1. Fractional Laplacian

- The fractional Laplacian operator is given by:
  \[ (-\Delta)^s u(x) := C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy \]
  for a certain normalizing constant \( C_{n,s} > 0 \).

- Notice that the operator \((-\Delta)^s\) is nonlocal (it uses information about \( u \) far from \( x \)).

- Also, \( \frac{d}{dx} \) is singular at the origin (not integrable), and thus it requires certain regularity of \( u \) near \( x \) in order to evaluate \((-\Delta)^s\).

- Because of the singularity at the origin, the operator \((-\Delta)^s\) "differentiates" in some sense the function \( u \), and this is why it is called an integro-differential operator.

- When \( s \in \left( \frac{1}{2}, 1 \right) \), then \((-\Delta)^s u(x)\) has to be understood in the principal value sense:
  \[ (-\Delta)^s u(x) = \lim_{\epsilon \to 0^+} \left[ \frac{\epsilon}{|y|^{n+2s}} \right] \int_{|y| > \epsilon} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy \]

An alternative option is to use the symmetry of \( \frac{1}{|y|^{n+2s}} \) and symmetrize the integral:

\[ (-\Delta)^s u(x) = C_{n,s} \frac{c}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \, dy \]

- Using this expression, one can check that if \( u \in C^2(O) \) and \( \nabla u \in L^p(O) \) then
  \[ |(-\Delta)^s u(x)| \leq C \int_{B_1} \frac{|\Delta u(y)|}{|y|^{n+2s}} \, dy + C \int_{|y| > 1} \frac{|\Delta u(y)|}{|y|^{n+2s}} \, dy \leq C M \|u\|_{L^p(O)} + C \|\nabla u\|_{L^p(O)} \]

**Exercise:** Show that if \( u \in C^{2s+\epsilon}(\mathbb{R}^n) \cap C^2(O) \), then \((-\Delta)^su \in C^{2\epsilon}(O)\), provided that \( \alpha \) and \( n+2s\alpha \) are not integers.

- In particular, \( u \in C^{2s+\epsilon}(\mathbb{R}^n) \) for some \( \epsilon > 0 \) is enough to evaluate \((-\Delta)^s u(x)\) (but not \( \nabla u \)).
11. Heuristic probabilistic motivation: from discrete to continuous long jump random walks

We will see how the fractional Laplacian arises in long jump random walks.

Let us consider the function $K: \mathbb{R}^n \to [0, \infty)$, satisfying

$$K(y) = K(-y)$$

and

$$\sum_{q \in \mathbb{Z}^n} K(q) = 1.$$

Given a small $h > 0$, we consider a random walk on the lattice $h \mathbb{Z}^n$.

After any time $t > 0$, a particle jumps from a point of $h \mathbb{Z}^n$ to any other point.

The probability with which the particle jumps from the point $h \overline{q}$ to the point $h \overline{p}$

is $K(h \overline{q} - h \overline{p}).$

(Notice that the particle may experience arbitrarily long jumps, though with a small probability.)

Consider a domain $\omega \subset \mathbb{R}^n$ and a given payoff function $g: \mathbb{R}^n \to \mathbb{R}$.

We call $u(x)$ the expected payoff the particle will get if it starts at $x \in \omega$. (The particle gets the payoff the first time it exits $\omega$)

Notice that $u(x) = g(x)$ in $\mathbb{R}^n \setminus \overline{\omega}$.

Moreover, if $x \in \omega$, then the expected payoff equals the sum of all expected payoffs of all possible positions $x + h q$, weighted by the probability of jumping from $x$ to $x + h q$,

$$u(x) = \sum_{q \in \mathbb{Z}^n} K(q) u(x + h q) \quad \text{if } x \in \omega$$
Recalling that \( \sum_{q \in \mathbb{Z}^n} K(q) = 1 \), we can write the previous identity as

\[ \sum_{q \in \mathbb{Z}^n} K(q)(u(x)-u(x+qh)) = 0 \]

The most canonical and simple choice of Kernel is a power

\[ K(y) = c|y|^{-n-2s} \quad \text{for} \quad y \neq 0 \]  
\[ \text{(and, say, } K(0) = 0) \]

with

\[ s \in (0,1) \]

The constant \( c \) is chosen so that \( \sum_{q \in \mathbb{Z}^n} K(q) = 1 \).

With this choice of the Kernel, (6) becomes

\[ \sum_{q \in \mathbb{Z}^n} \frac{u(x)-u(x+qh)}{|q|^{n+2s}} = 0 \]

Multiplying by an appropriate factor \( h^{2s} \), this is

\[ h^n \sum_{q \in \mathbb{Z}^n} \frac{u(x)-u(x+qh)}{|qh|^{n+2s}} = 0 \]

which is the approximating Riemann sum of

\[ \int_{\mathbb{R}^n} \frac{u(x)-u(x+y)}{|y|^{n+2s}} dy = 0 \quad \text{for } x \in \Omega. \]

Thus, in the limit \( h \to 0 \), the limiting stochastic process \( X_t \) will satisfy the following:

the expected payoff \( u(x) \) solves

\[ \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+2s}} dy = 0 \quad \text{for } x \in \Omega \]

\[ u(x) = g(x) \quad \text{for } x \in \partial \Omega. \]
Summarizing, the expected payoff \( u(x) \) solves the Dirichlet problem:

\[
\begin{align*}
(-\Delta)^s u & = 0 \quad \text{in } \mathbb{R}^n \\
\frac{\partial u}{\partial \nu} & = g \quad \text{on } \partial \mathbb{R}^n \cup L
\end{align*}
\]

for the fractional Laplacian,

\[
(-\Delta)^s u(x) = c_n s \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy
\]

As in case of local PDEs, if we consider running costs or expected exit times we are led to Dirichlet problems of the type:

\[
\begin{align*}
(-\Delta)^s u & = f \quad \text{in } L \\
\frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial L \cup \partial \mathbb{R}^n 
\end{align*}
\]

Important: The boundary conditions are in \( \mathbb{R}^n \setminus L \) instead of \( 2\mathbb{R}^n \).

Stochastic process with jumps \( \xrightarrow{\text{operator that is nonlocal}} \) \( \xrightarrow{\text{Dirichlet problem with boundary data}} \) \( \mathbb{R}^n \setminus L \)

Remark. The name fractional Laplacian comes from the fact that the Fourier symbol of the operator is \( |\xi|^{2s} \), so that \((-\Delta)^s\) is really the fractional power of the operator \(-\Delta\) in \( \mathbb{R}^n \).

In particular, we have the property \((-\Delta)^s \circ (-\Delta)^t = (-\Delta)^{s+t}\).
The fractional Laplacian is

\[
(\Delta)^s u(x) = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \left(2u(x) - u(x+y) - u(x-y)\right) \frac{dy}{|y|^{n+2s}} \quad \text{if } u \in C_c^\infty(\mathbb{R}^n).
\]

Let us find the Fourier symbol of \((\Delta)^s \) and check that it is \(\xi^{2s}\).

That is, we want to check that

\[
\mathcal{F}[ (\Delta)^s u] (\xi) = (\xi^{2s}) \mathcal{F}[ u].
\]

(where \(\mathcal{F}\) denotes Fourier transform)

Indeed, we have

\[
\mathcal{F}[ (\Delta)^s u] (\xi) = \mathcal{F} \left[ \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \left(2u(x) - u(x+y) - u(x-y)\right) \frac{dy}{|y|^{n+2s}} \right] =
\]

\[
= \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \mathcal{F} \left[ 2u(x) - u(x+y) - u(x-y) \right] \frac{dy}{|y|^{n+2s}} =
\]

\[
= \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \left( 2\mathcal{F}[u] - \mathcal{F}[u(\cdot+y)] - \mathcal{F}[u(\cdot-y)] \right) \frac{dy}{|y|^{n+2s}} =
\]

\[
= \left( C_{n,s} \int_{\mathbb{R}^n} (1 - \cos(\xi \cdot y)) \frac{dy}{|y|^{n+2s}} \right) \mathcal{F}[u] (\xi).\]

We only need to check that

\[
C_{n,s} \int_{\mathbb{R}^n} (1 - \cos(\xi \cdot y)) \frac{dy}{|y|^{n+2s}} = |\xi|^{2s}
\]

But this simply follows from the fact that

\[
\int (1 - \cos(\lambda \cdot y)) \frac{dy}{|y|^{n+2s}} = \left( C_{n,s} \int_{\mathbb{R}^n} (1 - \cos(\lambda \cdot y)) \frac{dy}{|y|^{n+2s}} \right) \frac{dy}{|y|^{n+2s}} \quad \lambda.
\]

is radially symmetric, and homogeneous of degree \(2s\): (Exercise).

\[
\int (\lambda^2) = C_{n,s} \int (1 - \cos(\lambda \cdot y)) \frac{dy}{|y|^{n+2s}} = C_{n,s} \int (1 - \cos(\lambda \cdot y)) \frac{dy}{|y|^{n+2s}} \quad \lambda^2s
\]

This means that

\[
\int (\xi^2) = K |\xi|^{2s} \quad \text{for some } K > 0, \text{ but the constant } C_{n,s} > 0 \text{ is chosen so that}
\]

\[
\int (\xi) = |\xi|^{2s}. \quad \text{[Exercise: Check that } C_{n,s} \text{ as above} \text{ and } C_{n,s} \text{ as before}.]
\]
1.2. Existence of solutions

For the Laplacian, the existence (and uniqueness) for the Dirichlet problem
\[ -\Delta u = f \text{ in } \mathbb{R}^n \quad u = g \text{ on } \partial \Omega \]
follows from Riesz representation theorem, once one has the appropriate ingredients.

- The energy functional associated to the problem is
  \[ \int_{\mathbb{R}^n} |\nabla u|^2 - \int_{\mathbb{R}^n} fu \]
  and the (weak) solution is the minimizer of the functional among functions \( u \in H^1(\mathbb{R}^n) \) with \( u = g \text{ on } \partial \Omega \).

