3. Fully nonlinear equations

3.1. Controlled diffusions

- We consider a random process $X_t^\xi$, with a control parameter $\delta: (\omega, \mathcal{F}) \rightarrow \Gamma$.
  
  The set $\Gamma$ is the set of all possible values of the parameter $\delta$.

- Depending on the choice of $\delta(\omega)$, we obtain different processes $X_t^\xi$.

- Given $\delta(\omega)$, we can consider the expected payoff

\[ E[g(X_t)] \]

where $\tau$ is the first time that $X_0^\xi$ hits $\partial D$.

- Now, we consider the following stochastic control problem:

  What is the best choice of the control $\delta(\omega)$ so that the expected payoff is maximum? And what is the maximum expected payoff?

- To answer these questions, we define

\[ \mathcal{U}(x) = \sup_{\delta \in \Gamma} E[g(X_0^\xi)] \]

- The value $\mathcal{U}(x)$ is to be chosen on the basis of observations of the controlled process $X_t^\xi$ before time $\tau$. (We don't know the future!). (Since the future is independent from past, we only care about the present!)

- Given a value $\delta(\omega) \in \Gamma$, the process $X_t^\xi$ (with a constant $\delta$) has infinitesimal generator $\mathcal{L}_\delta$. ($\mathcal{L}_\delta$ is a second order operator $\mathcal{L}_\delta = \sum_{ij} a_{ij} dX_t^i X_t^j$).
If $u_t = 0$ in $\mathbb{R}^N$, then the problem is

$$u_t = 0 \quad \text{in} \quad \mathbb{R}^N \times (0, T].$$

This means that $u$ is the maximum expected profit.

Moreover, we have

$$\sup_{x \in \mathbb{R}^N} u(x) = \lambda = \text{constant}.$$
Similar considerations within two-player stochastic games lead to the nonlinear operators

\[ I_u(x) = \inf_{\beta \in \mathcal{B}} \sup_{w \in \mathcal{W}} \mathbf{L}_{\beta}(u(x)) \]

- When running costs are considered, we get the problem

\[ I_u = 0 \quad \text{in} \quad \mathcal{D}; \quad u = g \quad \text{in} \quad \mathcal{R}\mathcal{S}\mathcal{D}. \]

\[ I_u = \sup_{g \in \mathcal{T}} (g \cdot u + g) \quad \text{(Stochastic control)} \]

\[ I_u = \inf_{\beta \in \mathcal{B}} \sup_{w \in \mathcal{W}} (\mathbf{L}_{\beta} u + C_{\psi}) \quad \text{(zero-sum games)} \]

Second order equations

- When \( L_g u \) or \( L_{\psi} u \) are second order uniformly elliptic operators of the form

\[ L_u(x) = \sum_{ij} a_{ij}(x) \partial^2 u(x), \quad \lambda \mathbf{I} \leq (a_{ij}) \leq \Lambda \mathbf{I} \]

- Then \( I_u \) is a fully nonlinear uniformly elliptic operator \( F(I_u) = 0 \).

- When \( I_u = \sup_{g \in \mathcal{T}} (L_g u + g) \) then \( F \) is convex.

- When \( I_u = \inf_{\beta \in \mathcal{B}} \sup_{w \in \mathcal{W}} (L_{\psi} u + C_{\psi}) \) then \( F \) is in general non-convex.

Extremal operators

- In any case, we have:

**Lemma.** Assume that all \( L_g \) and \( L_{\psi} \) belong to a class of linear operators \( \mathcal{L} \).

Let \( I_u = \sup_{g \in \mathcal{T}} (L_g u + g) \) or \( I_u = \inf_{\beta \in \mathcal{B}} \sup_{w \in \mathcal{W}} (L_{\psi} u + C_{\psi}) \).

Then,

\[ \inf_{v \in \mathcal{L}} I_v \leq I(u+v) - I_u \leq \sup_{v \in \mathcal{L}} I_v \]

**Proof.** We check one case:

\[ \sup_{g \in \mathcal{T}} (L_g u + g) - \sup_{g \in \mathcal{T}} (L_g u + g) \leq \sup_{w \in \mathcal{W}} L_w v \leq \sup_{v \in \mathcal{L}} I_v \]

For the case \( \psi \):

\[ L_{\psi} (u+v) \leq L_{\psi} u + \sup_{w \in \mathcal{W}} L_w v \rightarrow \inf_{v \in \mathcal{L}} \sup_{w \in \mathcal{W}} v \rightarrow I(u+v) \leq I(u) + \sup_{v \in \mathcal{L}} I_v \]
When $\mathcal{L}$ is the class of second order uniformly elliptic operators, these are called
Fucci operators

$$M_{\mathcal{L}}^+(D^2 u) = \sup_{\lambda \in (a_{ij})^{-1}, \lambda I} \left( \sum a_{ij} \partial_{ij} u \right)$$

$$M_{\mathcal{L}}^-(D^2 u) = \inf_{\lambda \in (a_{ij})^{-1}, \lambda I} \left( \sum a_{ij} \partial_{ij} u \right)$$

For a general class $\mathcal{L}$ of nonlocal operators, we define the extremal operators

$$M_{\mathcal{L}}^+ u = \sup_{\lambda \in \mathcal{L}} \lambda u \quad \text{and} \quad M_{\mathcal{L}}^- u = \inf_{\lambda \in \mathcal{L}} \lambda u$$

Notice that, thanks to the previous Lemma, if $I u$ is

$$I u = \sup_{\lambda \in \mathcal{L}} (\lambda u + s) \quad \text{or} \quad I u = \inf_{\lambda \in \mathcal{L}} \sup_{s \in \mathcal{L}} (\lambda u + c u)$$

with $s \in \mathcal{L}$ or $c \not\in \mathcal{L}$ for all $s \in \mathcal{L}$ or all $u \in \mathcal{L}$ and $c \not\in \mathcal{L}$, respectively, then

$$M_{\mathcal{L}}^-(u-v) \leq I u - I v \leq M_{\mathcal{L}}^+(u-v) \quad \bullet$$

We say that $I$ is elliptic with respect to the class $\mathcal{L}$ when $\bullet$ holds.

