The Proof Society Summer School and Workshop 2023

Booklet of Abstracts

10th - 14th July

Contents

Welcome Address

This booklet of abstracts witnesses the fifth edition of the Summer/Season School and affiliated Workshop of The Proof Society. During the 2017 Oberwolfach meeting Mathematical Logic: Proof Theory, Constructive Mathematics, various participants noticed that Proof Theory was lacking an overarching organisation as many other central areas in logic had such as Model Theory, Set Theory or Computability Theory. Thus The Proof Society (TPS) was conceived to address this need.

It was decided that the activities of TPS would initiate with a thread of main activities to later sort out its precise institutional status. Aside from the interruption due to COVID, yearly meetings were held in Ghent in 2018, Swansea in 2019, Madeira in 2021, Utrecht in 2022 and now, the fifth edition in Barcelona. This lustrum edition is a suitable moment to start defining the precise institutional status of The Proof Society. This will be the main agenda point for the AGM on Thursday, July 13 so that, after being conceived in 2017, hopefully The Proof Society will now be born in a formal sense.

Notwithstanding the lack of an institutional foundation, the view pronounced in Oberwolfach has proven to be visionary: Proof Theory indeed needed an overarching institution and The Proof Society has proven to provide a much needed service to its community. As such, we can safely pronounce that we are looking forward to the next lustrum edition of The Proof Society events. For the moment, we hope you will enjoy the current one.

On behalf of the organisers and program committee,

Joost J. Joosten and David Fernández-Duque

Summer School Tutorials

Proof Complexity of Propositional Resolution

Albert Atserias[1]

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Abstract

Propositional Resolution is at the core of modern SAT-solvers. These are highly optimized systems that routinely produce satisfying assignments or Resolution proofs of unsatisfiability for many formulas with hundreds and even thousands of variables. Resolution is also a proof system of theoretical interest for being the base of a hierarchy of proof systems of increasing reasoning power. This tutorial will set the focus on the theoretical aspects of propositional Resolution with an emphasis on its structural results. The first lecture will cover the size-width relationship of Ben-Sasson and Wigderson and its applications. The second lecture will cover lower bound methods including in the width method, the random restriction method, and the infinite model method. The third lecture will cover the recent result which states that automating Resolution is NP-hard: there is an algorithm that finds Resolution refutations in time polynomial in the shortest refutation if and only if P = NP.

Reflection algebras: an introduction

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Abstract

I will present an introduction to reflection principles and their use in the analysis of axiomatic systems. The idea of using reflection principles and their transfinite iterations to classify arithmetical sentences according to strength is due to A. Turing (1939). However, Turing also realized that there are serious difficulties associated with this approach, in particular, due to the lack of understanding how to distinguish 'canonical' from 'pathological' ordinal notation systems, now a well-known problem in proof theory. Later work due to Kreisel, Feferman, Schmerl and others revealed deep connections between Turing progressions and the results on proof-theoretic analysis of formal theories, however the subject remained technically demanding.

The aim of the tutorial is to outline the main ingredients of the approach to proof-theoretic analysis based on reflection algebras. From an abstract algebraic point of view, these structures are semilattices enriched by a family of monotone unary operators. The operators can be interpreted in the lattice of arithmetical theories as functions mapping a theory T to a theory axiomatized by a reflection principle for T. Within this framework it is possible to define in an abstract and general form appropriate canonical ordinal notation systems and the associated transfinite hierarchies of reflection principles. Rather than going very far, I aim at presenting basic results that would familiarize the participants with the specific notions involved.

Proof theory for ecumenical systems

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Abstract

Ecumenism can be understood as a pursuit of unity, where diverse thoughts, ideas, or points of view coexist harmoniously. In logic, ecumenical systems refer, in a broad sense, to proof systems for combining logics. One captivating area of research over the past few decades has been the exploration of seamlessly merging classical and intuitionistic connectives, allowing them to coexist peacefully.

In this tutorial, we will embark on a journey through ecumenical systems, drawing inspiration from Prawitz' seminal work [8]. We will begin by elucidating Prawitz' concept of "ecumenism" and present a pure sequent calculus version of his system. Building upon this foundation, we will expand our discussion to incorporate alethic modalities, leveraging Simpson's meta-logical characterization. This will enable us to propose several proof systems for ecumenical modal logics.

We will conclude our tour with some discussion towards a term calculus proposal for the implicational propositional fragment of the ecumenical logic, the quest of automation using a framework based in rewriting logic, and an ecumenical view of proof-theoretic semantics.

The tutorial is based in the following papers:

- Ecumenical modal logic by Sonia Marin, Luiz Carlos Pereira, Elaine Pimentel, and Emerson Sales [1]
- A pure view of ecumenical modalities by Sonia Marin, Luiz Carlos Pereira, Elaine Pimentel, and Emerson Sales [2]
- Separability and harmony in ecumenical systems by Sonia Marin, Luiz Carlos Pereira, Elaine Pimentel, and Emerson Sales [3]
- An ecumenical view of proof-theoretic semantics by Victor Nascimento, Luiz Carlos Pereira, and Elaine Pimentel [4]
- A rewriting logic approach to specification, proof-search, and meta-proofs in sequent systems by Carlos Olarte, Elaine Pimentel, and Camilo Rocha. [5]
- On an ecumenical natural deduction with stoup - part I: the propositional case by Luiz Carlos Pereira, and Elaine Pimentel [6]
- An ecumenical notion of entailment by Elaine Pimentel, Luiz Carlos Pereira, and Valeria de Paiva [7]

Acknowledgements

I want to thank my co-authors in this endeavour: Luiz Carlos Pereira, Valeria de Paiva, Sonia Marin, Emerson Sales, Delia Kesner, Mariana Milicich, Victor Nascimento and Carlos Olarte.

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MetaCoq, CertiCoq and Certified Extraction of Dependently-Typed Programs

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Abstract

In this lecture/tutorial, I will give an overview of the MetaCoq and CertiCoq projects, which together provide a formal specification and implementation of a large part of the Coq proof assistant's kernel. MetaCoq [3] includes the metatheory of the Calculus of Inductive Constructions at the basis of Coq and certified typechecking and erasure algorithms, all programmed in Coq itself using dependent types [2]. CertiCoq[1] is a certified compiler from the erased terms (an extended λ-calculus) down to C code. Throughout the lecture we will focus on the logical aspects of self-certification and on the proof-theoretical support for extraction.

Acknowledgements

This lecture presents joint work with the MetaCoq and CertiCoq teams.

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The Historical Origins of Proof Theory: Proofs in the 13th and 14th Centuries

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The aim of these three lectures is to introduce the student of modern proof theory to the historical origins of proof theory by looking at developments in the second "Golden Age" of logic, namely the 13th and 14th centuries in western Europe. The lectures presuppose no knowledge of medieval logic, and are organised as follows:

Lecture 1: Introduction

- What is "proof theory" when you don't have a syntax/semantics distinction?
- (Brief!) introduction to Latin syntax
- Basic background terms and concepts:
  - Syncategoremata and categoremata
  - Signification and supposition

Lecture 2: Basic rules

- Key authors: Sherwood, Burley, Buridan, Ockham
- Rules for propositional reasoning.
- "Proofs of propositions"
- Rules for quantificational reasoning.
  - Scope rules for quantifiers and negation
  - Compounded vs. divided readings

Lecture 3: Applying the rules

- Solving paradoxes
- "Obligational" disputations
- What can we learn from the medievals for modern proof theory?

Workshop Invited Talks

Automatability and Weak Automatability of Propositional Proof Systems

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1 Instructions

A Propositional Proof System (p.p.s.) is a polynomial time function whose range is the set of all propositional tautologies. The set of propositional tautologies, TAUT, is a coNP-complete set. For a given proof system, it is natural to ask if every tautology has a proof in that system polynomial in size of the tautology. If a system had this property, we would call it bounded. We don't think that there is such a proof system, because Cook-Reckhow show that there is a bounded propositional proof system if and only if the complexity classes NP and coNP are equal. As a consequence, if there was a bounded propositional proof system the classes P and NP would be equal, which we think very unlikely. Several p.p.s. like Resolution (Haken) have been proved not bounded, by exibiting a family of contradictions requiring exponential size refutations.

The question we will explore here is that of automatability and weak automatabiliy (Bonet-Pitassi-Raz). For the tautologies/contradictions that do have polynomial size proofs/refutations in some p.p.s., is there an algorithm that produces these proofs in time polynomial in the smallest refutation of that system? This question is very relevant to automated theorem proving. We will see that the answers are mostly negative.

Acknowledgements

Work from collaborations with Albert Atserias, Toniann Pitassi and Ran Raz, and conversations with Pavel Pudlák.

Bounded Arithmetic and a Consistency Result for NEXP $\not\subseteq$ P/poly

Sam Buss

University of California, San Diego

This talk will discuss the provability and unprovability of statements from computational complexity in theories of bounded arithmetic. A recent result, joint with Albert Atserias and Moritz Müller, shows that it is consistent with the second-order theory V^0_2 that nondeterministic exponential time (NEXP) does not have polynomial size circuits.

Ramsey-theoretic principles and proof size in second-order arithmetic

Leszek Kołodziejczyk[1]

[1]Institute of Mathematics, University of Warsaw

I will survey some results (obtained over the last few years by Katarzyna Kowalik, Tin Lok Wong, Keita Yokoyama, and myself) concerning the effect of Ramsey's theorem for pairs and its weaker versions on the size of proofs in fragments of second-order arithmetic. Time permitting, I will also try to connect these results to some more general facts concerning proof speedup, consistency statements, and interpretability. That part of the talk will be based on work in progress joint with Yokoyama.

Workshop Contributed Talks

Proof Equivalences in Constructive Modal Logic

Matteo Acclavio[1], Davide Catta[2], Federico Olimpieri[3], and Lutz Straßburger[4]

[1]University of Southern Denmark
[2]Télécom Paris
[3]University of Leeds
[4]INRIA-Saclay

Proof theory is the branch of mathematical logic whose aim is studying the properties of proofs, as well as their structure and invariants. For this purpose, the most used representations of proofs are based on tree-like data structures inductively defined using inference rules of a proof system. However, having formalisms able to represent proofs is not enough to define "what is a proof" since different derivations, or derivations in different proof systems, could represent the same abstract object.

A notion of *proof identity* is therefore required to define a proof as a proper mathematical entity [8]. Such a notion of identity is provided by delineating the condition under which two distinct formal representations of a proof represent the same logical argument. These conditions are often driven by semantic considerations (performing specific transformations on two derivations, they can be transformed to the same object) or intuitive ones (two derivations only differ for the order in which the same rules are applied to the same formulas).

Natural deduction is often considered a satisfactory formalism to represent proofs since it allows to define a more canonical representation of proofs with respect to *sequent calculus*: derivations in sequent calculus differing for some rules permutations are represented (*via* a standard translation) by the same derivation in natural deduction. Moreover, natural deduction provides a one-to-one correspondence between derivations and lambda-terms, as well as between derivations and winning innocent strategies.

In this talk, we discuss proof equivalence for constructive modal logic using recent results on new formalisms for proofs for the logic CK.

Constructive Modal Logic. Classical modal logics are obtained by extending *classical logic* with unary operators, called *modalities*, that qualify the truth of a judgment. The basic modalities are the $\Box$ (called *box*) and its dual operator $\Diamond$ (called *diamond*), which are usually interpreted as *necessity* and *possibility*. The work of Prawitz [13] initiate the investigation of the proof theory of modal logics extending intuitionistic logic, leading to numerous results on the topic and in particular, the Curry-Howard-Lambek correspondence has been extended to various constructive modal logics [5, 3, 11, 12]. Intuitionistic logic can be extended with modalities in different ways (for an overview see [14]): while in classical logic axioms involving only $\Box$ provide also description of the behavior of $\Diamond$, for intuitionistic logic this is no more the case since the duality of the two modalities does not hold anymore. This leads to different approaches. *Constructive modal logics* consider minimal sets of axioms to guarantee the definition of the behaviors of the $\Box$ and $\Diamond$ modalities. A second approach, referred to as *intuitionistic modal logic*, considers additional axioms in order to validate the Gödel-Gentzen translation [7]. In this talk we consider a minimal fragment of the constructive modal logic CK only containing the implication $\supset$ and the modality $\Box$ (see Figure 1). This fragment is enough to define types

$$\dfrac{}{a \vdash a}\,\mathsf{ax} \quad \dfrac{\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathsf{K} \quad \dfrac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\,\mathsf{C} \quad \dfrac{\Gamma \vdash B}{\Gamma, A \vdash B}\,\mathsf{W} \quad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\supset_{\mathsf{R}} \quad \dfrac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \supset B \vdash C}\,\supset_{\mathsf{L}} \quad \dfrac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge_{\mathsf{R}} \quad \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\,\wedge_{\mathsf{L}}$$

Figure 1: Sequent calculus rules for (the $\Diamond$-free fragment of) CK.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \vdash a}\,\mathsf{ax} \quad \dfrac{}{a \vdash a}\,\mathsf{ax}}{a, a \vdash a \wedge a}\,\wedge_{\mathsf{R}}}{a \vdash a \wedge a}\,\mathsf{C}}{c, a \vdash a \wedge a}\,\mathsf{W}}{\Box c, \Box a \vdash \Box(a \wedge a)}\,\mathsf{K} \quad \dfrac{}{b \vdash b}\,\mathsf{ax}}{\Box(a \wedge a) \supset b, \Box c, \Box a \vdash b}\,\supset_{\mathsf{L}}}{(\Box(a \wedge a) \supset b) \supset (\Box c \supset (\Box a \supset b))}\,3 \times \supset_{\mathsf{R}} \quad \rightsquigarrow$$

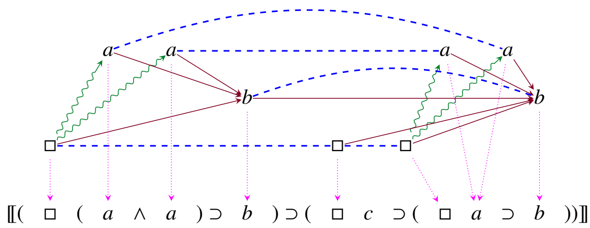


Figure 2: A sequent calculus derivation of the formula $(\Box(a \wedge a) \supset b, \Box c, \Box a) \supset (\Box c \supset (\Box a \supset b))$ and its corresponding combinatorial proof. The solid (red) edges represent the information encoded by the logical connective $\supset$. The squiggly (green) edges encodes the scope of modalities. The dashed (blue) edges represent the partition of vertices of G encoding the ax and K rules. The dotted downwards (pink) arrows represent the skew fibration f encoding the W and C rules.

for a λ-calculus with a Let constructor [3] allowing us to express modal λ-terms of the form $\mathsf{Let}\ M_1, \ldots M_n\ \mathsf{be}\ x_1, \ldots, x_n\ \mathsf{in}\ N$ which can be interpreted as an explicit substitution – for this reason we denote by $N\,[M_1, \ldots M_n / x_1, \ldots, x_n]_{\blacksquare}$.

Combinatorial Proofs for Constructive Modal Logic. In a recent work [2], we defined *combinatorial proofs* for constructive modal logic extending the previous work on combinatorial proofs for intuitionistic propositional logic [9], therefore defining a proof system (in the sense of [6]) enforcing a notion of proof equivalence on sequent calculus: two proofs are the same if they can be represented by the same combinatorial proof.

A combinatorial proof of a formula F is a graph homomorphism $f : G \to [\![F]\!]$ of a specific type between two directed graphs satisfying specific topological conditions. The directed graph $[\![F]\!]$ encodes a formula F with a syntax extending the ones of Hyghland-Ong arenas [10], while the graph G is the arena of a different formula F' enriched with additional edges carrying the information required to reconstruct a linear proof of F'. The homomorphism f allows us to ensure that this morphism encodes the "resource management part" of the proof, i.e., it collects the information about the instances of contraction and weakening in a sequent calculus derivation.

Game Semantics for Constructive Modal Logic. From the syntax of combinatorial proofs, we have developed a game semantics for constructive modal logic [1] relying on the connection between semantics on Hyghland-Ong arenas [10] and intuitionisitic combinatorial proofs provided in [9]. Our *winning innocent strategies* are defined by including additional conditions to address the additional constraints in proofs due to the presence of modalities. In particular, our games need to take into account the possibility of gather modalities in such a way instances of the K can correctly being used during proof search (see Figure 3 for examples of strategies respecting the winning conditions in intuitionistic logic, but encoding incorrect proofs in CK).

On Proof Equivalence in Constructive Modal Logic.

The works on combinatorial proofs and game semantics exposed a gap between the proof equivalences induced by the natural deduction ([5]) and winning innocent strategies ([1]) for the logic CK. This discrepancy cannot be observed in intuitionistic propositional logic where there are one-to-one correspondences between natural deduction derivations, lambda terms and innocent winning strategies (see Figure 2). In particular, in the logic CK we observe sequent calculus proofs which correspond to the same winning strategy but which cannot be represented by the same natural deduction derivation (or equivalently corresponding to different modal λ-

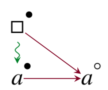
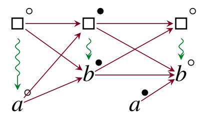
Arena	$[\![(\Box a)\supset a]\!] =$ 	$[\![(\Box a\supset\Box b)\supset\Box(a\supset b)]\!] =$ 
Winning Strategy	$\mathcal{S}_1 = \{\epsilon, a^\circ, a^\circ a^\bullet\}$	$\mathcal{S}_2 = \{\epsilon, b^\circ, b^\circ b^\bullet, b^\circ b^\bullet a^\circ, b^\circ b^\bullet a^\circ a^\bullet\}$
corresponding (failed) proof search	$\supset_R \dfrac{\dfrac{FAIL}{\Box a\vdash a}}{\vdash \Box a\supset a}$	$\supset_R \dfrac{\supset_L \dfrac{\mathsf{K}\dfrac{\dfrac{FAIL}{\vdash a}}{\vdash \Box^\circ a} \quad \mathsf{K}\dfrac{\supset_L\dfrac{\mathsf{W}\dfrac{\mathsf{ax}\dfrac{}{b\vdash b}}{b,a\vdash b}}{b\vdash a\supset b}}{\Box^\bullet b\vdash \Box^\circ(a\supset b)}}{\Box^\circ a\supset\Box^\bullet b\vdash\Box^\circ(a\supset b)}}{\vdash(\Box^\circ a\supset\Box^\bullet b)\supset\Box^\circ(a\supset b)}$

Figure 3: Examples of winning innocent strategies for arenas not corresponding to correct proofs in CK because K-rules cannot be applied in a correct way or prevent a successful proof search.