- To see that any \( C^2 \) weak solution solves the equation pointwise, we just integrate by parts
  \[ \int_{\mathbb{R}^n} \nabla u \cdot \nabla v = \int_{\mathbb{R}^n} -\Delta u \cdot v = \int_{\mathbb{R}^n} f v \quad \forall v \in H^1(\mathbb{R}^n) \]
  \[ \implies -\Delta u = f \text{ in } \mathbb{R}^n \]

- Let us next follow a similar strategy for the fractional Laplacian.

**Integration by parts**

**Prop.** Let \( u \) and \( v \) be \( C^2 \) functions, with \( u \neq 0 \) in \( \mathbb{R}^n \). Then,

\[ \int_{\mathbb{R}^n} (\Delta^s u) \cdot v \, dx = \frac{\gamma(s)}{2} \int_{\mathbb{R}^n} \frac{(u(x)-u(y)) \cdot (v(x)-v(y))}{|x-y|^{n+2s}} \, dx \, dy \]

**Proof:**

\[ \int_{\mathbb{R}^n} (\Delta^s u(x)) v(x) \, dx = \int_{\mathbb{R}^n} \left( \nabla \frac{\partial u}{\partial \nu}(x) \right) \cdot v(x) \, dx + \int_{\mathbb{R}^n} \frac{\partial u}{\partial \nu}(x) \, dS(x) \]

\[ = \frac{\gamma(s)}{2} \int_{\mathbb{R}^n} \left( \frac{(u(x)-u(y))}{|x-y|^{n+2s}} \right) v(x) \, dx \]

**Exercise.** Show that

\[ \frac{\gamma(s)}{2} \int_{\mathbb{R}^n} \frac{(u(x)-u(y)) \cdot (v(x)-v(y))}{|x-y|^{n+2s}} \, dx \, dy = \int_{\mathbb{R}^n} (\Delta^s u) \cdot v \, dx \]

- Notice that in Fourier side this is
  \[ \int_{\mathbb{R}^n} \mathcal{F}(u \Delta^s v) = \int_{\mathbb{R}^n} (\mathcal{F}u \mathcal{F}(\Delta^s v)) = \int_{\mathbb{R}^n} |\xi|^s \mathcal{F}u \mathcal{F}v \]
Weak solutions

- We say that \( u \) is a weak solution of
  \[
  \begin{cases}
  \Delta u = f \text{ in } \Omega \\
  u = g \text{ on } \partial \Omega
  \end{cases}
  \]

- The space of functions \( u \in L^2(\mathbb{R}^n) \) for which
  \[
  [u^2]_{H^1(\mathbb{R}^n)} := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2}} \, dx \, dy < \infty
  \]

  is called \( H^1(\mathbb{R}^n) \). The scalar product is
  \[
  (u, v)_{H^1(\mathbb{R}^n)} = \iint_{\mathbb{R}^n} u v + \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2}} \, dx \, dy
  \]

  - The norm is
    \[
    \|u\|_{H^1(\mathbb{R}^n)} = \sqrt{(u, u)_{H^1(\mathbb{R}^n)}}
    \]

- The space \( H^1(\mathbb{R}^n) \) is a Hilbert space.

Definition. We say that \( u \in H^1(\mathbb{R}^n) \) is a weak solution to
  \[
  \begin{cases}
  \Delta u = f \text{ in } \Omega \\
  u = g \text{ on } \partial \Omega
  \end{cases}
  \]

  if \( u = g \) on \( \partial \Omega \) and
  \[
  \frac{\partial^2}{2} \iint_{\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2}} \, dx \, dy = \iint_{\mathbb{R}^n} f \varphi \, dx
  \]

  for all \( \varphi \in C_c^\infty(\Omega) \) with \( \varphi \geq 0 \) in \( \mathbb{R}^n \).

- Notice that if \( u \in C^2 \) then we have
  \[
  \frac{\partial^2}{2} \iint_{\mathbb{R}^n} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2}} \, dx \, dy = \iint_{\mathbb{R}^n} \Delta u \varphi = \iint_{\mathbb{R}^n} f \varphi
  \]

  for all \( \varphi \in C_c^\infty(\mathbb{R}^n) \).

  and thus \( \Delta u = \varphi \) in \( \Omega \).
Notice also that, since \( \Omega = 0 \) in \( \mathbb{R}^n \), we have
\[
\frac{1}{2} \int_{\mathbb{R}^n} \frac{(u(x) - u(2))(u(x) - u(2))}{|x-2|^2} \, dx = \frac{1}{2} \int_{(\mathbb{R}^n \setminus \{0\}) \setminus (\mathbb{R}^n \setminus 2 \mathbb{R}^n)} \frac{(u(x) - u(2))(u(x) - u(2))}{|x-2|^2} \, dx.
\]

Thus, we don't really need \( \mathcal{E}(\mathbb{R}^n) \), but only
\[
\int_{(\mathbb{R}^n \setminus 2 \mathbb{R}^n) \setminus (\mathbb{R}^n \setminus 2 \mathbb{R}^n)} \frac{(u(x) - u(2))^2}{|x-2|^2} \, dx < \infty.
\]

This is important when \( g \) is not regular outside \( \Omega \), or it does not vanish at \( \infty \).

In this case, the right definition is:

**Definition.** We say that \( u \) is a weak solution of
\[
\begin{cases}
\Delta u = f & \text{in } \mathbb{R}^n \\
\mu = g & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
\]

if
\[
\int_{(\mathbb{R}^n \setminus 2 \mathbb{R}^n) \setminus (\mathbb{R}^n \setminus 2 \mathbb{R}^n)} \frac{(u(x) - u(2))^2}{|x-2|^2} \, dx < \infty
\]

and
\[
\frac{1}{2} \int_{(\mathbb{R}^n \setminus 2 \mathbb{R}^n) \setminus (\mathbb{R}^n \setminus 2 \mathbb{R}^n)} \frac{(u(x) - u(2))(u(x) - u(2))}{|x-2|^2} \, dx = \int_{\Omega} f \, dx.
\]

for all \( \mathcal{E}(\mathbb{R}^n) \) with \( \mu = 0 \) in \( \mathbb{R}^n \).

Note that when \( g \) satisfies
\[
\int_{\mathbb{R}^n} \frac{(g(x) - g(2))^2}{|x-2|^2} \, dx < \infty
\]

then the two definitions are the same.

However, it is important to allow \( g \) to be non-regular outside \( \Omega \).
The energy functional

- The energy functional associated to the problem is

\[
E(u) = \frac{c_n s}{4} \iint_\mathbb{R}^n \frac{(u(x) - u(z))^2}{|x-z|^m} \, dx \, dz - \int_{\mathbb{R}^n} f \, u \, dx
\]

for functions \( u \) satisfying \( u = g \) in \( \mathbb{R}^n \).

- When \( g \) satisfies \( \iint_{\mathbb{R}^n} \frac{g(x) - g(y)}{|x-y|^m} \, dx \, dy < \infty \), then we could take

\[
E(u) = \frac{c_n s}{4} \iint_\mathbb{R}^n \frac{(u(x) - u(z))^2}{|x-z|^m} \, dx \, dz - \int_{\mathbb{R}^n} f \, u \, dx
\]

(since the only difference between the two functionals would be a constant \( \iint_{\mathbb{R}^n} \frac{(g(x) - g(y))^2}{|x-y|^m} \, dx \, dy \) among \( u \) in \( \mathbb{R}^n \).

Proposition. If \( u \) minimizes the energy functional \( E(u) \), then it is a weak solution of \( (A) u = f \) in \( \mathbb{R} \)

\[ u = g \text{ in } \mathbb{R} \]

Proof. If \( u \) is a minimizer then for all \( \varphi \in \mathcal{D}(\mathbb{R}) \) such that \( \varphi \equiv 0 \) in \( \mathbb{R} \), we have

\[
E(u + \epsilon \varphi) \geq E(u), \quad \forall \epsilon > 0.
\]

Thus,

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon = 0} E(u + \epsilon \varphi) = 0
\]

But

\[
(u(x) + \epsilon \varphi(x) - u(x) - \epsilon \varphi(x))^2 = (u(x) - u(x))^2 + 2\epsilon (u(x)-u(x))(\varphi(x)-\varphi(x)) + \epsilon^2 (\varphi(x)-\varphi(x))^2
\]

and thus

\[
0 = \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} E(u + \epsilon \varphi) = \frac{c_n s}{2} \iint_{\mathbb{R}^n} \frac{(u(x) - u(z))(\varphi(x) - \varphi(z))}{|x-z|^m} \, dx \, dz - \int_{\mathbb{R}^n} f \, u \, dx
\]

This means that \( u \) is a weak solution.
Fractional Sobolev inequality and Poincaré inequality in $\Omega$

**Theorem.** For all $u \in H^s(\mathbb{R}^n)$ we have ($n > 2s$)

$$
\|u\|_{L^2(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} \frac{(u(x) - u(z))^2}{|x - z|^{n+2s}} \, dx \, dz
$$

$$
f = \frac{2n}{n-2s}
$$

**Proof.** (Next page)

**Corollary.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, for any $u \in H^s(\mathbb{R}^n)$ with $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$

we have

$$
\int_{\Omega} u^2 \leq C \int_{\mathbb{R}^n} \frac{(u(x) - u(z))^2}{|x - z|^{n+2s}} \, dx \, dz
$$

**Proof.**

$$
\|u\|_{L^2(\Omega)} \leq C \|u\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \leq C \|u\|_{L^{2n/(n+2s)}(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)}
$$

(Hölder) \hspace{1cm} (a \equiv 0, b \equiv 2s)

Existence of solutions

- We will prove existence of solutions only in case $g \equiv 0$ in $\mathbb{R}^n \setminus \Omega$.
- Notice, that when $g$ is regular enough, then we can extend it to a nice function in $\mathbb{R}^n$, and thus $\overline{u} = u - g$ solves

$$
\begin{cases}
-\Delta^s \overline{u} = \mathcal{F} - (-\Delta^s g) = \mathcal{F} \quad &\text{in } \Omega \\
\overline{u} = 0 & \text{in } \mathbb{R}^n \setminus \Omega
\end{cases}
$$