Notice that $M_{\mathcal{L}}^+$ and $M_{\mathcal{L}}^-$ are themselves fully nonlinear operators, and they are elliptic with respect to $\mathcal{L}$. 

\text{Endfooter}
3.2 Viscosity solutions

Definition: A function \( u : \mathbb{R}^n \rightarrow \mathbb{R} \), continuous in \( \overline{\Omega} \), is said to be a subsolution \( Iu \geq f \) in \( \Omega \) if the following happens:

For any test function \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) satisfying

- \( \psi \) is \( C^2 \) in a neighborhood of \( x \)
- \( \psi(x) = u(x) \)
- \( \psi \geq u \) in all of \( \mathbb{R}^n \)

Then \( I(\psi(x)) \geq f(x) \).

The definition of supersolution \( Iu \leq f \) in \( \Omega \)

is analogous.

\( u \) is a solution of \( Iu = f \) in \( \Omega \) if it is both a subsolution and supersolution.

• This definition only requires \( u \) to be continuous in \( \overline{\Omega} \).

Important properties of viscosity solutions

• Stability under uniform limits: Assume: \( \| \nabla \psi \|_{L^\infty(\Omega)} \leq C \), \( Iu_k \leq f_k \) in \( \Omega \) (in the viscosity sense)

\( u_k \rightarrow u \) locally uniformly in \( \Omega \),

\( u_k \rightarrow u \) a.e. in \( \mathbb{R}^n \),

\( f_k \rightarrow f \) locally uniformly in \( \Omega \)

Then, \( Iu \leq f \) in \( \Omega \) (in the viscosity sense).
2. Comparison principle: If $I$ is a fully nonlinear operator elliptic with respect to $\mathcal{L}$, (and $\mathcal{L}$ satisfies reasonable assumptions), then
\[
\begin{align*}
I(u) &> f \text{ in } \Omega \\Rightarrow \quad u > v \text{ in } \Omega, \\
I(v) &< f \text{ in } \Omega \quad \Rightarrow \quad v < u \text{ in } \Omega, \\
\end{align*}
\]

Idea: we have $0 \leq I(u) - I(v) \leq M^+(u-v) \text{ in } \Omega$, and $(u-v) \leq 0 \text{ in } \mathbb{R}^n$.

If $(u-v)(x) > 0$ for some $x \in \Omega$, then a contradiction because at a maximum we have $M^+(u-v) < 0$.

The actual proof must be done for viscosity solutions.

3. Existence of solutions: The existence of solutions to
\[
\begin{align*}
I(u) &= f \text{ in } \Omega \\
\end{align*}
\]

can be proved by using Perron's method (under reasonable assumptions on $\mathcal{L}$, $f$, and $\Omega$).

We will not prove these technical results, but we will use them.

By definition, viscosity solutions $u$ are continuous in $\overline{\Omega}$.

Question: If $f$ is "nice enough", what is the regularity of $u$ inside $\Omega$?

We will answer this question in the next sections.
3.3. Equations with bounded measurable coefficients: Harnack inequality and Hölder estimates

For second-order fully nonlinear equations, the main regularity results are the

If \( u \) is an viscosity soln of \( \begin{cases} F(D^2u) = 0 & \text{in } B_1 \end{cases} \) then

(a) \[ \|D^2u\|_{L^\infty(B_{1/2})} \leq C \|D^2u\|_{L^1(B_1)} \] for some small \( \varepsilon > 0 \).

(b) If in addition \( F \) is convex, then

\[ \|D^2u\|_{L^{2n}(B_{1/2})} \leq C \|D^2u\|_{L^n(B_1)} \] for some small \( \varepsilon > 0 \).

We will prove that if \( Lu \) is a fully nonlinear equation, then any viscosity solution \( u \) of \[ \begin{cases} Lu = 0 & \text{in } B_1 \end{cases} \] satisfies

\[ \|D^2u\|_{L^{2n}(B_{1/2})} \leq C \|D^2u\|_{L^n(B_1)} \].

For this, let us recall the strategy for second order equations:

Formally, if \( u \) solves \( F(Du) = 0 \) in \( B_1 \), and we differentiate the equation in the direction \( e \), we get

\[ \frac{\partial F}{\partial M_j} (Du(x)) \cdot D_j u(x) = 0 \quad \text{in } B_1, \]

where \( \frac{\partial F}{\partial M_j} \) is the derivative of \( F(M) \) with respect to the variable \( M_j \).

Since \( F \) is uniformly elliptic, then \( a_j(x) := \frac{\partial F}{\partial M_j} (Du(x)) \) satisfy

\[ \lambda \text{Id} \leq (a_j(x)) \leq \Lambda \text{Id}, \]

and hence \( a_j u \) solves:

\[ a_j(x) D_j D_k u = 0 \quad \text{in } B_1. \]

In other words, \( a_j u \) solves an equation with bounded measurable coefficients.
• Notice that the equation
  \[ a_{ij}(x)\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{in } B_1 \]
  is equivalent to
  \[
  M^+_{\lambda, A}(Dw) \geq 0 \quad \text{in } B_1 \]
  \[
  M^-_{\lambda, A}(Dw) \leq 0 \quad \text{in } B_1
  \]

  heuristically

  • Indeed, these two inequalities imply that for some choice of \( a_{ij}(x) \) we have
  \[ a_{ij}(x)\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{in } B_1. \]

  • Now, for solutions \( w \) of (4) we have a \( C^{0,1} \) estimate,

  \[ \|w\|_{C^{0,1}(B_{1/2})} \leq C\|w\|_{L^2(B_1)}, \]

  and since \( w \) was any derivative of \( u \), we find a \( C^{1,1} \) estimate for \( u \).

  • To make this argument rigorous, \( u \) is only continuous! we cannot differentiate the equation,

  we observe that \( w(x+h) \) solves the equation, too, and thus

  \[
  M^-_{\lambda, A}(w(x+h) - w(x)) \leq F(D^2 w(x+h)) - F(D^2 w(x)) \leq 0
  \]

  in \( B_{1/4} \).