Independent rules	$\dfrac{\Gamma_1,\Delta_1 \quad \dfrac{\Gamma_2,\Delta_2,\Delta_3 \quad \Gamma_3,\Delta_4}{\Gamma_2,\Gamma_3,\Delta_2,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma_1,\Delta_1 \quad \Gamma_1,\Delta_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Delta_2,}\rho_1 \quad \Gamma_3,\Delta_4}{\Gamma_1,\Gamma_2,\Gamma_3,\Sigma_1,\Sigma_2}\rho_2$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1}{\Gamma,\Sigma_1,\Sigma_2}\rho_2 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Delta_1,\Sigma_2}\rho_2}{\Gamma,\Sigma_1,\Sigma_2}\rho_1$ $\quad \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Delta_1,\Sigma_2}\rho_2}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_1 \equiv \dfrac{\dfrac{\Gamma,\Delta_1,\Delta_2}{\Gamma,\Sigma_1,\Delta_2}\rho_1 \quad \Gamma_2,\Delta_3}{\Gamma_1,\Gamma_2,\Sigma_1,\Sigma_2}\rho_2$
WC-interactions	$\dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{C}}{\Gamma,A\wedge B\vdash C}\wedge_{\mathsf{L}} \approx \dfrac{\dfrac{\Gamma,A,A,B,B\vdash C}{\Gamma,A\wedge B,A\wedge B\vdash C}2\times\wedge_{\mathsf{L}}}{\Gamma,A\vdash B}\mathsf{C}$ $\quad \dfrac{\dfrac{\Gamma\vdash C}{\Gamma,A,B\vdash C}2\times\mathsf{W}}{\Gamma,A\wedge B\vdash C}\wedge_{\mathsf{L}} \approx \dfrac{\Gamma\vdash C}{\Gamma,A\wedge B\vdash C}\mathsf{W}$ $\dfrac{\dfrac{\Gamma,A,A\vdash B}{\Gamma,A\vdash B}\mathsf{C}}{\Gamma,A,A\vdash B}\mathsf{W} \approx \Gamma,A,A\vdash B$ $\quad \dfrac{\dfrac{\Gamma,A\vdash B}{\Gamma,A,A\vdash B}\mathsf{W}}{\Gamma,A\vdash B}\mathsf{C} \approx \Gamma,A\vdash B$
Excising and Unfolding	$\dfrac{\Gamma\vdash A \quad \dfrac{\Delta\vdash C}{B,\Delta\vdash C}\mathsf{W}}{\Gamma,\Delta,A\supset B\vdash C}\supset_{\mathsf{L}} \approx_{ex} \dfrac{\Delta\vdash C}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{W}$ $\Vert$ $\dfrac{\Gamma\vdash A \quad \dfrac{\Delta,B,B\vdash C}{\Delta,B\vdash C}\mathsf{C}}{\Gamma,A\supset B\vdash C}\supset_{\mathsf{L}} \approx_{unf} \dfrac{\dfrac{\Gamma\vdash A \quad \dfrac{\Gamma\vdash A \quad \Delta,B,B\vdash C}{\Gamma,\Delta,A\supset B,B\vdash C}\supset_{\mathsf{L}}}{\Gamma,\Gamma,\Delta,A\supset B,A\supset B\vdash C}\supset_{\mathsf{L}}}{\Gamma,\Delta,A\supset B\vdash C}\mathsf{C}$
Structural vs K	$\dfrac{\dfrac{\Gamma\vdash A}{\Gamma,B\vdash A}\mathsf{W}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K} \approx_K \dfrac{\dfrac{\Gamma\vdash A}{\Box\Gamma\vdash\Box A}\mathsf{K}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{W}$ $\quad \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Gamma,B\vdash A}\mathsf{C}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{K} \approx_K \dfrac{\dfrac{\Gamma,B,B\vdash A}{\Box\Gamma,\Box B,\Box B\vdash\Box A}\mathsf{K}}{\Box\Gamma,\Box B\vdash\Box A}\mathsf{C}$

$$\approx_{\mathsf{ICP}} := (\equiv \cup \approx \cup \approx_{ex}) \quad\subsetneq\quad \approx_\lambda := \left(\approx_{\mathsf{ICP}} \cup \approx_{unf}\right) \quad\subsetneq\quad \approx_{\mathsf{WIS}} := (\approx_\lambda \cup \;\cup\; \approx_K)$$

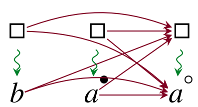
Figure 4: Rules permutations generating different proof equivalences over the sequent calculus for CK induced by combinatorial proofs from [2] ($\approx_{\mathsf{ICP}}$), modal lambda terms from [5] ($\approx_\lambda$), and winning innocent strategies from [1] ($\approx_{\mathsf{WIS}}$).

terms). By means of example, consider the terms $x\,[z/x]_\blacksquare$ and $x\,[z,w/x,y]_\blacksquare$ and their unique typing derivations (in the natural deduction system from [5, 11])

$$\Box\text{-subst}\,\dfrac{\mathsf{Id}\,\dfrac{}{z:\Box a, w:\Box b\vdash z:\Box a} \quad \mathsf{Id}\,\dfrac{}{x:a,y:b\vdash x:a}}{z:\Box a,w:\Box b\vdash x\,[z/x]_\blacksquare:\Box a} \qquad \Box\text{-subst}\,\dfrac{\mathsf{Id}\,\dfrac{}{z:\Box a,w:\Box b\vdash z:\Box a} \quad \mathsf{Id}\,\dfrac{}{z:\Box a,w:\Box b\vdash w:\Box b} \quad \mathsf{Id}\,\dfrac{}{x:a,y:b\vdash x:a}}{z:\Box a,w:\Box b\vdash x\,[z,w/x,y]_\blacksquare:\Box a} \tag{1}$$

According to the definitions of winning innocent strategies for CK from [1], both these two natural deduction derivations correspond to the to the same (unique) CK-winning innocent strategy $\{\epsilon, a^\circ, a^\circ a^\bullet\}$ over the arena

$$[\![\Box a,\Box b\vdash\Box a]\!] \quad = \tag{2}$$



Intuitively, the two terms $x\,[z/x]_\blacksquare$ and $x\,[z,w/x,y]_\blacksquare$ should be semantically equivalent since the explicit substitution in the term x is vacuous when applied to the variable y. That is, in presence of a rewriting rule performing the explicit substitution, we should have that both terms $x\,[z/x]_{\blacksquare}$ and $x\,[z, w/x, y]_{\blacksquare}$ reduce to the term z.

A New Modal Lambda Calculus for Constructive Modal Logic. We conclude this presentation by introducing a new modal λ-calculus for CK, where additional rewriting rules are introduced in order to to retrieve a one-to-one correspondence between terms in normal form and winning innocent strategies, that is, providing more canonical representatives for proofs with respect to natural deduction and modal λ-terms defined in the previous literature.

We show that the calculus is confluent and strongly normalizing. Moreover, we provide a typing system inspired by focused sequent calculi (see, e.g., [4]) allowing us to have a unique typing derivation for each term in normal form. This system allows us to establish a one-to-one correspondence between proofs of this calculi (therefore terms in normal form) and the winning strategies defined in [1].

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Ordinal Analysis of Labelled Kruskal's Theorem

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Kruskal's theorem [3] is one of the most famous and celebrated result in the theory of well quasi-orders with applications in many different areas such as mathematics, logic and computer science. All in all, Kruskal's theorem states that, given a w.q.o. set of labels Q, the set of finite trees over Q is a w.q.o. under a suitable embedding relation. In the remaining of this abstract, let us fix the following notation: by $\mathrm{KT}(\omega)$, we denote Kruskal's theorem for unlabelled trees, namely the statement "the set of unlabelled finite trees is a w.q.o."; by $\mathrm{KT}(n)$, Kruskal's theorem for unlabelled trees with branching degree n, i.e., finite trees in which each node has at most n successors; by $\mathrm{KT}_\ell(\omega)$, the standard aforementioned Kruskal's theorem for labelled trees, and by $\mathrm{KT}_\ell(n)$, Kruskal's theorem for labelled trees with branching degree n. A very thorough proof-theoretic analysis of $\mathrm{KT}(\omega)$ and $\forall n\, \mathrm{KT}(n)$ has been accomplished by Rathjen and Weiermann in [6], where the following equivalence result is obtained

$$\mathrm{RCA}_0 \vdash \forall n\, \mathrm{KT}(n) \leftrightarrow \mathrm{KT}(\omega) \leftrightarrow \mathrm{WO}(\vartheta\Omega^\omega).$$

Extending further the ordinal analysis of Kruskal's theorem, we have achieved the following ordinal estimations for $\mathrm{KT}_\ell(\omega)$ and $\forall n\, \mathrm{KT}_\ell(n)$,

$$|\mathrm{RCA}_0 + \mathrm{KT}_\ell(\omega)| = \vartheta(\Omega^{\omega+1}) \quad \text{and} \quad |\mathrm{RCA}_0 + \forall n\, \mathrm{KT}_\ell(n)| = \vartheta(\Omega^\omega + \omega).$$

In both these calculations, a key step is to convert Kruskal's theorem in some equivalent Well Ordering Principle $\mathrm{WOP}(\mathfrak{g})$; roughly speaking, given an ordinal function $\mathfrak{g}$, $\mathrm{WOP}(\mathfrak{g})$ amounts to the statement "$\forall X\ [\mathrm{WO}(X) \to \mathrm{WO}(\mathfrak{g}(X))]$" with $\mathrm{WO}(X)$ standing for "X is a well-ordering". Such principles have already been studied in literature [1, 5, 7]. For $\mathrm{KT}_\ell(\omega)$, we can then rely on a result due to Arai [2, Theorem 3]; whereas for $\forall n\, \mathrm{KT}_\ell(n)$, a suitable extension of Arai's result is previously obtained.

For sake of completeness, we highlight how another possible approach is given by a result due to Pakhomov and Walsh [4, lemma 3.8], which allows to move from a Well Ordering Principle to a Π^1_1-equivalent Well Ordering Rule; namely, under some side conditions, $\mathrm{ACA}_0 + \mathrm{WOP}(\mathfrak{g}) \equiv_{\Pi^1_1} \mathrm{ACA}_0 + \frac{\mathrm{WO}(\alpha)}{\mathrm{WO}(\mathfrak{g}(\alpha))}$.

Finally, for what concerns future works, the next step amounts to establish, as already done for $\mathrm{KT}(\omega)$ in [6], the proof-theoretic strength of $\mathrm{KT}_\ell(\omega)$ and $\forall n\, \mathrm{KT}_\ell(n)$ in terms of reflection principles; our actual conjectures, due respectively to F. Pakhomov and A. Freund, are the following:

$$\begin{aligned} \mathrm{RCA}_0 &\ \vdash\ \mathrm{KT}_\ell(\omega) \leftrightarrow \Pi^1_2\text{-}\omega RFN(\Pi^1_2\text{-}BI_0 \restriction \Pi^1_3) \\ \mathrm{RCA}_0 &\ \vdash\ \forall n\, \mathrm{KT}_\ell(n) \leftrightarrow \Pi^1_2\text{-}RFN(\Pi^1_2\text{-}BI_0) \end{aligned}$$

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Regular Resolution Effectively Simulates Resolution (Abstract)

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Abstract

Regular resolution is a refinement of the resolution proof system where no variable is resolved upon more than once along any path in the proof. It is known that there are sequences of formulas that require exponential-size proofs in regular resolution while admitting polynomial-size proofs in resolution. Thus, with respect to the usual notion of a simulation, regular resolution is separated from resolution. An alternative, and weaker, notion for comparing proof systems is that of an "effective simulation," which allows the translation of the formula along with the proof when moving between proof systems. We prove that regular resolution is equivalent to resolution under effective simulations.

1 Introduction

Proof complexity studies the sizes of proofs[1] in propositional proof systems. A common kind of question is to compare the strengths of different systems. This comparison is performed typically with respect to the notion of a "simulation" [10]. A system P *simulates* another system Q if any Q-proof of a formula can be converted, with at most a polynomial increase in size, into a P-proof of the same formula. An alternative, and weaker, notion of a simulation is the following, which is arguably more natural from an algorithmic point of view (see [14] for a discussion).

Definition 1 ([11, 14])**.** Let P and Q be two proof systems for a class $\mathcal{C}$ of propositional formulas. For a formula Γ, let $|\Gamma|$ denote its size. Then P *effectively simulates* Q if there exists a function $f\colon \mathcal{C} \times \mathbb{N} \to \mathcal{C}$ such that the following hold.

- The formula $f(\Gamma, m)$ can be output in time polynomial in $|\Gamma| + m$ and it is satisfiable if and only if Γ is.
- When m is at least the size of the shortest Q-proof of Γ, the formula $f(\Gamma, m)$ has a P-proof of size polynomial in $|\Gamma| + m$.

Effective simulations exist in several cases where either no simulation is known or a separation exists [2, 6, 9, 12, 14, 7, 13, 8]. This work focuses on resolution [5, 15], which is a proof system for refuting the satisfiability of propositional formulas in conjunctive normal form (CNF). We view formulas in CNF as sets of clauses. Resolution consists of

[1]Throughout this work, by "proof" we mean a refutation of satisfiability.

a single inference rule that allows *resolving* two premises of the forms $A \vee x$ and $B \vee \neg x$ upon the variable x to derive the conclusion $A \vee B$. A *resolution proof* of a formula Γ is a sequence that ends with the empty clause such that each item in the sequence is either a clause in Γ or derived by resolving two earlier clauses in the sequence. We view a proof as a directed acyclic graph with vertices that represent the clauses in the proof and directed edges from the premises of each inference to its conclusion. A resolution proof is *regular* [16] if no variable is resolved upon more than once along any path in the proof. Regular resolution does not simulate resolution [1]. We prove that this is not the case under effective simulations.

2 Main result

Theorem 2. *Regular resolution effectively simulates resolution.*

To prove Theorem 2, we use the following transformation, where each $W[x,i]$ is a new variable, representing an "annotated version" of the variable x. We identify $W[x,m]$ with x. For each $1 \le i \le m$, we identify $W[\neg x,i]$ with the literal $\neg W[x,i]$.

$$f(\Gamma,m) \coloneqq \Gamma \wedge \bigwedge_{x\in\mathrm{var}(\Gamma)} \bigwedge_{1\le i<m} [(W[x,i] \vee \neg W[x,i+1]) \wedge (\neg W[x,i] \vee W[x,i+1])].$$

We show that there exists a polynomial p such that if Γ has a resolution proof Π of size m, then $f(\Gamma,m)$ has a regular resolution proof of size $p(m)$. The intuition for forming the regular resolution proof is that the new variables $W[x,i]$ are equivalent to x, and the proof Π can be turned into a regular proof by replacing literals p with $W[p,i]$, letting i decrease as the proof progresses. In this way, multiple resolutions upon a variable x are replaced by resolutions upon variables $W[x,i]$, with i decreasing along paths in the proof so that no $W[x,i]$ is resolved upon twice on any path. Details of the proof will be given in the full version of this paper.

As a technical point of interest, the size parameter m is often not needed in effective simulations. That is, the function f in Definition 1 depends only on Γ for almost all of the known effective simulations. (The only exception that we know of is [8, Lemma 2.2].) In our simulation, it is necessary that f have access to the parameter m to ensure that there are sufficiently many variables $W[x,i]$. However, the use of "size" is not completely necessary; indeed, it suffices for f to depend on the *height* of the proof (i.e., the length of the longest path in the proof graph). It is an open question whether the dependence of f on the size or height can be eliminated.

Theorem 2 has some interesting consequences, given in the next corollaries. A proof system P is *closed under restrictions* if for every partial assignment ρ, any P-proof of Γ can be converted, with at most a polynomial increase in size, into a P-proof of the formula $\Gamma|_\rho$, where $\Gamma|_\rho$ is obtained by evaluating Γ under ρ and simplifying. Similarly, a system is *closed under variable substitutions* if the same holds for every variable substitution. Regular resolution is closed under restrictions, and resolution is closed under both restrictions and variable substitutions. Theorem 2 implies the following corollary via the fact that regular resolution is separated from resolution.

Corollary 3. *Regular resolution is not closed under variable substitutions.*

A proof system P is *automatizable* if there exists an algorithm A that, given Γ, produces a P-proof of Γ in time polynomial in $s + |\Gamma|$, where s is the size of the shortest

P-proof of Γ. We say P is *weakly automatizable* if the algorithm A is allowed to output a proof in some other system. Since effective simulations give reductions between the automatizability properties of proof systems, we have the following by Theorem 2.

Corollary 4. *If resolution is not weakly automatizable, then neither is regular resolution.*

An analogous result is known for (strong) automatizability: Atserias and Müller [3] recently proved that automating resolution is NP-hard. It was observed afterwards that the same result holds also for regular resolution. However, the extension of their result to regular resolution is a nontrivial step and requires an inspection of their proof. (See also the preprint by Bell [4] for a detailed writeup.) In contrast, Corollary 4 works in a black-box fashion.

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Finite labeled trees ordered by non-inf-preserving embeddings

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1 Abstract

We study the well-quasi-order (wqo) consisting of the set of finite trees with internal and leaf labels coming from arbitrary wqo's P and Q respectively, ordered by homomorphic embeddability which respects the order of the labels. This is a variant of the usual Kruskal ordering but without infima preservation. We calculate the precise maximal order types of these wqo's — in the style of De Jongh and Parikh[1], and Schmidt[8] — as a function of the maximal order types of P and Q. In doing so, we sharpen some recent results of Harvey Friedman and Andreas Weiermann[3]. This also helps to calibrate the reverse mathematical strength of certain well-foundedness assertions and obtain natural combinatorial independence results.

Nash-Williams proved that arbitrary transfinite sequences using finitely many elements from a well-quasi-ordered set are also well-quasi-ordered[6], but the proof does not offer immediate information about the maximal order type. Erdos and Rado previously proved this for the specific case of sequences of length ω^n using a more concrete approach[7]. Our results lead to precise bounds for transfinite sequences of length less than ω^ω, using the correspondence between the set of finite leaf-labeled trees and indecomposable transfinite sequences of finite range with length less than ω^ω.

These embeddings seem to have been first defined by Montalban[5], and specific instances of this wqo were considered by Marcone and Montalban to study a limited form of Fraisse's Conjecture[4]. An almost-equivalent ordering was very recently used by Anton Freund to show the reverse mathematical strength of certain statements about better-quasi-orders[2].

Acknowledgements

The motivation to look at leaf-labeled trees originally came from Fedor Pakhomov, who observed a correspondence with indecomposable transfinite sequences. Andreas Weiermann was also working on this problem independently, and is a leading reference in the field of well-quasi-orders and maximal order types.

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On the computational expressivity of (circular) proofs with fixed points

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Cyclic proofs are an emerging topic of proof theory that is attracting increasing interest in the literature. This area originates (in its modern guise) in the context of the modal μ-calculus [16, 8], serving as an alternative framework to manipulate least and greatest fixed points, and hence to model inductive and coinductive reasoning as well as (co)recursion mechanisms.