- This allows us to reduce to the case $g \equiv 0$ (when $g$ is regular enough).
Proof. Since

\[ |u(x)| \leq |u(x)| - |u(x)| + |u(x)| \]

then

\[ |u(x)| \leq \int_{B_r(x)} |u(x)| - |u(x)| + |u(x)| \, dx. \]

By Hölder inequality,

\[ \int_{B_r(x)} |u(x)| - |u(x)| + |u(x)| \, dx \leq \left( \int_{B_r(x)} (u(x)| - |u(x)| + |u(x)|)^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{B_r(x)} \frac{(u(x)| - |u(x)| + |u(x)|)^2}{1 + x^2 \lambda^2} \, dx \right)^{\frac{1}{2}} \leq r^{\frac{n}{2}} \left( \int_{R^n} \frac{c^n}{1 + x^2 \lambda^2} \, dx \right)^{\frac{1}{2}}. \]

Again, by Hölder inequality,

\[ \int_{B_r(x)} |u(x)| - |u(x)| + |u(x)| \, dx \leq \left( \int_{B_r(x)} (u(x)| - |u(x)| + |u(x)|)^2 \, dx \right)^{\frac{1}{2}} \leq r^{\frac{n}{2}} \left( \int_{R^n} \frac{c^n}{1 + x^2 \lambda^2} \, dx \right)^{\frac{1}{2}}. \]

Thus,

\[ |u(x)| \leq r^{\frac{n}{2}} \left( \int_{R^n} \frac{c^n}{1 + x^2 \lambda^2} \, dx \right)^{\frac{1}{2}} + r^{\frac{n}{2}} \left( \int_{R^n} u(x)^2 \, dx \right)^{\frac{1}{2}}. \]

Minimizing the RHS with respect to \( r > 0 \), we get

\[ |u(x)| \leq C \left( \int_{R^n} \frac{|u(x)|}{1 + x^2 \lambda^2} \, dx \right)^{\frac{1}{2}} \left( \int_{R^n} u(x)^2 \, dx \right)^{\frac{1}{2}}. \]

\( \star \) Taking \( q = \frac{2n}{n-2s} \), and raising both sides to power \( q \), we get

\[ |u(x)|^q \leq C \left( \int_{R^n} \frac{|u(x)|}{1 + x^2 \lambda^2} \, dx \right)^{\frac{2n}{n-2s}} \left( \int_{R^n} u(x)^{2s} \, dx \right)^{\frac{2s}{n}}. \]

\( \star \) Integrating over \( x \), we get the desired result.
Theorem. Given $f \in L^1(\mathbb{R})$, there exists a unique weak solution $u \in H^1(\mathbb{R}^n)$ of
\[ \begin{cases} \Delta u + f &= 0 \\ u &= 0 \text{ on } \partial \Omega \end{cases} \]

Proof. Define
\[ X := \{ u \in H^1(\mathbb{R}^n) : u \equiv 0 \text{ on } \partial \mathbb{R}^n \} \subseteq H^1(\mathbb{R}^n) \]

Thanks to the Poincaré inequality, $X$ is a Hilbert space with the scalar product
\[ (u, v)_X = \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2}} \, dx \, dy. \]

- Notice that the weak formulation is then
\[ (u, v)_X = \int_{\Omega} f(x) v(x) \, dx \quad \forall v \in X. \]

- The existence and uniqueness of weak solutions follows then from
  \textit{Riesz representation theorem}.

\underline{Remark. (Viscosity solutions)}

We showed how to prove existence of weak solutions.

On the other hand, one could prove existence of \underline{viscosity solutions} by Perron's method.

When $f \in C^0(\mathbb{R})$, then solutions (weak or viscosity) are pointwise or classical solutions,
and thus they are the same.
1.3. Maximum Principle

- Recall that

\[
(-\Delta)^2 u(x) = C_N \int_{\mathbb{R}^n} (u(x) - u(x+y)) \frac{dy}{|y|^{n+2}} = C_N \int_{\mathbb{R}^n} \frac{2(u(x) - u(x+y))}{|y|^{n+2}} \frac{dy}{|y|^{n+2}}
\]

- It follows from this definition the following:

**Proposition (Maximum Principle).** Assume that \( u \in C^2 \) satisfies \( (-\Delta)^2 u = 0 \) in \( \Omega \subset \mathbb{R}^n \).

Then, \( u \) cannot attain a global maximum inside \( \Omega \) unless \( u \) is constant in all of \( \mathbb{R}^n \).

In other words,

\[
\max_{\mathbb{R}^n} u = \max_{\Omega} u
\]

**Proof:** Assume \( x_0 \in \Omega \) and \( u(x_0) \geq u(z) \) for all \( z \in \mathbb{R}^n \) (\( x_0 \) is a global maximum).

Then,

\[
(-\Delta)^2 u(x_0) = C_N \int_{\mathbb{R}^n} (u(x_0) - u(x+\eta)) \frac{dy}{|y|^{n+2}} \geq 0,
\]

with equality if and only if \( u(x_0) = u(z) \) a.e. in \( \mathbb{R}^n \).

Thus, if \( (-\Delta)^2 u(x_0) = 0 \) and \( u \) has a global maximum at \( x_0 \), then \( u \equiv c \) in \( \mathbb{R}^n \).

- More generally, we have the following:

**Proposition.** Assume that \( u \in C^2 \) satisfies

\[
(-\Delta)^2 u \leq 0 \text{ in } \Omega
\]

\[
\text{in } \mathbb{R}^n \setminus \Omega.
\]

Then, \( u \leq 0 \) in \( \Omega \).

**Proof:** If \( u \) attains positive values in \( \Omega \), then it has a maximum in \( \Omega \).

By the same argument as before, we get \( u \equiv c \). Since \( u \leq 0 \) in \( \mathbb{R}^n \), then the constant must be negative.
Let us next prove the same maximum principle but for weak solutions:

**Proposition.** Let \( \Omega \) be any bounded domain, and let \( u \) be the weak solution of

\[
\begin{align*}
\Delta u &= f \text{ in } \Omega, \\
\frac{1}{|x|_\infty^2} \int_{|x|_\infty^{2}} dx &= g \text{ in } \mathbb{R}^n. 
\end{align*}
\]

Then, \( f \geq 0 \) in \( \Omega \) \( \Rightarrow \) \( u \geq 0 \) in \( \Omega \).

**Proof.** Recall that \( u \) is a weak solution if \( u \in \mathbb{R}^n \), and

\[
\int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))(f(x)-f(y))}{|x-y|_{\infty}^2} \, dx \, dy = \int_{\Omega} f \cdot \phi \, dx
\]

for all \( \phi \in C^1(\Omega) \) with \( \phi \geq 0 \) in \( \mathbb{R}^n \).

Write \( u = u^+ - u^- \), where \( u^+ = \max\{u, 0\} \), \( u^- = \max\{-u, 0\} \).

Take \( \phi = u^- \), and assume \( u^- \) is not identically zero. (Notice \( \phi \geq 0 \) in \( \mathbb{R}^n \).)

Then, since \( f \geq 0 \), we have

\[
\int_{\Omega} f \cdot \phi \geq 0.
\]

On the other hand,

\[
\int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))(f(x)-f(y))}{|x-y|_{\infty}^2} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|_{\infty}^2} \, dx \, dy + \\
+2 \int_{\Omega} \int_{\Omega} \frac{(u(x)-u(x))(u^- (x))}{|x-y|_{\infty}^2} \, dx \, dy
\]

Moreover, notice that \( (u^+(x)-u^+(y))(u^+(x)-u^+(y)) \leq 0 \), and thus

\[
\int_{\Omega} \int_{\Omega} \frac{(u(x)-u(y))(u^+(x)-u^+(y))}{|x-y|_{\infty}^2} \, dx \, dy \leq -\int_{\Omega} (u^+(x)-u^+(y))^2 \, dx \frac{1}{|x-y|_{\infty}^2} \leq 0
\]

Note that the strict inequality is because we are assuming \( u^- \neq 0 \).

Since \( g \geq 0 \), then

\[
\int_{\Omega} \int_{\Omega} \frac{(u(x)-g(y))(u^+(x)-u^+(y))}{|x-y|_{\infty}^2} \, dx \, dy = -\int_{\Omega} \int_{\Omega} \frac{|u^-|^2 + g(x)u^- (x)}{|x-y|_{\infty}^2} \, dx \, dy \leq 0
\]
Putting the previous inequalities together, we have shown that

$$\iint_{\mathbb{R}^2} (u(\mathbf{x}) - u(\mathbf{y})) (\Phi(\mathbf{x}) - \Phi(\mathbf{y})) \, dx \, dy < 0$$

and this contradicts $$\int_\Omega \Phi > 0.$$
- As a consequence of the previous result, we find:

**Corollary (Comparison principle)** If \( u_2 \in C^2 \) and \( u_1 \in C^2 \) satisfy

\[
\begin{align*}
(-\Delta)^2 u_1 &= f_1 \text{ in } \mathbb{R}^n, \\
\Delta u_1 &= g_1 \text{ in } \mathbb{R}^n, \\
\end{align*}
\]

Then

\[
\begin{align*}
\frac{f_1}{f_1} &\geq \frac{f_2}{f_2} \\
\text{and} \\
g_1 \geq g_2
\end{align*}
\]

\[\implies u_1 \geq u_2\]

**Proof.** Use the previous proposition with \( u = u_2, -u_1 \).

### 1.4 (Fundamental solution in \( \mathbb{R}^n \))

- Recall that for the Laplace operator \( -\Delta \), its inverse operator is given by the Riesz potential.

  Indeed, a solution of \(-\Delta u = f(x) \text{ in } \mathbb{R}^n\) with \( f \) decaying at infinity is given by

  \[
u(x) = c \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-2}} \, dz \quad \text{(when } n \geq 3)\]

  In other words,

  \[ (-\Delta)^{-1} f := c \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-2}} \, dz \quad \text{(Riesz potential).} \]

- The function \( \frac{1}{|y|^{n-2}} \) is the fundamental solution of the Laplace operator.
For the fractional Laplacian, we have

\[ (-\Delta)^s f(x) = c \int_{\mathbb{R}^n} \frac{f(z)}{|x-z|^{n-2s}} \, dz \quad \text{for } n > 2s \]

It is called the Riesz potential of order \(2s\).