  This means that

  \[ w(x) = \frac{w(x+h) - w(x)}{h^2} \]

  solves

  \[
  M^+_{\lambda, A}(Dw) \geq 0 \quad \text{in } B_{1/4} \]
  \[
  M^-_{\lambda, A}(Dw) \leq 0 \quad \text{in } B_{1/4}
  \]

  in the viscosity sense.

  • Using the estimate for equations with bounded meas. coeff., we first find that \( u \in C^{0,1} \), then

  \[ C^{0,1}, \ldots, \] until we reach \( u \in C^{2,0} \) and \( u \in C^{3,0} \) for some small \( \epsilon > 0. \)
\[ M^-_\infty(u(x) - u(x)) \leq I(u(x)) - I(u(x)) \leq M^+_\infty(u(x) - u(x)) \text{ in } B_{1}\ | \leq 0 \]

Thus, the function \( w(x) = \frac{u(x) - u(0)}{h^1} \) solves \(
\begin{align*}
M^+ w &\geq 0 \text{ in } B_{1} \\
M^- w &< 0 \text{ in } B_{1}
\end{align*}
\)

This is (heuristically) equivalent to
\[
\frac{1}{2} \int_{\mathbb{R}^n} \left[ (w(x+y) + w(x-y) - 2w(x)) \right] K(x,y) \, dy
\]
for some kernel \( K(x,y) \) (with no regularity in \( x \) !).

The class \( \mathcal{L} \) that we will consider is
\[
\mathcal{L} \::= \left\{ \text{operators } L(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left( w(x+y) + w(x-y) - 2w(x) \right) K(x,y) \, dy \text{ with kernels } K \text{ satisfying } \frac{1}{|y|^{n+2s}} \leq K(y) \leq \frac{A}{|y|^{n+2s}} \right\}
\]

To prove \( C^{2\alpha} \) regularity of solutions of a \( C^\alpha \) estimate for solutions of
\[
\begin{align*}
M^+_\infty u &\geq 0 \text{ in } B_1 \\
M^-\infty u &< 0 \text{ in } B_1
\end{align*}
\]

To prove such \( C^\alpha \) estimate we will first establish a Harnack inequality for such equation.
Remark. For the class of kernels $\mathcal{K}$, we have the following explicit expressions for $M_{\mathcal{K}}^\pm$:

\[
M_{\mathcal{K}}^+ u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \Lambda(u(xy) + u(x-y) - 2u(x)) + \frac{dy}{|y|^{n+2s}} + \frac{1}{2} \int_{\mathbb{R}^n} \Lambda(u(xy) + u(x-y) - 2u(x)) - \frac{dy}{|y|^{n+2s}}.
\]

\[
M_{\mathcal{K}}^- u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \Lambda(u(xy) + u(x-y) - 2u(x)) + \frac{dy}{|y|^{n+2s}} - \frac{1}{2} \int_{\mathbb{R}^n} \Lambda(u(xy) + u(x-y) - 2u(x)) - \frac{dy}{|y|^{n+2s}}.
\]

Such expressions follow from

\[
M_{\mathcal{K}}^\pm u(x) = \sup_{\|\varepsilon\|_{\mathcal{H}} = \frac{1}{2}} \left\{ \int_{\mathbb{R}^n} \Lambda(u(xy) + u(x-y) - 2u(x)) \varepsilon(y) \, dy \right\}.
\]

( It is not very important to have these expressions, but it is sometimes useful.)
Theorem. Assume that \( M_0^2 u \leq C_0 \) in \( B_1 \) and it solves

\[
\int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^2s} \, dx \leq C \left( \inf_{B_{\frac{1}{2}}} u + C_0 \right)
\]

in the viscosity sense. Then,

\[
M_0^{-} u \leq C_0 \text{ in } B_{\frac{1}{2}}
\]

The theorem says: if \( u \geq 0 \) is a supersolution, then the infimum is controlled (by below) by the \( L^1 \) norm of \( u \).

Thus, if \( u \geq 0 \) is a supersolution in \( B_1 \), it cannot be close to 0 in \( B_{\frac{1}{2}} \) unless it is close to 0 everywhere (in an \( L^1 \) sense).

Proof. Let \( b \in C_0^\infty (B_3) \) be such that \( 0 \leq b \leq 1 \) and \( b \equiv 1 \) in \( B_{\frac{1}{2}} \).

Let \( t > 0 \) be the maximum value for which \( u \geq t b \).

Notice that \( t \leq \inf_{B_{\frac{1}{2}}} u \)

Moreover, since \( u \) and \( b \) are continuous in \( B_1 \), there is \( x_0 \in B_{\frac{1}{2}} \) such that \( u(x_0) = t b(x_0) \).

Now, on the one hand we have

\[
M_0^-(u - tb) \leq M_0^- u - tM_0^- b \leq C_0 + Ct \quad \text{in } B_{\frac{1}{2}}.
\]

where \( C = \|M_0^- b\|_{L^1(B_{\frac{1}{2}})} \).

On the other hand, since \( (u - tb) \geq 0 \) in \( \mathbb{R}^n \) and \( (u - tb)(x_0) = 0 \), then

\[
M_0^-(u - tb)(x) = \frac{1}{2} \int_{\mathbb{R}^n} \lambda \left( t - (x - y) \cdot \nabla t \right) \nabla^2 t \, dy \\
\geq C \int_{\mathbb{R}^n} \frac{t(x,y) + t(x,y')}{1 + |x-y|^2s} \, dy - Ct \int_{\mathbb{R}^n} \frac{x(t)}{1 + |x|^2s} \, dx - C_t
\]
Combining the previous identities, we get
\[
C \int_{\mathbb{R}^n} \frac{\mu(z)}{|1 + |z|^2|} \, dz - C_2 t \leq C_0 + C t
\]
\[
\times C(t+C_0)
\]
Using that \( t \leq \inf_{B_{1/4}} \), we are done.

