HUGE REFLECTION

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Abstract. We study *Structural Reflection* beyond *Vopěnka's Principle*, at the level of almost-huge cardinals and higher, up to rank-into-rank embeddings. We identify and classify new large cardinal notions in that region that correspond to some form of what we call *Exact Structural Reflection* (ESR). Namely, given cardinals $\kappa < \lambda$ and a class $\mathcal{C}$ of structures of the same type, the corresponding instance of ESR asserts that for every structure A in $\mathcal{C}$ of rank λ, there is a structure B in $\mathcal{C}$ of rank κ and an elementary embedding of B into A. Inspired by the statement of *Chang's Conjecture*, we also introduce and study sequential forms of ESR, which, in the case of sequences of length ω, turn out to be very strong. Indeed, when restricted to Π_1-definable classes of structures they follow from the existence of $I1$-embeddings, while for more complicated classes of structures, e.g., Σ_2, they are not known to be consistent. Thus, these principles unveil a new class of large cardinals that go beyond $I1$-embeddings, yet they may not fall into *Kunen's Inconsistency*.

1. Introduction

Given a class[1] $\mathcal{C}$ of structures[2] of the same type and a cardinal κ, the principle of *Structural Reflection*[3] SR holds at κ for $\mathcal{C}$ if for every structure A in $\mathcal{C}$, there exists some $B \in \mathcal{C} \cap V_\kappa$ and an elementary embedding of B into A. Different forms of SR have been investigated in [Bag, Bag12, BCMR15, BV16, BW, Lüc], yielding canonical characterizations of large cardinals in different regions of the large cardinal hierarchy. For example, results in [Bag12] and [BCMR15] use Magidor's classical characterization of supercompact cardinals from [Mag71] to show that the existence of such a cardinal is equivalent to the validity of the principle SR for all classes of structures definable by Π_1-formulas without parameters.

The principles of structural reflection considered so far correspond to large cardinals up to *Vopěnka's Principle*, stating that every proper class of structures of the same type contains a structure that is elementary embeddable into another structure in the given class. The validity of this principle can be shown to be equivalent to the existence of cardinals witnessing SR for every class of structures (see [Bag12]). In this paper, we shall study principles of structural reflection that correspond to large cardinal notions stronger than Vopěnka's Principle, up to rank-into-rank embeddings, and beyond. These principles are given by variations of the following *exact* form of SR:

Definition 1.1 (Exact Structural Reflection)**.** *Given infinite cardinals $\kappa < \lambda$ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ denote the assertion that for every $A \in \mathcal{C}$ of rank λ, there exists some $B \in \mathcal{C}$ of rank κ and an elementary embedding from B into A.*

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[1]We work in ZFC. So, proper classes are always definable, with set parameters.

[2]Throughout this paper, the term *structure* refers to structures for countable first-order languages. The *cardinality* (respectively, the *rank*) of a structure is defined as the cardinality (respectively, the rank) of its domain.

[3]A detailed discussion of this principle can be found in [Bag].

Given a definability class Γ (i.e., one of Σ_n or Π_n, for some $n < \omega$) and a class P, we introduce the following variations of the above definition that will allow us to formulate our results in a compact way:

(i) We let $\Gamma(P)$-ESR(κ, λ) denote the statement that ESR$_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures of the same type that is Γ-definable with parameters in P.
(ii) We let $\Gamma(P)$-ESR(κ) denote the statement that $\Gamma(P)$-ESR(κ, λ) holds for some cardinal $\lambda > \kappa$.
(iii) We let $\Gamma(P)^{ic}$-ESR(κ, λ) and $\Gamma(P)^{ic}$-ESR(κ) denote the restrictions of the respective principles to classes of structures that are Γ-definable with parameters in P *and* are closed under isomorphic copies.