The function \( \frac{1}{|x|^{n-2s}} \) is the fundamental solution of the fractional Laplacian \((-\Delta)^s\).

- Classical embedding theorems for Riesz potentials yield the following:

\[ \quad \| \frac{1}{|x|^{2s+\alpha}} \|_{L^p(\mathbb{R}^n)} \leq C \left( t^{-s} \left( 1 + t^{n-2s} \right) \right) \]

provided that \( \alpha \in (0,1) \) and \( 2s+\alpha \) is not an integer.

- Essentially, this means that \((-\Delta)^s\) regularizes up to \(2s\) derivatives.

*(The proof of \(\circ\) is very similar to that of the Laplacian, we will not prove it.)*

**Remark: (Hölder spaces)**

When \( \beta \) is not an integer, we will denote \( C^\beta \) the Hölder space of order \( \beta \).

That is,

\[ C^\beta := C^{\beta,0} \text{ if } \beta \in (0,1) \]

\[ C^\beta := C^{\beta,1} \text{ if } \beta \in (1,2) \]

\[ C^\beta := C^{\beta,\gamma} \text{ if } \beta \in (m,m+1) \]

We used this notation in \(\circ\).
1.5. Poisson Kernel and mean value property for s-harmonic functions

- For the Laplace operator, we have the explicit Poisson kernel for a ball:

\[ \Delta u = 0 \text{ in } B_r \quad \Rightarrow \quad u(x) = c \int_{\partial B_r} \frac{g(\xi) (|x-\xi|^2)}{|x-\xi|^{n+2}} dS \quad \text{for } x \in B_r. \]

- This, in turn, yields to the mean value property

\[ \Delta u = 0 \text{ in } B_r \quad \Rightarrow \quad u(x) = \frac{1}{|B_r|} \int_{\partial B_r} u \quad \text{for any } x \in B_r \text{ and } r > 0 \text{ such that } B(x) \subset B_r. \]

- Moreover, integrating in r one gets

\[ u(x) = \frac{1}{R^n} \int_{\partial B_1} u \quad \text{whenever } B(x) \subset B_1. \]

- Another application of the Poisson Kernel is that harmonic functions are \( C^\infty \).

- Let us next see what happens for the fractional Laplacian.

**Poisson Kernel for \((-\Delta)^s\) in a ball**

**Theorem (Poisson Kernel in a ball):**

\[ (-\Delta)^s u = 0 \text{ in } B_r \quad \Rightarrow \quad u(x) = c \int_{\mathbb{R}^n \setminus B_r} \frac{2(\xi) (|x-\xi|^2)^s}{(|x|^2-1)^2 \cdot |x-\xi|^{n+2s}} d\xi \]

- We will not prove the result, it requires really long computations.

- Let us use this result to show some properties of s-harmonic functions.
[Corollary] Assume \((\Delta)^s u = 0\) in \(B_1\). Then, \(u\) is \(C^s\) inside \(B_1\).

Proof. We have the representation
\[
  m(x) = c \int \frac{u(z) \cdot (x-u(z))^s}{(|x-z|^2)^{n/4}} \, dz.
\]
The dependence on \(x\) in the right-hand side is only on the term \(\frac{\sigma}{|x-z|^n}\).
This term is \(C^s\) for \(x\) inside \(B_1\) (since \(|x| > 1\)).

Thus, we can just differentiate under the integral sign as many times as desired, to get that \(u \in C^s(B_\delta)\) (but not up to the boundary).

Exercise. Assume \((\Delta)^s u = 0\) in \(B_1\). Then,
\[
  |D^k u(0)| \leq C^{k-k_1} ||u||_{L^1(\mathbb{R}^n)}.
\]
In particular, \(u\) is analytic.

Proof. \[
  |D^k u(0)| = \left| c \int \frac{\sigma(z) \cdot D^k \left(\frac{1}{|x-z|^1}\right)(0)}{(|x-z|^2)^{n/4}} \, dz \right| \leq C ||u||_{L^1(\mathbb{R}^n)} \int \frac{C \cdot k_1}{(|x-z|^2)^{n/4}} \, dz \leq C^{k-k_1} ||u||_{L^1(\mathbb{R}^n)}.
\]

[Corollary] Assume \((\Delta)^s u = 0\) in \(\Omega \subset \mathbb{R}^n\). Then, \(u\) is \(C^s\) inside \(\Omega\).

Proof. Let \(\delta(x) \subset \Omega\). Rescaling and translating, the corollary we get \(u \in C^s(\delta(x))\).

Since this can be done for any ball \(B(x) \subset \Omega\), \(u\) is \(C^s\) inside \(\Omega\).

Remark. By rescaling the Poisson kernel in \(B_1\), we find the Poisson Kernel in \(B_\delta\):
\[
  (\Delta)^s u = 0 \text{ in } B_\delta, \quad u = g \text{ in } \mathbb{R}^n \setminus B_\delta \quad \Rightarrow \quad m(x) = c \int \frac{g(z) \cdot (r^2-|x-z|^2)^s}{(r^2-|x-z|^2)^{n/4}} \, dz.
\]
Mean value property for \(s\)-harmonic functions

- Using the Poisson kernel in \(B_r\), we find that

\[
(A)^s u = 0 \quad \text{in} \quad B_r \quad \Rightarrow \quad u(0) = c \int_{\mathbb{R}^n \setminus B_r} \frac{u(x)}{(|x-r|^2)^{n/2}} \, dx. 
\]

- In particular, this yields the following:

\[
\text{Proposition (Mean value property)}: \quad \text{If} \quad (A)^s u = 0 \quad \text{in} \quad \Omega, \quad \text{then for every} \quad \xi \in \partial \Omega \\
\quad \text{we have} \\
\quad u(0) = c \int_{\mathbb{R}^n \setminus B_r} \frac{r^{2s} \cdot u(x)}{(|x-r|^2)^{n/2} \cdot |x|^n} \, dx. 
\]

- This is the analogous of \(u(x) = f(x)\) for harmonic functions.

\[
\text{Corollary.} \\
\quad \text{There exists a function} \quad u_\Omega(x) \quad \text{such that,} \\
\quad (A)^s u = 0 \quad \text{in} \quad B_1 \quad \Rightarrow \quad u(0) = \int_{\mathbb{R}^n} u(x) u_\Omega(x) \, dx. 
\]

Moreover, the function \(u_\Omega(x)\) satisfies

\[
\frac{C}{1+|x|^{n+2s}} \leq u_\Omega(x) \leq \frac{C}{1+|x|^{n+2s}} 
\]

\[ \text{Proof.} \]

\[
u(0) = \int_0^1 n \, r^{n-1} u(0) \, dr = c \int \int_{\mathbb{R}^n \setminus B_r} \frac{r^{2s+n-2s} u(x)}{(|x-r|^2)^{n/2} |x|^n} \, dx \, dr = \cdots = \int u(x) u_\Omega(x) \, dx. 
\]

Exercise
\[
\mu(t) = \frac{g_0}{\pi} \int_{R^3 \setminus B_r} \frac{r^{2s}}{(y^2 + r^2)^{n/2}} \, dy
\]

Multiply by \( r^n \) (just in case), and integrate in \( r \):

\[
\frac{\mu(t)}{r^{n+1}} = \mu(0) \int_0^1 r^n \, dr = c_n \int_0^1 \frac{r^{2n+2s} \mu(y)}{(y^2 + r^2)^{n/2}} \, dy = \infty
\]

\[
= c_n \int_0^1 \frac{dy}{(y^2 + r^2)^{n/2}} \int_0^r \frac{r^{2n+2s} \mu(y)}{y^{2n+1}} \, dy + c_n \int_0^1 \frac{dy}{y^{2n+1}} \int_0^r \frac{r^{2n+2s} \mu(y)}{(y^2 + r^2)^{n/2}} \, dy
\]

\( (I_1) \)

\( (I_2) \)

\[
(I_1) = \int_0^1 \frac{\mu(y)}{y^{2n}} \, dy \int_0^r \frac{r^{2s+n} \mu(y)}{(y^2 + r^2)^{n/2}} \, dr = \int_0^1 \frac{\mu(y)}{y^{2n+1}} \, dy \int_0^r \frac{r^{2s+n} \mu(y)}{(y^2 + r^2)^{n/2}} \, dr
\]

\[
(I_2) = \int_0^1 \frac{\mu(y)}{y^{2n+1}} \, dy \int_0^r \frac{r^{2s+n} \mu(y)}{(y^2 + r^2)^{n/2}} \, dr = \int_0^1 \frac{\mu(y)}{y^{2n+1}} \, dy \int_0^r \frac{r^{2s+n} \mu(y)}{(y^2 + r^2)^{n/2}} \, dr
\]

\[
y = n - 1
\]

\[
\frac{\mu(0)}{n} = c_n \int_{R^3 \setminus B_r} \left( \int_0^1 \frac{t^{2s+n-1}}{(1-t)^{n+1}} \, dt \right) \mu(y) \, dy + c_n \int_{B_r} \left( \int_0^1 \frac{t^{2s+n-1}}{(1-t)^{n+1}} \, dt \right) \mu(y) \, dy
\]

\[
\approx \int_{R^3 \setminus B_r} \frac{\mu(y)}{a + |x|^{2s+n}} \, dy
\]

\[
\mu(y) = \begin{cases} 
1 & \text{if } |y| \leq 1 \\
\frac{1}{|y|^{2s+n}} & \text{if } |y| > 1 
\end{cases}
\]
Corollary (Harnack inequality) If \( u > 0 \) in \( \mathbb{R}^n \) and \( (\Delta)^s u = 0 \) in \( B_1 \) then

\[
\sup_{B_r} u \leq C \inf_{B_r} u.
\]

Moreover, both quantities \( \sup_{B_r} u \) and \( \inf_{B_r} u \) are comparable to

\[
\int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{2s}} \, dx.
\]

Proof. By the previous Corollary, we know that

\[
C^{-1} \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{2s}} \, dx \leq u(0) \leq C \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{2s}} \, dx.
\]

Notice that for the lower bound we used that \( u > 0 \) in \( \mathbb{R}^n \).