**Hölder Regularity Estimate**

- Using the weak Harnack inequality, we now want to prove that solutions of
  \[
  M^+ \mu \geq C_0 \text{ in } B_1, \quad M^- \mu \leq -C_0 \text{ in } B_1
  \]
  satisfy
  \[
  \mu_{\sup}(B_{3/4}) \leq C \left( C_0 + \mu_{\inf}(B_{1/2}) \right)
  \]
  for some small \( \alpha > 0 \).

- Let us see first a sketch of the proof in the local case:

  - Assume we have a weak Harnack
    \[
    \frac{\mu_{\sup}}{\inf_{B_1}} \leq C_0
    \]
  - To prove a Hölder estimate, it suffices to show that
    \[
    \left[ \sup_{B_1} \mu \right] \leq C \left( \inf_{B_1} \mu \right)
    \]
  - To show this, assume \( \sup_{B_1} \mu = 1 \), and \( \inf_{B_1} \mu = -1 \).

  Assume also \( \mu \geq 0 \) (if \( \mu \leq 0 \), just take \( -\mu \) instead of \( \mu \)).

  - Then the function \( w = \mu + 1 = \mu + \inf_{B_1} \mu \) satisfies \( w \geq 0 \) in \( B_1 \) and thus
    \[
    0 < C \leq \int_{B_1} (w+1) \geq \int_{B_1} w \leq C \inf_{B_1} w = C \left( \inf_{B_1} \mu + 1 \right)
    \]
  Thus,
  \[
  \inf_{B_1} \mu \geq c - 1, \quad \text{for some } c > 0.
  \]
  This yields
  \[
  \inf_{B_{1/2}} \mu \geq \sup_{B_{1/2}} \mu - \inf_{B_{1/2}} \mu \leq 1 - (c - 1) = 2 - c = 2(1 - \alpha) = (1 - \alpha) \inf_{B_1} \mu
  \]
We now want to do the same but with our half Harnack inequality which requires $u > 0$ in $\mathbb{R}^n$ (instead of $B_2$).

Thus, the non-locality affects this iterative procedure, and the proof needs to be done carefully in the non-local case.

**Lemma.** Assume $\|u\|_{L^1(B_1)} \leq 1$, and with $r_0$ small enough.
Assume $\varepsilon > 0$ is small enough, and $u$ solves

$$
\begin{cases}
M^+ u > -\varepsilon & \text{in } B_1 \\
M^- u \leq \varepsilon & \text{in } B_1
\end{cases}
$$

- If $\int_{B_1} u > 0$ then $u \geqsort 1 - \varepsilon$ in $B_{2r}$.
- If $\int_{B_1} u < 0$ then $u \leqsort 1 - \varepsilon$ in $B_{2r}$. (If $\int_{B_1} u < 0$ then $u \leqsort 1 - \varepsilon$ in $B_{2r}$.)

**Proof.** Assume $\int_{B_1} u > 0$, and take $w = (u+1)^+$.
Notice that $w > 0$ in $\mathbb{R}^n$, and $w = (u+1)^+ + (u+1)^-$. Let $L$ be any operator with kernel $\frac{2}{|y|^{n+2s}} \leq \frac{A}{|y|^{n+2s}}$.

Then, in $B_{2r}$ we have $Lw(x) = Lu(x) + L(u+1)^-(x) = (u(x) + \int_{\mathbb{R}^n}(u(x))(x+y)_-K(y))dy$.

And since $0 \leq (u+1)^- \leq 2(2|x||\nabla u| + 2|x||u|)^2 \frac{A}{|y|^{n+2s}}$ if $|x|$ is small enough.

$$
\int_{B_{2r}} L(u+1)^- \leq 4 \int_{B_{2r}} (1-1)^- \frac{A}{|y|^{n+2s}} \leq \varepsilon
$$

Since $M^+ u > -\varepsilon$ and $M^- u \leq \varepsilon$, then we get $M^+ w > -2\varepsilon$ in $B_{2r}$.

$$
\int_{B_{2r}} M^+ w \geq -2\varepsilon
$$

$$
\int_{B_{2r}} M^- w \leq 2\varepsilon
$$

$$
\int_{B_{2r}} w > 1 - \varepsilon
$$
We will show by induction that, for some small $\varepsilon > 0$, we have
\[
\text{osc } u \leq 2^{-\frac{K\varepsilon}{2}}
\]

- More precisely, we will construct sequences $a_k$ and $b_k$ such that
  \[b_k \leq m(x) \leq a_k \text{ in } B^{-k}_1\]
  \[a_k - b_k = 2^{-K\varepsilon} \leq \varepsilon \leq a_k \leq b_{k+1} \leq \cdots \leq a_{k+1} \leq \cdots \leq a_0 = \alpha \leq \beta_k \leq a_k \]

We construct those sequences by induction. For $K \leq 0$, let $a_k = \frac{1}{2}$ and $b_k = -\frac{1}{2}$.

For $K > 0$, just take $b_k = -\frac{1}{2}$, $a_k = b_k + 2^{-K\varepsilon}$.

- Assume now we have those sequences $a_j, b_j$ for $j \leq k$, and let us construct $a_{k+1}, b_{k+1}$.

Let $m = \frac{a_k + b_k}{2}$ and notice $|m - m| \leq \frac{1}{2} 2^{-K\varepsilon}$ in $B^{-k}_1$.

Let $V(x) = 2^{-K\varepsilon} (u(2^{-k}x) - m)$.

Then, \[\|V\|_{L^p(B_1)} \leq 1\] and $V$ solves
\[
M^+ V - E(2^{-K\varepsilon}) 2^{K\varepsilon} \text{ in } B_1
\]
\[
M^- V \leq E(2^{-K\varepsilon}) 2^{K\varepsilon} \text{ in } B_1
\]

Since $\varepsilon \leq 2^{K\varepsilon}$, \(\varepsilon\) is small, then
\[
M^+ V \geq -E \text{ in } B_1
\]
\[
M^- V \leq E \text{ in } B_1
\]

In order to use the Lemma, we need \[\|V\| \leq 2/2x^k - 1 \text{ for } k \leq 1\] let us prove this.