Cyclic proof theory has been investigated in many settings, such as first-order inductive definitions [3], Kleene algebras [7], automata [9], continuous cut-elimination and linear logic [12, 1], arithmetic [17], Gödel's system T [6, 13], implicit complexity [5].

In this paper we study the computational strength of $\mu\mathsf{LJ}$ and its circular presentation $\mathsf{C}\mu\mathsf{LJ}$, which are extensions of intuitionistic logic with least and greatest fixed points introduced by Clairambault in [4]. More specifically, we show that the functions on natural numbers representable in $\mu\mathsf{LJ}$ and $\mathsf{C}\mu\mathsf{LJ}$ are exactly those provably total in $\mu\mathsf{PA}$, a first-order arithmetic with generalised inductive definitions (see, e.g., [14]). Our fundamental theorem will be established via a series of inclusions comparing the computational expressivity of $\mu\mathsf{LJ}$ and $\mathsf{C}\mu\mathsf{LJ}$ with various theories of arithmetic:

$$\mu\mathsf{PA} \overset{(i)}{\subseteq} \mu\mathsf{HA} \overset{(ii)}{\subseteq} \mu\mathsf{LJ} \overset{(iii)}{\subseteq} \mathsf{C}\mu\mathsf{LJ} \overset{(iv)}{\subseteq} \Pi^1_2\text{-}\mathsf{CA}_0 \overset{(v)}{\subseteq} \mu\mathsf{PA}$$

We first prove Π^0_2-conservativity of $\mu\mathsf{PA}$ over its intuitionistic version, $\mu\mathsf{HA}$, by standard double-negation translations (i). Secondly, we show that the provably recursive functions of $\mu\mathsf{HA}$ are representable in $\mu\mathsf{LJ}$ using realisability techniques (ii). Thirdly, we show a simulation result relating $\mu\mathsf{LJ}$ and $\mathsf{C}\mu\mathsf{LJ}$ (iii). The most relevant contribution of this paper is the inclusion (iv), where we formalise a totality argument for circular proofs in Π^1_2-CA_0, the subsystem of second-order arithmetic with Π^1_2-comprehension and set induction. In particular, the totality argument is based on hereditary recursive models. We conclude by leveraging on a recent result by Möllerfeld in [15], who showed that Π^1_2-CA_0 is arithmetically conservative over $\mu\mathsf{PA}$ (v). Along the way we develop some novel reverse mathematics for the Knaster-Tarski fixed point theorem.

The above methods extend to other fixed point logics, such as $\mu\mathsf{LL}$ (i.e., linear logic with least and greatest fixed points) [10, 11] and its multiplicative-additive fragment $\mu\mathsf{MALL}$ [2, 1]. As a future work, we are planning to investigate the computational strength of notable subsystems of $\mu\mathsf{LJ}$, such as those restricting fixed points to parameter-free formulas or to strictly positive formulas.

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On computing the complexity of μMALL proof systems

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1 Proof systems for fixed point logics

First introduced to capture inductive definitions, fixed point logics have had several applications over the years. In order to define the language of a fixed point logic, one introduces explicit fixed point construct(s) and takes the closure under these construct(s) thus obtaining a richer language. For example, a popular choice is two operators μ and ν which are duals of each other and depict the least and greatest fixed points respectively. The (multi)modal μ-calculus [8], extensions of first order logic with various fixed point operators, Kleene Algebra (and its extensions) are some well-studied fixed point logics.

1.1 Wellfounded systems

In order to design sequent calculi for fixed point logics, there are some fundamental design choices to be made. For instance, one can employ inference rules that explicitly express the (co)induction invariant:

$$\frac{F(S) \vdash S}{\mu x.F(x) \vdash S}\,(\mu) \qquad ; \qquad \frac{S \vdash F(S)}{S \vdash \nu x.F(x)}\,(\nu)$$

However, sequent calculi with explicit (co)induction do not have the subformula property. This poses a major challenge when it comes to proof search since one has to essentially guess induction invariants. A more robust and natural alternative formalisation of inductive reasoning is *implicit* induction, which avoids the need for explicitly specifying (co)induction invariants. This formalism generally recovers true cut elimination but at the cost of infinitary axiomatisation of the fixed points.

There are two approaches to implicit (co)induction. The first approach is to consider infinitary wellfounded derivations which use a ω-rule with infinitely many premises of finite approximations of a fixed point.

$$\frac{\vdash \top \quad \vdash F(\top) \quad \vdash F^2(\top) \quad \dots}{\vdash \nu x.F(x)}\,(\nu)$$

1.2 Non-wellfounded systems

The second approach is to define a *non-wellfounded* and/or a *circular* proof system with finitely branching inferences [12, 13]. Such systems potentially admit greater proof-theoretic expressivity while, at the same time, reinforcing connections between these logics and automata theory. However, when considering all possible non-wellfounded derivations (aka *pre-proofs*), the resulting system is inconsistent. In particular one can derive the empty sequent.

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu x.x}\,(\mu)}{\vdash \mu x.x}\,(\mu) \quad \dfrac{\dfrac{\vdots}{\vdash \nu x.x}\,(\nu)}{\vdash \nu x.x}\,(\nu)}{\vdash}\,(\mathsf{cut})$$

Therefore, a global progress criterion is imposed to sieve the logically valid proofs from the unsound ones. Typically, it requires that every infinite branch is supported by some thread tracing some formula in a bottom-up manner and witnessing infinitely many progress points of a coinductive property. Furthermore, in this non-wellfounded setting, termination of the cut-elimination procedure shall be replaced by *productivity i.e.* that arbitrarily large prefixes of the result can be computed in a finite number of steps. The aforementioned progress condition is a sufficient condition for the productivity of cut-elimination which restores the subformula property.

On the other hand, on account of their infinitude, non-wellfounded proofs cannot be communicated or checked in finite time. Consequently, we consider a fragment of non-wellfounded derivations *viz.* that of derivation trees with finitely many distinct subtrees, known as *circular*, or cyclic, derivations. Therefore, instead of giving an infinite proof, we give a finite description of an infinite proof, formulated in the meta-theory.

A natural question to ask at this point is if these systems all prove the same set of theorems [3, 4]. This is heavily dependent on the base logic since the availability of structural rules or modal constructs induce subtle differences. For instance, the modal μ-calculus coincides on all systems whereas in Kleene Algebras (where the base logic is substructural) all the various systems are different [6, 9].

2 The situation in linear logic

In linear logic, the use of structural rules like contraction and weakening is carefully controlled (available only to formulas of a certain form). We consider multiplicative-additive linear logic(MALL) as our base logic. where neither do we have contraction and weakening nor are they derivable. The language of MALL closed under least and greatest fixed point operators is called $\mu\mathsf{MALL}$. We set up the notation: $\mu\mathsf{MALL}^{\mathsf{ind}}$and $\mu\mathsf{MALL}_\omega$ are the wellfounded systems with explicit (co)induction [2] and the ω-rule respectively; $\mu\mathsf{MALL}^\omega$ and $\mu\mathsf{MALL}^\infty$ are the circular and non-wellfounded systems [1] respectively. It is known that $\mu\mathsf{MALL}^{\mathsf{ind}} \subseteq \mu\mathsf{MALL}^\omega \subseteq \mu\mathsf{MALL}^\infty$ however it is not known if the inclusions are strict. We emphasize that $\mu\mathsf{MALL}_\omega$ has not been studied before and it is easy to see that $\mu\mathsf{MALL}^{\mathsf{ind}} \subseteq \mu\mathsf{MALL}_\omega$. Our first main result is $\mu\mathsf{MALL}^\infty \subseteq \mu\mathsf{MALL}_\omega$ which we prove by first defining an infitinary rewriting of non-wellfounded that produces a wellfounded proof in the limit.

The provability of $\mu\mathsf{MALL}$ systems have, in general, not received serious introspection. $\mu\mathsf{MALL}^{\mathsf{ind}}$has been established as a foundation for model checking and $\mu\mathsf{MALL}^\omega$ can be seen as natural type system for (co)recursive programs. Therefore, the decidability and complexity of the provability of $\mu\mathsf{MALL}$ systems are just not of mathematical intrigue but have deep implications.

3 Complexity results

If μMALL is restricted to formulas without greatest fixed points, then the various systems coincide to a wellfounded system which we call $\mu\mathsf{MALL}^*$. We show that provability in $\mu\mathsf{MALL}^*$ is undecidable. Our second result is that the complexity of $\mu\mathsf{MALL}^\infty$ provability is much higher – in fact it is in the analytical hierarchy. As a consequence of these results, we can show that $\mu\mathsf{MALL}^\omega \subsetneq \mu\mathsf{MALL}^\infty \subsetneq \mu\mathsf{MALL}_\omega$. Some of these results were obtained in [5, 7].

3.1 Lower bound on $\mu\mathsf{MALL}^*$ provability

Full linear logic was shown to be undecidable in [10, 11] by a reduction from the reachability problem in an *and-branching* two counter machine without zero-test. We use this fact to prove the undecidability of $\mu\mathsf{MALL}^*$ using a standard encoding of the exponential modalities by fixed point formulas of the following form:

$$[?F] = \mu X.\bot \oplus [F] \oplus (X \parr X) \quad ; \quad [!F] = \nu X.\mathbf{1} \,\&\, [F] \,\&\, (X \otimes X)$$

This encoding is not known to be faithful. In other words, it is not known if $[F]$ is provable in $\mu\mathsf{MALL}^{\mathsf{ind}}$or $\mu\mathsf{MALL}^\omega$, then F is provable in full linear logic. However, the reduction in [10, 11] uses only ? so the encoding is indeed in $\mu\mathsf{MALL}^*$. We show that the encoding is faithful for !-free linear logic formula which allows to conclude that $\mu\mathsf{MALL}^*$ is Σ^0_1-complete. $\mu\mathsf{MALL}^\omega$ and $\mu\mathsf{MALL}^{\mathsf{ind}}$ inherits the Σ^0_1 hardness of $\mu\mathsf{MALL}^*$. Since both are systems of finitely presentable proofs that are recursively checkable, Σ^0_1-membership is also immediate and consequently, we have that $\mu\mathsf{MALL}^{\mathsf{ind}}$and $\mu\mathsf{MALL}^\omega$ are Σ^0_1-complete.

3.2 Lower bounds on $\mu\mathsf{MALL}^\infty$ provability

We reduce from decision problems of Minsky machines to obtain our hardness results. In [5, 7], we encoded the non-halting of Minsky machines to obtain that $\mu\mathsf{MALL}^\infty$ provability is Π^0_1-hard. In this talk, we extend that result: we encode the emptiness of Büchi Minsky machines to obtain that $\mu\mathsf{MALL}^\infty$ provability is Σ^1_1-hard.

The idea of the encoding is to mimic runs of a Minsky machine in the infinite branches of the proof. In order to show that this encoding is faithful, we need to restrict the proof search space by considering cut-free focussed proofs.

The hardness results has several consequences *viz.* we are able to obtain that $\mu\mathsf{MALL}^\omega \subsetneq \mu\mathsf{MALL}^\infty \subsetneq \mu\mathsf{MALL}_\omega$. Observe that this proof is apparently non-constructive in the sense that we do not explicitly exhibit sequents in $\mu\mathsf{MALL}^\infty \setminus \mu\mathsf{MALL}^\omega$ and $\mu\mathsf{MALL}_\omega \setminus \mu\mathsf{MALL}^\infty$. However, the argument can indeed be constructivised using recursion-theoretic techniques, namely the notion of *productive functions*.

4 Towards the exact complexity of $\mu\mathsf{MALL}^\infty$

We will present a technique to obtain an upper bound for $\mu\mathsf{MALL}^\infty$ provability. We construe the provability in $\mu\mathsf{MALL}^\infty$ as a turn-based game between two players Prover and Denier. The idea is that a play corresponds to a branch in a potential proof so winning plays are either ones which end in an axiom or ones which go on forever but satisfy the progress condition. Therefore, proving a formula amounts to asking if the Prover has a winning strategy. By analytical determinacy, that is tantamount to Denier having a losing strategy which gives us a Π^1_2 upper bound for $\mu\mathsf{MALL}^\infty$ provability. The exact complexity remains open.

In conclusion, in this talk, we plan to (i) introduce the ω-branching system of $\mu\mathsf{MALL}$ (ii) report the complexity results obtained in [5, 7] (iii) strengthen those results and outline their consequences (iv) and finally to have feedback from the Proof Society audience.

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On intuitionistic diamonds (and lack thereof)

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A variety of intuitionistic versions of modal logic K have been proposed. A widespread misconception (even appearing in the literature) is that all these logics coincide on their $\Box$-only (i.e. $\Diamond$-free) fragment, suggesting some robustness of '$\Box$-only intuitionistic modal logic'. However, in this work, we show that this is not true, by consideration of negative translations from classical modal logic: Fischer Servi's IK proves strictly more $\Diamond$-free theorems than Fitch's CK, and indeed iK, the minimal $\Box$-normal intuitionistic modal logic.

On the other hand we show that the smallest extension of iK by a normal $\Diamond$ is in fact conservative (over $\Diamond$-free formulas). To this end, we develop a novel proof calculus based on nested sequents for intuitionistic propositional logic due to Fitting. Along the way we establish a number of new metalogical results for various related logics.

Context

Usual (propositional) modal logic extends the language of classical propositional logic by two modalities, $\Box$ and $\Diamond$, informally representing 'necessity' and 'possibility' respectively. This informality is made precise by its well-known relational semantics. This semantics gives rise to the so-called 'standard translation', allowing us to distill the normal modal logic K as a well-behaved fragment of the first-order logic (FOL).

Notably, in the classical setting $\Box$ and $\Diamond$ are De Morgan dual, just like $\forall$ and $\exists$: we have that $\Diamond A = \neg\Box\neg A$. However, in light of the association with FOL above, one would naturally expect an intuitionistic counterpart of modal logic not to satisfy any such reduction. In particular, while the usual axiomatisation of K simply extends propositional logic by the axiom and rule,

$$k: \ \Box(A \to B) \to (\Box A \to \Box B) \qquad\qquad nec: \ \frac{A}{\Box A}$$

notice that such an axiomatisation cannot be adequate for an intuitionistically based version of modal logic, which does not admit inter-reducibility of $\Box$ and $\Diamond$. In particular, such an axiomatisation tells us nothing about $\Diamond$.

The pursuit of a reasonable definition for an 'intuitionistic' modal logic goes back decades, including (but not limited to) works such as [9, 4, 2, 3, 15, 12, 16, 7, 8, 1, 14, 6, 10, 13, 18, 19]. See [17] or [11] for a survey.

To understand these, let us first list the following (classical) consequences of the k axiom above:

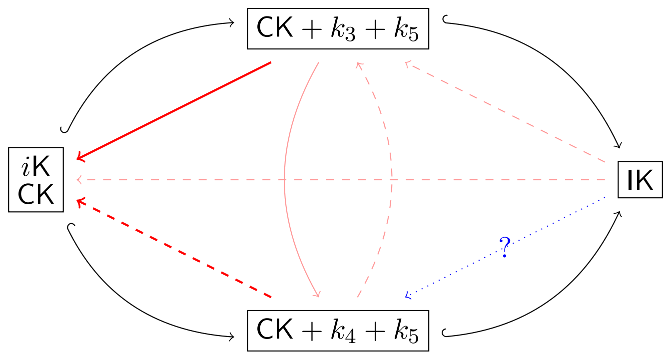

Figure 1: Comparison of $\diamond$-free fragments. A solid arrow $a \to b$ denotes inclusion $a_\square \subseteq b_\square$, a dashed arrow $a \dashrightarrow b$ denotes non-inclusion $a_\square \not\subseteq b_\square$. All new results of this work are indicated in red, where the faded arrows are consequences of the non-faded ones. The dotted blue ? arrow remains open, as far as we can tell from the literature.

$$
\begin{array}{ll}
k_1: & \square(A \to B) \to (\square A \to \square B) \\
k_2: & \square(A \to B) \to (\diamond A \to \diamond B) \\
k_3: & \diamond(A \lor B) \to (\diamond A \lor \diamond B) \\
k_4: & (\diamond A \to \square B) \to \square(A \to B) \\
k_5: & \diamond\bot \to \bot
\end{array}
$$

Definition 1. iK is the extension of intuitionistic propositional logic (IPL) by *nec* and k_1, and CK further extends iK by k_2. IK is the extension of IPL by *nec* and all the axioms above.

It seems that Fitch [9] was the first one to propose a way to treat $\diamond$ in an intuitionistic setting by considering CK which has since become prominent in modal type theory. CK enjoys a rather natural proof-theoretic formulation that simply adapts the sequent calculus for K according to the usual intuitionistic restriction: each sequent may have just one formula on the RHS. What is more, cut-elimination for this simple calculus is just a specialisation of the classical case. An immediate consequence is the conservativity of CK over iK over the $\diamond$-free fragment.

IK was introduced by Plotkin and Stirling in [14] and is equivalent to the one proposed by Fischer Servi [8], or even by Ewald [6] in the context of intuitionistic tense logic. In [17], Simpson gives logical arguments in favour of IK, namely as a logic that corresponds to intuitionistic FOL along the same standard translation that lifts K to classical FOL. The price to pay, however, is steep: there is no known cut-free sequent calculus complete for IK. On the other hand, Simpson demonstrates how the relational semantics of classical modal logic may be leveraged to recover a labelled sequent calculus, exemplifying the utility of pursuing an 'intuitionistic standard translation'. The cut-elimination theorem, this time, specialises the cut-elimination theorem for intuitionistic FOL.

Contribution

In this work we resolve the misconception that the $\diamond$-free fragments of iK, CK, IK coincide. We show that IK (even CK + k_4 + k_5) validates the Gödel-Gentzen translation from K, but that CK (and so iK) does not, separating the two logics on $\diamond$-free formulas. In fact, the simplest such separation we could find is:

$$\neg\neg\Box\bot \to \Box\bot$$

An important question at this point is whether it is even *possible* to conservatively extend $i\mathsf{K}$ by a normal $\Diamond$, or whether such an extension forces new $\Diamond$-free theorems. More precisely: is $\mathsf{CK} + k_3 + k_5$ conservative over $i\mathsf{K}$? To answer this (positively) we give a new system for the logic, extending a nested system for IPL by modalities and proving a cut-elimination result. We argue that the system we give for $\mathsf{CK} + k_3 + k_5$ is of natural interest in its own right, not only being a conservative extension of $i\mathsf{K}$ by a normal $\Diamond$, but yet again distilled from a natural intuitionistic calculus by the addition of modalities, just like $i\mathsf{K}, \mathsf{CK}, \mathsf{IK}$ before.

All our results are summarised in Figure 1.

