developed by Kim and Pillay in [7], [6] and [8]. See [9] for historical background. The book [13] of Wagner is now the main reference. In the last five years the foundations of simple theories have been simplified after some new ideas from Shami, Shelah and Buechler and Lessmann. Here we add further simplifications from [3] and systematize all the basic results up to the Independence Theorem and type-definability of equality of Lascar strong types. The references contain the sources for all the results. We do not give credits for every single result but we give complete proofs of everything without previous assumptions except the usual ones in Model Theory.

We use the standard notation and conventions. Everywhere T is a complete theory with infinite models in a first-order language and $\mathfrak{C}$ is its monster model.

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2 Lascar strong types

Definition 1 *A relation R is bounded if for some cardinal κ there is not a sequence $(a_i : i < \kappa)$ such that $\neg R(a_i, a_j)$ for all $i < j < \kappa$. The relation is finite if this*

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bound κ is in fact a natural number. Observe that for definable relations finiteness is equivalent to boundedness.

Remark 2.1 *Any intersection of a bounded number of bounded relations is a bounded relation.*

Proof. Let $(R_l : l < \lambda)$ be a sequence of bounded relations. For all $l < \lambda$ let κ_l be a bound for R_l and let $\kappa = \lambda + \sup\{\kappa_l : l < \lambda\}$. Assume that there are $(a_i : i < (2^\kappa)^+)$ such that$\neg R(a_i, a_j)$ for all $i < j < (2^\kappa)^+$, where $R = \bigcap_{l<\lambda} R_l$. By Erdös-Rado for some $l < \lambda$ there is a subset $I \subseteq (2^\kappa)^+$ of cardinality κ^+ such that $\neg R_l(a_i, a_j)$ for all $i < j$ in I. This contradicts the choice of κ_l. $\square$

Definition 2 *An equivalence relation is A-invariant if its classes are preserved under automorphisms of the monster model fixing A pointwise. Since every A-invariant equivalence relation is definable by a disjunction (maybe infinite) of types over A, there is a bounded number of them. Therefore the intersection of all A-invariant bounded equivalence relations is again an A-invariant bounded equivalence relation. We say that the sequences a, b have the same Lascar strong type over A and we write* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *if a and b are equivalent in this least A-invariant bounded equivalence relation.*

Definition 3 *Let x, y be finite tuples of variables of the same length. We say that the formula $\theta(x, y)$ is thick if it defines a reflexive and symmetric relation which is finite. For any set A and for any sequences of variables x, y of the same length, the set of all thick formulas over A in (finite subtuples of) the variables x, y will be* $\mathrm{nc}_A(x, y)$. *For every natural number n,* $\mathrm{nc}^n_A(x, y)$ *is the type* $\exists y_1 \dots y_{n-1}(\mathrm{nc}_A(x, y_1) \wedge \mathrm{nc}_A(y_1, y_2) \wedge \dots \wedge \mathrm{nc}(y_{n-1}, y))$.

Remarks 2.2
1. *The conjunction and the disjunction of thick formulas are thick formulas.*
2. *Any consequence of a thick formula is a finite formula.*
3. *If $\varphi(x, y)$ is finite then, $x = y \vee (\varphi(x, y) \wedge \varphi(y, x))$ is thick.*

Lemma 2.3 *For any $a \neq b$,* $\models \mathrm{nc}_A(a, b)$ *if and only if a, b start an infinite A-indiscernible sequence.*

Proof. If a, b start an infinite A-indiscernible sequence, then it is clear that $\models \theta(a, b)$ for any finite formula $\theta(x, y)$ over A. Now assume $\models \mathrm{nc}_A(a, b)$. Me may also assume that $a \neq b$. Let $p(x, y) = \mathrm{tp}(ab/A)$. By Ramsey's Theorem and compactness, to prove that a, b start an infinite A-indiscernible sequence it is enough to check that there is an infinite sequence $(a_i : i < \omega)$ such that $\models p(a_i, a_j)$ for all $i < j < \omega$. For this we have to prove for any $\varphi \in p$, the consistency of $\{x_i \neq x_j : i < j <$

$\omega\} \cup \{\varphi(x_i, x_j) : i < j < \omega\}$. In case a, b are infinite sequences we choose a coordinate where they are different to express inequality. If this set of formulas is inconsistent, then $\neg\varphi(x, y)$ is finite and therefore $(x = y \vee (\neg\varphi(x, y) \wedge \neg\varphi(y, x)) \in \mathrm{nc}_A(x, y)$. Hence $\models \neg\varphi(a, b)$, a contradiction. □

Proposition 2.4 *Equality of Lascar strong types over A is the transitive closure of the relation of starting an A-indiscernible sequence. Hence it is defined by the infinite disjunction $\bigvee_n \mathrm{nc}_A^n(x, y)$.*

Proof. Let E be this transitive closure. It is an A-invariant equivalence relation. Since the relation of starting an infinite indiscernible sequence is defined by a set of finite formulas, it is bounded. Hence its transitive closure E is also bounded. From this it follows that equality of Lascar strong types is contained in E. For the other direction it suffices to show that if a, b start an infinite A-indiscernible sequence then $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$. Let κ be a bound for the number of classes in the relation of equality of Lascar strong types over A. We can choose an A-indiscernible set starting with a, b of length bigger than κ. If $\mathrm{Lstp}(a/A) \neq \mathrm{Lstp}(b/A)$ then by A-invariance $\mathrm{Lstp}(a'/A) \neq \mathrm{Lstp}(b'/A)$ for any two different a', b' in the indiscernible sequence, which contradicts the choice of κ. □

Lemma 2.5

1. *If $\mathrm{nc}_A(a, b)$, then there is a model $M \supseteq A$ such that $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$.*
2. *If for some model $M \supseteq A$ $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$, then $\mathrm{nc}_A^2(a, b)$.*