- Take $k \leq 1$, and let $j \geq 0$ such that $2^j \leq k \leq 2^{j+1}$. Then, by inductive hypothesis,
  \[V(x) = 2^{-K\varepsilon} (u(2^{-k}x) - m) \leq 2^{-K\varepsilon} (a_{k-j} - b_{k-j} - m) \leq 2^{-K\varepsilon} (a_{k-j} - b_{k-j} + 6_k - m) \leq 2^{K\varepsilon} 2^{-K\varepsilon} = 1 \leq 2/2x^k - 1 \]

Place by the previous Lemma we have \inf V \geq \Theta - 1$, or equivalently,
\[
\left\{ \Theta \geq 2(\Theta - 1) 2^{-K\varepsilon} \text{ in } B_{2^{-k+1}} \right\} \longrightarrow \left\{ b_k \geq \frac{1}{2} 2^{-K\varepsilon} \leq m < a_k \text{ in } B_{2^{-k+1}} \right\}
\]

Taking $\xi > 0$ so that we have $a_{k+1} = a_k$ and $b_{k+1} = b_k + 2^{K\varepsilon} 2^{-K\varepsilon} - 1$.
By the half Harnack inequality, (recall $w \geq 0$ in $\mathbb{R}^n$) we get:

$$\int_{\mathbb{R}^n} \frac{w(x)}{1 + |x|^{n+2s}} \, dx \leq C \left( \inf_{\mathbb{B}_k} w + \varepsilon \right).$$

In particular,

$$\int_{\mathbb{B}_k} w \leq C \left( \inf_{\mathbb{B}_k} w + \varepsilon \right).$$

Now,

$$\int_{\mathbb{B}_k} w = \int_{\partial(\mathbb{B}_k)} (u + 1) \geq \int_{\partial(\mathbb{B}_k)} 1 \geq \sigma(\mathbb{B}_k),$$

so

$$\sigma = C(\mathbb{B}_k) \quad \varepsilon - \inf_{\mathbb{B}_k} w = \inf_{\mathbb{B}_k} u + 1.$$

This means that

$$\inf_{\mathbb{B}_k} u \geq \varepsilon - 1.$$

---

We next deduce the $C^*$ estimate from the previous Lemma.

**Theorem.** Assume that $u$ solves

$$M^+ u \geq -C_0 \quad \text{in} \quad \mathbb{B}_k,$$

$$M^- u \leq C_0 \quad \text{in} \quad \mathbb{B}_k.$$

Then,

$$\|u\|_{C^* (\mathbb{B}_k)} \leq C \left( \|u\|_{L^2 (\mathbb{B}_k)} + C_0 \right).$$

**Proof.** We need to show

$$|u(x) - u(y)| \leq C|x-y|^s$$

for all $x, y \in \mathbb{B}_k$.

We will show it for $x = 0$ (it is the same at all points). For this, notice that dividing $u$ by $2\log |x|$,$$
\frac{M^+ u \geq -C_0 \log |x| \quad \text{in} \quad \mathbb{B}_k',}{\frac{M^- u \leq C_0 \log |x| \quad \text{in} \quad \mathbb{B}_k'}.}$

Taking $r$ small so that $C_0 \log r \leq \varepsilon$, we may assume

that

$$M^+ u \geq -\varepsilon \quad \text{in} \quad \mathbb{B}_k'$$

$$M^- u \leq \varepsilon \quad \text{in} \quad \mathbb{B}_k'.$$
Remark: We have \( M^{*} u \geq -C_0 \) in \( B_1 \) \( \Rightarrow \) \( \|u\|_{L^\infty(\Omega)} \leq C(1 + \|u\|_{W^{1,p}(\Omega)}) \).

(Harnack inequality)

We have proved the half-Harnack inequality for supersolutions, and showed the \( C^\alpha \) estimate for equations with bounded measurable coefficients.

We now show the other half Harnack (for subsolutions), and this will yield the "full" Harnack.

**Theorem** Assume that \( \begin{array}{cc}
M^+ u \geq -C_0 \text{ in } B_1 \\
& \text{(subsolution)}
\end{array} \)

in the viscosity sense. Then

\[
\sup_{B_{1/2}} u \leq C \left( \int_{B_1} \frac{1}{1 + |x|^2} \, dx + C_0 \right)
\]

**Proof.** Dividing \( u \) by a constant, we may assume

\[
M^+ u = -1 \quad \text{and} \quad \int_{B_1} \frac{\max u}{1 + |x|^2} \, dx \leq 1.
\]

Let us consider the minimum value of \( t \) such that

\[
\min u \leq t(1 - |x|^2)^n \text{ in } B_1
\]

Then, since \( u \) is continuous in \( B_1 \), there must be \( x_0 \in B_1 \) such that \( \min u(x_0) = t(1 - |x|^2)^n \),

We want to show that \( t \) cannot be very large. For this, let \( \gamma = (1 - 4\varepsilon_1) \) and \( r = \frac{1}{2} \),

and let us estimate \( \int_{B_1} |u - u(x_0)|^2 \, dx \).

Let \( A = B_1 \cap \{ u > u(x_0) \} \). Since \( \int_{B_1} u \leq 2 \) then

\[
1A \leq 2 \left( \frac{2}{u(x_0)} \right) = 4t^{-1}A^n.
\]

In particular

\[
\int_{B_1} u \leq C \int_{B_1} \frac{1}{|u(x_0)|} \, dx \leq C t^{-1} |B_1|
\]

If \( t \) is large, then \( A \) can cover only a small portion of \( B_1(x_0) \).
Let us now estimate $|\{ u < M^+ R \} \cap B_r(x) |$ by using the half Harnack inequality for super-solutions.

More precisely, we estimate

$$|\{ u < M^+ R \} \cap B_r(x) |$$

for $\theta > 0$ small.