Using the *Downward Löwenheim–Skolem Theorem*, it is easy to see that ESR$_{\mathcal{C}}(\kappa, \lambda)$ holds for every countable first-order language $\mathcal{L}$, every elementary class $\mathcal{C}$ of $\mathcal{L}$-structures and all uncountable cardinals $\kappa < \lambda$ with $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$ (see Proposition 2.1 below). In contrast, we shall see that the above principles for externally defined classes are quite strong, for they correspond to large cardinals in the region between supercompact and rank-into-rank embeddings, and beyond. Below is a summary of the main results.

First, we discuss our results for classes of structures closed under isomorphic copies. While an easy application of Σ_1-absoluteness shows that the principle $\Sigma_1(V_\kappa)^{ic}$-ESR(κ, λ) holds for all uncountable cardinals $\kappa < \lambda$ with $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$ (Proposition 3.1), we prove that the principle Π^{ic}_1-ESR(κ, λ) already implies the existence of a $<\lambda$-supercompact cardinal less than or equal to κ (Lemma 4.9). Moreover, for singular cardinals κ, our results show that the validity of principles of the form $\Pi_1(V_\kappa)^{ic}$-ESR(κ, λ) is equivalent to the existence of cardinals below κ possessing certain degrees of supercompactness. In particular, it turns out that singular limits of supercompact cardinals can be characterized through exact structural reflection for Π_1-definable classes closed under isomorphic copies. Namely, we have the following equivalences:

Theorem 1.2. *The following statements are equivalent for every singular cardinal κ:*

(i) *κ is a limit of supercompact cardinals.*
(ii) *$\Pi_1(\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ.*
(iii) *$\Sigma_2(V_\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ.*

In order to state an analogous result for more complicated classes of structures, we have to introduce a weak form of the notion of $C^{(n)}$-*extendibility* from [Bag12]. Recall that a cardinal κ is λ-*extendible* for some ordinal $\lambda > \kappa$ if there is an ordinal η and an elementary embedding $j : V_\lambda \to V_\eta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Following [Bag12], for every $n < \omega$, we let $C^{(n)}$ denote the Π_n-definable closed unbounded class of all of ordinals α such that V_α is a Σ_n-elementary substructure of V. Given cardinals $\kappa < \lambda$ and $n < \omega$, the cardinal κ is λ-$C^{(n)}$-*extendible* if there is an elementary embedding $j : V_\lambda \to V_\nu$ for some cardinal ν with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. In addition, we say that κ is $C^{(n)}$-*extendible* if it is λ-$C^{(n)}$-extendible for all (equivalently, for a proper class of) $\lambda > \kappa$ (see [Bag12, Section 3]).

The following weaker form of $C^{(n)}$-extendibility will allow us to prove a version of Theorem 1.2 for classes of structures of complexity greater than Σ_2.

Definition 1.3. *Given ordinals $\mu < \lambda$ and a natural number n, a cardinal $\kappa \leq \mu$ is $[\mu, \lambda)$-$C^{(n)}$-*extendible *if there exist $\nu \in C^{(n+1)} \cap [\mu, \lambda)$ and an elementary embedding $j : V_\lambda \to V_\eta$, for some η, with $\mathrm{crit}(j) = \kappa$, $j(\mu) \geq \lambda$ and $j(\nu) \in C^{(n)}$. In addition, we say that κ is $[\mu, \infty)$-$C^{(n)}$-extendible if it is $[\mu, \lambda)$-$C^{(n)}$-extendible for a proper class of ordinals λ.*

It is easy to see that extendible cardinals κ are $[\mu, \infty)$-$C^{(1)}$-extendible for all $\mu \geq \kappa$. Using the fact that the requirement "$j(\kappa) > \lambda$" can be omitted in the definition of extendibility (see [Kan03, Proposition 23.15]), we can also see that a cardinal κ is extendible if and only if it is $[\mu, \infty)$-$C^{(1)}$-extendible for some $\mu \geq \kappa$. We will later show that every $C^{(n)}$-extendible cardinal is $[\mu, \infty)$-$C^{(n)}$-extendible, for every $\mu \geq \kappa$ (Proposition 4.4).