Proof. For *1* fix a model $M \supseteq A$ and an infinite A-indiscernible sequence I starting with a, b. By Ramsey's Theorem and compactness we can obtain another infinite A-indiscernible sequence $(a_i : i < \omega)$ where $\mathrm{tp}(ab/A) = \mathrm{tp}(a_0 a_1/A)$ and $\mathrm{tp}(a_i/M) = \mathrm{tp}(a_j/M)$ for all $i < j < \omega$. This implies that $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$. For *2* we assume that a, b have the same type over some model $M \supseteq A$ and we show that for any thick formula $\theta(x, y)$ over A, $\models \exists z(\theta(a, z) \wedge \theta(b, z))$. Let n be the maximal length of a sequence $a_1, \ldots, a_n$ such that $\models \neg\theta(a_i, a_j)$ for all $i < j \leq n$. We can find such $a_1, \ldots, a_n$ in M. For some $i, j \leq n$, $\models \theta(a, a_i)$ and $\models \theta(b, a_j)$. Since a, b have the same type over M we may take $i = j$. □

Proposition 2.6 *Equality of Lascar strong types over A is the transitive closure of the relation of having the same type over a model containing A.*

Proof. Clear by Proposition 2.4 and Lemma 2.5. □

Definition 4 *The group* $\mathrm{Autf}(\mathfrak{C}/A)$ *of* strong automorphisms over A *of the monster model* $\mathfrak{C}$ *is the subgroup of* $\mathrm{Aut}(\mathfrak{C}/A)$ *generated by the automorphisms fixing a small submodel containing A.*

Corollary 2.7 $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *if and only if* $f(a) = b$ *for some* $f \in \mathrm{Autf}(\mathfrak{C}/A)$.

Proof. It follows from Proposition 2.6. $\square$

Corollary 2.8 *If* $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ *then for any* c *there is some* d *such that* $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd/A)$

Proof. Choose $f \in \mathrm{Autf}(\mathfrak{C}/A)$ such that $f(a) = b$ and put $d = f(c)$. $\square$

3 Dividing and forking

Definition 5 *The formula* $\varphi(x, a)$ *divides over the set* A *with respect to* k *(*$2 \leq k < \omega$*) if there is an infinite sequence* $I = (a_i : i < \omega)$ *of realizations of* $\mathrm{tp}(a/A)$ *such that* $\{\varphi(x, a_i) : i < \omega\}$ *is* k*-inconsistent, i.e, every subset of* k *elements is inconsistent. We may always assume that* I *is* A*-indiscernible. We may also assume that* $a = a_0$*. Finally, in place of* ω *we may choose any infinite linear ordering. The formula* $\varphi(x, a)$ *divides over* A *if it divides over* A *with respect to some* k.

Remarks 3.1

1. *If* $\varphi(x, a)$ *divides over* A *with respect to* k *and* $\psi(x, b) \vdash \varphi(x, a)$*, then* $\psi(x, b)$ *divides over* A *with respect to* k *too.*
2. *If* $a \in \mathrm{acl}(A)$*, then* $\varphi(x, a)$ *does not divide over* A.

Definition 6 *The set of formulas* $\pi(x, a)$ *divides over the set* A *if it implies a formula* $\varphi(x, b)$ *which divides over* A*. We may always assume that* $b = a$ *and that* $\varphi(x, y)$ *is a conjunction of formulas in* $\pi(x, y)$.

Remarks 3.2

1. *If* $\pi(x, a)$ *is inconsistent, it divides over* A.
2. $\pi(x, a)$ *divides over* A *iff for some infinite* A*-indiscernible sequence* $(a_i : i < \omega)$ *with* $a_0 = a$*, the set of formulas* $\bigcup_{i<\omega} \pi(x, a_i)$ *is inconsistent.*
3. $\varphi(x, a)$ *divides over* A *iff the set* $\{\varphi(x, a)\}$ *divides over* A.
4. $\mathrm{acl}(A) = \{a : \mathrm{tp}(a/Aa)$ *does not divide over* $A\}$

Definition 7 *The set of formulas* $\pi(x, a)$ *forks over* A *if for some* n *there are formulas* $\varphi_1(x, a_1), \ldots, \varphi_n(x, a_n)$ *such that* $\pi(x, a) \vdash \varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ *and every* $\varphi_i(x, a_i)$ *divides over* A*. The formula* $\varphi(x, a)$ *forks over* A *if the set* $\{\varphi(x, a)\}$ *forks over* A.

The advantage of forking over dividing lies in that we can make sure that non-forking types can be extended while it is not clear whether we can do it in the case of dividing.

Remarks 3.3 *1. If $\pi(x,a)$ divides over A, then it forks over A.*

2. If $\pi(x,a)$ is finitely satisfiable in A, then it does not fork over A.

3. $\pi(x,a)$ forks over A iff a conjunction of formulas in $\pi(x,a)$ forks over A.

4. If $\pi(x,a)$ does not fork over A, then it can be extended to a complete type over a which does not fork over A. Any complete type over a extending the partial type $\pi(x,a) \cup \{\neg\varphi(x,a) : \varphi(x,a)$ forks over $A\}$ does the job.

Next lemma turns out to be very useful. From it we can prove a result on pairs which anticipates some version of transitivity for nondividing (Proposition 3.5).