For $x \in B_{dr}(x)$ we have $1 \times |x|_1 + \theta r = 1 - d + \theta d/2$, so

$$M(x) \leq c (1 - |x|_1)^n \leq c (d - \theta d/2)^n \leq c d^{-n} (1 - \theta^2)^n = \mu(x) (1 - \theta^2)^n$$

Let

$$V(x) = (1 - \theta^2)^n \mu(x) - \mu(x)$$

and notice that $M V \leq 1$ (super-solution) since $M^+ \mu \geq 1$.

We want to use the half Harnack, and for this we need to consider

$$W = V^+$$

We need to estimate $M^+ W \in B_{dr/2}(x)$, using that

$$V^+(x) = (1 - \theta^2)^n \mu(x) - \mu(x)$$

$$\leq \int 0 \quad \text{in } B_{dr}(x)$$

$$\leq \int \mu(x) \quad \text{in } R \setminus B_{dr}(x)$$

We get

$$M^+ W \leq M^+ V + \varepsilon \int_{R \setminus B_{dr/2}(x)} \frac{\mu(x)}{1 + |x - z|^{n+2s}} \, dz \leq 1 + C(\theta) \int_{R \setminus B_{dr/2}(x)} \frac{\mu(x)}{1 + |x - z|^{n+2s}} \, dz \leq C(\theta) \int R^n \, dz = C(\theta) n^{-2s}$$

in $B_{dr/2}(x)$.

By the half Harnack for super-solutions (rescaled to the ball $B_{dr/2}(x)$) we get

$$r^{-n} \int_{B_{r/2}(x)} \inf_{B_{r/2}(x)} W \leq C \left( \inf_{B_{r/2}(x)} W + (\theta)^{2s} \cdot C(\theta)^{n+2s} \right) \leq C \left( \inf_{B_{r/2}(x)} W + (\theta)^{n} \right).$$
Now, since \( W(x) = (1 - g(x))^+ \mathcal{W}(x) - \mathcal{W}(x)^+ \)

then

\[ \left| \{ u < M g(x) \} \cap B_{B_2} \right| \leq \cdots \]

**Corollary** If \( u \geq 0 \) in \( \mathbb{R}^n \) solves

\[ M^+ u \geq 0 \quad \text{in} \quad B_1 \]
\[ M^{-} u \leq 0 \quad \text{in} \quad B_1 \]

then

\[ \left\{ u \leq \sup_{B_{B_2}} \text{inf}_{B_{B_2}} \right\} \]

Proof.

Since \( u \) is a superolution, then

\[ \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^2} \, dx \leq \text{inf}_{B_{B_2}} \]

since \( u \) is a subolution

\[ \sup_{B_{B_2}} u \leq C \int_{\mathbb{R}^n} \frac{u(x)}{1 + |x|^2} \, dx \]
\( C_{1,\alpha} \) estimates for fully nonlinear equations

Let

\[
\begin{cases} 
Iu := \inf \sup \{ L_{\alpha}(u) \} \\
\text{or} \\\nIu := \sup \{ L_{\alpha}(u) \} 
\end{cases}
\]

be a fully nonlinear nonlocal operator, with for all being kernels

\[
\frac{1}{|y|^m2^s} \leq K(y) \leq \frac{A}{|y|^m2^s} \quad 0 < \lambda \leq A, \quad se(\mathfrak{L}).
\]

Recall that

\[
M^{-}(u-v) \leq Iu - IV \leq M^{+}(u-v)
\]

Now, we want to show that if \( u \) solves \([Iu = 0 \quad in \quad B_{2}]\) then

\[
\|u\|_{C^{1,\alpha}(B_{2})} \leq C\|u\|_{L^{r}(B_{2})}
\]

for some small \( \varepsilon > 0 \).

(Can this be the case? We need either \( 1 + \alpha > 2s \) or further assumptions on the kernels!)

The easiest approach to show the estimate would be the following:

\[
\begin{align*}
Iu &= 0 \quad \Rightarrow \quad M^{-}u \geq 0 \quad \Rightarrow \quad M^{-}u \leq 0 \\
&\quad \Rightarrow \quad \|u\|_{C^{1,\alpha}(B_{2})} \leq \|u\|_{L^{r}(B_{2})} \\
&\quad \Rightarrow \quad \frac{\|u^{(1)}(h) - u^{(1)}(x)\|_{L^{\infty}(B_{h^\alpha})}}{h^{1/\alpha}} \leq C\frac{\|u^{(1)}(h) - u^{(1)}(x)\|_{L^{\infty}(B_{h^\alpha})}}{h^{1/2\alpha}} \\
&\quad \Rightarrow \quad \frac{\|u^{(1)}(h) - u^{(1)}(x)\|_{L^{\infty}(B_{h^\alpha})}}{h^{1/\alpha}} \leq C\frac{\|u^{(1)}(h) - u^{(1)}(x)\|_{L^{\infty}(B_{h^\alpha})}}{h^{1/2\alpha}} \\
&\quad \Rightarrow \quad \cdots \longrightarrow \frac{u^{(1)}(h) - u^{(1)}(x)}{h^{1/\alpha}} \in L^{\infty} \rightarrow \frac{u^{(1)}(h) - u^{(1)}(x)}{h^{1/\alpha}} \in C^{1,\alpha} \rightarrow u \in C^{1,\alpha}
\end{align*}
\]

However, we were cheating here!
At each step we should truncate the functions, since

\[ u \in L^\infty_0(R^n) \rightarrow u \in C^0(B_R) \rightarrow \frac{u(x+h) - u(x)}{|h|^a} \in L^\infty_0(B_R) \text{ but not } L^\infty_0(R^n)! \]

In the next step we would need \( \frac{u(x+h) - u(x)}{|h|^a} \in L^\infty_0(R^n) \) in the iteration.

By just truncating the function at each step, the above proof works provided that one assumes that

\[ \left| \nabla K(y) \right| \leq \frac{C}{|y|^{n+2+a}} \]

Thus, kernels must be \( C^2 \) outside the origin for this argument to work.

To avoid this, the proof must be done by blow-up and compactness, as we did in case of linear equations.