Using this notion, the above characterization of singular limits of supercompact cardinals now generalizes in the following way:

Theorem 1.4. *For every $n > 0$, the following statements are equivalent for every singular cardinal κ:*

(i) *κ is a limit of $[\kappa, \infty)$-$C^{(n)}$-extendible cardinals.*
(ii) *$\Pi_{n+1}(\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ.*
(iii) *$\Sigma_{n+2}(V_\kappa)^{ic}$-ESR$(\kappa, \lambda)$ holds for a proper class of cardinals λ.*

In particular, Theorem 1.4 shows that a singular cardinal κ is a limit of extendible cardinals if and only if $\Pi_2(\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ.

In combination with results from [Bag12] and [BCMR15], the methods developed in the proof of Theorem 1.4 also allow us to conclude that exact structural reflection for classes of structures closed under isomorphisms holding at singular cardinals does not imply the existence of large cardinals stronger than Vopěnka's Principle. In fact, Vopěnka's Principle can be characterized through the validity of principles of the form $\Pi_n(V_\kappa)^{ic}$-ESR(κ, λ).

Theorem 1.5. *Over the theory* ZFC*, the following schemes of sentences imply each other:*

(i) *Vopěnka's Principle.*
(ii) *For every class $\mathcal{C}$ of structures of the same type that is closed under isomorphic images, there is a cardinal κ with the property that* ESR$_{\mathcal{C}}(\kappa, \lambda)$ *holds for all $\lambda > \kappa$.*
(iii) *For every natural number $n > 0$, there exists a proper class of cardinals κ with the property that $\Pi_n(V_\kappa)^{ic}$-ESR(κ, λ) holds for all $\lambda > \kappa$.*

In contrast to the above results, both the validity of the principle Π_1^{ic}-ESR at a regular cardinal and the validity of the principle Π_1-ESR at some cardinal turn out to imply the existence of large cardinals stronger than Vopěnka's Principle, e.g. almost huge cardinals. The large cardinal properties introduced below will allow us to capture the strength of these forms of exact structural reflection. Their definition is motivated by results in [Lüc] that provide a characterization of *shrewd cardinals* (introduced by Rathjen in [Rat95]) through a variation of Magidor's classical characterization of supercompactness in [Mag71] and similar characterizations of n-hugeness in [HLN19, Section 6]. In the following, we will say that a set M is $\Pi_n(P)$*-correct* for some natural number $n > 0$ and a class P if all Π_n-formulas with parameters in $M \cap P$ are absolute between V and M.

Definition 1.6. *Given a natural number $n > 0$, an infinite cardinal κ is* weakly n-exact *for a cardinal $\lambda > \kappa$ if for every $A \in V_{\lambda+1}$, there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in \operatorname{ran}(j)$. If we further require that $j(\operatorname{crit}(j)) = \kappa$, then we say that κ is* weakly parametrically n-exact for λ.

Observe that if κ is weakly parametrically 1-exact for λ, then κ and λ are both inaccessible. The following result shows how weakly n-exact cardinals are connected to principles of exact structural reflection for Π_n-definable classes of structures.

Theorem 1.7. *The following statements are equivalent for all cardinals κ and all natural numbers $n > 0$:*

(i) *κ is the least regular cardinal such that Π_n^{ic}-ESR(κ) holds.*
(ii) *κ is the least cardinal such that Π_n-ESR(κ) holds.*
(iii) *κ is the least cardinal such that $\Pi_n(V_\kappa)$-ESR(κ) holds.*
(iv) *κ is the least cardinal that is weakly n-exact for some $\lambda > \kappa$.*
(v) *κ is the least cardinal that is weakly parametrically n-exact for some $\lambda > \kappa$.*

In the case of Σ_{n+1}-definable classes of structures, the large cardinal principles corresponding to the different forms of exact structural reflection are the following:

Definition 1.8. *Given a natural number n, an infinite cardinal κ is* n-exact *for some cardinal $\lambda > \kappa$ if for every $A \in V_{\lambda+1}$, there exists a cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in \operatorname{ran}(j)$. If we further require that $j(\operatorname{crit}(j)) = \kappa$ holds,*[4] *then we say that κ is* parametrically n-exact *for λ.*

[4]Even though X need not be transitive, we still define $\operatorname{crit}(j)$ as the least ordinal moved by j, which exists since j is not the identity on the ordinals as $j(\kappa) = \lambda$.