Lemma 3.4 *Those following are equivalent.*

1. $\mathrm{tp}(a/Ab)$ *does not divide over A.*
2. *For every infinite A-indiscernible sequence I such that $b \in I$, there is $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible.*
3. *For every infinite A-indiscernible sequence I such that $b \in I$, there is $J \equiv_{Ab} I$ such that J is Aa-indiscernible.*

Proof. The equivalence of *2* and *3* follows by conjugation. It is clear that *3* implies *1*. We prove that *1* implies *2*. We may assume that A is empty, that $I = (b_i : i < \omega)$ and that $b = b_0$. Let $p(x,b) = \mathrm{tp}(a/b)$ and let $\Gamma(x,(x_i : i < \omega))$ be a set of formulas expressing that $(x_i : i < \omega)$ is x-indiscernible. It will be enough to prove that $p(x,b) \cup \Gamma(x,(b_i : i < \omega))$ is consistent. By *1* $q(x) = \bigcup_{i<\omega} p(x,b_i)$ is consistent. Let $c \models q$ and let Γ_0 a finite subset of Γ. By Ramsey's Theorem, there is an order preserving $f : \omega \to \omega$ such that $\models \Gamma_0(c,(b_{f(i)} : i < \omega))$. By indiscernibility $(b_i : i < \omega) \equiv (b_{f(i)} : i < \omega)$ and therefore we can find c' such that $c'(b_i : i < \omega) \equiv c(b_{f(i)} : i < \omega)$. Clearly $c' \models q(x) \cup \Gamma_0(x,(b_i : i < \omega))$. □

Proposition 3.5 *If $\mathrm{tp}(a/B)$ does not divide over $A \subseteq B$ and $\mathrm{tp}(b/Ba)$ does not divide over Aa, then $\mathrm{tp}(ab/B)$ does not divide over A.*

Proof. It is an easy application of Lemma 3.4. □

Proposition 3.6 *If $\varphi(x,a)$ divides over A with respect to k and $\mathrm{tp}(b/Aa)$ does not divide over A, then $\varphi(x,a)$ divides over Ab with respect to k.*

Proof. Let $I = (a_i : i < \omega)$ be an infinite A-indiscernible sequence such that $a = a_0$ and $\{\varphi(x,a_i) : i < \omega\}$ is k-inconsistent. By Lemma 3.4 there is $J \equiv_{Aa} I$ which is Ab-indiscernible. Then J witnesses that $\varphi(x,a)$ divides over Ab with respect to k. □

4 The tree property and simplicity

This section can be skipped if one chooses to define simple theories as the theories where every complete type does not fork over a subset of cardinality at most $|T|$ of its domain (Proposition 4.10). With this definition it is straightforward that a type does not fork over its domain. Here we present the proofs of the equivalence of this definition with some others, like not having the tree property, types not dividing over a small subset of its domain or finiteness of rank $D(p, \varphi, k)$.

Definition 8 *$\varphi(x, y)$ has the tree property with respect to k ($2 \leq k < \omega$) if there is a tree $(a_s : s \in \omega^{<\omega})$ such that for all $\eta \in \omega^\omega$, the branch $\{\varphi(x, a_{\eta \restriction n}) : n < \omega\}$ is consistent and for all $s \in \omega^{<\omega}$ the set $\{\varphi(x, a_{s^\frown i}) : i < \omega\}$ is k-inconsistent. By compactness it is easy to obtain a corresponding tree $(a_s : s \in \kappa^{<\lambda})$ for any cardinals κ, λ.*

Lemma 4.1 *Let α be an ordinal number, $(\varphi_i(x, y_i) : i < \alpha)$ a sequence of formulas and $(k_i : i < \alpha)$ a sequence of natural numbers $k_i \geq 2$. The following are equivalent.*

1. *There is a tree $(a_s : s \in \omega^{<\alpha})$ such that for all $\eta \in \omega^\alpha$, the branch $\{\varphi_i(x, a_{\eta \restriction i+1}) : i < \alpha\}$ is consistent and for all $i < \alpha$ and $s \in \omega^i$, the set $\{\varphi_i(x, a_{s^\frown j}) : j < \omega\}$ is k_i-inconsistent.*

2. *There is a sequence $(a_i : i < \alpha)$ such that $\{\varphi_i(x, a_i) : i < \alpha\}$ is consistent and for every $i < \alpha$, $\varphi_i(x, a_i)$ divides over $\{a_j : j < i\}$ with respect to k_i.*

Moreover in 1 we can add that all the branches $(a_{\eta \restriction i} : i < \alpha)$ have the same type.

Proof. Assume first that the tree is given. By compactness we may obtain a corresponding tree $(a_s : s \in \lambda^{<\alpha})$ for a very big cardinal λ. By induction on $\beta \leq \alpha$ we can show then that there is such a tree with the additional property that for all $i < \beta$ all $\eta \in \lambda^i$ and all $j, l < \lambda$, $\mathrm{tp}(a_{\eta^\frown j}/(a_{\eta \restriction h} : h \leq i)) = \mathrm{tp}(a_{\eta^\frown l}/(a_{\eta \restriction h} : h \leq i))$. Apply this with $\beta = \alpha$. Any branch in the resulting tree is a dividing sequence as required. For the other direction, fix the dividing sequence $(\varphi_i(x, a_i) : i < \alpha)$ and construct the tree inductively with the additional property that every branch is isomorphic to the initial segment of the chain of the same level. It is enough to show how to extend a given branch and it is clear how to do it using the fact that the formulas in the sequence always divide. Observe that the parameters a_s in the tree play no role at all if $s \in \omega^i$ and i is either 0 or a limit ordinal. $\square$

Definition 9 *A dividing chain for $\varphi(x, y)$ is a sequence $(a_i : i < \alpha)$ such that $\{\varphi(x, a_i) : i < \alpha\}$ is consistent and for every $i < \alpha$, $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$. If it divides with respect to k_i, we say that it is a dividing chain with respect to $(k_i : i < \alpha)$. We say that $\varphi(x, y)$ divides α times (with respect to $(k_i : i < \alpha)$) if there is a dividing chain of length α for $\varphi(x, y)$ (with respect to $(k_i : i < \alpha)$).*

Remarks 4.2 *1. $\varphi(x, y)$ divides ω times with respect to k iff it has the tree property with respect to k.*

2. If $\varphi(x, y)$ divides n times with respect to k for every $n < \omega$, then it divides α times with respect to k for every ordinal α.