The proof would be essentially the same, but the Liouville theorem we need is the following:

\[ \text{There is a small } \delta > 0 \text{ such that:} \]

**Theorem.** Assume that \( U \in C_0(R^n) \) is a solution of

\[ \begin{cases} M^+ U > 0 \text{ in } R^n \\ M^- U < 0 \text{ in } R^n \end{cases} \]

and that it satisfies

\[ |U(x)| \leq C(1 + |x|^{\delta}) \quad \text{for all } x \in R^n. \]

Then, \( U \) is constant.

**Proof.** Let \( \delta > 0 \) be the one for which we have the estimate

\[ M^W \geq 0 \quad M^W \leq 0 \]

\[ [W]_{C^0(B_\delta)} \leq \frac{C}{|W|_{L^\infty(R^n)}} \]

Let \( \delta > 0 \) be any number \( 0 < \delta < \delta \).

Let \( (w(x) = R^{-\delta} U(Rx)) \) with \( R > 1 \). Then, we have

\[ |w(x)| \leq CR^\delta (1 + |R|x)^{\delta} \leq C(1 + |x|^{\delta}) \]

with \( C \) independent of \( R > 1 \).
In particular, we have
\[ \|u\|_{L^2(B_R)} \leq \int_{\mathbb{R}^n} \frac{C(\alpha \|u\|_{L^2(\mathbb{R}^n)})}{1 + \|x\|^2} \, dx \leq C \quad \text{(since } \alpha < 2s) . \]

Thus, we have
\[ [u]_{C^s(\mathbb{R}^n)} \leq C \|u\|_{L^2(\mathbb{R}^n)} \leq C \quad \text{(for every } R > 1) . \]

Now, this means
\[ [u]_{C^s(B_{R/2})} \leq R^{-s} [u]_{C^s(\mathbb{R}^n)} \leq CR^{n-s} \rightarrow 0 \quad \text{as } R \to \infty . \]

This yields \[ [u]_{C^s(\mathbb{R}^n)} = 0, \quad \text{so } u \text{ is constant}. \]

Using this Liouville theorem, we get the following:

**Theorem.** Let \( \alpha > 0 \) be given by the previous result. Let \( I \) be a fully nonlinear operator,
\[ I[u] = \inf_{\mathcal{M} \in \mathbb{R}^n} \sup_{\mathcal{M} \in B \subset \mathbb{R}^n} (L[u]) \]
with \( L \) being nonlinear operators with kernels
\[ \frac{s}{1 + |x|^{2s}} \leq (Kx) \leq \frac{A}{1 + |x|^{2s}} . \]

If \( s > \frac{n}{2} \) we assume in addition that
\[ \frac{[K]}{C^{1 + x - 2s}(B_R)} \leq \frac{C}{r^{n+1+a}} \]

Then, any solution \( u \) of \( \overline{\text{I}u} = 0 \text{ in } B \), satisfies
\[ \|u\|_{C^s(\mathbb{R}^n)} \leq C \|u\|_{L^2(\mathbb{R}^n)} . \]

**Proof.** Exercise. (using the previous Liouville theorem, the proof is very similar to the one we did for linear equations).
Boundary regularity for fully nonlinear nonlocal equations

- Recall: For the fractional Laplacian, we showed that solutions "look like" $c^{d/2}$, where $c(x) = \text{dist}(x, \partial \Omega)^d$. Let $\lambda = \infty$ in $\Omega$.

- Moreover, we saw that the same happens for all operators with kernels

\[
\frac{\lambda}{\mu^{1/d}} < \kappa(x) < \frac{\Delta}{\mu^{1/2}} \quad \text{with } \kappa(x) \text{ homogeneous}
\]

- For such operators, we saw that

\[
\begin{align*}
\text{Let } f \in \mathcal{C}^0_\text{loc} & \quad \Rightarrow \quad [u(x) = c_d \delta(x) + o(\|x-x_0\|^{1/d})] \quad (\text{Theorem}) \\
\text{for } f \in \mathcal{C}^0_\text{loc} & \quad \Rightarrow \quad [u(x) = c_d \delta(x) + o(\|x-x_0\|^{1/d})] \quad \text{whenever } f \in \mathcal{C}^0_\text{loc} \quad \text{and } \Omega \text{ is } C^2.
\end{align*}
\]

- This was equivalent to $\frac{\lambda}{d} \in \mathcal{C}^0_\text{loc}.$

- Furthermore, when $f \in \mathcal{C}^0_\text{loc}$, $\Omega$ is $C^2$, and the $\kappa(x)$ is $C^1$ outside the origin, then $u \in \mathcal{C}^1_\text{loc}.$

- For fully nonlinear equations, one has the following:

**Theorem**: Assume $\Omega$ is $C^2$, and $[\text{Let } f \in \mathcal{C}^0_\text{loc} \Rightarrow [u(x) = c_d \delta(x) + o(\|x-x_0\|^{1/d})] \quad \text{with } f \text{ being nonlocal operators with kernels}

\[
\frac{\lambda}{\mu^{1/d}} < \kappa(x) < \frac{\Delta}{\mu^{1/2}} \quad \text{with } \kappa(x) \text{ homogeneous and } [\kappa(x)]_{C^0(\mathbb{R}^d)} \leq \frac{\kappa}{\mu \text{ small}}.
\]

Let $u$ be any solution of

\[
\begin{align*}
\text{Let } f \in \mathcal{C}^0_\text{loc} & \quad \Rightarrow \quad [u(x) = c_d \delta(x) + o(\|x-x_0\|^{1/d})] \\
\text{Let } f \in \mathcal{C}^0_\text{loc} & \quad \Rightarrow \quad [u(x) = c_d \delta(x) + o(\|x-x_0\|^{1/d})] \\
\end{align*}
\]

They

\[
\begin{align*}
\|u\|_{C^{\alpha, \alpha}(\mathbb{R}^d)} & \leq C \|f\|_{C^0(\mathbb{R}^d)} \\
\end{align*}
\]

This is an estimate of order $2\alpha$:

\[
\begin{align*}
u(x) = c_d \delta(x) + o(\|x-x_0\|^{1/d}) + o(\|x-x_0\|^{2\alpha})
\end{align*}
\]

- In the limit $\alpha \to 1$, it says that "$u$ is $C^2$ on the boundary."
To show this theorem, we define

\[ m^* \omega := \sup_{L \in \Omega} L \omega \]

where \( L \) is an operator with \( \frac{3}{2} \leq |K| \leq 4 \) and \( \frac{2}{3} \leq \frac{1}{\eta} \leq \frac{3}{2} \) (and \( K \) is homogeneous).