Related communications

The Gödel-Gentzen translation from K to IK, and consequent separation of $i\mathsf{K}, \mathsf{CK}$ and IK on $\Diamond$-free formulas was announced in a blog post [5], on which some of the content of this abstract is based. There Alex Simpson commented that the separation of $\mathsf{CK}, i\mathsf{K}$ and IK was already communicated to him by Carsten Grefe in 1996, whose minimal separating formula was $(\neg\Box\bot \to \Box\bot) \to \Box\bot$. Note that this separation is weaker and less general than ours. In the same discussion it was mentioned that the $\Diamond$-free fragment of IK was not finitely axiomatisable, which would separate IK from $\mathsf{CK} + k_4 + k_5$. We could not find this result in the literature, nor could we easily verify it independently.

In the same post there was significant discussion with Nicola Olivetti and Tiziano Dalmonte about the status of $\mathsf{CK}+k_3+k_5$, with no definitive resolution about its $\Diamond$-free fragment. Our conservativity result over $i\mathsf{K}$ resolves its status.

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Towards Intuitionistic Gödel-Löb Logic (with a capital 'I')

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The field of intuitionistic modal logics studies intuitionistic counterparts of classical modal logics. Classically, $\Box$ and $\Diamond$ are dual operators, but this is not necessarily true intuitionistically, resulting in different lines of research. We observe a stabilization of notation of such logics in recent years based on which we identify the following lines:

- *intuitionistic* modal logics (often prefixed by a small i), e.g., iK (early studied in, e.g., [4, 20], see [12] for an overview[1]). These logics are defined over the language with only $\Box$.
- *Constructive* modal logics (often prefixed by a capital C), e.g., CK (introduced in [3][2], and further studied in [13]). These logics are defined over $\Box$ and $\Diamond$ and their $\Box$-fragment typically coincides with its small i version.
- *Intuitionistic* modal logics (often prefixed by a capital I), e.g., IK (defined in [6] and investigated in details in [16]). These logics are defined over the language with $\Box$ and $\Diamond$. They typically validate the *standard translation* into first-order logics, intuitionistically, and the addition of excluded middle yields their classical counterpart. Their $\Box$-fragment extends the small i version [5].

This work is concerned with versions of Gödel-Löb logic GL, the provability logic of Peano Arithmetic [18]. From the first intuitionistic viewpoint, intuitionistic Gödel-Löb logic iGL is sound (but not complete) with respect to the provability logic of Heyting Arithmetic, see e.g. [11]. In this work-in-progress, we investigate an Intuitionistic version of GL, along the third viewpoint. We provide several natural Intuitionistic characterizations starting from the classical framework, both proof-theoretic and semantic, and show that these characterizations coincide.

1 GL: semantics and a new non-wellfounded calculus

From a semantic point of view, GL is sound and complete with respect to models (W, R, V) where R is transitive and conversely wellfounded. From a proof-theoretic point of view, logic GL is characterized in several sequent-like calculi, such as the sequent calculus in [2], cyclic sequent system in [15], and labelled system in [14].

Note that the labelled system for GL in [14] is non-standard as it modifies the usual labelled rules for $\Box$ and $\Diamond$. Such modification is somewhat necessitated by the fact that

[1] In these references, iK is denoted as $\mathbf{IntK}_\Box$, $\mathbf{HK}_\Box$, and K^{i}, respectively. For the current development of small i notation we provide some recent examples [10, 1, 7].

[2] Where it is originally called IK.

converse wellfoundedness is not even first-order definable. Instead, we take inspiration from *non-wellfounded proof theory*, where (co)induction principles are devolved to the proof structure rather than explicit rules or axioms. Employing a correctness condition essentially identical to that from Simpson's cyclic arithmetic [17], we define a 'standard' labelled calculus GL which is non-wellfounded, denoted LGL^∞.

2 Towards IGL: semantics and a non-wellfounded calculus

Logic iGL is sound and complete with respect to *birelational models* $(W, \leq, R, V)$ such that $(\leq; R) \subseteq R$ and R is transitive and conversely wellfounded [19]. The valuation V is *persistent*, i.e. monotone in $\leq$. To interpret the $\Diamond$, the models for iGL are too restrictive. In this work we adopt the same frame conditions as [16], i.e., $(R^{-1}; \leq) \subseteq (\leq; R^{-1})$ and $(R; \leq) \subseteq (\leq; R)$, and further require R to be transitive and $(R; \leq)$ to be conversely wellfounded. We call these *birelational IGL-models.*

One can also view (Intuitionistic) modal logic as a fragment of (Intuitionistic) predicate logic under the standard translation, cf. [16]. In this sense, we obtain another Intuitionistic reading of GL, by interpreting $(R; \leq)$-termination within a predicate Kripke model. This yields an Intuitionistic version of GL via models, henceforth called *predicate IGL-models.* It is not hard to see that the induced logic includes that of birelational IGL-models.

Proof-theoretically, to obtain intuitionistic/Intuitionistic versions of classical modal logics, it typically suffices to restrict a 'standard' calculus, to having one formula on the right of a sequent. In this way, starting from the classical system in [2] one obtains a calculus for iGL [8] and an intuitionistic version of the classical cyclic system from [15] is presented in [9]. To this end, labelled systems admitting independent treatments of $\Box$ and $\Diamond$ have been fruitful to define Intuitionistic calculi [16]. We can similarly restrict our classical calculus LGL^∞ for GL to one formula on the right and obtain the labelled non-wellfounded calculus ILGL^∞. Soundness for both aforementioned classes of models is readily established via an infinite descent argument by contradiction that is now standard in non-wellfounded proof theory. Towards a completeness-via-proof-search result, we also consider a multi-conclusion version of this calculus, mILGL^∞.

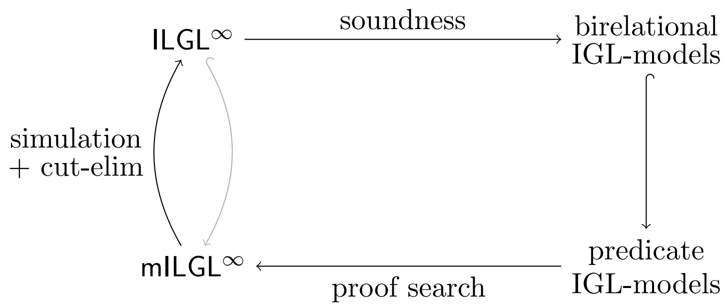

Figure 1: Summary of main results for Intuitionistic modal logics considered in this work. All arrows denote inclusions of modal logics, so the four characterisations coincide.

3 Main results and future work

For the classical system LGL^{∞}, we provide a cut-elimination procedure and show soundness and completeness with respect to Kripke models of GL.

This cut-elimination procedure extends to the intuitionistic counterpart ILGL^{∞} of LGL^{∞}, which allowed us to show that mILGL^{∞} and ILGL^{∞} are equivalent.

We provide a predicate-IGL-countermodel construction from a failed proof search in mILGL^{∞}. Due to the nature of the correctness criteria in non-wellfounded proof theory, navigation of the proof search space is driven by appealing to a determinacy result for the corresponding 'proof search game', yielding a canonical 'strategy', in turn inducing a countermodel. As a consequence, both mILGL^{∞} and ILGL^{∞} are modal complete for predicate-IGL-models.

Our results are summarised in Figure 1. In short, the four proposed Intuitionistic versions of GL coincide. We could arguably designate the resulting logic 'IGL'.

In future work we would like to establish an explicit axiomatisation for the logic introduced. At the same time it would be pertinent to investigate the complexity of our logic, given our hitherto non-finitary-presentations. Finally, we would like to examine the role of our logic as a logic of provability in appropriate models of Heyting Arithmetic.

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To prove or not to prove that IS4

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Intuitionistic modal logics are formed by introducing modalities to intuitionistic propositional logic (IPL) rather than classical propositional logic. There are many options available for defining these systems, which has led to several different versions of intuitionistic modal logics. In this context, we adopt the approach taken by Fischer Servi [1] and Plotkin and Stirling [5], which was investigated in detail by Simpson [6]. Similar to the classical case, logics in the intuitionistic modal family can be established by defining axioms or, equivalently, frame conditions on the class of models. Despite the fact that the decidability of most intuitionistic modal logics in S5-cube has been proven [6], one notable exception is the logic IS4, also known as intuitionistic S4. Since its introduction in Simpson's PhD thesis in 1994 [6], the question of its decidability has remained unresolved. This problem is finally solved positively: we show that IS4 is decidable.

The logic IS4 is formulated in the language $A ::= \bot \mid a \mid (A \wedge A) \mid (A \vee A) \mid (A \supset A) \mid \Box A \mid \Diamond A$ (note that, unlike the classical case, modalities $\Box$ and $\Diamond$ are independent). Its axiom system is obtained by extending any standard axiom system for IPL with

k_1:	$\Box(A \supset B) \supset (\Box A \supset \Box B)$	k_2:	$\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$		
k_3:	$\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$	k_4:	$(\Diamond A \supset \Box B) \supset \Box(A \supset B)$	k_5:	$\Diamond\bot \supset \bot$
$\mathsf{4}$:	$(\Diamond\Diamond A \supset \Diamond A) \wedge (\Box A \supset \Box\Box A)$	t:	$(A \supset \Diamond A) \wedge (\Box A \supset A)$		

and the standard necessitation rule. As its classical counterpart, its Kripke frames are reflexive and transitive, but in the so-called *birelational semantics*:

A ***birelational model*** $\mathcal{M}$ for IS4 is a quadruple $\langle W, R, \leq, V \rangle$ of a set $W \neq \varnothing$ of ***worlds*** equipped with two preorders (i.e., a reflexive and transitive relations) — an ***accessibility relation*** R and future relation $\leq$ — and a ***valuation function*** $V : W \to 2^{\mathcal{A}}$ satisfying:

(F_1) For all $x, y, z \in W$, if xRy and $y \leq z$, there exists $u \in W$ s.t. $x \leq u$ and uRz.
(F_2) For all $x, y, z \in W$, if $x \leq z$ and xRy, there exists $u \in W$ s.t. zRu and $y \leq u$.
(M) If $w \leq w'$, then $V(w) \subseteq V(w')$.

Forcing $\Vdash$ for atomic formulas is determined by the valuation function: $\mathcal{M}, w \Vdash a$ iff $a \in V(w)$, with $\mathcal{M}, w \nVdash \bot$. It is recursively extended to all formulas (propositional clauses are standard):

$\mathcal{M}, w \Vdash A \supset B$	iff	for all w' with $w \leq w'$, if $\mathcal{M}, w' \Vdash A$, then $\mathcal{M}, w' \Vdash B$;
$\mathcal{M}, w \Vdash \Box A$	iff	for all w' and u with $w \leq w'$ and $w'Ru$, we have $\mathcal{M}, u \Vdash A$;
$\mathcal{M}, w \Vdash \Diamond A$	iff	there exists u such that wRu and $\mathcal{M}, u \Vdash A$.

Theorem ([1, 5])**.** A formula A is a theorem of IS4 if and only if A is valid in every birelational model for IS4.

Our proof of decidability of IS4 is proof-theoretical, and employs the fully labelled sequent calculus of [4]. If the proof search is successful in finding a proof, the formula in question is derivable. Otherwise, a failed proof search provides sufficient information to construct a countermodel. The difficulties in applying this method to IS4 are not new either. In practice, naive proof search for a logic that has transitive Kripke frames typically does not terminate. There have been several terminating calculi proposed for both IPL (w.r.t. transitive $\leq$) and S4 (w.r.t. transitive R) in the literature, with most of them including loop-check strategies to stop the naive proof search. Loop-check mechanisms detect repetitions of the same formulas within a sequent or throughout a branch. Repetitions are bound to happen thanks to the subformula property, which ensures a global bound on the number of sequents that can appear in a proof search. When a repetition is detected, proof search stops, and a countermodel can be constructed by emulating the algorithm loop by an appropriate R-loop for S4 or $\leq$-loop for IPL.

The unique challenges of IS4 are due to the fact that the two sources of repetition, i.e., transitivity of the $\leq$-relation and the R-relation, can interact, creating a possibility of a proof search neither terminating nor repeating any sequents. To overcome this problem we use a fully labelled sequent calculus (see [3, 4]) that employs relational atoms for both binary relations R and $\leq$, which enables us to represent R-loops on a sequent level. We incorporate several loop-checks into the proof search algorithm by adding new rules for creating R-loops. This R-loop-enabled proof search still does not guarantee sequent repetition, forcing us to formulate a more complex loop-check condition with respect to $\leq$-loops: the proof search is stopped if the latest sequent can be *simulated* by an earlier sequent. The soundness of the new R-loop-creating rules is proved by a non-trivial *unfolding* algorithm that converts R-loop-enabled derivations into proper loop-free derivations by creating multiple duplicates of each loop. Thus, this loop-augmented proof search provides a decision procedure for IS4.

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Implicit Commitments, Reflection and Believability

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Abstract

In this talk, we analyze a formal theory of implicit commitments Bel(Th) introduced in [1]. Let us recall that the implicit commitments of a formal theory Th are sentences independent from the axioms of Th, whose acceptance is implicit in the acceptance of Th. The examples of implicit commitments of the theory Th given in the literature consist of various expressions of the soundness of Th such as the consistency statement Con(Th), or uniform reflection principles over Th, that is a collection RFN(Th):

$$\{\forall x(\mathrm{Prov}_{\mathrm{Th}}(\phi(\dot{x})) \to \phi(x)) : \phi(x) \in \mathcal{L}_{\mathrm{Th}}.\}$$

The idea of implicit commitments and reflection principles in particular proved to be very interesting from the perspective of the foundations of mathematics. Recently, the reflection principles became the object of intense research also in the philosophy of mathematics. It is widely believed that providing the philosophical justification for the transition from accepting Th to the acceptance of, say, uniform reflection over Th is one of the fundamental problems of the epistemology of mathematics (see [2], [1], [5]).

In [1] the phenomenon of implicit commitments was studied from the epistemological perspective through the lenses of the formal theory of mathematical belief – the believability theory Bel(Th). In our approach, we assume that the theory Th includes a basic theory of syntax, namely Kálmar Elementary Arithmetic EA. Bel(Th) consists of a theory Th with the axioms for the language $\mathcal{L}_{\mathrm{Th}} \cup \{\mathrm{B}\}$, where B is a fresh unary predicate, together with the following axioms for B:

- (BTh): $\forall\phi\ (\mathrm{Prov}_{\mathrm{Th}}(\phi) \to \mathrm{B}(\phi))$,
- (MP): $\forall\phi,\psi\ (\mathrm{B}(\phi \to \psi) \to (\mathrm{B}(\phi) \to \mathrm{B}(\psi)))$,
- (ωR): $\forall\phi\ (\mathrm{B}(\forall x \mathrm{B}(\phi(\dot{x})) \to \mathrm{B}(\forall x\phi))$.

Moreover, the system is closed under the rule (NEC) $\dfrac{\varphi}{\mathrm{B}(\varphi)}$. The internal theory of Bel(Th) is defined as $\mathcal{I}(\mathrm{Bel(Th)}) := \{\phi : \mathrm{Bel(Th)} \vdash \mathrm{B}(\phi)\}$.

The idea behind Bel(Th) is that it makes the implicit commitments of Th explicit by proving their believability. Although the theory is a conservative extension of Th, its internal theory $\mathcal{I}(\mathrm{Bel(Th)})$ proves ω-many iterations of uniform reflection over Th, as was shown in [1].

In the talk, we present a proof-theoretic analysis of this approach and compare it to other main theories of implicit commitments. Our main result is the exact upper bound of the proof-theoretic strength of $\mathcal{I}(\mathrm{Bel(Th)})$ in terms of iterations of uniform reflection principles. We also prove that this result generalizes to the transfinite iterations of $\mathrm{Bel}^{\alpha}(\mathrm{Th})$.

Definition. *We define iterations of believability over* Th *as follows:*

- $\mathrm{Bel}^1(\mathrm{Th}) := \mathrm{Bel}(\mathrm{Th})$,
- $\mathrm{Bel}^{<\lambda}(\mathrm{Th}) := \bigcup_{\alpha\in\lambda} \mathrm{Bel}^{\alpha}(\mathrm{Th})$.
- $\mathrm{Bel}^{\lambda}(\mathrm{Th}) := \mathrm{MP} + \omega R + \forall\phi \in \mathcal{L}_B \forall\alpha < \lambda\ (\mathrm{Prov}_{\mathrm{Bel}^{\alpha}(\mathrm{Th})}(\mathrm{B}(\phi)) \to \mathrm{B}(\phi))$.

The following theorem is our main result.

Theorem. $\mathcal{I}(\mathrm{Bel}^{<1+\alpha}(\mathrm{Th})) \equiv_{\mathcal{L}_{\mathrm{Th}}} \mathrm{RFN}^{<\omega\cdot\alpha}(\mathrm{Th})$

To prove this theorem, we define a sequence of theories τ_α, which approximates the internal theory of Bel(Th) with a restricted number of applications of (NEC) rule. Later on, we interpret τ_α in $\mathrm{RFN}^{\alpha}(\mathrm{Th})$. The methods used in the proof are purely syntactic and can be formalized in EA. We use Löb's Theorem and formalizability of the basic syntactic facts in EA. This yields the following results.

Theorem. $\mathcal{I}(\mathrm{Bel}(\mathrm{Th})) + \neg\mathrm{B}(0 = 1)$ *is definable in* $\mathrm{RFN}^{<\omega}(\mathrm{Th})$.

Corollary. *Every model of* $\mathrm{RFN}^{<\omega}(\mathrm{Th})$ *is expandable to a model of* $\mathcal{I}(\mathrm{Bel}(\mathrm{Th}))$.

Moreover, one can show that the interpretation used in the proof of definability is feasible, which yields the following result:

Theorem. $\mathcal{I}(\mathrm{Bel}(\mathrm{Th}))$ *has at most polynomial speed-up over* $\mathrm{RFN}^{<\omega}(\mathrm{Th})$.

We use these results about the believability theory to contrast it with the other main approach to the formal rendering of the implicit commitments, which uses axiomatic truth theories to prove the implicit commitments of the base theory (proposed e.g. in [3] and [4]). We argue that, from the philosophical perspective, the above definability results favor the believability approach over the truth-theoretic approaches.

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The structure of the definability relation between definitions of truth

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A theory T is a *theory of truth* for a language $\mathcal{L}$ if and only if there exists a formula Θ such that for each sentence $\sigma \in \mathcal{L}$, the theory T proves $\Theta(\ulcorner\sigma\urcorner) \leftrightarrow \sigma$. The well-known theorem of Alfred Tarski states that a reasonably strong theory cannot be a theory of truth for its language.