3. If $\varphi(x, y)$ divides ω_1 times, then for some $k < \omega$, $\varphi(x, y)$ divides ω times with respect to k.

Definition 10 *T is simple if in T there are no formulas with the tree property. This is clearly equivalent to the non existence of formulas which divide ω times with respect to some k and also to the non existence of formulas which divide ω_1 times.*

Proposition 4.3 *The following conditions are equivalent to the simplicity of T.*

1. *For every type $p \in S(A)$ in finitely many variables there is a $B \subseteq A$ such that $|B| \leq |T|$ and p does not divide over B.*
2. *There is some cardinal κ such that for every type $p \in S(A)$ in finitely many variables there is a $B \subseteq A$ such that $|B| \leq \kappa$ and p does not divide over B.*
3. *There is no increasing chain $(p_i(x) : i < |T|^+)$ of types $p_i(x) \in S(A_i)$ in finitely many variables such that for every $i < |T|^+$, p_{i+1} divides over A_i.*
4. *For some cardinal κ there is no increasing chain $(p_i(x) : i < \kappa)$ of types $p_i(x) \in S(A_i)$ in finitely many variables such that for every $i < \kappa$, p_{i+1} divides over A_i.*

Proof. Simplicity implies *1*, since if $p \in S(A)$ divides over every subset of A of cardinality $\leq |T|$, then we can inductively construct a sequence of formulas $(\varphi_i(x, y_i) : i < |T|^+)$ and a sequence $(a_i : i < |T|^+)$ of parameters $a_i \in A$ such that $\varphi_i(x, a_i) \in p$ and $\varphi(x, a_i)$ divides over $\{a_j : j < i\}$. Clearly one formula $\varphi(x, y)$ appears ω_1 times in the sequence and this contradicts simplicity. It is clear that *1* implies *2* and that *3* implies *4*. To show that *1* implies *3*, observe that if the increasing chain $(p_i(x) : i < |T|^+)$ is given and we set $A = \bigcup A_i$ and $p = \bigcup p_i$, then $p(x) \in S(A)$ divides over every subset of A of cardinality $\leq |T|$. The same argument proves *4* from *2*. It remains only to show simplicity from *4*. If T is not simple, then some formula $\varphi(x, y)$ divides κ times. Let $(a_i : i < \kappa)$ a witness of this. Let a be a realization of $\{\varphi(x, a_i) : i < \kappa\}$, let $A_i = \{a_j : j < i\}$ and let $p_i = \mathrm{tp}(a/A_i)$. The chain $(p_i : i < \kappa)$ contradicts point *4*. □

Definition 11 *Let $\Delta = \{\varphi_i(x, y_i) : i = 1, \ldots, n\}$ and let $2 \leq k < \omega$. For any partial type $\pi(x, a)$ we define the rank $D(\pi(x, a), \Delta, k)$ as the supremum (an ordinal or ∞) of the ordinals α such that $D(\pi(x, a), \Delta, k) \geq \alpha$ according to the following recursive definition. If we want we can agree that inconsistent sets have rank -1.*

1. $D(\pi(x,a),\Delta,k) \geq 0$ *iff* $\pi(x,a)$ *is consistent.*

2. $D(\pi(x,a),\Delta,k) \geq \alpha+1$ *iff for some* $\varphi(x,y) \in \Delta$ *there is a sequence* $(a_i : i<\omega)$ *such that* $\{\varphi(x,a_i) : i<\omega\}$ *is* k*-inconsistent and for every* $i<\omega$, $D(\pi(x,a)\cup\{\varphi(x,a_i)\},\Delta,k) \geq \alpha$.

3. *For limit* β, $D(\pi(x,a),\Delta,k) \geq \beta$ *iff* $D(\pi(x,a),\Delta,k) \geq \alpha$ *for all* $\alpha<\beta$.

Remarks 4.4 1. *If* $\pi(x,a) \vdash \pi'(x,a')$, $\Delta \subseteq \Delta'$ *and* $k \leq k'$, *then* $D(\pi(x,a),\Delta,k) \leq D(\pi'(x,a'),\Delta',k')$

2. *If* $\pi(x,a)$, $\sigma(x,b)$ *are equivalent partial types, then* $D(\pi(x,a),\Delta,k) = D(\sigma(x,b),\Delta,k)$.

Proof. *2* follows from *1* and to prove *1* one shows by induction on α that

$$D(\pi(x,a),\Delta,k) \geq \alpha \Rightarrow D(\pi'(x,a'),\Delta',k') \geq \alpha.$$

□

Lemma 4.5 $D(\pi(x,a),\Delta,k) \geq n$ *iff there is a sequence* $(\varphi_i(x,y_i) : i<n)$ *of formulas in* Δ *and parameters* $(a_i : i<n)$ *such that for each* $i<n$, $\pi(x,a) \cup \{\varphi_i(x,a_i) : i<n\}$ *is consistent and for all* $i<n$, $\varphi_i(x,a_i)$ *divides over* $a\cup\{a_j : j<i\}$ *with respect to* k.

Proof. By Lemma 4.1. □

Proposition 4.6 T *is simple iff for all finite* Δ *and all* k, $D(x=x,\Delta,k)<\omega$.