Note that, if $m \leq n < \omega$ and κ is (parametrically) n-exact for λ, then κ is also (parametrically) m-exact for λ. Moreover, standard arguments show that, if κ is parametrically 0-exact for λ, then both κ and λ are inaccessible cardinals. In addition, it is easily seen that, given $0 < n < \omega$, if κ is (parametrically) n-exact for λ, then it is also weakly (parametrically) n-exact for λ. Finally, we will later show that, if a cardinal κ is weakly parametrically $(n+1)$-exact for some λ and $0 < n < \omega$, then it is also parametrically n-exact for λ (Proposition 7.2).

The equivalence of the existence of n-exact cardinals with ESR principles for Σ_n-definable classes of structures is given in the following theorem.

Theorem 1.9. *The following statements are equivalent for all cardinals κ and all natural numbers $n > 0$:*

(i) *κ is the least cardinal such that Σ_{n+1}-ESR(κ) holds.*
(ii) *κ is the least cardinal such that $\Sigma_{n+1}(V_\kappa)$-ESR(κ) holds.*
(iii) *κ is the least cardinal that is n-exact for some $\lambda > \kappa$.*
(iv) *κ is the least cardinal that is parametrically n-exact for some $\lambda > \kappa$.*

The above results allow us to uniquely place the large cardinal properties introduced in Definitions 1.6 and 1.8 into a linear hierarchy. More specifically, if κ is a cardinal satisfying the equivalent statements of Theorem 1.9 for some $0 < n < \omega$, then there exists a cardinal $\mu < \kappa$ satisfying the equivalent statements of Theorem 1.7 for the same natural number n (Lemma 7.1). In addition, if $0 < n < \omega$ and κ satisfies the statements listed in Theorem 1.7 for $n + 1$, then there is a cardinal $\mu < \kappa$ satisfying the statements of Theorem 1.9 (see Lemma 5.8) for n. These results should be compared with the corresponding statements for the principle SR, showing that SR for Π_n-definable classes of structures is equivalent to SR for Σ_{n+1}-definable classes ([Bag12, Section 4]).

Exact cardinals are very strong, consistency-wise. In Section 8 we give lower and upper bounds for their consistency strength, and we also prove they imply the existence of well-known large cardinals in the upper ranges of the large-cardinal hierarchy. Recall that a cardinal κ is *almost huge* (see [Kan03]) if it is the critical point of an elementary embedding $j : V \to M$, with M transitive and closed under sequences of length less than $j(\kappa)$. Given such an embedding j, we then say that κ is *almost huge with target $j(\kappa)$*. If κ is either parametrically 0-exact for λ, or weakly parametrically 1-exact for λ, then many cardinals smaller than κ are almost huge with target κ (Corollary 8.2). As for upper bounds, while every huge cardinal (with target some λ) is weakly parametrically 1-exact (for the same λ), the least huge cardinal κ is not 1-exact for any $\lambda > \kappa$ (Propositions 8.4 and 8.6). A strong consistency upper bound is provided by an $I3$-embedding (see [Kan03, §24]), for if $j : V_\delta \to V_\delta$ is such an embedding, then in V_δ a proper class of cardinals are parametrically n-exact for unboundedly-many λ, for every n (Proposition 8.7). A much lower upper bound, namely an almost 2-huge cardinal, is given in Proposition 8.9 for the consistency of weakly parametrically n-exact cardinals, all $n > 0$.