In this talk, we focus on finitely axiomatizable theories of truth (called *definitions of truth*) for the language of Peano Arithmetic extending $\mathrm{I}\Delta_0 + \mathrm{Exp}$. For simplicity, we assume that all considered theories are expressed in languages extending $\mathcal{L}_{\mathrm{PA}}$ by relational symbols only.

Having two theories of truth S and T, we say that S *defines* T iff we can assign to every non-arithmetic n-ary symbol $\mathfrak{R}$ of $\mathcal{L}_T$ an n-ary formula $\Theta_{\mathfrak{R}}$ of $\mathcal{L}_S$ in such a way that S proves every axiom of T with $\Theta_{\mathfrak{R}}$ substituted for each occurrence of $\mathfrak{R}$ for each symbol $\mathfrak{R}$ – in other words, S directly and conservatively over $\mathcal{L}_{\mathrm{PA}}$ interprets T. It can be seen that a definability relation constitutes a preorder on the theories of truth ($S \geq T$ iff S defines T).

Using a method developed by Fedor Pakhomov and himself, Albert Visser showed that there is no minimal element in the definability preorder among definitions of truth. Combining that method with some truth-theoretic techniques, we prove that the order generated by that preorder is a distributive lattice which embeds every countable distributive lattice.

Joint work with Mateusz Łełyk.

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Surprising or Predictable?
Weak Systems Have Hard Theorems.

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Given a proof system, how can we specify the "hardness" of its theorems? One way to tackle this problem is taking the lengths of proofs as the corresponding hardness measure. Following this route, we call a theorem hard when even its shortest proof in the system is "long" in a certain formal sense. Finding hard theorems in proof systems for classical logic has been an open problem for a long time. However, in recent years as significant progress, many super-intuitionistic and modal logics have been shown to have hard theorems. In this talk, we will extend the aforementioned result to also cover a variety of weaker logics in the substructural realm. We show that there are theorems in the usual calculi for substructural logics that are even hard for the intuitionistic systems.

In technical terms, for any proof system **P** at least as strong as Full Lambek calculus, **FL**, and polynomially simulated by the extended Frege system for some infinite branching super-intuitionistic logic, we present an exponential lower bound on the proof lengths. More precisely, we will provide a sequence of **P**-provable formulas $\{A_n\}_{n=1}^{\infty}$ such that the length of the shortest **P**-proof for A_n is exponential in the length of A_n. The lower bound also extends to the number of proof-lines (proof-lengths) in any Frege system (extended Frege system) for a logic between FL and any infinite branching super-intuitionistic logic. Finally, in the classical substructural setting, we will establish an exponential lower bound on the number of proof-lines in any proof system polynomially simulated by the cut-free version of $\mathbf{CFL_{ew}}$.

To be able to present the results formally, we need some ingredients. Let us start with defining substructural logics. For simplicity, we provide hard formulas for $\mathsf{FL_e}$. However, there are also hard theorem for the weaker logic FL [2]. The language we use is $\{0, 1, \wedge, \vee, *, \to\}$. Uppercase Greek letters denote multisets of formulas, and lower case Greek letters represent formulas. Consider the following sequent calculus:

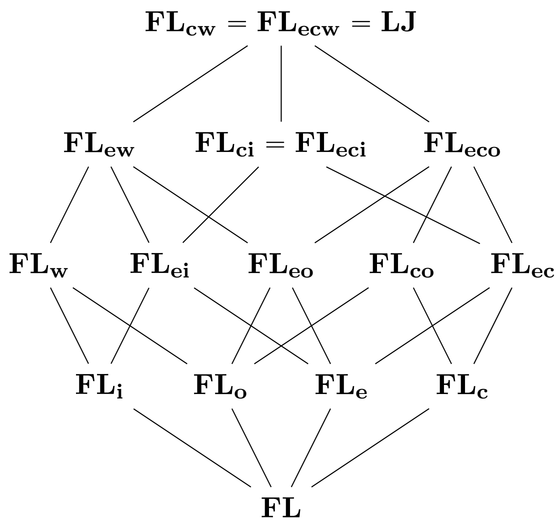
$$\varphi \Rightarrow \varphi \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, 1 \Rightarrow \Delta}\,(1w) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow 0, \Delta}\,(0w)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi * \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Sigma \Rightarrow \psi, \Lambda}{\Gamma, \Sigma \Rightarrow \varphi * \psi, \Delta, \Lambda}$$



$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Sigma, \psi \Rightarrow \Lambda}{\Gamma, \Sigma, \varphi \to \psi \Rightarrow \Delta, \Lambda} \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Sigma, \varphi \Rightarrow \Lambda}{\Gamma, \Sigma \Rightarrow \Delta, \Lambda}\ (cut)$$

The sequent calculus $\mathbf{FL_e}$ is the single-conclusion version of the sequent calculus presented above and $\mathbf{CFL_e}$ is the multi-conclusion version. The structural rules are as usual:

Weakening rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (i) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ (o)$$

Contraction rules:

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ (Lc) \qquad \frac{\Gamma \Rightarrow \varphi, \varphi, \Delta}{\Gamma \Rightarrow \varphi, \Delta}\ (Rc)$$

Adding these rules to the sequent calculi defined, result in various substructural calculi. It is worth mentioning that if we consider uppercase Greek letters to be *sequences* of formulas instead of multisets, i.e., the exchange rule is not present, then, we can introduce two implication-like connectives $\backslash$ and $/$, and include their respective rules. This system is called **FL**. The figure on top of this page shows the web of the sequent calculi between the full Lambek calculus **FL** and **LJ**, the usual sequent calculus for the intuitionistic logic IPC. Some other sequent calculi for which our result holds for are listed in Table 1.

Second, let us define Frege systems. They are the most natural calculi for propositional logic. A *(Frege) rule* is an expression of the form $\dfrac{\varphi_1, \dots, \varphi_k}{\varphi}$ where $\varphi_1, \dots, \varphi_k, \varphi$ are propositional formulas. Let $\mathbf{P}$ be a finite set of rules. A $\mathbf{P}$-proof of φ from a set of assumptions X, denoted by $X \vdash_{\mathbf{P}} \varphi$, is $\varphi_1, \dots, \varphi_m = \varphi$ such that each $\varphi_i \in X$, or is inferred from some φ_j, $j < i$, by a substitution instance of rule in $\mathbf{P}$. The formulas φ_i are called *lines* of the proof.

A finite set of rules, $\mathbf{P}$, is called a *Frege system* for a logic L when

Table 1: Some sequent calculi with their definitions.

Sequent calculus	Definition
RL	$\mathbf{FL} + (0 \Leftrightarrow 1)$
CyFL	$\mathbf{FL} + (\varphi\backslash 0 \Leftrightarrow 0/\varphi)$
DFL	$\mathbf{FL} + (\varphi \wedge (\psi \vee \theta) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta))$
$\mathbf{P_nFL}$	$\mathbf{FL} + (\varphi^n \Leftrightarrow \varphi^{n+1})$
psBL	$\mathbf{FL_w} + \{(\varphi \wedge \psi \Leftrightarrow \varphi * (\varphi\backslash\psi)), (\varphi \wedge \psi \Leftrightarrow (\psi/\varphi) * \varphi)\}$
HA	$\mathbf{FL_w} + (\varphi \Leftrightarrow \varphi^2)$
DRL	$\mathbf{RL} + (\varphi \wedge (\psi \vee \theta) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \theta))$
IRL	$\mathbf{RL} + (\varphi \Rightarrow 1)$
CRL	$\mathbf{RL} + (\varphi * \psi \Leftrightarrow \psi * \varphi)$
GBH	$\mathbf{RL} + \{(\varphi \wedge \psi \Leftrightarrow \varphi * (\varphi\backslash\psi)), (\varphi \wedge \psi \Leftrightarrow (\psi/\varphi) * \varphi)\}$
Br	$\mathbf{RL} + (\varphi \wedge \psi \Leftrightarrow \varphi * \psi)$

(1) $\mathbf{P}$ is strongly sound: if $\varphi_1, \dots, \varphi_n \vdash_{\mathbf{P}} \varphi$, then $\varphi_1, \dots, \varphi_n \vdash_{\mathsf{L}} \varphi$,

(2) $\mathbf{P}$ is strongly complete: if $\varphi_1, \dots, \varphi_n \vdash_{\mathsf{L}} \varphi$, then $\varphi_1, \dots, \varphi_n \vdash_{\mathbf{P}} \varphi$.

Third, and finally, we give a characterization of superintuitoinistic logics of infinite branching. Consider the following superintuitionistic (si) logics:

$$\mathsf{KC} = \mathsf{IPC} + \neg p \vee \neg\neg p \quad , \quad \mathsf{BD_n} = \mathsf{IPC} + BD_n$$

where IPC is the intuitionistic logic and $BD_0 := \bot$ and $BD_{n+1} := p_n \vee (p_n \to BD_n)$. Jeřábek in [3] proved the following interesting theorem that a superintuitionistic logic L has infinite branching iff $\mathsf{L} \subseteq \mathsf{BD_2}$ or $\mathsf{L} \subseteq \mathsf{KC} + \mathsf{BD_3}$.

Now, let us give a sketch of how to prove the lower bound. In order to do so, we have to provide a sequence of formulas provable in $\mathbf{FL_e}$, such that every proof of them are long. This task requires two steps. The first step, which is the main task, is providing a sequence of $\mathbf{FL_e}$-tautologies. To achieve this goal we change the existing hard intuitionistic tautologies in a suitable way that they become provable in $\mathbf{FL_e}$, but remain hard. The next step, which is the easier part, is proving that these tautologies are hard. To do so, we use the canonical translation of the language of $\mathbf{FL_e}$ to the language of IPC, i.e., sending $\{0, 1, *\}$ to $\{\bot, \top, \wedge\}$, respectively and the other connectives to themselves. It is easy to see that this transformation takes polynomial time.

Let us mention the form of the hard intuitionistic tautologies. The following formulas, $\Theta_{n,k}$, are hard for IPC and they are negation-free and $\bot$-free. Small Roman letters denote atomic formulas and the formulas α_n^k and β_n^{k+1} are monotone, i.e., only consist of atoms, $\wedge$, $\vee$.

$$\Theta_{n,k} := \bigwedge_{i,j} (p_{i,j} \vee q_{i,j}) \to$$

$$[(\bigwedge_{i,l} (s_{i,l} \vee s'_{i,l}) \to \alpha_n^k(\bar{p}, \bar{s}, \bar{s'})) \vee (\bigwedge_{i,l} (r_{i,l} \vee r'_{i,l}) \to \beta_n^{k+1}(\bar{q}, \bar{r}, \bar{r'}))]$$

The result by Hrubeš [1] and Jeřábek [3] is the following theorem:

Theorem. The formulas $\Theta_{n,k}$ are IPC-tautologies and require IPC-Frege proofs with $2^{n^{\Omega(1)}}$ lines, for $k = \lfloor \sqrt{n} \rfloor$.

In the following we see the form of the hard $\mathsf{FL_e}$ tautologies:

$$\Theta^*_{n,k} := [\mathop{\circledast}_{i,j}((p_{i,j} \wedge 1) \vee (q_{i,j} \wedge 1))] \;\; \rightarrow$$

$$\big[(\mathop{\circledast}_{i,l}((s_{i,l} \wedge 1) \vee (s'_{i,l} \wedge 1)) \rightarrow \alpha^k_n) \;\vee (\mathop{\circledast}_{i,l}((r_{i,l} \wedge 1) \vee (r'_{i,l} \wedge 1)) \rightarrow \beta^{k+1}_n)\big]$$

Now, we have all the ingredients to formally state our result:

Theorem. [2] The formulas $\Theta^*_{n,k}$ are $\mathsf{FL_e}$-tautologies. Moreover, for any substructural logic L and any superintuitionistic logic of infinite branching M such that $\mathsf{FL_e} \subseteq \mathsf{L} \subseteq \mathsf{M}$, the formulas $\Theta^*_{n,k}$ require L-Frege proofs with $2^{n^{\Omega(1)}}$ lines, for $k = \lfloor \sqrt{n} \rfloor$.

The concrete application of the theorem follows:

Corollary. Let $S \subseteq \{e, c, i, o\}$, and L be $\mathsf{FL_S}$, or any of the logics of the sequent calculi in Table 1. Then the length of every proof of Θ^*_n in any (extended) Frege system for L is exponential in n.

Let us end with the following question: what happens in the case of the classical versions of the above substructural logics? They are not included in IPC and hence our method does not work. However, for their cut-free versions we have the following theorem.

Theorem. The length of every proof of Θ^*_n in the sequent calculi $\mathbf{CFL}^-_{\mathbf{e}}$, $\mathbf{CFL}^-_{\mathbf{ei}}$, $\mathbf{CFL}^-_{\mathbf{eo}}$, and $\mathbf{CFL}^-_{\mathbf{ew}}$ is exponential in n, where the " $-$ " means without the cut rule.

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Implementation of Anaphora Resolution Using the Refine Tactic of Coq

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1 Introduction

Dependent type theory has been used to develop a theory of natural language semantics for explaining various linguistic phenomena [7, 6]. Among the state-of-the-art theories of natural language semantics using dependent types (e.g., [3]), Dependent Type Semantics (DTS) (e.g., [1]) uses a proof-theoretic procedure to explain meaning. Following the paradigm of anaphora resolution via proof construction in [5], DTS provides a procedure for resolving an anaphoric expression by type checking including proof search, where proof search is conducted in the natural deduction style. The current version of this procedure is based on *underspecified types*, which were introduced recently in [1] for analysis of not only anaphora but also other linguistic phenomena.

Anaphora resolution in the previous version of DTS, which does not include underspecified types, has been implemented as an automated procedure in [2, 4], but the automated proof search in the procedure is partial due to the undecidability of dependent type theory. This suggests that, to implement the current version of DTS, a hybrid approach of manual proof search and automated proving can be used if full automation is not required.

The proof assistant Coq[1] provides a framework for such a hybrid approach. The core of Coq is the dependent type theory called Predicative Calculus of Cumulative Inductive Constructions, so the basic ingredients of DTS such as dependent function types and dependent pair types are included in Coq. Moreover, Coq's built-in tactics and its tactic language Ltac enable proofs to be manipulated both interactively and automatically in order to formalize anaphora resolution in DTS and natural language inference using the result of the anaphora resolution as a premise.

By using Coq, we aim to implement the procedure of DTS for resolving anaphora and making natural language inference from the result of the anaphora resolution. We first define underspecified types as inductive types in Coq. Next, we show that the `refine` tactic with these types can simulate the aforementioned procedure of DTS.

2 Anaphora Resolution by Dependent Types in Coq

In DTS, anaphora resolution is the process of constructing potential denotations of an anaphoric expression using preceding contexts. When the antecedent of an anaphoric

[1] "The Coq Proof Assistant." Accessed May 3, 2023. `https://coq.inria.fr/`.

expression is determined, it provides the semantic representation of the text. The framework of DTS enables to rephrase anaphora resolution as proof search: a task of searching the proof object, i.e., the denotation of an anaphoric expression as the goal from the premises formed by the preceding sentences. To perform proof search for anaphoric expression, we utilize both Coq's built-in tactics and newly defined tactics using Ltac. Coq's tactics are tools for progressively completing proofs, acting like functions that take a proof state as an input and produce the next proof state as an output. For instance, applying the `intro` tactic with the goal $A \to B$ allows us to assume $x : A$ and transform the goal to B, resulting in a change in the proof state as shown below.

$$\begin{array}{c} \vdots \\ ? : A \to B \end{array} \quad \xrightarrow{\texttt{intro.}} \quad \frac{\begin{array}{c} x : A \\ \vdots \\ ? : B \end{array}}{\lambda x.? : A \to B}$$

Among Coq's built-in tactics, the `refine` tactic is crucial to our implementation. By using the `refine` tactic, the goal is partially resolved: if there are incomplete parts represented by placeholders or _, it is necessary to fill those placeholders, and these tasks are set as the subgoals. Suppose that the goal is $A \wedge B$ and a proof term $t : A$ is already obtained. Applying `refine (conj t _)` to this proof state partially constructs a proof of $A \wedge B$ by resolving the problem to prove A with $t : A$, leaves a placeholder for a proof of B, and sets the subgoal to prove B, as shown below:

$$\begin{array}{c} \vdots \\ ? : A \wedge B \end{array} \quad \xrightarrow{\texttt{refine (conj } t \texttt{ \_).}} \quad \frac{\begin{array}{c} \vdots \\ t : A \end{array} \quad \begin{array}{c} \vdots \\ ? : B \end{array}}{\texttt{conj } t \ ? : A \wedge B}$$

To implement the anaphora resolution in DTS, we simulate underspecified types and the proof search procedure by means of user-defined inductive types and the `refine` tactic: we define an underspecified type as an inductive type `aspT A a B`, which consists of a type `A`, a term `a` of type `A`, and a type `B`.

```
Inductive aspT (A : Type) (a : A) (B : Type) : Type :=
  resolve : B -> aspT A a B.
```

An anaphoric expression is then represented by an `aspT` type and a placeholder `?[asp]` denoting a subgoal in proof search. The task of identifying the antecedent is treated as that of using the `refine` tactic to search for a proof term to fill the placeholder.

Our implementation of the anaphora resolution in DTS can be explained using the following example. Consider a discourse consisting of (1a) *A man entered.* followed by (1b) *He whistled.* The implementation consists of the steps below:

(i) Represent the anaphoric expression *He* with an `aspT` type and a placeholder `?[asp]`. The expressions (1a') and (1b') below are the semantic representations of (1a) and (1b), respectively, where *He* is represented by `projT1 ?asp` in (1b').

(1a') `{x:entity & {_: man x & enter x}}`

(1b') `aspT {x:entity & man x} ?[asp] (whistle (projT1 ?asp))`

The semantic representation of the whole discourse is obtained by composing (1a') and (1b') through the formation of a dependent pair type:

(1c) `{u:{x:entity & {_: man x & enter x}} &`
`aspT {x:entity & man x} ?[asp] (whistle (projT1 ?asp))}`

The goal of anaphora resolution in this case is to show that (1c) is well-formed, that is, (1c) is indeed a type. In particular, we show that (1b') is a type under the assumption `u:{x:entity & {_: man x & enter x}}`.