Now, for every \( x \in B_{\frac{1}{2}} \), we have that \( B_{\frac{1}{2}}(x) \subset B_1 \) and thus \( (\Delta)^s u = 0 \) in \( B_{\frac{1}{2}}(x) \).

This means that

\[
\sup_{B_{\frac{1}{2}}(x)} u \leq C \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{2s}} \, dx = C \int_{\frac{1}{2} + |x|^{2s}} \frac{u(x)}{1 + |x|^{2s}} \, dx.
\]

Since \( 2 + |x|^{2s} \) is comparable to \( 2 + |x|^{2s} \), then

\[
C^{-1} \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{2s}} \, dx \leq u(x) \leq C \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^{2s}} \, dx.
\]

Since the upper and lower bounds do not depend on \( x \), both \( \inf \) and \( \sup \) are comparable to \( \int \frac{u(x)}{1 + |x|^{2s}} \, dx \), as claimed.

Remark. Classical Harnack fails!

Remark 2. All functions are \( s \)-harmonic up to a small error!
1.6. Extension problem for the fractional Laplacian

1. Let \( u: \mathbb{R}^n \to \mathbb{R} \) be a \( C^2 \)-bounded function, and let us consider the operator \( Lu \) defined as follows:

2. Let \( w(x,y) \) be the harmonic extension of \( u(x) \) in \( \mathbb{R}^{n+1}_+ \), that is, the solution of

\[
\begin{align*}
\Delta_y w &= 0 \quad \text{in} \quad \{ y > 0 \} \\
W(x,0) &= u(x) \quad \text{on} \quad \{ y = 0 \}
\end{align*}
\]

(The function \( w \) has an explicit expression in terms of the Poisson kernel of a half-space.)

3. Then, the operator \( Lu \) is defined as

\[ Lu = -2w(x,0) \]

4. This is the Dirichlet-to-Neumann operator.

Given a function \( u: \mathbb{R}^n \to \mathbb{R} \), \( Lu \) is a new \( \frac{1}{2} \) superharmonic function in \( \mathbb{R}^n \).

5. Notice that \( Lu \) is one derivative less regular than \( u \). Also, \( Lu \) is nonlocal, in the sense that depends on the values of \( u \) in all of \( \mathbb{R}^n \). Moreover, \( Lu \) has maximum principle.

6. It turns out that

\[ Lu = (-\Delta)^{\frac{1}{2}} u \]

Indeed, let us see what is \( L(u) \): Define \( v = Lu = -2w(x,0) \), where \( w \) is the harmonic extension of \( u \). Then, the harmonic extension of \( v \) is just \(-2w\)!

Thus, \( Lv = -2(2w) = -3w \)

But since \( \Delta_y w = \Delta w + \Delta_x w = 0 \), then \( Lv = 3w = -\Delta w(x,0) = -\Delta u(x) \).

Thus, \( L(u) = -\Delta u \), so that \( Lu = (-\Delta)^{\frac{1}{2}} u \).
For the fractional Laplacian \((\Delta)^s\) with \(s \in (0,1)\), we have an analogous extension problem.

- Given \(u: \mathbb{R}^n \to \mathbb{R}\), smooth and bounded, we consider the extension \(\tilde{u}(x, y)\) given by

\[
\begin{align*}
\text{div}(y^{1-2s} \nabla \tilde{u}(x, y)) &= 0 \quad \text{in} \quad \mathbb{R}^{m+2} \\
\tilde{u}(x, 0) &= u(x) \quad \text{on} \quad \mathbb{R}^n
\end{align*}
\]

- This is a degenerate weighted PDE when \(s < \frac{1}{2}\), singular when \(s > \frac{1}{2}\), and Laplace equation when \(s = \frac{1}{2}\).

- The extension \(\tilde{u}(x, y)\) has an explicit representation in terms of the Poisson Kernel

\[
P_s(x, y) = C_{n,s} \frac{y^{2s}}{(1|x-y|^2)^{\frac{n+2s}{2}}}
\]

given by

\[
\tilde{u}(x, y) = \int_{\mathbb{R}^n} P_s(x, y, \xi) u(\xi) d\xi
\]

- The relation with the fractional Laplacian is given by

\[
-\lim_{y \to 0} \int_{\mathbb{R}^n} y^{1-2s} \partial_{x} \tilde{u}(x, y) = \mathcal{C}_{n,s} (-\Delta)^s u(x)
\]

- This can be checked either by using the explicit expression of the Poisson kernel, or by Fourier transform.

Notice

\[
\begin{bmatrix}
\Delta_{x,y} \tilde{u} + \frac{1-2s}{y} \partial_y \tilde{u} = 0 \quad \text{in} \quad \mathbb{R}^{m+2} \\
\end{bmatrix}
\]

( equivalent to \(\text{div}(y^{1-2s} \nabla \tilde{u}) = 0 \quad \text{in} \quad \mathbb{R}^n\) )
Some explicit solutions in 1-D

Proposition. The function \( (x_4)_s^4 \begin{cases} x^s & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \) satisfies \( (-\Delta)^s (x_4)_s^4 = 0 \) in \( \{x > 0\} \).

Proof. The function \( r^s \cos \left( \frac{\theta}{2} \right) \) (in polar coordinates) is a solution to
\[
\nabla \cdot (r^{s+2} \nabla \tilde{u}) = 0 \text{ in } \{y > 0\} \\
\tilde{u} = (x_4)_s^4 \text{ on } \{y = 0\}
\]

Thus, \( (-\Delta)^s (x_4)_s^4 = -\lim_{\mu \to 0} r^s \left( \sin \theta \right)^{s+2} \tilde{u}(\mu \cdot r) \)

\[
= -\lim_{\mu \to 0} r^s \left( \sin \theta \right)^{s+2} \tilde{u}(\mu \cdot r) \\
= -\lim_{\mu \to 0} r^s \left( \sin \theta \right)^{s+2} \tilde{u}(\mu \cdot r) \\
= -\lim_{\mu \to 0} r^s \left( \sin \theta \right)^{s+2} \tilde{u}(\mu \cdot r) \\
= -\lim_{\mu \to 0} r^s \left( \sin \theta \right)^{s+2} \tilde{u}(\mu \cdot r)
\]

Exercise

An explicit solution in \((-1,1)\) is given by the following:

Proposition. The function \( u_6(x) = C_6 (1-x^2)_s^5 \) satisfies
\( (-\Delta)^s u_6 = 1 \) in \((-1,1)\)
\( u_6 = 0 \) in \( \mathbb{R} \setminus (-1,1) \)

This is one of the few explicit solutions for \((-\Delta)^s\) in bounded domains.

(The proof consists of long computations, so we will not do it.)
1.7 Regularity estimates

- We have seen several qualitative and quantitative properties of (Δ)^s:
  - Maximum principle and Harnack inequality
  - Poisson Kernel and some explicit solutions
  - Extension problem

- We have also seen that s-harmonic functions are smooth, and the relation between fractional Laplacian and Riesz potentials in \( \mathbb{R}^n \).

- We next turn our attention to regularity estimates: If \( u \) solves the Dirichlet problem
  \[
  (-\Delta)^s u = f \text{ in } \Omega \\
  u = g \text{ in } \mathbb{R}^n \setminus \Omega
  \]
  then what is the regularity of \( u \) inside \( \Omega \)? And what about regularity up to the boundary?

- We will next answer these questions.

**Interior regularity**

Using previous results we saw on s-harmonic functions and Riesz potentials, we next prove:

**Theorem.** Let \( u \in C^{s,\alpha}(\mathbb{R}^n) \) be a solution to \((-\Delta)^s u = f\) in \( B_2 \). Then,

\[
|u|_{C^{s+\alpha,\alpha}(B_2)} \leq C \left( H^1 L^s(B_1) + H^s L^1(B_2) \right),
\]

wherever \( s \) and \( s+\alpha \) are not integers.

**Proof.** Let \( f \) be a function in \( C^0(\mathbb{R}^n) \) with compact support, and with \( f \equiv f \) in \( B_2 \), and with \( \|f\|_{L^s(\mathbb{R}^n)} \leq \|f\|_{L^s(B_1)} \).

Let
\[
W = I_{2s}(f) = c_{s,s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2s}} \, dy,
\]

which satisfies \((-\Delta)^s W = f\) in \( \mathbb{R}^n \).
By the results we stated for reverse potentials, we have
\[ \|\mathcal{W}_{c+2s}(R^n)\| \leq C(\|\mathcal{F}f\|_{L^2(B_2^n)} + \|\mathcal{W}_{c}(R^n)\|) \]

Since \( f \) has support in \( B_2 \), then
\[ \|\mathcal{W}(x)\| \leq C\int_{B_2} \frac{\|\mathcal{F}f\|_{L^2(B_2^n)}}{|x-y|^{n+2s}} \, dy \leq C\|\mathcal{F}f\|_{L^2(B_2^n)} \rightarrow \|\mathcal{W}(x)\| \leq C\|\mathcal{F}f\|_{L^2(B_2^n)} \]

Combining this with \( \|\mathcal{W}_{c+2s}(R^n)\| \leq C\|\mathcal{F}f\|_{L^2(B_2^n)} \), we find
\[ \|\mathcal{W}_{c+2s}(R^n)\| \leq C\|\mathcal{F}f\|_{L^2(B_2^n)} \]

Let now \( \nu = \mu - \lambda_1 \) and notice that
\[ (\Delta)^v \nu = (\Delta)^\mu \mu - (\Delta)^\lambda \lambda = f - f = 0 \quad \text{in} \quad B_1 \]

Moreover,
\[ \|\mathcal{W}_{c}(R^n)\| \leq \|\mathcal{W}_{c}(R^n)\| + \|\mathcal{W}_{c+2s}(R^n)\| \leq C\|\mathcal{F}f\|_{L^2(B_2^n)} \]

Therefore, \( \nu \in L^\infty(B_2^n) \) is an \( s \)-harmonic function in \( B_2 \), and thus satisfies
\[ \|\mathcal{W}_\beta(B_2^n)\| \leq C(\|\mathcal{W}_{c+2s}(R^n)\|) \]

For any \( \beta > 0 \). Taking \( \beta = c+2s \), we find
\[ \|\mathcal{W}_{c+2s}(B_2^n)\| \leq \|\mathcal{W}_{c+2s}(B_2^n)\| + \|\mathcal{W}_{c+2s}(B_2^n)\| \leq C(\|\mathcal{F}f\|_{B_2^n} + C\|\mathcal{W}_{c}(B_2^n)\|) \leq C(\|\mathcal{F}f\|_{B_2^n} + \|\mathcal{W}_{c}(B_2^n)\|) \]
• This is the main interior regularity estimate for the fractional Laplacian.