Proof. By Lemma 4.5 it is clear that it holds if we state for sets Δ consisting in only one formula. The general case can be established by a standard coding of Δ-types in φ-types or by noticing that $D(x=x,\Delta,k) \leq \sum_{\varphi\in\Delta} D(x=x,\varphi,k)$. □

Remark 4.7 *For all* $\pi(x,a)$, Δ, k *and* $(\varphi(x,a_i) : 1\leq i\leq n)$,

$$D(\pi(x,a)\cup\{\bigvee_{i=1}^{n}\varphi(x,a_i)\},\Delta,k) = \max_{1\leq i\leq n} D(\pi(x,a)\cup\{\varphi(x,a_i)\},\Delta,k)$$

Proof. Let $\alpha = D(\pi(x,a)\cup\{\bigvee_{i=1}^{n}\varphi(x,a_i)\},\Delta,k)$ and let $\alpha_i = D(\pi(x,a)\cup\{\varphi(x,a_i)\},\Delta,k)$. By Remarks 4.4 $\alpha \geq \max_{1\leq i\leq n}\alpha_i$. By Lemma 4.5 if $\alpha \geq n$ then for some i, $\alpha_i \geq n$. Hence $\max_{1\leq i\leq n}\alpha_i \geq \alpha$. □

Lemma 4.8 *Let* $\Delta = \{\varphi_1(x,y_1),\ldots,\varphi_n(x,y_n)\}$, $D(\pi(x,a)\restriction A,\Delta,k)<\omega$ *and* $\pi(x,a) \vdash \varphi_1(x,a_1)\vee\ldots\vee\varphi_n(x,a_n)$ *where every* $\varphi(x,a_i)$ *divides over* A *with respect to* k. *Then* $D(\pi(x,a),\Delta,k) < D(\pi(x,a)\restriction A,\Delta,k)$.

Proof. By Remarks 4.7 and 4.4, $D(\pi(x,a),\Delta,k) \leq D(\pi(x,a) \upharpoonright A \cup \{\varphi_i(x,a_i)\},\Delta,k)$ for some i. Let $m = D(\pi(x,a) \upharpoonright A \cup \{\varphi_i(x,a_i)\},\Delta,k)$. By Lemma 4.5 there is a sequence $(\psi_j(x,z_j) : j < m)$ of formulas in Δ and a sequence $(b_j : j < m)$ such that $\pi(x,a) \upharpoonright A \cup \{\varphi_i(x,a_i)\} \cup \{\psi_j(x,b_j) : j < m\}$ is consistent and every $\psi_j(x,b_j)$ divides over $A \cup \{a_i\} \cup \{a_l : l < j\}$ with respect to k for all $j < m$. Again by Lemma 4.5, the sequence $\varphi_i(x,a_i), \psi_0(x,b_0), \ldots, \psi_{m-1}(x,b_{m-1})$ witnesses that $D(\pi(x,a) \upharpoonright A,\Delta,k) \geq m+1$. □

Proposition 4.9 *Simplicity is also equivalent to the conditions in Proposition 4.3 if we replace forking for dividing.*

Proof. Point *4* from Proposition 4.3 stated for forking implies its original version with dividing. The arguments in the proof of Proposition 4.3 showing that *1* implies *2* and *3* and that any of *2* and *3* implies *4* adapt to its version with forking. Moreover it is pretty clear that *3* implies *1* in any version. Hence it will be enough to prove that simple theories verify point *3* in this new version for forking. Here is where we use $D(\pi,\Delta,k)$-rank. Assume $(p_i(x) : i < |T|^+)$ is a increasing chain of types $p_i(x) \in S(A_i)$ such that p_{i+1} forks over A_i for all $i < |T|^+$. This means that for all $i < |T|^+$ we can find some $\varphi_1^i(x), \ldots, \varphi_{n_i}^i(x)$ such that $p_{i+1}(x) \vdash \varphi_1^i(x) \vee \ldots \vee \varphi_{n_i}^i(x)$ and each $\varphi_j^i(x)$ divides over A_i with respect to some $k_{ji} < \omega$. Clearly we may assume that there are $n,k < \omega$ and some $\varphi_1(x,y_1),\ldots,\varphi_n(x,y_n)$ such that for all $i < |T|^+$ there are tuples $a_1^i,\ldots,a_n^i \in A_{i+1}$ for which $\varphi_i(x,a_j^i) = \varphi_j^i(x)$ and moreover $k = k_{j,i}$. Let $\Delta = \{\varphi_1(x,y_1),\ldots,\varphi_n(x,y_n)\}$. By Lemma 4.8 $D(p_i(x),\Delta,k) > D(p_{i+1}(x),\Delta,k)$ for all $i < |T|^+$, which is a contradiction. □

Corollary 4.10 *If T is simple and $p(x) \in S(A)$, then p does not fork over A.*

Proof. By Proposition 4.9. □

5 Independence and Morley sequences

Definition 12 *We say that A is independent of B over C (written $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$) if for every finite sequence $a \in A$, $\mathrm{tp}(a/BC)$ does not fork over C.*

Remarks 5.1

1. *$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff $A \mathop{\smile\!\!\!\!\!\!\mid}_C CB$.*
2. *If $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ for all finite $a \in A$, $b \in B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.*
3. *If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $B' \subseteq B$, then $A \mathop{\smile\!\!\!\!\!\!\mid}_{CB'} B$.*
4. *If $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, $A' \subseteq A$ and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!\mid}_C B'$.*

Definition 13 *Let X be a linearly ordered set. The sequence $(a_i : i \in X)$ is A-independent if for every $i \in X$, $a_i \mathop{\smile\!\!\!\!\!\!\bigm|}_A \{a_j : j < i\}$. A Morley sequence over A is a sequence $(a_i : i \in X)$ which is A-independent and A-indiscernible. It is said to be a Morley sequence in the type p if every a_i realizes p.*

Remark 5.2 *Let $(a_i : i \in X)$ be A-independent. If Y, Z are subsets of X such that $Y < Z$, then $\mathrm{tp}((a_i : i \in Z)/A(a_i : i \in Y))$ does not divide over A.*