(ii) Use the `refine` tactic to set the subgoal of filling the placeholder `?[asp]` by a proof. This corresponds to identifying the denotation of the anaphoric expression *He*.

$$\dfrac{\overset{\vdots}{\texttt{\{x:entity \& man x\}} : \mathsf{type}} \qquad \texttt{?[asp]} : \texttt{\{x:entity \& man x\}} \qquad \overset{\texttt{?asp} : \texttt{\{x:entity \& man x\}}}{\overset{\vdots}{\texttt{(whistle (projT1 ?asp))}}}}{\texttt{aspT \{x:entity \& man x\} ?[asp] (whistle (projT1 ?asp))} : \mathsf{type}}$$

(iii) Resolve the anaphora by constructing a proof that fills the placeholder `?[asp]`.

$$\dfrac{\overset{\vdots}{\texttt{\{x:entity \& man x\}} : \mathsf{type}} \qquad \overset{\vdots}{\texttt{a} : \texttt{\{x:entity \& man x\}}} \qquad \overset{\texttt{?asp} : \texttt{\{x:entity \& man x\}}}{\overset{\vdots}{\texttt{(whistle (projT1 ?asp))}}}}{\texttt{aspT \{x:entity \& man x\} a (whistle (projT1 ?asp))} : \mathsf{type}} \quad \xrightarrow{\text{resolution}}$$

$$\dfrac{\overset{\vdots}{\texttt{\{x:entity \& man x\}} : \mathsf{type}} \qquad \overset{\vdots}{\texttt{a} : \texttt{\{x:entity \& man x\}}} \qquad \overset{\overset{\vdots}{\texttt{a} : \texttt{\{x:entity \& man x\}}}}{\overset{\vdots}{\texttt{(whistle (projT1 a))}}}}{\texttt{aspT \{x:entity \& man x\} a (whistle (projT1 a))} : \mathsf{type}}$$

(iv) Register the result of anaphora resolution as a theorem in Coq.

The anaphora resolution, i.e., the proof search in this example is processed by using the assumption `u:{x:entity & {_: man x & enter x}}`, and we obtain the type below as its result:

(1d) `aspT {x:entity & man x} (projT1 u , projT1 (projT2 u))`
`(whistle (projT1 (projT1 u , projT1 (projT2 u))))`

3 Inference from the Results of Anaphora Resolution

Several tasks can be performed on a text that has undergone anaphora resolution. In this section, we perform natural language inference as the most fundamental task for such a text. To see how the result of anaphora resolution is used in the subsequent inferences, consider the inference (2) whose assumption is the discourse formed by (1a) and (1b):

(2) A man entered. He whistled. $\Longrightarrow$ A man whistled.

Note that the consequence *A man whistled* cannot be deduced without the anaphora resolution of the pronoun *He* to the antecedent *a man*. We replace the assumption of (2) with the result of anaphora resolution obtained by the procedure above. Then, by applying the `destruct` tactic to the `aspT` type (1d) occurring in the result of the anaphora resolution, we are able to retrieve a proof that *a man who entered whistled.* The `destruct` tactic is available here thanks to our definition of `aspT` types as inductive types, and the current instance of this tactic can be considered as the elimination rule of `aspT`. Indeed, the `destruct` tactic corresponds to the application of an elimination rule in proof search, as the following instance concerning the $\vee$-elimination rule illustrates:

$$\begin{array}{c} x : A \vee B \\ \vdots \\ ? : C \end{array} \xrightarrow{\texttt{destruct}\ x.} \dfrac{x : A \vee B \quad \begin{array}{c} y : A \\ \vdots \\ ? : C \end{array} \quad \begin{array}{c} z : B \\ \vdots \\ ? : C \end{array}}{\texttt{match}\ x\ \texttt{with | or\_introl}\ y\ \texttt{=>}\ ?\ \texttt{| or\_intror}\ z\ \texttt{=>}\ ?\ \texttt{end} : C}$$

To sum up, we implement anaphora resolution in DTS and natural language inference from its result as follows: first, `aspT` types are defined as inductive types in Coq to provide anaphoric expressions with their semantic representations. Then, we use the `refine` tactic to resolve anaphoric expressions, and apply the `destruct` tactic to make inference from the result of anaphora resolution. Our implementation shows that natural language semantics in terms of dependent type theory can be developed in a proof-theoretic manner by using Coq's tactics for proof search.

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Proof Disbalancing for Proving Complexity Results: Towards Infinitary Focusing

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Infinitary action logic, denoted by $\mathbf{ACT}_\omega$, was introduced by W. Buszkowski and E. Palka [4] as the algebraic logic of $*$-continuous residuated Kleene lattices (action lattices). Compared to the usual, finitary version of action logic $\mathbf{ACT}$ [15, 8], $\mathbf{ACT}_\omega$ has the following two distinctive features.

First, $\mathbf{ACT}_\omega$ is not recursively enumerable, which makes it inevitable to use infinitary proof theory for this logic. Buszkowski and Palka formulate $\mathbf{ACT}_\omega$ as a calculus with the ω-rule. In general, logics defined by calculi of this form belong to the Π_1^1 complexity class (see [11]). For $\mathbf{ACT}_\omega$, however, Palka proved a Π_1^0 upper bound, which is much more modest. (This is connected to the finite model property.) The lower complexity bound, proved by Buszkowski, is Π_1^0-hardness, which still ensures that $\mathbf{ACT}_\omega$ is not recursively enumerable. An extension of $\mathbf{ACT}_\omega$ with the exponential modality is Π_1^1-complete [11]; in this abstract, we present a commutative version of this result.

The second feature of $\mathbf{ACT}_\omega$ is the existence of a good (though infinitary) Gentzen-style sequent formulation, with cut elimination. (For $\mathbf{ACT}$, this is not the case.) Such a calculus opens the path to further development of structural proof theory for $\mathbf{ACT}_\omega$ and its variations. We shall perform some initial steps in this direction.

The axiomatization of $\mathbf{ACT}_\omega$ is based on the multiplicative-additive Lambek calculus $\mathbf{MALC}$ [12, 6]. This calculus, in its turn, can be viewed as a non-commutative variant of intuitionistic linear logic [1]. This motivates extending $\mathbf{ACT}_\omega$ with other elements of Girard's linear logic [5]. In this presentation, we shall consider the commutative variant of $\mathbf{ACT}_\omega$, extended by the linear logic exponential modality [11]. This system will be denoted by $\mathbf{!CommACT}_\omega$.

Formulae of $\mathbf{!CommACT}_\omega$ are built from variables and constants $\mathbf{0}$ and $\mathbf{1}$ using the following binary operations: $\otimes$ (multiplicative conjunction), $\&$ (additive conjunction), $\oplus$ (additive disjunction), $\multimap$ (linear implication), and the following unary ones: ! (exponential) and * (Kleene star). Sequents of $\mathbf{!CommACT}_\omega$ are expressions of the form $\Pi \to B$, where B is a formula and Π is a (finite) multiset of formulae. (In general, Greek letters will denote multisets of formulae and Latin ones stand for individual formulae.)

Axioms and inferences rules of $\mathbf{!CommACT}_\omega$ are as follows. The core of the calculus is the commutative version of $\mathbf{MALC}$:

$$\frac{}{x \to x}\ \mathrm{Id},\ x \text{ is a variable} \qquad \frac{}{\Gamma, \mathbf{0} \to B}\ \mathbf{0}\mathrm{L} \qquad \frac{\Gamma \to B}{\Gamma, \mathbf{1} \to B}\ \mathbf{1}\mathrm{L} \qquad \frac{}{\to \mathbf{1}}\ \mathbf{1}\mathrm{R}$$

$$\frac{\Gamma, A, B \to C}{\Gamma, A \otimes B \to C}\ {\otimes}\mathrm{L} \qquad \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \otimes B}\ {\otimes}\mathrm{R} \qquad \frac{\Pi \to A \quad \Gamma, B \to C}{\Gamma, \Pi, A \multimap B \to C}\ {\multimap}\mathrm{L}$$

$$\frac{\Gamma, A \to C}{\Gamma, A \& B \to C} \quad \frac{\Gamma, B \to C}{\Gamma, A \& B \to C}\ \&\mathrm{L} \qquad \frac{\Pi \to A \quad \Pi \to B}{\Pi \to A \& B}\ \&\mathrm{R} \qquad \frac{A, \Pi \to B}{\Pi \to A \multimap B}\ {\multimap}\mathrm{R}$$

$$\frac{\Gamma, A \to C \quad \Gamma, B \to C}{\Gamma, A \oplus B \to C}\ \oplus\mathrm{L} \qquad \frac{\Pi \to A}{\Pi \to A \oplus B} \quad \frac{\Pi \to B}{\Pi \to A \oplus B}\ \oplus\mathrm{R}$$

Next, the rules for ! and * are as follows:

$$\frac{\Gamma, A, \to B}{\Gamma, !A \to B}\ !\mathrm{L} \qquad \frac{!A_1, \dots, !A_n \to B}{!A_1, \dots, !A_n \to !B}\ !\mathrm{R} \qquad \frac{\Gamma \to B}{\Gamma, !A \to B}\ !\mathrm{W} \qquad \frac{\Gamma, !A, !A \to B}{\Gamma, !A \to B}\ !\mathrm{C}$$

$$\frac{\left(\Gamma, A^n \to B\right)_{n=0}^{\infty}}{\Gamma, A^* \to B}\ {}^*\mathrm{L} \qquad \frac{\Pi_1 \to A \quad \dots \quad \Pi_n \to A}{\Pi_1, \dots, \Pi_n \to A^*}\ {}^*\mathrm{R},\ n \geqslant 0$$

Finally, the cut rule appears in the following form:

$$\frac{\Pi \to A \quad \Gamma, A \to B}{\Gamma, \Pi \to B}\ \mathrm{Cut}$$

Cut is eliminable, via a commutative modification of the argument from [11].

Our aim is to prove the commutative variant of the complexity result from [11]:

Theorem 1. *The derivability problem in* $\mathbf{!CommACT}_\omega$ *is* Π_1^1*-complete.*

Here the upper bound is proved by the same argument as in [11]. The interesting one is the lower bound. The proof idea for lower bound is a combination of the constructions by Lincoln et al. [14] and by Kozen [9]. Namely, we encode Minsky machines (which are, unlike Turing machines, suitable for commutative encoding), and via this encoding we represent well-foundedness of recursively defined graphs.

We consider Minsky machines with 4 counters (two for input/output, and two for internal usage), denoted by $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$. The machine has two kinds of instructions, INC and JZDEC, with the following meaning and formulae which encode them. Here $r \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ is a counter and $p, q, q_0 \in Q$ are states of the machine.

$I = \mathrm{INC}(p, r, q)$	from state p, move to q and increase r by 1	$A_I = p \multimap (q \otimes a)$
$I = \mathrm{JZDEC}(p, r, q_0, q)$	from state p: if $r = 0$, move to q_0 if $r > 0$, move to q and decrease r by 1	$A_I = p \multimap (q_0 \oplus z_r)$ $\&(p \otimes r) \multimap q$

Elements of $Q \cup \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, z_{\mathsf{a}}, z_{\mathsf{b}}, z_{\mathsf{c}}, z_{\mathsf{d}}\}$ here are variables of $\mathbf{!CommACT}_\omega$ (z_r's are "pseudo-states," used for zero-checks). For a given Minsky machine $\mathtt{M}$ let $!\Psi_{\mathtt{M}} = \{!A_{I_1}, \dots, !A_{I_m}\}$, where $\{I_1, \dots, I_m\}$ is the set of instructions of $\mathtt{M}$. Also let

$$Z = (z_{\mathsf{a}} \otimes \mathsf{b}^* \otimes \mathsf{c}^* \otimes \mathsf{d}^*) \oplus (z_{\mathsf{b}} \otimes \mathsf{a}^* \otimes \mathsf{c}^* \otimes \mathsf{d}^*) \oplus (z_{\mathsf{c}} \otimes \mathsf{a}^* \otimes \mathsf{b}^* \otimes \mathsf{d}^*) \oplus (z_{\mathsf{d}} \otimes \mathsf{a}^* \otimes \mathsf{b}^* \otimes \mathsf{c}^*).$$

Now let $\mathtt{M}$ define a binary relation (directed graph) on the set of natural numbers as follows (see Kozen [9]):

- if there is no edge from n to m, then $\mathtt{M}$, starting from state s with $\mathsf{a} = n$, $\mathsf{b} = m$, $\mathsf{c} = \mathsf{d} = 0$, reaches state r with $\mathsf{a} = \mathsf{b} = \mathsf{c} = \mathsf{d} = 0$;

- if such an edge exists, then $\mathtt{M}$, starting from the same configuration, reaches state t with $\mathsf{a} = m$, $\mathsf{b} = \mathsf{c} = \mathsf{d} = 0$.

The set $\mathsf{WF}_{\mathtt{M}}$ is the set of all natural k such that there is no infinite path starting from k in the graph defined by $\mathtt{M}$.

Theorem 2. *The sequent* $!\Psi_{\mathtt{M}}, !(t \multimap (s \otimes \mathsf{b}^*)), t, \mathsf{a}^n \to r + Z$ *is derivable in* **!CommACT**$_\omega$ *if and only if* $n \in \mathsf{WF}_{\mathtt{M}}$.

This theorem immediately yields the desired lower bound, since the problem "given $\mathtt{M}$, determine whether $0 \in \mathsf{WF}_{\mathtt{M}}$" is a well-known Π^1_1-complete problem.

The "if" part in Theorem 2 is easy. The execution of $\mathtt{M}$ is naturally simulated by a derivation in **!CommACT**$_\omega$ (cf. [10]). Each time we use $t \multimap (s \otimes \mathsf{b}^*)$ in order to fill b with an arbitrary m (infinite branching by the ω-rule). Next, we simulate the execution of $\mathtt{M}$. If it reaches r, on top we get a derivable sequent $!\Psi_{\mathtt{M}}, !(t \multimap (s \otimes \mathsf{b}^*)), r \multimap r + Z$. In the case of state t, we get $!\Psi_{\mathtt{M}}, !(t \multimap (s \otimes \mathsf{b}^*), t, \mathsf{a}^m \to r + Z$, and restart the process from m, which is connected to n by an edge (thus also $m \in \mathsf{WF}_{\mathtt{M}}$). The well-foundedness condition $n \in \mathsf{WF}_{\mathtt{M}}$ guarantees well-foundedness of the resulting infinite derivation.

The "only if" direction is harder, since the derivation may go in various ways, even if it is cut-free. We pursue the syntactic approach (unlike Kozen [9]), which is based on applying certain transformations to the given cut-free derivation of $!\Psi_{\mathtt{M}}, !(t \multimap (s \otimes \mathsf{b}^*)), t, \mathsf{a}^n \to r + Z$. This process is called *proof disbalancing.* After disbalancing, the derivation is exactly in the form suitable for proving $n \in \mathsf{WF}_{\mathtt{M}}$ by transfinite induction.

The idea of disbalancing is inspired by *focusing* techniques for linear logic proofs [2, 13] and their usage for proving the "from derivation to computation" direction when encoding computations in linear logic systems [7]. Focusing is based on exchanging rule applications in the derivation: e.g., invertible rules like $\otimes$L may be propagated downwards. Focusing provides a discipline of such transformations, so that the resulting derivation achieves better structural properties than the original one.

Focusing is performed via induction on proof structure. We aim towards extending focusing techniques to the infinitary setting, with the ω-rule. In this case, induction becomes transfinite, and one should be very careful about its peculiarities. (Another version of infinitary focusing, for non-well-founded proofs rather than the ones with ω-rules, is developed in [3].) This abstract presents so-called disbalancing transformations, which provide a discipline of proof structure in a specific case, where some rules are forbidden. Not being a full-power focusing itself, proof disbalancing, on one hand, is rather simple, and on the other hand, makes the proof structure suitable for proving the "only if" direction in Theorem 2.

Definition 1. Consider a cut-free derivation in **!CommACT**$_\omega$ which does not use rules $\multimap$R and !R. Such a derivation is called *disbalanced* if the following holds.

1. No left rule is applied above a right rule.
2. Derivations of left premises of $\multimap$L consist only of right rules.
3. If a sequent in the derivation includes $B \otimes C$, $B \oplus C$, B^*, $\mathbf{1}$, or $\mathbf{0}$ as one of the elements of its left-hand side, then rule immediately above this sequent is a rule which introduces one of those formulae.

The second condition explains the term "disbalanced." The "main branch" of the derivation, which uses left rules, always turns right at $\multimap$L, and the tree itself is far from a balanced one.

Theorem 3. *If a sequent is derivable in* **!CommACT**$_\omega$ *which does not use rules* $\multimap$R*,* !R*, and* Cut*, the it has a disbalanced derivation.*

The condition of forbidden rules is a syntactic one, due to polarized subformula property, and it is true for the sequent in Theorem 2. As noticed above, transfinite induction on a disbalanced derivation yields the "only if" direction in Theorem 2.

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(The philosophy of) translating first-order natural deduction proofs

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This talk will focus on a translation of proofs between first-order theories, specifically in the classical first-order natural deduction calculus. There, given a natural deduction proof $\mathcal{D}$ in theory T_1, a translation of this proof in a theory T_2 should contain connected subproofs from the translated premises to the translated conclusion of each rule application in $\mathcal{D}$. A natural tool to help us get this result is the interpretation translation of [3]. An interpretation $i : \mathsf{T}_1 \to \mathsf{T}_2$ consists of a translation function F, mapping predicates and function symbols of $\mathcal{L}_{\mathsf{T}_1}$ onto formulas of $\mathcal{L}_{\mathsf{T}_2}$, and a formula δ giving the domain of the interpretation. Let $\delta_{\lambda_{\mathcal{D}}}$ stand for the conjunction $\delta(x_1) \wedge ... \wedge \delta(x_n)$ for each free variable x_j $(1 \leq j \leq n)$ occurring in a proof $\mathcal{D}$. For a slight adaptation of the interpretation translation i from Visser, we will show how for each proof $\Gamma \vdash_{\mathsf{T}_1} \varphi$ (referred to by $\mathcal{D}$), we can get a translation $\Gamma^i, \delta_{\lambda_{\mathcal{D}}} \vdash_{\mathsf{T}_2} \varphi^i$. This proceeds by induction on the length of the derivation of φ, where the rules $\forall$E and $\exists$I are given extra attention, as term translations need to be treated as a special case.

A proof translation result has technical value, but also philosophical value — we suggest for instance that it has value for the property of 'purity of proof' (see e.g. [2, 1]). Purity of proof restricts the methods of a proof to those that in some sense intrinsically belong to a theorem. Here, notions that are thought extraneous to a theorem are excluded from a proof. We suggest that, if a natural deduction proof is pure in T_1, a translated version in T_2 still has a secondary level of purity. For purity purposes, we require the proof translation to additionally restrict itself to certain 'good' syntax — meaning that nothing besides translated T_1-syntax and descriptions of domain elements can occur in the proof. We also comment on the limitation of proof translations for philosophical values of proof.

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Reduction of arithmetical completenesses

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1 Abstract

In one hand we have fascinating first-order mathematical logics which are sophisticated, undecidable and wild. On the other hand we have propositional logics, including modal logics, which are tame and decidable. Then an arithmetical completeness of a propositional logic L for a first-order theory T, say $\mathcal{AC}(\mathsf{L},\mathsf{T})$, states that L is complete for *arithmetical interpretations* in T. The notion of arithmetical interpretations could vary, however a common fact in all of them is that the interpretation of atomic letters are considered to be sentences in the first-order language and boolean connectives commutes with interpretations. In the case of modal language with $\Box$ as its unary modal operator, $\Box$ usually is interpreted as the T-provability predicate.