Finally, in Section 9, we show how the principle ESR(κ, λ) can be strengthened to encompass increasing sequences of cardinals of length at most ω, instead of a single cardinal λ, in order to obtain principles of structural reflection that are much stronger, implying the existence of many-times huge cardinals or even $I3$-embeddings. The formulation of these stronger sequential ESR principles is motivated by the observation that the principle Π_1-ESR(κ, λ) directly implies the instance

$$(\lambda, \kappa) \twoheadrightarrow (\kappa, {<}\kappa)$$

of *Chang's Conjecture*, i.e. every structure A in a countable language with domain λ has an elementary substructure B of cardinality κ with $|B \cap \kappa| < \kappa$. The definition of our sequential ESR principles will then directly imply that higher versions of Chang's Conjecture hold for the respective cardinals.

We then also strengthen, accordingly, the notions of weakly exact and exact cardinals to obtain large cardinal properties that correspond to the new sequential ESR principles and show that much of the theory developed for ESR(κ, λ) can be generalized to this stronger context. In particular, we obtain exact equivalences for the least cardinals witnessing the

sequential forms of ESR and the corresponding sequential forms of weakly exact and exact cardinals (Theorems 9.8 and 9.11). As for determining the position of these large cardinals in the large cardinal hierarchy, we show, on the one hand, that the existence of a weakly 1-exact or a 0-exact cardinal for a sequence of cardinals of length $n+1$ implies the existence of smaller n-huge cardinals. On the other hand, every n-huge cardinal is weakly parametrically n-exact for some sequence of cardinals of length n (Proposition 9.3). Also, if κ is the critical point of an $I1$-embedding (see [Kan03, §24]), then it is weakly parametrically 1-exact for a sequence of cardinals of length ω.

Many questions remain, and some of them are addressed in the last section of the article. Most interesting is the problem of determining the exact strength of the sequential forms of ESR. We know that these principles, in the case of sequences of length ω, are very strong, so much so that even when restricted to Σ_2-definable classes of structures we don't know them to be consistent. This makes the study of such principles both challenging and exciting, for they appear to constitute a new class of large-cardinal principles that go beyond $I1$-embeddings, yet they may not fall into *Kunen's Inconsistency*.

2. Isomorphism-Closed classes

We start by studying instances of the principle ESR for classes of structures closed under isomorphic copies. Notice that if a class $\mathcal{C}$ of structures of the same type is Σ_n-definable (with or without parameters) for some $n > 0$, then the closure of $\mathcal{C}$ under isomorphic copies is also Σ_n-definable (with the same parameters, if any).

Proposition 2.1. *Given uncountable cardinals $\kappa < \lambda$, the following statements are equivalent for every class $\mathcal{C}$ of structures of the same type that is closed under isomorphic copies:*

(i) $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$.
(ii) *For every structure B in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$, there exists an elementary embedding of a structure A in $\mathcal{C}$ into B such that the cardinality of A is contained in the interval $[\mathrm{cof}(\kappa), \beth_\kappa]$.*

Proof. Assume that (i) holds and fix a structure B in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$. Then we can pick an injection i from the domain of B into V_λ such that the set

$$\{\gamma < \lambda \mid \mathrm{ran}(i) \cap (V_{\gamma+1} \setminus V_\gamma) \neq \varnothing\}$$

is unbounded in λ. Let B_0 denote the isomorphic copy of B induced by i. Then B_0 is a structure in $\mathcal{C}$ of rank λ and our assumptions yield an elementary embedding of a structure A in $\mathcal{C}$ of rank κ into B_0. But this allows us to conclude that the cardinality of A is contained in the interval $[\mathrm{cof}(\kappa), \beth_\kappa]$, and there exists an elementary embedding of A into B.

Now, assume that (ii) holds and fix a structure B in $\mathcal{C}$ of rank λ. Then the cardinality of B is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$ and our assumption yields an elementary embedding of a structure A in $\mathcal{C}$ into B whose cardinality is contained in the interval $[\mathrm{cof}(\kappa), \beth_\kappa]$. Pick an injection i from the domain of A into V_κ with the property that the set

$$\{\alpha < \kappa \mid \mathrm{ran}(i) \cap (V_{\alpha+1} \setminus V_\alpha) \neq \varnothing\}$$

is unbounded in κ, and let A_0 denote the isomorphic copy of A induced by i. Then A_0 is a structure in $\mathcal{C}$ of rank κ and there exists an elementary embedding of A_0 into B. □