Proof. It can be assumed that Z is finite. An induction on $|Z|$ using Lemma 3.5 gives easily the result. $\square$

Lemma 5.3 *If $p(x) \in S(B)$ does not fork over $A \subseteq B$, there is a Morley sequence $(a_i : i < \omega)$ in p over A which is moreover B-indiscernible. If p is not algebraic, the sequence is infinite, i.e, $a_i \neq a_j$ for $i < j < \omega$.*

Proof. Using the extension Lemma (item 4 in Remarks 3.3) one can construct a sequence $(a_i : i < \omega)$ of realizations a_i of p such that $a_i \mathop{\smile\!\!\!\!\!\!\bigm|}_A B\{a_j : j < i\}$. Observe that this property is type definable over B (because p is complete) and implies A-independence. Hence compactness and Ramsey's Theorem give us a Morley sequence over A in p which is moreover B-indiscernible. $\square$

Proposition 5.4 *Let T be simple. The formula $\varphi(x, a)$ divides over A iff for every infinite Morley sequence $(a_i : i < \omega)$ over A in $\mathrm{tp}(a/A)$, $\{\varphi(x, a_i) : i < \omega\}$ is inconsistent.*

Proof. Without loss of generality $A = \emptyset$. Assume that $\varphi(x, a)$ divides over $\emptyset$ but for some infinite Morley sequence the consistency fails. Let X be a linearly ordered set isomorphic to the reverse order of the cardinal $|T|^+$. By compactness there is an infinite Morley sequence $a_X = (a_i : i \in X)$ in $\mathrm{tp}(a)$ such that $\{\varphi(x, a_i) : i \in X\}$ is consistent. Let c realize this type. By simplicity there is $Y \subseteq X$ of cardinality at most $|T|$ such that $\mathrm{tp}(c/a_X)$ does not fork over $a_Y = (a_i : i \in Y)$. By choice of the order of X we can find $i \in X$ such that $i < Y$. By Lemma 5.2 $\mathrm{tp}(a_Y/a_i)$ does not divide over $\emptyset$. Since $\varphi(x, a_i)$ divides over $\emptyset$, by Proposition 3.6 it divides over a_Y. But $\mathrm{tp}(c/a_X)$ contains $\varphi(x, a_i)$ and hence it divides (and forks) over a_Y, a contradiction. $\square$

Proposition 5.5 *Let T be simple. A partial type $\pi(x, a)$ divides over A iff it forks over A.*

Proof. We may assume $\pi(x, a)$ is a formula $\varphi(x, a)$ and that a is not algebraic over A. Assume $\varphi(x, a)$ does not divide over A but it implies a disjunction $\varphi_1(x, a_1) \vee \ldots \vee \varphi_n(x, a_n)$ where every $\varphi(x, a_i)$ divides over A. Let $(a^j a_1^j \ldots a_n^j : j < \omega)$ be an infinite Morley sequence over A in $\mathrm{tp}(aa_1 \ldots a_n/A)$ (a nonalgebraic type). Then

$(a^j : j < \omega)$ is an infinite Morley sequence over A in $\mathrm{tp}(a/A)$. By Proposition 5.4, there exists a realization c of $\{\varphi(x, a^j) : j < \omega\}$. For every $j < \omega$ there is some i such that c realizes some $\varphi_i(x, a_i^j)$. By Ramsey's Theorem there is some i such that for a cofinal subset $X \subseteq \omega$, c realizes every $\varphi(x, a_i^j)$ with $j \in X$. By indiscernibility, $\{\varphi(x, a_i^j) : j < \omega\}$ is consistent and then by Proposition 5.4 $\varphi_i(x, a_i)$ does not divide over A since $(a_i^j : j < \omega)$ is an infinite Morley sequence over A in $\mathrm{tp}(a_i/A)$. □

Proposition 5.6 *In a simple theory independence is a symmetric relation, i.e, $A \mathop{\smile\!\!\!\!\!|}_C B$ implies $B \mathop{\smile\!\!\!\!\!|}_C A$.*

Proof. It is enough to prove that if $\mathrm{tp}(a/Cb)$ does not fork over C, then $\mathrm{tp}(b/Ca)$ does not divide over C. We may assume that $\mathrm{tp}(a/C)$ is not algebraic. By Lemma 5.3 there is an infinite Morley sequence $I = (a_i : i < \omega)$ in $\mathrm{tp}(a/C)$ which is Cb-indiscernible and starts with $a_0 = a$. Let $\varphi(x, y, z)$ be a formula and $c \in C$ such that $\models \varphi(a, b, c)$. We will show that $\varphi(x, a, c)$ does not divide over C. By indiscernibility of I over Cb we know that $\models \varphi(a_i, b, c)$ for all $i < \omega$. Hence $\{\varphi(a_i, y, c) : i < \omega\}$ is consistent. Since $(a_i c : i < \omega)$ is a Morley sequence in $\mathrm{tp}(ac/C)$, by Proposition 5.4 we conclude that $\varphi(a, y, c)$ does not divide over C. □

Proposition 5.7 *In a simple theory independence is a transitive relation, i.e, whenever $B \subseteq C \subseteq D$, $A \mathop{\smile\!\!\!\!\!|}_B C$ and $A \mathop{\smile\!\!\!\!\!|}_C D$, then $A \mathop{\smile\!\!\!\!\!|}_B D$.*

Proof. It is a direct consequence of Proposition 5.6, Lemma 3.5 and Proposition 5.5. □

Corollary 5.8 *Let T be simple. If I is an ordered set an $(a_i : i \in I)$ is an A-independent sequence, then $a_i \mathop{\smile\!\!\!\!\!|}_A \{a_j : j \neq i\}$ for all $i \in I$.*