During past 5 decades, several arithmetical completeness results were obtained. As far as we know, the first such arithmetical completeness result is due to D. de Jongh [1] which proves $\mathcal{AC}(\mathsf{IPC},\mathsf{HA})$, the arithmetical completeness of propositional intuitionistic logic for the Heyting's Arithmetic. Another important result is due to R. Solovay [3], in which he proves $\mathcal{AC}(\mathsf{GL},\mathsf{PA})$, the arithmetical completeness of Gödel-Löb logic GL for first-order Peano Arithmetic PA.

In this talk, we define a notion of *propositional reduction* for arithmetical completenesses [2]. Intuitively, propositional reduction of $\mathcal{AC}(\mathsf{L}_1,\mathsf{T}_1)$ to $\mathcal{AC}(\mathsf{L}_2,\mathsf{T}_2)$ (in other words we say that the arithmetical completeness $\mathcal{AC}(\mathsf{L}_2,\mathsf{T}_2)$ is harder than the one for $\mathcal{AC}(\mathsf{L}_1,\mathsf{T}_1)$) means that one may prove arithmetical completeness $\mathcal{AC}(\mathsf{L}_1,\mathsf{T}_2)$ via purely propositional argument, given that we already have $\mathcal{AC}(\mathsf{L}_2,\mathsf{T}_2)$.

Then we show that the arithmetical completeness of some provability logics are *harder* than others. And finally, as a witness for not having a trivial notion of reducibility, we show that some provability logics are *strictly harder* than others (thanks to the argument provided by F. Pakhomov).

Let $\mathsf{PL}(\mathsf{T},\mathsf{U})$ and $\mathsf{PL}_\Sigma(\mathsf{T},\mathsf{U})$ respectively indicate the provability logic and Σ_1-provability logic of T ralative in U. We show that arithmetical completeness of $\mathsf{PL}_\Sigma(\mathsf{HA},\mathbb{N})$ is harder than the arithmetical completeness of many relative provability logics including $\mathsf{PL}_\Sigma(\mathsf{HA},\mathsf{HA})$, $\mathsf{PL}(\mathsf{HA},\mathsf{HA})$, $\mathsf{PL}_\Sigma(\mathsf{PA},\mathsf{PA})$, $\mathsf{PL}(\mathsf{PA},\mathsf{PA})$ and $\mathsf{PL}(\mathsf{PA},\mathbb{N})$. Finally we show that the arithmetical completeness of $\mathsf{PL}(\mathsf{PA},\mathsf{PA})$ is not easier than the one for $\mathsf{PL}_\Sigma(\mathsf{PA},\mathsf{PA})$ and $\mathsf{PL}_\Sigma(\mathsf{HA},\mathbb{N})$ (thanks to the argument provided by F. Pakhomov).

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Reflection and Induction for subsystems of HA

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By a classical result of Leivant and Ono [1, 2], the subsystem $I\Pi_n$ of PA is equivalent to the scheme of uniform reflection $\mathrm{RFN}_{\Pi_{n+2}}(\mathrm{EA})$ over elementary arithmetic EA. In the present paper, we study the correspondence between the schemes of induction and reflection for subsystems of Heyting arithmetic HA.
In an intuitionistic setting, complexity classes of formulas behave quite differently than over classical logic. Underpinning this, we show by an application of realizability that reflection over prenex formulas $\mathrm{RFN}_{\Pi_\infty}(i\mathrm{EA})$ is equivalent over intuitionistic elementary arithmetic $i\mathrm{EA}$ to just $\mathrm{RFN}_{\Sigma_1}(i\mathrm{EA})$ or the totality of hyperexponentiation. More generally, for any class $\Gamma \supseteq \Sigma_1$ of formulas, we have an equivalence between $\mathrm{RFN}_{\Pi_\infty\Gamma}(i\mathrm{EA})$ and $\mathrm{RFN}_{\Gamma}(i\mathrm{EA})$. This phenomenon does not have any counterpart in classical logic where Π_∞ exhausts all arithmetical formulas.
As our main result, we show that a suitable generalization of the result by Leivant and Ono holds true intuitionistically. We show for some natural classes Γ of formulas that the principle of induction $I\Gamma$ for Γ is equivalent over $i\mathrm{EA}$ to the reflection principle $\mathrm{RFN}_{\forall^b(\Gamma\to\Gamma)\to\Gamma}(i\mathrm{EA})$. Here $\forall^b(\Gamma \to \Gamma) \to \Gamma$ denotes the class of formulas of type $\forall x < N.\, (\phi(x) \to \psi(x)) \to \theta$ with $\phi, \psi, \theta \in \Gamma$ and $N \in \mathbb{N}$. This appears as the natural class containing the induction axioms for Γ. Note that classically, for $\Gamma = \Pi_n$, (the universal closure of) this class is just equivalent to Π_{n+2}, in harmony with the classical result.

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There are no minimal essentially undecidable theories

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1 Instructions

In their book Mostowski, Tarski, and Robinson [2] generalize Gödel's First Incompleteness Theorem by showing that a very weak arithmetical theory Q (Robinson's arithmetic) is *essentially undecidable*. That is that all its consistent extensions are undecidable. Of course, in particular this implies that any consistent extension of Q is incomplete.

In this talk we present theorem from [1] asserting that in fact there are no interpertability minimal essentially undecidable theories. We provide a relatively simple proof of the result by employing a classical recursion-theoretic theorem (due to Mostowski and Rogers) that the set of indices of decidable c.e. sets is Σ_3-complete.

Acknowledgements

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Combinatorial Flows

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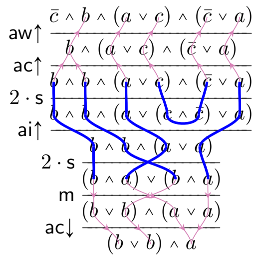
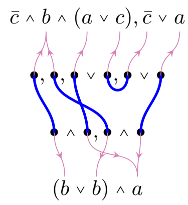
Combinatorial flows are an extension of *Combinatorial proofs* [5] which form a canonical proof presentation that (1) comes with a polynomial correctness criterion, (2) is independent of the syntax of proof formalisms (like sequent calculi, tableaux systems, resolution, etc.), and (3) can handle proof compression mechanisms like cut and substitution, and their elimination.

The main innovation of combinatorial proofs is the global separation of the *linear* part and the *resource management* part of a proof. Below is an example showing how a combinatorial proof (right below) can be extracted from a deep inference derivation (left below). It is well-known that the global separation in combinatorial proofs comes at the cost of a size explosion of the proof as it corresponds to proof normalization.

$$\begin{array}{rc} & \bar{c} \wedge b \wedge (a \vee c) \wedge (\bar{c} \vee a) \\ \cline{2-2} \mathsf{aw}\uparrow & b \wedge (a \vee c) \wedge (\bar{c} \vee a) \\ \cline{2-2} \mathsf{ac}\uparrow & b \wedge b \wedge (a \vee c) \wedge (\bar{c} \vee a) \\ \cline{2-2} 2 \cdot \mathsf{s} & b \wedge b \wedge (a \vee (c \wedge \bar{c}) \vee a) \\ \cline{2-2} \mathsf{ai}\uparrow & b \wedge b \wedge (a \vee a) \\ \cline{2-2} 2 \cdot \mathsf{s} & (b \wedge a) \vee (b \wedge a) \\ \cline{2-2} \mathsf{m} & (b \vee b) \wedge (a \vee a) \\ \cline{2-2} \mathsf{ac}\downarrow & (b \vee b) \wedge a \end{array}$$





A recurring theme of finding suitable canonical proof representation is tracing formulas in a derivation. Atomic flows [3] are an example of such tracing by completely detaching the flow from the derivation. However, atomic flows have two major drawbacks. First, *we cannot read back a proof from an atomic flow.* They lose too much information about the proof and there is no polynomial correctness criterion.[1] Second, *yanking is not possible in atomic flows.* One of the main advantages of coherence graphs or string diagrams is that they can abstract away from superfluous "bends" which is not possible in atomic flows.The reason lies in the interference of contraction with cut elimination.

The idea behind *combinatorial flows* is to use the two colors of combinatorial proofs inside the atomic flows, to distinguish between the linear parts (multiplicative) that can be yanked (blue) and the resource management parts (additive) that cannot be yanked (purple). This is less restrictive than in combinatorial proofs, as the two parts can be composed freely — there is no global separation between the linear part and the resource management.

Figure 1 shows each rule in the deep inference system and their translations into colored flowboxes. To achieve combinatorial flows from a proof, we first translate each instance of an inference rule to a colored flowbox. Next, we can compose flowboxes vertically or horizontally only if two flowboxes have the same color, they can be composed into a single one.

There are two approaches studied for normalization, *global rewriting* and *local rewriting*. An example of local rewriting is the normalization for atomic flows [2]. The problem with this local rewriting is that it does not terminate [3, 4, 8]. In sequent calculus on the other hand, cut elimination is managed by *global* rewriting where the whole subproofs can be duplicated or deleted within one step.

[1]Das has shown in [1] that no such criterion is feasible, under the assumption that integer factoring is hard for *P/poly*.

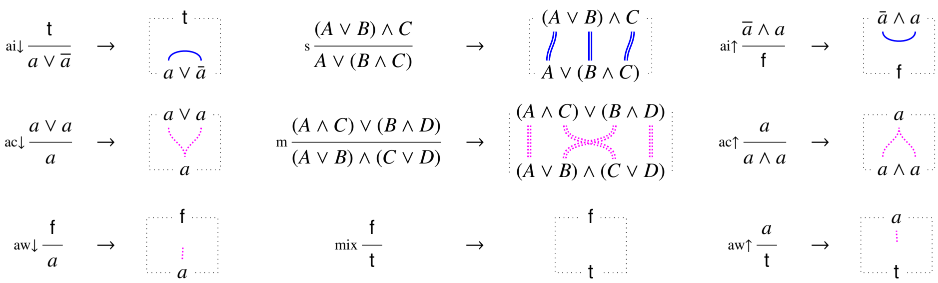


Figure 1: Inference rules of system SKS and their translation into flowboxes

The normalization for combinatorial flows uses a mixed approach: as local as possible, with global steps in a locally bounded scope. This reduction steps are shown in the combinatorial proof setting in [7]. Each normalization is local in the sense that we are running normalizations on the two flowboxes concerned with the cut rule but not the whole proof. But at the scope of a flowbox the normalization steps are acting globally. It is easy to show that normalization is terminating.

Combinatorial flows have been presented at WoLLIC 2022 [6]. This means that in this talk we would present partly published work [6], and partly work in progress (normalization).

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Succinctness of the fixed point theorem for GL

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A classical result of provability logic is the fixed point theorem, proved independently by D. de Jongh and G. Sambin [3] with various proof methods for it ever since. Its statement is the following: Given a modal formula $\phi(p)$ that is modalized for p — i.e. every occurrence of p in ϕ occurs within the scope of a $\Box$ — there is a formula σ without p occurring in it such that $\mathsf{GL} \vdash \phi(\sigma) \leftrightarrow \sigma$. In fact, this fixed point is unique under equivalence over GL. The formula σ can in fact be effectively constructed from ϕ using the following method from G. Sambin [3, 4]: First call ϕ m-decomposable iff there is some formula $\psi(q_1, \dots q_m)$ with fresh variables $q_1, \dots q_m$ and formulae $\chi_1(p), \dots \chi_m(p)$ such that $\phi(p) = \psi(\Box\chi_1(p), \dots, \Box\chi_m(p))$. Next, since every modalized formula is m-decomposable for some m, we inductively construct σ based on the m-decomposability of ϕ. So assuming we have a fixed point for every m-decomposable formula, we get the fixed point of an $n+1$-decomposable formula ϕ as follows:

- $\phi = \psi(\Box\chi_1(p), \dots, \Box\chi_{m+1}(p))$;
- Let $\phi_i = \psi(\Box\chi_1(p), \dots, \Box\chi_{i-1}, \top, \Box\chi_{i+1}, \dots, \Box\chi_{m+1}(p))$ which by IH has a fixed point σ_i;
- Finally $\sigma = \psi(\Box\chi_1(\sigma_1), \dots, \Box\chi_{m+1}(\sigma_{m+1}))$.

Following this construction, we can get a rough upper bound for succinctness of the fixed point σ relative to the original formula ϕ of the scale of $|\sigma| \leq n^{O(n)}$ where $n = \|\phi\|$. However there was no known succinctness lower bound.

The methods that we use to obtain a succinctness lower bound are those of formula-size games that were developed in the setting of Boolean function complexity by Razborov [1] and in the setting of first-order logic and some temporal logics by Adler and Immerman [2]. By now, the formula-size games have been adapted to a host of modal logics and used to obtain lower bounds on modal formulas expressing properties of Kripke models. These methods work by selecting a formula ϕ of a language $\mathcal{L}$ and two sets of models $\mathcal{A}, \mathcal{B}$ that are separable by ϕ. Then the game is setup and played with rules according to a language $\mathcal{L}'$. Once the game is concluded, we obtain a formula ψ in $\mathcal{L}'$ equivalent to ϕ and the size of ψ can be calculated by a careful analysis of the game on the sets $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{L}_\Diamond$ be the standard modal language (with an irreflexive modality) and $\mathcal{L}_{\diamondplus}$ be the language which instead includes a reflexive modality as primitive. In the case of GL, P. Iliev and D. Fernández-Duque have derived an exponential ($2^{O(n)}$) succinctness lower bound for $\mathcal{L}_{\diamondplus}$ over $\mathcal{L}_\Diamond$ in GL. The sequence of formulas they used were defined inductively as:

- $\phi_1 = p_1$;
- $\phi_{n+1} = \Diamond(p_{n+1} \wedge \phi_n)$,

and have the property that any equivalent formula in $\mathcal{L}_\Diamond$ has size $2^{O(n)}$. These can be linearly reformulated into formulas in $\mathcal{L}_\Diamond$ with an equivalent fixed point by

$$\psi_n = \bigwedge_{i<n} (p_{i+1} \to p_i \vee \Diamond(p_i \wedge x))$$

Then, for a fixed point θ for $\psi_n(x)$, we have that $(p_n \wedge \theta) \vee \Diamond(p_n \wedge \theta)$ is equivalent to ϕ_n. Thus it also gives a lower bound for the size of the fixed point for ψ_n over GL frames. We then obtain the following.

Theorem 1. *There exists a sequence of formulas $(\psi_n)_{n<\omega}$ linear in n such that any fixed point in $\mathcal{L}_\Diamond$ for ψ_n over GL has size $2^{O(n)}$.*

We expand this succinctness lower bound in the following sense, we write formulas of $\mathcal{L}$ whose fixed point in $\mathcal{L}_{\Diamond\diamonddot}$ (i.e., the bi-modal logic with a reflexive and an irreflexive modality) is of the scale $2^{O(n)}$. This is done with formulas expressing a kind of tree embeddability into our model. With this, we may improve upon Theorem 1 as follows.

Theorem 2. *There exists a sequence of formulas $(\gamma_n)_{n<\omega}$ linear in n such that any fixed point in $\mathcal{L}_{\Diamond\diamonddot}$ for γ_n over GL has size $2^{O(n)}$.*

Acknowledgements

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Neutral free logic: Proof theory and its applications

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1 Background and motivation

Free logics are a family of first-order logics which came about as a result of examining the existence assumptions of classical logic [9, 14, 15, 16]. What those assumptions are varies, but the central ones are that (i) the domain of interpretation is not empty, (ii) every name denotes exactly one object in the domain and (iii) the quantifiers have existential import.

Free logics reject the claim that names need to denote in (ii). *Positive* free logic concedes that some atomic formulas containing non-denoting names (including self-identity) are true, *negative* free logic treats them as uniformly false, and *neutral* free logic as taking a third value. There has been a renewed interest in analyzing proof theory of free logic in recent years [20, 25, 10, 11], based on intuitionistic logic in [20] as well as classical logic in [25], there for the positive and negative variants.

While the latter streamlines the previous [2, 24] presentation of free logics and offers a more unified approach to the variants under consideration, this unification comes with a caveat. Namely, it does not cover neutral free logic, since there is some lack of both clear formal intuitions on the semantic status of formulas with empty names, as well as a satisfying account of the conditional in this context. So in this paper we continue that project and discuss extending those results to this third major variant of free logics.

1.1 Proof theory of Neutral free logic(s)

Naturally, one might take different approaches to tackling this issue. E.g., one way to acknowledge the neutral phenomena, but avoid them by reducing them to positive free logic, are *supervaluations* [30, 29]. Obviously (as the title of the paper might suggest), it is not our goal here to avoid neutral free logic, but as we will see the approach we suggest is general enough to accommodate that option as well.

Still, one needs to start somewhere, and weak and strong Kleene logics [12, 13] have been identified as the obvious options [27]. Since the former is somewhat more involved, we tackle that first, following a suggestion from [19]. Recently, a sequent calculus has been proposed for quantified weak Kleene logics [7], allowing for the usual array of structural properties. The system there is a five-sided calculus, with the fifth side introduced to account for crispness of formulas (formulas are crisp when they are either true or false), specifically in order to deal with the falsity conditions of the universal quantifier – these, for weak Kleene logics, require all instances to be crisp.

However, in neutral free logic, quantification is limited to the extension of the predicate $E!$ (commonly, though not uniformly, read as 'exists'), for which every atom containing it is crisp, and which determines the crispness of every other atom (namely, $P(t_1...t_n)$ is crisp iff $E!t_i$ for $1 \leq i \leq n$). Consequently, if for some t_i such that $E!t_i$ the instantiated formula $A[t_i/x]$ is not crisp, then for any such t_i it is not crisp (intuitively, it is due to some term other than t_i in A that A is not crisp). Therefore, it follows from $A[t_i/x]$ being false that it is crisp for every t_i s.t. $E!t_i$, and therefore the additional crispness condition is not required.

This enables us to drop the fifth side in adopting the rules of [7], making it an interesting (and unusual) case where the free version of a logic is a simplification of the base logic it departs from. Reshuffling of the presentation then yields the sequents of the form

$$\Gamma \mid \Gamma' \Rightarrow \Delta \mid \Delta',$$

essentially a slight notational variation of the generalized propositional sequent calculi for weak and strong Kleene in [11] (see also [3, 5]), extended to quantification (which is then modified to represent free logic quantification). Several advantages of this mode of presentation are discussed in [11], and these carry over to the present approach. Notably, it facilitates easy transitions between two intuitive readings of the sequents, implicational (if everything in Γ is true and everything in Γ' non-false, then either something in Δ is non-false or something in Δ' is true), providing an easy connection to the notions of strict and tolerant validity [4] and negation-conjunctive (it is not the case that everything in Γ is true, everything in Δ is false, everything in Γ' is non-false and everything in Δ' is non-true), enabling legible interpretations of the rules, including initial sequents and cuts. Moreover, use of the structural symbol $|$, instead of labels in the style of [8], allows easier extension into labelled modal calculi [23].