Then, if \( \omega \) is as in the theorem, we will have

\[ m^* (\omega - \nu) \leq \omega - \nu \leq m^* (\omega - \nu) \]

For such operators \( m^* \) (with homogeneous kernels), we have the following.

**Theorem.** Assume \( \Omega \) is \( C^2 \) and \( m^* \mu \geq -C_0 \) in \( \Omega \)

\[ m^* \mu \leq C_0 \text{ in } \Omega \]

Then,

\[ \left( \frac{1}{C_0} d^i \chi_{(\Omega \cap B_R)} \right) \leq C \left( \mu \left| \omega \right| \log \left( \frac{1}{\mu \left| \omega \right|} \right) + \mu \right) \]

For some small \( \mu > 0 \)

This is the analogue of the \( C^2 \) interior estimate for equations with bounded measurable coefficients.

In this case, however, the boundary helps, and the estimate is of order \( \log \) on the boundary:

\[ \left( \mu \chi_{(\Omega \cap B_R)} \right) \leq C_2 d^i \chi_{(\Omega \cap B_R)} + O(d^i \log d^{21/2}) \]

This result is proved by a boundary Harnack type estimate, and an iteration that improves the oscillation of \( \mu \omega \) in dyadic balls:

\[ b_k \leq \mu \omega \leq a_k \text{ in } B_{2^{-k}}, \text{ with } b_k \leq b_{k+1} \leq a_k \leq a_{k+1} \]

Then, the estimate for fully nonlinear equations is proved by a blow-up and compactness argument (much more delicate than the interior...).
Boundary regularity for non-homogeneous kernels

Question: What happens with the boundary regularity for non-homogeneous kernels?

Let us try to answer this question in dimension 1. (In $\mathbb{R}^n$ would be similar.)

If we denote $M^+$ and $M^-$ the extremal operators

$$M^+ w = \sup_{\frac{x}{y}} \frac{L w}{y^{n+1}}, \quad \text{and} \quad M^- w = \inf_{\frac{x}{y}} \frac{L w}{y^{n+1}}.$$  

then, these operators are scale invariant of order $2s$.

Namely, if $w_r(x) = w(rx)$ then $(M^+ w_r)(x) = r^{2s} (M^+ w)(rx)$.

Thus, if we consider the 4D functions $(x_t)^\beta$ with $0 < \beta < 2s$

we will have that the functions $M^+(x_t)^\beta$ and $M^-(x_t)^\beta$ will be homogeneous of degree $\beta - 2s$. In particular,

$$M^+(x_t)^\beta = c_1(\beta) (x_t)^{\beta - 2s} \quad \text{in} \quad (0, \infty),$$

$$M^-(x_t)^\beta = c_2(\beta) (x_t)^{\beta - 2s} \quad \text{in} \quad (0, \infty).$$

For the fractional Laplacian, we showed that $(-\Delta)^s (x_t)^3 = 0$ in $(0, \infty)$.

For the operators $M^+$ and $M^-$ we have the following:

- The constants $c_1(\beta)$ and $c_2(\beta)$ are continuous in $\beta$ for $\beta \in (0, 2s)$.

- We have $\lim_{\beta \to 0} c_1(\beta) < 0$ and $\lim_{\beta \to 2s} c_2(\beta) = +\infty$.

- In particular, there exist $\beta_1$ and $\beta_2$ such that $c_1(\beta_1) = c_2(\beta_2) = 0$, i.e.,

- such $\beta_1$ and $\beta_2$ satisfy $0 < \beta_1 < s < \beta_2 < 2s$.
This is very different to what happens for homogeneous kernels:

- **Homogeneous kernels in \( \mathbb{R}^n \)**
  \[
  u(x) = f(x) \quad \text{for} \quad x \in \Omega
  \]
  \[
  u = 0 \quad \text{on} \quad \partial \Omega
  \]
  and if \( F \geq 0 \) then \( u \leq C e^{Ct} \).

- **General kernels in \( \mathbb{R}^n \) and \( \mathbb{R}^+ \)**
  \[
  \int_0^t \int_{\Omega} F(\tau, x) u(x) \, dx \, d\tau
  \]
  Solutions do not look like \( d^2 \).
  In general, we only have \( u \leq C e^{Ct} \) and if \( F \geq 0 \) then \( u \leq C t^\alpha \), but \( \alpha < \frac{1}{2} \).

This is very different to the local case \( F(Bu) = 0 \), all solutions are \( C^2(\Omega) \) and in particular they look like \( d(x) \).

This finishes the chapter on fully nonlinear equations.

Some results that we did not cover and could be good for presentations:

- \( C^{2,\alpha} \) estimate for convex equations (\( Iu = \sup \{ f \mid \text{convex} \} \))
  \( \rightarrow \) Hitting

- Parabolic equations \( u_t - \Delta u = 0 \) in \( B_1 \times (0,1) \)
  \( \rightarrow \) Hamilton

- Equations localized in space-time:
  \[
  u(t,x) = \int_0^\infty \int_{\Omega} (u(\tau, x+y) - u(x)) K(s,y) \, dy \, d\tau
  \]
  \( K(s,y) \approx \frac{1}{(t+s)^{n+\frac{2s}{s}}} \)

- Equations with kernels
  \[
  K(s,y) \approx (t+s)^{-n-\frac{2s}{s}}
  \]
  or
  \[
  K(s,y) \approx (t+s)^{-n-2s} \log(t+s)
  \]
  or other types of scaling conditions.