Proof. By induction on n it is easy to show that for all different $i_1, \ldots, i_{n+1} \in I$, $a_{i_{n+1}} \mathop{\smile\!\!\!\!\!|}_A a_{i_1}, \ldots, a_{i_n}$. For the inductive case one uses symmetry and Lemma 3.5.□

6 The independence theorem

Lemma 6.1 *Let T be simple. If $(a_i : i < \omega + \omega)$ is an infinite A-indiscernible sequence, then $(a_i : \omega \leq i < \omega + \omega)$ is a Morley sequence over $A\{a_i : i < \omega\}$.*

Proof. Let $I = (a_i : i < \omega)$. Clearly $(a_i : \omega \leq i < \omega + \omega)$ is AI-indiscernible. It suffices to show that it is AI-independent. Let X be a finite subset of $\{i : \omega \leq i < \omega + \omega\}$ an let $i < \omega + \omega$ be greater than every element in X. By symmetry it will be enough to check that $a_X \mathop{\smile\!\!\!\!\!|}_{AI} a_i$, where $a_X = (a_j : j \in X)$. But this is clear since by A-indiscernibility $\mathrm{tp}(a_X/AIa_i)$ is finitely satisfiable in I. □

Proposition 6.2 *Let T be simple. If $\varphi(x, a)$ does not divide over A and a, b start an infinite A-indiscernible sequence, then $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over A.*

Proof. Let us first assume that a, b start an infinite Morley sequence $(a_i : i < \omega)$ over A. Since $\varphi(x, a)$ does not divide over A, $\{\varphi(x, a_i) : i < \omega\}$ is consistent. If $b_i = a_{2i}a_{2i+1}$, $(b_i : i < \omega)$ is an infinite Morley sequence in $\mathrm{tp}(ab)$ over A. Since $\{\varphi(x, a_{2i}) \wedge \varphi(x, a_{2i+1}) : i < \omega\}$ is consistent, by Proposition 5.4, $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over A. Now let us consider the general case, where a, b start an infinite A-indiscernible sequence $(a_i : i < \omega)$. Choose $(b_i : i < \omega)$ such that $ab \perp_A (b_i : i < \omega)$ and the sequence $(b_i : i < \omega)^\frown(a_i : i < \omega)$ is A-indiscernible. By Lemma 6.1 $(a_i : i < \omega)$ is a Morley sequence over $B = A \cup \{b_i : i < \omega\}$. Since $\varphi(x, a)$ does not divide over B also, by the first case considered we know that $\varphi(x, a) \wedge \varphi(x, b)$ does not divide over B. By Proposition 3.6 this conjunction does not divide over A. □

Remark 6.3 *The proof of Proposition 6.2 generalizes to show that if $\pi(x, a)$ does not divide over A and $I \ni a$ is an infinite A-indiscernible sequence, then $\bigcup_{b \in I} \pi(x, b)$ does not divide over A.*

Lemma 6.4 *Let T be simple. If a, b start an infinite A-indiscernible sequence and $c \perp_{Aa} b$, then for some d, the extended sequences ac, bd start an infinite A-indiscernible sequence also.*

Proof. Assume $A = \emptyset$. Let $c \perp_a b$ and assume $I = (a_i : i < \omega)$ is an infinite indiscernible sequence with $a = a_0$ and $b = a_1$. Since $(a_n : n \geq 1)$ is a-indiscernible and $c \perp_a b$, by Lemma 3.4 there is an ac-indiscernible sequence $(a'_n : n \geq 1)$ such that $(a_n : n \geq 1) \equiv_{ab} (a'_n : n \geq 1)$. Thus we may assume that $a_n = a'_n$ for all $n \geq 1$. Let $c_0 = c$ and choose for $n \geq 1$ some c_n such that

$$ca_0a_1 \ldots \equiv c_na_na_{n+1} \ldots$$

Since $(a_n : n \geq 1)$ is ac-indiscernible, $cab \equiv caa_m$. Hence $cab \equiv c_na_na_{n+m}$, i.e., in the sequence $(c_na_n : n < \omega)$ all triangles $c_na_na_{n+m}$ have the same type $p(x, y, z) = \mathrm{tp}(cab)$. By Ramsey's Theorem there is an indiscernible sequence $(d_nb_n : n < \omega)$ where all triangles $d_nb_nb_{n+m}$ satisfy $p(x, y, z)$. Clearly we may assume that $c = d_0$, $a = b_0$ and $b = b_1$. Take $d = d_1$. □

Proposition 6.5 *Let T be simple and assume that $\varphi(x, a) \wedge \psi(x, b)$ does not fork over A. If b, b' start an infinite A-indiscernible sequence and $a \perp_{Ab} b'$, then $\varphi(x, a) \wedge \psi(x, b')$ does not fork over the A.*

Proof. Apply Lemma 6.4 finding c such that $ba, b'c$ start an infinite A-indiscernible sequence. By Proposition 6.2, $\varphi(x, a) \wedge \psi(x, b) \wedge \varphi(x, c) \wedge \psi(x, b')$ does not fork over A. In particular $\varphi(x, a) \wedge \psi(x, b')$ does not fork over A. □

Corollary 6.6 *Let T be simple and assume that $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A. If $\mathrm{Lstp}(b/A) = \mathrm{Lstp}(b'/A)$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_A bb'$, then $\varphi(x,a) \wedge \psi(x,b')$ does not fork over A.*

Proof. Assume $A = \emptyset$. Find $b_1, \ldots, b_n$ such that $b = b_1$, $b' = b_n$ and b_i, b_{i+1} start an infinite indiscernible sequence. Let a' be such that $a' \equiv_{bb'} a$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_{bb'} b_1, \ldots b_n$. By Proposition 6.5 we see that $\varphi(x,a') \wedge \psi(x,b_i)$ does not fork over the empty set for all $i \leq n$. Hence $\varphi(x,a) \wedge \psi(x,b')$ does not fork over the empty set . $\square$