• When $f$ is not $C^0(B)$ but only $C^0(B)$, the previous argument yields $u \in C^{s+\varepsilon}(B_\varepsilon)$ for all $\varepsilon > 0$. (We will see this in detail in next Chapter.)

• The previous regularity estimate is for functions that are bounded in $\mathbb{R}^n$.

We next give a Corollary for functions that may have some growth at infinity.

We will denote

$$\|w\|_{L^2_w(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \frac{|w(x)|^2}{1 + |x|^{n+2s}} \, dx \right)^{\frac{1}{2}}$$

Recall that this quantity already appeared in the Heisenberg inequality for the fractional Laplacian.

Remark: This is the minimum integrability required so that $C^0(B)$ makes sense.

$$\begin{align*}
(C^0)^\star w(x) & = c \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x - y|^{n+2s}} \, dy \\
& = c \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|y|^{n+2s}} \, dy + \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|y|^{n+2s}} \, dy
\end{align*}$$

$$\begin{align*}
& = c \left( \int_{R^n} \frac{w(x)}{|x|^{n+2s}} \, dx \right) \underset{\text{comparable to}}{\leq} \int_{\mathbb{R}^n} \frac{|w(y)|}{1 + |y|^{n+2s}} \, dy =: \|w\|_{L^\infty_w(\mathbb{R}^n)}.
\end{align*}$$

Corollary: Let $u \in L^2_w(\mathbb{R}^n)$ be a solution of $C^0 u = f$ in $B_1$. Then,

$$\|u\|_{C^{s+\varepsilon}(B_\varepsilon)} \leq C \left( \|f\|_{L^2_w(\mathbb{R}^n)} + \|u\|_{L^\infty_w(B_1)} + \|u\|_{L^2_w(\mathbb{R}^n)} \right)$$

provided that $s$ and $\varepsilon$ are not integers.
Proof. We consider $\vec{u} = \mu \vec{x}_{B_2}$ and see what is the equation that it satisfies.

First, notice that

$$(\Delta)^5 \vec{u} = \frac{(\Delta)^5 \vec{u} - (\Delta)^5 (\mu \vec{x}_{B_2})}{f - h}$$

We want to see that $f \in C^4(B_2)$. In order to apply the previous theorem, we need to prove that $h \in C^4(B_2)$, namely

$$\|h\|_{C^4(B_2)} \leq C \|u\|_{L^1(\mathbb{R}^n)}$$

for $x, \bar{x} \in B_2$

Let us show this. Take $x \in B_2$, then

$$h(x) = c \int_{\mathbb{R}^n} \frac{(\mu \vec{x}_{B_2}) (\vec{x}) - (\mu \vec{x}_{B_2}) (\bar{z})}{1|x-\bar{z}|^{n+2s}} d\bar{z} = -c \int_{\mathbb{R}^n} \frac{\mu(\bar{z})}{1|x-\bar{z}|^{n+2s}} d\bar{z}$$

and thus,

$$(h(x) - h(\bar{x})) \leq c \int_{B_2^c} \mu(\bar{z}) \left( \frac{1}{1|x-\bar{z}|^{n+2s}} - \frac{1}{1|\bar{x}-\bar{z}|^{n+2s}} \right) d\bar{z}$$

Now, we can differentiate under the integral sign in $\mathbb{R}^n$ and get

$$|D^{k} h(x)| \leq c \int_{B_2^c} \frac{|\mu(\bar{z})|}{1|x-\bar{z}|^{n+2s+k}} d\bar{z} \leq c \int_{B_2^c} \frac{|\mu(\bar{z})|}{1+|\bar{z}|^{n+2s+k}} d\bar{z} \leq C \mu \|u\|_{L^1(\mathbb{R}^n)}$$

In particular, we get

$$\|h\|_{C^4(B_2)} \leq C \|u\|_{L^1(\mathbb{R}^n)}$$

(An alternative way would be to estimate the RHS in $\mathbb{R}^n$.)
Boundedness of weak solutions

- We next prove the following:

**Proposition.** Let $\Omega$ be any bounded domain, and $u$ be the weak solution of

$$(-\Delta)u = f \text{ in } \Omega, \quad u = g \text{ on } \partial \Omega,$$

with $f \in L^p(\Omega)$ and $g \in L^q(\partial \Omega)$. Then,

$$\|\nabla u\|_{L^2(\Omega)} \leq \|g\|_{L^q(\partial \Omega)} + \|f\|_{L^p(\Omega)}.$$

**Proof.** Let $B_R$ be a large ball in $\mathbb{R}^n$ such that $\Omega \subset B_R$.

Let $\varphi \in C_c^\infty(B_R)$ and such that

$$0 \leq \varphi \leq 1 \text{ in } \mathbb{R}^n, \quad \varphi \equiv 1 \text{ in } \Omega.$$

Then, for each $x \in \Omega$, we have $\varphi(x) = \max_{\mathbb{R}^n} \varphi$, and thus

$$(-\Delta)^s \varphi (x) = \frac{\omega_s}{n} \int_{\mathbb{R}^n} \frac{\varphi(x) - \varphi(y)}{|x-y|^{n+2s}} \, dy \geq c_s \int_{\mathbb{R}^n} \frac{dy}{|y|^{n+2s}} \geq c \frac{dy}{|y|^{n+2s}} = c \frac{dy}{|y|^{n+2s}} = \psi(x).$$

Hence, $\varphi$ satisfies

$$(-\Delta)^s \varphi \geq c > 0 \text{ in } \Omega$$

$$\varphi = 1 \text{ in } \Omega$$

$$\varphi > 0 \text{ in } \mathbb{R}^n.$$

Let now $v(x) = \|g\|_{L^q(\partial \Omega)} + \frac{1}{c_s} \|f\|_{L^p(\Omega)} \cdot \varphi(x)$.

Then,

$$(-\Delta)^s v \geq \|g\|_{L^q(\partial \Omega)} \text{ in } \Omega$$

$$v \geq \|g\|_{L^q(\partial \Omega)} \text{ in } \mathbb{R}^n.$$

In particular,

$$(-\Delta)^s v \geq (\Delta)^s u \text{ in } \Omega$$

$$v \geq u \text{ in } \mathbb{R}^n.$$

Thus,

$$\|\nabla u\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} \leq \|g\|_{L^q(\partial \Omega)} + \frac{1}{c_s} \|f\|_{L^p(\Omega)}.$$
Recall 1D example

- Solutions are smooth inside $\Omega$ (by interior regularity), but not up to the boundary (by the previous example).

In view of the example, it looks natural to conjecture that $u \in C^{1,1}(\Omega)$. This is what we will prove in this section:

\[ \begin{cases} (\Delta)^5 u = f \text{ in } \Omega, & \\
0 \text{ in } \mathbb{R}^n \setminus \Omega, & \\
\Omega \text{ is a } C^2 \text{ domain} \end{cases} \implies \|u\|_{C^{1,1}(\Omega)} \leq C\|f\|_{L^2(\Omega)} \]

- To prove this, the strategy is to first prove that $\|u\| \leq C d^s$, where $d(x) = \text{dist}(x, \partial \Omega)$, and then combine this with interior estimates to get $u \in C^{1,1}(\Omega)$.

- To establish $\|u\| \leq C d^s$ in $\Omega$, we will need to construct suitable barriers.

**Barriers**

\[ \text{Proposition. For any } e \in S^{n-1}, \text{ the function } u(x) = (x \cdot e)_+^s \text{ satisfies} \]

\[ \begin{cases} (\Delta)^5 u = 0 \text{ in } \{x \cdot e > 0\} & \\
0 \text{ in } \{x \cdot e \leq 0\} & \\
\end{cases} \]

**Proof.** Recall that $(\Delta)^5$ solves $(\Delta)^5 u = 0 \text{ in } \mathbb{R}^n$, using this, we find

\[
\int_{\mathbb{R}^n} \frac{2u(x) - u(x+e) - u(x-e)}{r^{n+2s}} \, dr \, ds = \int_{\mathbb{R}^n} \frac{2u(x) - u(x+re) - u(x-re)}{r^{n+2s}} \, dr \, ds
\]

(polar coordinates)

\[
= \frac{1}{2} \int_{S^n} \left( \int_0^\infty \left( \frac{2u(x) - u(x+re) - u(x-re)}{r^{n+2s}} \right) \, dr \right) \, d\sigma
\]
Now, using that \( \mu(x) = (x \cdot e)^s \), we have

\[
2u(x) - \mu(x+r) - \mu(x-r) = 2(x \cdot e)^s - (x \cdot e + (x \cdot e) r)^s - (x \cdot e - (x \cdot e) r)^s
\]

Therefore, for any \( \sigma \in \mathbb{R}^n \) with \( x \cdot e \leq 0 \), we have

\[
\int_0^\infty \left[ 2u(x) - \mu(x+r) - \mu(x-r) \right] dr = \int_0^\infty \left\{ (x \cdot e)^s - (x \cdot e + t)^s - (x \cdot e - t)^s \right\} dt \left[ \frac{\sigma \cdot e}{r} \right]^{s-1} \frac{dr}{r^{s+2s}} = 0
\]

(since \( \Delta^s (x \cdot e)^s = 0 \) in \( \mathbb{R}^+ \)).

