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Learning seminar on Quadratic Chabauty

The final goal of the seminar is to understand the use of quadratic Chabauty in [BDMTV]. Two intermediate goals will occupy the first 3 of the 4 sessions of the seminar:

- In order to motivate the problem addressed in [BDMTV], in talks 1,3,5 we want to understand Mazur's determination of the set of rational points of the modular curves $X_0(N)$.
- In order to explain some background for the techniques used in [BDMTV], in talks 2,4,6 we want to study Coleman's approach to the Chabauty method.

Talk 1. (February 22 at S3 of UB; Enric Florit) Quickly cover sections 1 and 2 of [Maz] with the objective of showing why Thm.1 reduces to Thm. 2 (if possible, comment on Kubert's input only mentioned in Mazur's paper ([Kub, IV. 1. 2]). Of Section 3, state the proposition on p. 122 and explain how Axiom 3 holds from the Theorem of Herbrand-Kummer.

Talk 2. (February 22 at S3 of UB; Ignasi Sánchez) Let X be a curve over $\mathbb{Q}$. Chabauty's theorem states that, if the rank of the Jacobian J of X is less than the genus of X, then $X(\mathbb{Q})$ is finite. A crucial observation is that in that case there exists a continuous homomorphism $J(\mathbb{Q}_p) \to \mathbb{Q}_p$ such that the group of global points $J(\mathbb{Q})$ is contained in the kernel. In [Col] Coleman makes Chabauty's method effective. Explain *loc.cit.* in detail.

Talk 3. (April 27 in Cardedeu; Xavier Guitart) Cover the argument spreading from page 125 to page 133 of [Maz]: that is, assuming that Axiom 2 holds for the Eisenstein quotient complete the proof of the proposition on p. 122 of [Maz]. The validity of Axiom 2 for the Eisenstein quotient will checked in Talk 5.

Talk 4. (April 27 in Cardedeu; Marc Masdeu) The easiest case in which Chabauty's condition on the rank fails is that of an elliptic curve of rank 1. In [BB] Balakrishnan–Besser replace the linear p-adic logarithm of Chabauty's argument by a quadratic map. More precisely, they use a p-adic height pairing in order to determine the set of integral points of the elliptic curve. Explain Section 1 and 2 of [BB] as well as the necessary background on local heights from [CG]. Given an overview over the theory of Coleman integration necessary for Section 3 (see for example [Bes2, Section 1.4]).

Talk 5. (May 31 at C1/366 of UAB; Francesc Fité) Prove that Axiom 2 of the proposition on page 122 of [Maz] holds for the Eisenstein quotient. For this, instead of following [Maz, Section 4], follow [MS].

Talk 6. (May 31 at C1/366 of UAB; Santiago Molina) [BB] Explain Section 3 to 5 of [BB] in as much detail as possible. If time permits, discuss one of the examples presented in Section 6.

Talk 7. (June 28 in Montserrat; Lennart Gehrmann) Explain Besser's approach to Coleman integration using Tannakian formalism (see[Bes1] and [Bes2]).

Talk 8. (June 28 in Montserrat; Jan Vonk) Give an overview of the strategy and ingredients used in the proof of the main theorem of [BDMTV].

References

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[Bes2] A. Besser, Heidelberg Lectures on Coleman Integration. In: Stix, J. (eds) *The Arithmetic of Fundamental Groups. Contributions in Mathematical and Computational Sciences*, vol 2. Springer, Berlin, Heidelberg, 2012.

[BB] A. Besser, J. Balakrishnan, *Coleman-Gross height pairings and the p-adic sigma function*, J. Reine Angew. Math. 698 (2015), 89-104.

[BDMTV] N. Dogra, J. S. Müller, J. Tuitman, J. Vonk, *Explicit Chabauty-Kim for the split Cartan modular curve of level 13*, Ann. of Math., 189, No. 3 (May 2019), 885-944.

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[Maz] B. Mazur, *Rational points on modular curves*, Antwerp V, 1977.

[CG] R. Coleman and B. Gross. p-adic heights on curves. In J. Coates, R. Greenberg, B. Mazur, and I. Satake, editors, *Algebraic number theory*, volume 17 of *Advanced Studies in Pure Mathematics*, pages 73–81. Academic Press, Boston, MA, 1989.

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