We show that the usual structural properties [22], and some new ones [7], hold for both strong and weak versions. Ultimately, we obtain proof-theoretically well behaved systems of quantified free logics where the only difference is in the choice of the propositional base, with identical quantifier rules in either variant (this is not the case in the non-free version), furthering the previous goal of greater unification.

1.2 Further work

However, we do not aim to stop there and in the final part of the paper we discuss how this very general framework can be used to incorporate further logics. As already mentioned, these include the possibility to incorporate supervaluations, and furthermore include a series of logics, from positive and negative free (and classical) ones to the very general first-degree entailment, FDE [1, 28].

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Rule-Elimination Theorems

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Cut-elimination theorems constitute one of the most important class of theorems of proof theory and have many important consequences. Since Gentzen's proof of the cut-elimination theorem for the system **LK**, introduced in [3], several other proofs of the theorem has been proposed (see [4]). Even though the techniques of these proofs can be modified to sequent systems other than **LK**, they are essentially of a very particular nature; each of them describes an algorithm to transform a given proof to a cut-free proof.

Cut-elimination can, however, be seen as the *elimination* of a certain rule. One may, therefore, ask the same question for any rule in any sequent system. We can begin this investigation with the following questions.

(1) What makes the elimination of cut possible in **LK**? Do the other rules play any part?

(2) Is it possible to characterize sequent systems for which cut-elimination holds?

(3) Is it possible to give necessary and sufficient conditions of eliminating *any* rule from a given sequent system? What does a 'rule' mean? What are we supposed to understand by a 'sequent system'?

Unfortunately, the algorithmic proofs of the cut-elimination theorems hardly shed any light on issues like the above, primarily due to their heavy dependence on the syntactic structures of the rules.

We, therefore, consider rules abstractly, within the framework of logical structures familiar from *universal logic* in the sense of [1]. A logical structure is a pair of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is a set and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$. In particular, $\mathscr{L}$ can be a set of 'sequents', and $\vdash$ can be defined so that, given a sequent system $\mathbf{S}$, $\emptyset \vdash_{\mathbf{S}} \Gamma \implies \Delta$ holds whenever there is a proof of the sequent $\Gamma \implies \Delta$ in $\mathbf{S}$. We can thus connect the theory of logical structures from universal logic with proof-theory.

One of the goals of universal logic, according to [2], is "to clarify the fundamental concepts of logic and to construct general proofs." In this paper, our aim is to clarify the essence of the so-called "elimination theorems" and construct general proofs of the same. The strategy for achieving this is as follows: we first give a non-algorithmic proof of the cut-elimination theorem for the propositional fragment of **LK**. From this proof, we abstract the essential features of the argument and define something called *normal sequent structures relative to a particular rule.* We then prove a version of the rule-elimination theorem for these. Finally, we define the notion of *abstract sequent structures* and point out the essential features that made the proof of the rule-elimination theorem for the normal sequent structures work. This paves the way towards formulating the most general version of the RULE-elimination theorems. We then show that for abstract sequent structures, the RULE-elimination theorem also has a converse: thus answering question (3) above.

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Tree rewriting system for RC

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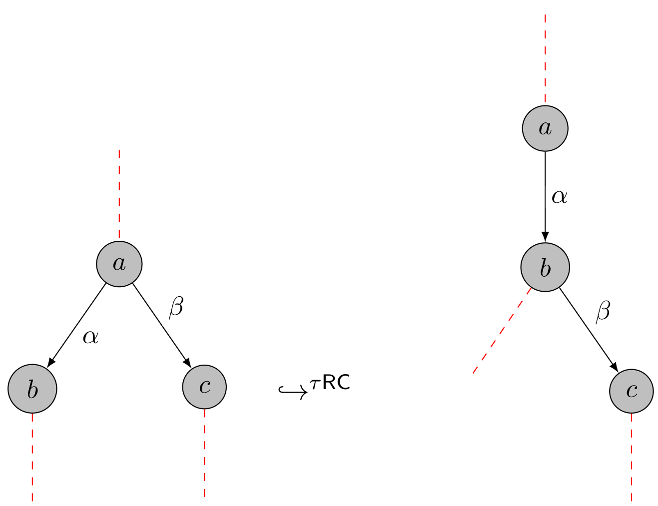
Reflection Calculus (RC) is a propositional sequent logic for the strictly positive fragment of the polymodal logic GLP. More concretely, it deals with the language of strictly positive formulae $\mathcal{L}^+$ which are built-up from $\top$ and propositional variables using conjunction and diamond modalities. As a matter of fact, RC helps with several interesting applications of provability logic. In particular, RC is complete with respect to the arithmetical interpretation of associating modalities with reflection principles.

This paper is a work in progress whose aim is the design of a calculus of structures being a tree rewriting system for RC, denoted by τRC. It is based on embedding strictly positive formulae into the defined class of rooted labeled Kripke trees, denoted by $^{\mathsf{Kripke}}\mathsf{Tree}$. This embedding is given by the canonical tree representation of formulae presented by Beklemishev (see [3]). The canonical tree representation is an operator inductively defined over $\mathcal{L}^+$ mapping formulae to $^{\mathsf{Kripke}}\mathsf{Tree}$. Furthermore, canonical trees seen as treelike Kripke models are shown to be Kripke complete with respect to RC. Therefore, the defined tree rewriting system τRC consists of an abstract rewriting system for the class of rooted labeled Kripke trees and six rewriting rules simulating derivation in RC.

$$\langle\alpha\rangle\varphi \wedge \langle\beta\rangle\psi \vdash_{\mathsf{RC}} \langle\alpha\rangle(\varphi \wedge \langle\beta\rangle\psi),\ \alpha > \beta$$

(a) J axiom for RC



(b) J rewriting rule ($\alpha > \beta$) for τRC

There are six rewriting rules in $\tau\mathsf{RC}$ of three kinds: structural, conjunctive and modality rewriting rules. Firstly, structural rewriting rules manage label and edge elimination in the tree. Then, conjunctive rewriting rules eliminate and duplicate parts of the trees. Finally, modality rewriting rules simulate two concrete axioms of RC: monotonicity axiom ($\langle\alpha\rangle\varphi \vdash_{\mathsf{RC}} \langle\beta\rangle\varphi$, $\alpha > \beta$) and the J axiom.

The tree rewriting system is proven to be sound and complete with respect to the RC proof system modulo certain equivalence relation $\sim_\wedge$. For this purpose, an inverse operator of the tree embedding is given. As usual, $\sim_\wedge$ describes the syntactic structure for our calculus by stating associativity and commutativity for conjunction and the unit equations $\top \wedge \varphi = \varphi$ and $p \wedge p = p$.

As an application, we show the *Reflection conjecture* about certain class of formulae, the R-formulae, over the fragment $\mathcal{L}_0^+/\sim_\wedge$ of $\mathcal{L}^+/\sim_\wedge$ with modalities restricted to 0.

Theorem (Reflection conjecture). If $\mathsf{R}^n(\varphi) \vdash_{\mathsf{RC}} \psi$, then $\mathsf{R}^n(\varphi) \vdash_{\mathsf{RC}} \langle 0\rangle\psi$ for every $\varphi, \psi \in \mathcal{L}_0^+/\sim_\wedge$ such that the modal depth of ψ is smaller than n.

Thus, $\tau\mathsf{RC}$ is indeed a cut-free system for RC providing an effective provability tool for RC from a new deep inference approach. Furthermore, this project is expected to help on the study of several aspects of RC such as the subformula property result and admissibility of rules. On the other hand, exploring the different notions of confluence related to the abstract rewrite system $\tau\mathsf{RC}$ could also open up interesting outcomes to the modal logic.

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A natural combinatorial principle that is weak over weak theories yet strong over strong theories

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Better quasi orders (henceforth bqos) are a strengthening of the notion of well quasi order. Even if their definition is more complicated, the former enjoy nice closure properties, that make them, in a way, easier to work with than the latter: this feature made bqos an instrumental tool in proving landmark results like Nash-Williams' theorem and Laver's theorem. From the reverse mathematical point of view, the study of bqos is an interesting area still full of open questions.

In this talk, we will focus on a property of non-bqos, the so-called *minimal bad array lemma*, and in particular one version of it that we will call MBA^-. In particular, we will show that MBA^- has a very odd behavior when it comes to its reverse-mathematical strength, namely

- over ATR_0, MBA^- can bee seen to be equivalent to the very strong principle of Π^1_2-comprehension (see [2]), yet
- over ACA_0, MBA^- does not imply ATR_0 (as shown in [1]).

This is joint work with Anton Freund, Alberto Marcone, and Fedor Pakhomov.

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The Proof-Theoretic Criteria of Logic and the Logicality of Arithmetic

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1 Introduction

Initial questions about the logicality of arithmetic occurred without a systematic criteria for what is logical. Since then there has been a systematic approach offered by the model-theoretic invariance criteria [5]. By this standard arithmetic, and most of mathematics, is logical. In this talk, I lay out the proof-theoretical criteria for logicality. This is found in the work of Belnap [1] and Došen [2]. This approach can be summed up by the title of Došen's paper "logical constants as punctuation marks". As it is not sufficient to have proof rules to meet the proof-theoretic criteria, the question of whether arithmetic meets it is open. The talk will argue that induction does not meet the proof-theoretic criteria and that this is due to it encoding structural information which is more than mere punctuation can do.

A warning before we continue: there are many systems of arithmetic used for many different purposes. Here we are concerned with the logicality of the first-order induction axiom.

2 Proof-Theoretic Criteria of Logic

In contrast with the model theoretical criteria which makes strong metaphysical assumptions, the proof-theoretic criteria holds that a connective is logical if it does not tell you anything about the area it is applied to. This is one way of spelling out the idea of logic as subject-neutral or general.

Concervativity Belnap [1] proposes that a proof rule should not allow you to prove any nonlogical formulas that you can't otherwise. Formally this is the proposal that the proof rules should be conservative over the base theory. While an interesting proposal, the completeness of Robinson's arithmetic Q for closed atomic formulas make it inappropriate for our question.[1]

Harmony Harmony is a property of the introduction rules that let you infer a statement with a logical operator in it and the elimination rules that let you derive consequences of statements containing that same operator. We say that an elimination rule is in harmony with the introduction role if you can't get any more information via the elimination rule then you put in via the introduction rule. We will consider the possibility of providing harmonious rules for arithmetic. (In particular, rules that have an inversion principle.)

[1]Robinson's arithmetic is the axiomatization of arithmetic without induction.

3 The Logicality of Arithmetic

Induction as 'all numbers' introduction The first approach we consider is where induction is treated as the introduction rule for quantification over all numbers. This is how it is treated in the standard axioms of PA. Steinberger [6] points out that there are general results about the impossibility of normalisation in Peano's arithmetic that suggest there is no way to produce harmonious rules.[2] There are of course other normalization results for the natural numbers.[3] Most famously we have Gentzen's proof of the consistency of arithmetic. However, these theorems use modified notions of a normal proof that do not require an inversion principle.

The concern can be put in formally as follows: induction allows us to claim that every number has a property based on zero having it and the successor of any number that has it having it. The pared elimination rule claims that everything in the domain has this property in doing so it encodes structural information about what the numbers are namely zero and the successor. But this goes beyond mere punctuation. The harmonious elimination rule for induction as an introduction rule only allows one to conclude facts about zero and its successors and so does not provide information about the numbers in general.

Induction as 'is a number' elimination There is an alternative view of induction not as an introduction rule but as an elimination rule. This approach is inspired by the treatment of arithmetic in MLTT [4]. Take 'is a number' as the potentially logical predicate and give it the introduction rules '0 is a number' and the schematic 't is a number $\longrightarrow$ st is a number'. It then follows that induction is an elimination rule with the inversion property.

This is a very nice set of rules which interestingly allow the recovery of most of the rules of PA as theorems. The problem is that it does not follow that zero has no predecessor. Because the axioms are compatible not just with models of the natural numbers but also models of cyclic numbers. So these harmonious rules capture not the predicate 'is a number' but a weaker 'is a number-like object' which allows for more structures than just $\mathbb{N}$.

Structure So when we gave induction as an introduction rule, the elimination rule needed was too strong. When we gave induction as an elimination rule, it was too weak. In both cases, the difficulty relates to capturing an omega sequence.[4] This suggests that the proof-theoretic criteria provides a more nuanced diagnosis of the relationship between mathematics and logic than the model-theoretic criteria. This allows for an explanation of what is right about saying arithmetic is logical (it is structural) and what is wrong (induction cannot be treated as saying nothing about the domain it is in).

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[2]E.g. [7, §10.4.12].
[3]E.g. [3].
[4]Those wondering about nonstandard models, recall that all such models have a 'core' of an omega sequence.

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Completeness Proof of Strict Finitistic Predicate Logic

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Strict finitism, in Crispin Wright [5]'s and our sense, is a constructive standpoint that a statement holds iff it is verifiable in practice, and a number is acceptable iff it is constructible in practice. Thus it is more severe than intuitionism that uses the notion of possibility 'in principle'. Strict finitistic logic is meant to be the system of reasoning according to this standpoint, and Wright in [5] gave a sketch of its Kripke-style semantics in his strict finitistic metatheory. We have reconstructed it in the classical metatheory, and investigated further. We will in this talk provide the semantics and a proof system for it, and present a completeness proof.

The strict finitistic semantics does not greatly differ from that of intuitionistic predicate logic **IQC**. A model represents all possible histories of a human agent's actual verifications, and *validity* is being forced at every node. The condition of implication is similar to that of **IQC**: $k \models A \to B$ iff for any $k' \geq k$, if $k' \models A$, then there is a $k'' \geq k'$ such that $k'' \models B$. Thus strict finitistic implication is intuitionistic implication with a 'time-gap'; and we can regard it as practical implication in the sense that if A is verified, then so is B soon after A. While we impose no restriction on the length of the time-gap, we consider k'' 'within the agent's reach' by virtue of being in the frame. We write $\sim A$ for $A \to \bot$. Then $k \models \sim A$ has the same condition as intuitionistic negation.

More significant is the difference made by negation and quantification. Strict finitistic negation stands for practical unverifiability, and the condition is that $k \models \neg A$ iff $l \not\models A$ for all l. The formula $\neg A$ is *global* in the sense that its satisfiability implies its validity. Our quantifiers basically quantify only over the constructed objects. To formally demarcate them in the domain of discourse, we use the 'existence predicate' E of **IQCE**, which was first introduced by Dana Scott ([2]; cf. [3, pp.50-6] and [1]). We set all nodes' domains to be a constant nonempty set D, and require that the terms over which are quantified satisfy E. But we do not require this for the terms occurring in a negated formula. Since negated formulas are statements of unverifiability, $\neg P(a)$ can hold without $E(a)$; and $\exists x \neg P(x)$ should only mean that something in the domain of discourse is never verified to be P. Thus we employ two modes of quantification, 'global' and 'local'. First we define the class GN of the *global negative* formulas by

- $N ::= \bot \mid \neg\,\mathrm{Form}[\mathcal{L}] \mid N \wedge N \mid N \vee N \mid N \to N \mid \forall x N \mid \exists x N.$

Then, a term t is *occurring globally in* A if t occurs in a GN subformula of A. We set

($\forall$) (global) if x occurs in A only globally, then $k \models_W \forall x A$ iff for all $d \in D$, $k \models_W \top \to A[\overline{d}/x]$, (local) otherwise $k \models_W \forall x A$ iff for any $d \in D$, $k \models_W E(\overline{d}) \to A[\overline{d}/x]$,

($\exists$) (global) if x occurs in A only globally, then $k \models_W \exists x A$ iff there is a $d \in D$ such that $k \models_W A[\overline{d}/x]$, (local) otherwise $k \models_W \exists x A$ iff there is a $d \in D$ such that $k \models_W E(\overline{d}) \wedge A[\overline{d}/x]$ ($\overline{d}$ being the name of d).

The 'local' conditions are the quantification conditions of **IQCE** taken from [1], and thus our quantification is the mixture of that of **IQC** and **IQCE**.

This way, the basic stance of strict finitistic quantification is generalised: quantification is only over constructed objects, except the case of a statement of unverifiability. For instance, while (i) $\exists x(\neg P(x) \to \neg P(x))$ is valid, (ii) $\exists x(\sim P(x) \to \sim P(x))$ is not. (i) is obtained from that $\neg P(\overline{d}) \to \neg P(\overline{d})$ for some $d \in D$. Since this states an implicational relationship between two statements of unverifiability, the object d it speaks of does not have to be constructed. On the other hand, (ii) comes from that $\sim P(\overline{d}) \to \sim P(\overline{d})$ for some d. Certainly this formula is valid for any closed term, but it has to be about a specific object d when we existentially quantify, and therefore we require that it is speaking of some constructed object.

We note that the other valid formulas include $\sim\sim(A \vee \sim A)$, $\neg A \vee \neg\neg A$, $((A \to B) \to A) \to A$, $\sim\sim A \to A$, $\forall x E(x)$ and $\forall x \sim\sim A \to \sim\sim \forall x A$. While Modus Ponens $(A \to B, A/B)$ and $B \vee \neg B$ do not in general hold, they do if B is *stable*, i.e., $k \models \sim\sim B$ implies $k \models B$ for all k.

We define our natural deduction system **NSF** mainly by (i) all rules of **IQC** and the quantification rules of **IQCE** with the distinction of the two modes, (ii) $\neg$-introduction rules and (iii) the rule for formulas with stability. We will explain our method of the completeness proof, while comparing it with the 'prevalence' case and the case of **IQC**. An atomic P is *prevalent* if for any k, there is a $k' \geq k$ such that $k' \models P$. A *prevalent model* is one where E and all satisfiable atomic formulas are prevalent. Conceptually, the histories in such a model are all homogeneous: any verifiable statement is verified in the future of any point in a history.

Out proof method is in the Henkin-style, and the proof is rather simple in the prevalent case. For the general case, we make use of the standard method for **IQC** (cf. [4, pp.169-72] and [3, pp.87-9]). We take care of implication and local universal quantification, by making a countable tree of theories isomorphic to the tree of the finite sequences of the natural numbers; and prove the truth lemma that states that $B \in k$ iff $k \models B$. However, we need significant modifications for negation and global universal quantification. This is ultimately due to the fact that $\{\sim P, \neg\neg P\}$ is consistent. This set stands for a situation where P is never verified in the future, but is verified in another possible history; and to cover all alternative histories, we will make countably many instances of the countable tree.

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