This gives us

\[
\int_{\mathbb{R}^n} \frac{2u(x) - \mu(x+y) - \mu(x-y)}{|y|^{n+2s}} dy = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \left( \int_{\mathbb{R}^n} \left[ 2u(x) - \mu(x+r) - \mu(x-r) \right] dr \right) dy = 0
\]

This is the first and most simple barrier in \( \mathbb{R}^n \). It is a \( 1D \) function in \( \mathbb{R}^s \).

Notice that with this barrier we can already show that \( u \in C_0^s \) in convex domains \( \Omega \subset \mathbb{R}^n \).

Indeed, if \( u \) solves \( \Delta^s u = 0 \) in \( \Omega \) (with \( f \in L^2(\Omega) \) and \( \Omega \) convex), then we do

\[
\Delta^s u = 0 \quad \text{in} \quad \Omega - \mathbb{R}^n
\]

We let \( \xi \in C^\infty_0(\mathbb{R}^n) \) be a function with compact support outside \( \Omega \) and with \( \xi \geq 0 \) in \( \mathbb{R}^n \). (but \( \xi \not\equiv 0 \)).

Then, we have \( \Delta^s \xi \leq -c_0 \) in \( \Omega \), for some \( c_0 > 0 \).

Now, for each \( x \in \Omega \) define \( x^* \in \mathbb{R}^n \) such that \( x - x^* = d(x) \) (the closest point to \( x \) on \( 2\Omega \)).

We define \( \phi(x) = (\phi(x-x^*) \cdot e \cdot e)^s \) (a translation of our \( 1D \) barrier), which solves \( \Delta^s \phi = 0 \) in \( \mathbb{R}^n \) (since \( \phi \in C_0^s \{ x-x^* \} \) by convexity).

Take now

\[
W = C_1 \phi \ast C_2 \xi
\]

with \( C_2 \) large enough so that \( \Delta^s W \geq \frac{1}{h^s} \) in \( \Omega \), and \( C_1 \) large enough so that \( W \geq 0 \) in \( \mathbb{R}^n \).
Then, we have
\[ (-\Delta)^{\frac{n}{2}} u \leq (M^\prime(x))^\frac{n}{2} \leq (-\Delta)^{\frac{n}{2}} w \quad \text{in } \mathbb{R} \]
\[ u \equiv 0 \leq w \quad \text{in } \mathbb{R}^n. \]

In particular,
\[ u(x) \leq C_1 \phi(x) = C_1 |x-x_0|^\frac{n}{2} = C_1 d(x), \]

since this can be done for any \( x \in \mathbb{R} \), we get \( u \leq C d^\frac{n}{2} \text{ in } \mathbb{R} \).

Replacing \( u \) by \( -u \), we get \( |u| \leq C d^\frac{n}{2} \text{ in } \mathbb{R} \).

Essentially, the idea was to use \( \phi \) as a barrier for \( u \), but we had to modify \( \phi \) because of the right-hand side (RHS) of the equation.

Now, we want to do the same but in general \( C^2 \)-domains \( \Omega \) (not necessarily convex).

For this, we need to construct a barrier of this type:

In this way we will touch \( \Omega \) from outside with balls, and this barrier will lead to \( u \leq C d^\frac{n}{2} \).

In \( B_1 \), we have an explicit solution:

\[
\begin{align*}
\text{Prop. Solution in } B_1 & \quad \text{The function } u(x) = |x|^{-\frac{n}{2}} \left( 1 - |x|^2 \right)^\frac{n}{2} \quad \text{solves } \\
& \quad \begin{cases}
(-\Delta)^{\frac{n}{2}} u = K \quad \text{in } B_1 \\
u = 0 \quad \text{in } \mathbb{R}^n,
\end{cases}
\end{align*}
\]

where \( K > 0 \) is a constant.

Proof: Exercise (deduce it from the case \( n=1 \)).
However, for the construction of a barrier in $\mathbb{R}^n \setminus \mathcal{B}_1$, we have no explicit solution, and such construction is more complicated. This is what we will do next.

**Superduation in $\mathbb{R}^n \setminus \mathcal{B}_1$**

\[\text{Lemma. Let } \mathcal{E} \subseteq (\mathcal{E}_0, \mathcal{E}_1). \text{ Then, the function } \psi(x, \epsilon) \in \mathbb{R} \text{ satisfies}
\]
\[
\begin{cases}
(-\Delta)^s \psi(x, \epsilon) - \epsilon^{s-\sigma} \psi(x, \epsilon) = 0 \\
\psi(x, \epsilon) = 0
\end{cases}
\]

with $\psi < 0$.

**Proof.** We prove it in case $n=1$ (the proof is almost the same).

Since the function $\psi(x) = (-\Delta)^{s-\sigma} \psi(x)$ is homogeneous of degree $s-\sigma$, the $(-\Delta)^{s-\sigma} \psi$ will be homogeneous of order $(s-\sigma) - 2s = \sigma - s$ (recall that $(-\Delta)^s$ is of order $2s$).

Thus,

\[(-\Delta)^{s-\sigma} \psi(x) = \psi(x) \text{ in } (0, \infty),
\]

for some $\psi \in \mathbb{R}$. We need to check the sign of $\psi$.

We slide the function $(x-h)^s$ from the right until we touch $(x-h)^{s-\sigma}$. Namely consider $(x-h)^s$. For $h$ large, this function is below $(x-h)^{s-\sigma}$. We make $h$ small until they touch at one point $x_0$. Then,

\[
\psi(x_0) = \psi(x_0) - \psi(x_0) < (\Delta)^s \psi(x_0) = 0,
\]

and thus $\psi < 0$.

**Remark.** Here, we used that if $V \equiv W \in \mathbb{R}^n$ and $v(x) = w(x)$, then with strict inequality unless $V \equiv W$ a.e. in $\mathbb{R}^n$.  

\[
(-\Delta)^s \psi(x) = \int_{\mathbb{R}^n} [\psi(x) - \psi(x-h)] (-\Delta)^s \psi(x-h) \, dx \leq \int_{\mathbb{R}^n} \left[ \frac{1}{2} |v(x) - v(x-h)|^2 \right] \psi(x) \, dx \leq \int_{\mathbb{R}^n} \left[ \frac{1}{2} |v(x) - v(x-h)|^2 \right] \psi(x) \, dx.
\]
We next show the following.

**Lemma.** Let $B_1$ be the unit ball in $\mathbb{R}^n$, and $u(x) = (x_1 - 1)^{5+\varepsilon}$. Then,

- If $\varepsilon = 0$, \[ 0 \leq -\varepsilon \frac{\partial}{\partial x_1} u \leq C_{\varepsilon} \left( \log |x_1 - 1| + 1 \right) \quad \text{in} \quad B_2 \setminus B_1. \]
- If $\varepsilon > 0$, \[ -\varepsilon \frac{\partial}{\partial x_1} u \geq C_{\varepsilon} (x_1 - 1)^{5+\varepsilon - 5} \quad \text{in} \quad B_2 \setminus B_1. \]

**Proof.** For $x \in \mathbb{R}^n$, we denote $x = (x', x_n)$. To show the lemma, we compute $(-\Delta) u(x)$ where $x_0 = (0, t, 1)$, $t \in (0, 1)$.

To estimate $(-\Delta) u(x_0)$, we subtract the $d$-function $u(x) = (x_n - 1)^{5+\varepsilon}$ which satisfies

\[ (-\Delta)^2 u(x_0) = \begin{cases} 0 & \text{if } \varepsilon = 0 \\ C_{\varepsilon} e^{-5} & \text{if } \varepsilon > 0 \end{cases} \quad \text{(with } C_{\varepsilon} \text{ as)} \]

Note that $u \equiv 0$ in $\mathbb{R}^n$ and $u(x_0) = u(x_0')$ for all $p > 0$.

and that, for $1 \ll t$,

\[ 0 \leq \begin{cases} (x_n - 1)_+ - (p + t)^{5+\varepsilon}_+ \leq C |y|^2 \\ \text{(Exercise)} \end{cases} \]

Thus,

\[ 0 \leq (-\Delta)^2 (x_n - 1)_+ \leq \begin{cases} C \varepsilon e^{-5} & \text{for } \varepsilon \in \mathbb{R}^n \\ C |y|^{2+\varepsilon} & \text{for } \varepsilon \in \mathbb{R}^n \setminus B_2 \\ C |y|^{5+\varepsilon} & \text{for } \varepsilon \in \mathbb{R}^n \setminus B_2 \end{cases} \]

Therefore,

\[ \Delta u \geq (-\Delta)^2 (x_n - 1)_+ \quad \text{in} \quad B_2 \setminus B_1 \]

\[ \geq \begin{cases} -C (\log |x_1 - 1| + 1) & \text{if } \varepsilon = 0 \\ -C & \text{if } \varepsilon > 0 \end{cases} \]
Using now that
\[
(-\Delta)^s u(x) = (-\Delta)^s (u(x)) + (-\Delta)^s (v(x)),
\]
we find
\[
-C(\delta \rho_{\rho(x)}) \leq (-\Delta)^s u(\rho(x)) \leq 0 \quad \text{if } \rho \leq 0,
\]
\[
-C \rho^s \leq (-\Delta)^s u(\rho(x)) \leq C \rho^s \quad \text{if } \rho > 0,
\]
with \(C < 0\) (by previous lemma).

Using the previous lemma, we can now construct a supersolution.

Prop. (Supersolution). Let \(se(\delta)\). There exists \(\delta > 0\) and a radial function \(v(\rho)\) such that
\[
(-\Delta)^s v \geq 1 \quad \text{in } B_{\delta + \epsilon} \setminus \overline{B_\delta},
\]
\[
v = 0 \quad \text{in } B_\delta,
\]
\[
0 \leq v \leq C(1 + \|x\|)^{-s} \quad \text{in } \mathbb{R}^n \setminus B_{\delta + \epsilon},
\]
\[
1 \leq v \leq C \quad \text{in } \mathbb{R}^n \setminus B_{\delta + \epsilon},
\]
\[
(\text{Supersolution})
\]

**Proof.** Let
\[
v(x) = \begin{cases} 2(1 + \|x\|)^{-s} & \text{in } B_{\delta + \epsilon} \setminus \overline{B_\delta}, \\
1 & \text{in } \mathbb{R}^n \setminus B_\delta, \end{cases}
\]
with \(\epsilon \in (0, \delta)\) (for example, \(\epsilon = \delta/2\)).