Lemma 6.7 *Let T be simple. Let κ be a cardinal number bigger than $|T| + |A|$. If $(a_i : i < \kappa)$ is A-independent and the length of every a_i is smaller than κ, then for any a of length smaller than κ there is some $i < \kappa$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A a_i$.*

Proof. By choice of κ, there is a proper subset $B \subsetneq \{a_i : i < \kappa\}$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_B \{a_i : i < \kappa\}$. Take $a_i \notin B$. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_B a_i$ and, by Corollary 5.8, $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A B$. By symmetry and transitivity, $a \mathop{\smile\!\!\!\!\!\!\mid}_A a_i$. $\square$

Lemma 6.8 *Let T be simple. For any a, A and $B \supseteq A$ there is a' such that $\mathrm{Lstp}(a'/A) = \mathrm{Lstp}(a/A)$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_A B$.*

Proof. Let κ be a cardinal bigger than $|T| + |B|$ and bigger than the length of a. We may assume that $\mathrm{tp}(a/A)$ is not algebraic. Let $(a_i : i < \kappa)$ be a Morley sequence in $\mathrm{tp}(a/A)$. By Lemma 6.7 there is some $i < \kappa$ such that $B \mathop{\smile\!\!\!\!\!\!\mid}_A a_i$. Clearly, $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(a_i/A)$. $\square$

Lemma 6.9 *Let T be simple and $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$. For any c, B there is some d such that $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd/A)$ and $d \mathop{\smile\!\!\!\!\!\!\mid}_{Ab} B$.*

Proof. By Corollary 2.8 there is some d' such that $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd'/A)$ and by Corollary 2.7, there is a strong automorphism $f \in \mathrm{Autf}(\mathfrak{C}/A)$ such that $f(ac) = bd'$. By Lemma 6.8 there is some d such that $\mathrm{Lstp}(d/Ab) = \mathrm{Lstp}(d'/Ab)$ and $d \mathop{\smile\!\!\!\!\!\!\mid}_{Ab} B$. Again by Corollary 2.7 there is some $g \in \mathrm{Aut}(\mathfrak{C}/Ab)$ such that $g(d') = d$. It follows that $g \circ f \in \mathrm{Autf}(\mathfrak{C}/A)$ and $g \circ f(ac) = bd$. Hence $\mathrm{Lstp}(ac/A) = \mathrm{Lstp}(bd/A)$. $\square$

Corollary 6.10 (Independence Theorem) *Let T be simple and $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$. If there are c, d such that $\models \varphi(c,a)$, $c \mathop{\smile\!\!\!\!\!\!\mid}_A a$, $\models \psi(d,b)$, $d \mathop{\smile\!\!\!\!\!\!\mid}_A b$, and $\mathrm{Lstp}(c/A) = \mathrm{Lstp}(d/A)$, then $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A.*

Proof. Using Lemma 6.9, choose $b' \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} ab$ such that $\mathrm{Lstp}(cb'/A) = \mathrm{Lstp}(db/A)$. Then $\models \varphi(c,a) \wedge \psi(c,b')$ and $c \mathop{\smile\!\!\!\!\!\!\mid}_A ab'$. Therefore $\varphi(x,a) \wedge \psi(x,b')$ does not fork over A. Since $a \mathop{\smile\!\!\!\!\!\!\mid}_A bb'$ by Corollary 6.6, $\varphi(x,a) \wedge \psi(x,b)$ does not fork over A. $\square$

Remarks 6.11 *1. The version of the Independence Theorem for partial types $\pi(x,a)$, $\sigma(x,b)$ in place of formulas $\varphi(x,a)$, $\psi(x,b)$ follows in a straightforward way. It can be also generalized easily to any ordered sequence of types $(\pi_i(x,a_i) : i \in I)$ if the corresponding sequence $(a_i : i \in I)$ of parameters is independent over A.*

2. The Independence Theorem for types over a model follows also easily from the stated version.

Proposition 6.12 *Let T be simple. If $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$, then a,b start a Morley sequence over A.*

Proof. Let $p = \mathrm{tp}(ab/A)$. We prove first that there is an infinite A-independent sequence $(a_i : i < \omega)$ such that $\models p(a_i,a_j)$ for all $i < j < \omega$. The sequence is constructed inductively starting with $a_0 = a$ and $a_1 = b$. To get a_n we need to prove that $\bigcup_{i<n} p(x,a_i)$ does not fork over A. But this is clear by the generalized version of the Independence Theorem. Now, once we have this A-independent sequence we still need to make it A-indiscernible. But this can be done by Ramsey's Theorem and compactness if one notices that the fact that $a_0,\ldots,a_n \mathop{\smile\!\!\!\!\!\!\,|}_A a_{n+1}$ can be expressed by a type over A since the type $\mathrm{tp}(a_{n+1}/A)$ is given. $\square$

Proposition 6.13 *If T is simple, then $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ if and only if there is some c such that a,c start an infinite A-indiscernible sequence and b,c start an infinite indiscernible sequence over A.*

Proof. Assume $\mathrm{Lstp}(a/A) = \mathrm{Lstp}(b/A)$ and find with Lemma 6.8 some c such that $\mathrm{Lstp}(c/A) = \mathrm{Lstp}(a/A)$ and $c \mathop{\smile\!\!\!\!\!\!\,|}_A ab$. By Proposition 6.12 a,c start an infinite Morley sequence over A and b,c start an infinite Morley sequence over A. $\square$

Corollary 6.14 *If T is simple, then equality of Lascar strong types over A is type definable over A by $\exists z(\mathrm{nc}_A(x,z) \wedge \mathrm{nc}_A(y,z))$.*

Proof. Clear, by Proposition 6.13. $\square$

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