By previous lemma, we have in \(B_{\delta + \epsilon} \setminus \overline{B_\delta}\)
\[
(-\Delta)^s v(x) \geq -C(1 + \|x\| + \|x\|^{-s}) + C(1 + \|x\|)^{-s} - C \quad \text{in } B_{\delta + \epsilon} \setminus \overline{B_\delta},
\]
with \(C > 0\). Thus, if \(\delta > 0\) is small, we have
\[
(-\Delta)^s v(x) \geq 1 \quad \text{in } B_{\delta + \epsilon} \setminus \overline{B_\delta},
\]
as desired.
Boundary regularity

Proposition. Let \( \Omega \) be any \( C^2 \) and bounded domain. Let \( u \) be the weak solution of
\[
\begin{align*}
\Delta u &= f \text{ in } \Omega \\
\mathbf{n} \cdot \nabla u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]
with \( f \in L^2(\Omega) \). Let \( d(z) = \text{dist}(z, \partial \Omega) \). Then,
\[
|u(z)| \leq C(\|f\|_{L^2(\Omega)}) d(z) \quad \text{in } \Omega,
\]
with \( C \) depending only on \( \Omega \) and \( f \).

Proof. Since \( \Omega \) is \( C^2 \), there exists \( \rho > 0 \) such that any point of \( \partial \Omega \) can be touched from outside with a ball of radius \( \rho \) contained in \( \Omega \). (Receding \( \rho \) if necessary, we may assume that \( \rho = 1 \).)

Thus, we only need to prove \( |u(x)| \leq C(\|f\|_{L^2(\Omega)}) d(z) \quad \text{for } x \in \Omega \text{ close to } \partial \Omega. \)

Let \( \delta \) be small, and let \( \rho \) be the recombination constructed before,
\[
\begin{align*}
\Delta \rho &\geq 1 \quad \text{in } B_{2\delta} \setminus B_{\delta} \\
\rho &= 0 \quad \text{in } B_{\delta} \\
\rho &\geq 1 \quad \text{in } \Omega \setminus B_{2\delta} \\
\rho &\leq C(\frac{1}{\delta}, \|f\|_{L^2(\Omega)}) \quad \text{in } B_{2\delta} \setminus B_{\delta}
\end{align*}
\]

For any \( x \in \Omega \) such that \( d(z) > \delta \), let \( \rho(x) = \epsilon \) be such that \( x + \epsilon \rho = \rho(x) \) (projection of \( x \) on \( \partial \Omega \)). Let \( z \) be the center of the ball \( B_{\rho(z)}(z) \subset \Omega \) such that \( x \in B_{\rho(z)}(z) \).

Then, the function \( W(x) = \rho(\rho(x)) \) satisfies:
\[
\begin{align*}
\Delta W &= \Delta \rho \geq 1 \quad \text{in } (B_{3\delta} \setminus B_{2\delta}) \setminus \partial \Omega \\
W &\geq 0 \quad \text{in } \Omega \setminus \partial \Omega \\
W &\geq C(\frac{1}{\delta}, \|f\|_{L^2(\Omega)}) \quad \text{in } \Omega \setminus B_{\rho(z)}(z)
\end{align*}
\]
Thus, \( w = w \) in \( \Omega \), and in particular \( w(x) \geq 0 \).

This means that \( w(x) \leq C(1, \|f\|_{L^2(\Omega)}) d(z) \), as desired.
Theorem. Let \( \Omega \) be any \( C^2 \) and bounded domain, and \( u \) be the weak solution to
\[
\Delta u = f \text{ in } \Omega,
\]
with \( f \in L^2(\Omega) \). Then,
\[
\|u\|_{C^2(\Omega)} \leq C \|f\|_{L^2(\Omega)},
\]
with \( C \) depending only on \( \Omega \) and \( \Delta u \).

Proof: Let \( x_1, x_2 \in \Omega \), and let us prove that
\[
\|u(x_1) - u(x_2)\| \leq C \|f\|_{L^2(\Omega)} |x_1 - x_2|.
\]

Noticing that we used \( f \leq Cd^2 \) in \( \Omega \), proceed before.

- If \( \frac{r}{2} < |x_1 - x_2| \), then \( B_{\frac{r}{2}}(x_2) \subset \Omega \). Therefore, any function \( \tilde{u}(x) = u(x_1 + r \xi) \)
  satisfies:
  \[
  \tilde{u} \in C^2(B_1),
  \]
  \[
  (\Delta)^2 \tilde{u} = f, \quad \text{ in } B_1.
  \]
  Notice that \( u(x_1) - u(x_2) \) is finite, \( |x_1 - x_2| \leq C \|f\|_{L^2(\Omega)} |x_1 - x_2|.

- In particular, \( \|u\|_{C^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \).
Remarks. The main part of the proof was $L^\infty \leq C d^s$. Once this was proved, we had two cases, depending on how close are $x_1$ and $x_2$, relative to their distance to $\partial \Omega$.

- When they are very close to each other, and far from $\partial \Omega$, then $u \in C^s$ by interior estimates.
- When they are very close to each other, but closer to the boundary, then we just used $L^\infty \leq C d^s$.
- When they are very close to $\partial \Omega$, but even closer to each other, we needed to combine (rescaled) interior estimates with $L^\infty \leq C d^s$.

A minor modification of the proof of the previous theorem yields the following:

**Proposition:** Assume $\Omega$ is $C^2$ in $B_R$, and $u$ satisfies

\[
\Delta^s u = f \text{ in } \Omega \cap B_R,
\]

\[
\mu = 0 \text{ in } \Omega \cap \partial B_R.
\]

Then,

\[
\|u\|_{C^s(\Omega \cap B_{R/2})} \leq C \left( \|\Delta^s u\|_{L^2(\Omega \cap B_{R/2})} + \|f\|_{L^2(\Omega \cap B_{R/2})} \right)
\]

with $C$ depending on $s$ and $\Omega$.

**Exercise:** Show this Proposition.

We next show that, when $\mu > 0$, then $C d^s \leq \mu \leq C d^s$ in $\Omega$ for all $C^2$ domains $\Omega$. 
Subsolution and Hopf's Lemma

Proposition (Hopf's Lemma): Let \( \Omega \) be an \( C^2 \)-domain in \( \mathbb{R}^n \), and assume that \( u \) satisfies

\[
\begin{align*}
(-\Delta) u &\leq 0 \text{ in } \Omega \\
\n &\text{and } u = 0 \text{ on } \partial \Omega.
\end{align*}
\]

Then, either \( u \equiv 0 \) in \( \mathbb{R}^n \), or \( \exists c \geq 0 \) such that

\[
u \equiv cd^\frac{n}{n-2} \text{ in } \Omega.
\]

for some \( c > 0 \).

To show this, we need:

Lemma (Subsolution): There is a function \( \psi \) satisfying

\[
\begin{align*}
(-\Delta) \psi &\leq -1 \text{ in } B_1 \setminus B_{\frac{1}{4}} \\
\psi &> 0 \text{ in } B_1 \\
\psi &\geq (g - 1)_+ \text{ in } B_{\frac{1}{2}} \\
\psi &\equiv 0 \text{ on } \partial B_1.
\end{align*}
\]

Proof: We know that \( v(x) = (g - 1)_+ \) satisfies

\[
(-\Delta) v = K \text{ in } B_1 \\
v \equiv 0 \text{ on } \partial B_1, \quad \text{with } K > 0.
\]

Let \( q \in C^2(B_{\frac{1}{2}}) \) such that \( \int_{B_{\frac{1}{2}}} q = 1 \), \( q > 0 \).

Then,

\[
(-\Delta) \psi \leq -c < 0 \text{ in } B_1 \setminus B_{\frac{1}{4}}.
\]

Thus, \( (g + (a - 1)_+)^2 + \psi \geq 0 \) satisfies

\[
(-\Delta) \psi = K - \psi \leq -1 \text{ in } B_1 \setminus B_{\frac{1}{4}},
\]

if \( g > 0 \) is large enough.

Proof of proposition. By maximum principle, either \( u \equiv 0 \) or \( u > 0 \) in \( \Omega \).

Rescaling if necessary, for every \( x \notin \Omega \) there is \( z \in \Omega \) such that \( g(y) \leq x \) and \( x^* \in \partial \Omega \).

Since \( u > 0 \) in \( \ Omega \), \( u \equiv 0 \) on \( \partial \Omega \). Using the subsolution, we find

\[
u(x) \geq cd^\frac{n}{n-2} \text{ for all } x \text{ in the segment from } z \text{ to } x^*.
\]

Thus, \( u \geq cd^\frac{n}{n-2} \) in \( \{d \geq \frac{1}{2}\} \), and we are done.
Summary of Chapter 1

\[(\Delta)^s u(x) = c_s \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy = \frac{c_s}{2} \int_{\mathbb{R}^n} \frac{\partial^2 u(x)}{|x-y|^{n+2s}} \, dy\]

[\((\Delta)^s \) is nonlocal, of order 2s, translation/rotation/scale invariant.]

- Maximum and comparison principle (global):

\[\text{if } s < \frac{n}{2}, \text{ then } \exists M, m \geq 0 \text{ such that } M \Delta^s u \geq m \Delta^s u \text{ in } \mathbb{R}^n\]

- Existence and uniqueness of weak solutions

- Regularity:

\[\Delta^s u = f \in C^0(B_1) \Rightarrow u \in C^{2s+m}(B_{1/2})\]

\[\Delta^s u = 0 \text{ in } B_1 \Rightarrow u \in C^{2s-\varepsilon}(B_1) \text{ for all } \varepsilon > 0\]

- Harnack inequality:

\[\Delta^s u = 0 \text{ in } B_1 \Rightarrow 0 \leq u \leq C_0 |B_1|^{-1/2s} \text{ in } B_{1/2}\]

- Boundary regularity:

\[\Delta^s u = f \text{ on } \partial B_1 \Rightarrow \partial u \in C^{s-\varepsilon}(\partial B_1) \text{ for all } \varepsilon > 0\].