Corollary 2.2. *Let $\kappa < \lambda$ be inaccessible cardinals and let $\mathcal{C}$ be a class of structures of the same type that is closed under isomorphic copies. Then $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds if and only if for every structure $B \in \mathcal{C}$ of cardinality λ, there exists an elementary embedding of a structure $A \in \mathcal{C}$ of cardinality κ into B.* □

Corollary 2.3. *Let $\mathcal{C}$ be a class of structures of the same type that is closed under isomorphic copies and let $\kappa < \mu < \lambda$ be infinite cardinals with the property that $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds.*

(i) *If $\mathrm{cof}(\mu) \leq \mathrm{cof}(\kappa)$, then $\mathrm{ESR}_{\mathcal{C}}(\mu, \lambda)$ holds.*
(ii) *If $\mathrm{cof}(\mu) \geq \mathrm{cof}(\lambda)$, then $\mathrm{ESR}_{\mathcal{C}}(\kappa, \mu)$ holds.*

Proof. Since $\mathrm{cof}(\mu) \leq \mathrm{cof}(\kappa)$ implies $[\mathrm{cof}(\kappa), \beth_\kappa] \subseteq [\mathrm{cof}(\mu), \beth_\mu]$ and $\mathrm{cof}(\mu) \geq \mathrm{cof}(\lambda)$ implies $[\mathrm{cof}(\mu), \beth_\mu] \subseteq [\mathrm{cof}(\lambda), \beth_\lambda]$, both statements follow directly from Proposition 2.1. □

3. Low complexities

In this section, we study exact structural reflection for Σ_1-definable classes of structures. In the case of classes closed under isomorphic copies, these principles are provable in ZFC.

Proposition 3.1. *If κ is an uncountable cardinal, then the principle $\Sigma_1(V_\kappa)^{ic}$-ESR(κ,λ) holds for every cardinal $\lambda > \kappa$ with $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$.*

Proof. Fix a Σ_1-formula $\varphi(v_0, v_1)$ and $z \in V_\kappa$ such that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ is a class of structures of the same type and pick a structure B in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$. Let B_0 be an isomorphic copy of B in $H_{\beth^+_\lambda}$ and pick an elementary substructure X of $H_{\beth^+_\lambda}$ of cardinality $\beth_\kappa$ with $V_\kappa \cup \{\kappa, B_0\} \subseteq X$. Let $\pi : X \to M$ denote the induced transitive collapse. Since $\pi(z) = z$, Σ_1-absoluteness now implies that $\varphi(\pi(B_0), z)$ holds and hence $\pi(B_0)$ is an element of $\mathcal{C}$. Moreover, our construction ensures that $\pi(B_0)$ has cardinality at most $\beth_\kappa$ and, since B_0 has cardinality at least $\mathrm{cof}(\lambda) \geq \mathrm{cof}(\kappa)$ and $\pi(\mathrm{cof}(\kappa)) = \mathrm{cof}(\kappa)$, we know that $\pi(B_0)$ has cardinality at least $\mathrm{cof}(\kappa)$. Finally, using the inverse collapse π^{-1}, it is easy to see that there exists an elementary embedding of $\pi(B_0)$ into B. By Proposition 2.1, the above computations yield the desired conclusion. □

In contrast with the previous Proposition, the principle ESR$_{\mathcal{C}}(\kappa,\lambda)$ for some $\kappa < \lambda$ and all Σ_0-definable (without parameters) classes $\mathcal{C}$ of structures of the same type (so, no closure under isomorphic copies required), has considerable large-cardinal strength and fails in Gödel's constructible universe L.

Lemma 3.2. *If Σ_0-ESR(κ) holds for some uncountable cardinal κ, then $a^\#$ exists for every real a.*

Proof. Let $\mathcal{L}$ denote the first-order language that extends the language $\mathcal{L}_\in$ of set theory by a binary predicate symbol $\dot{E}$, a constant symbol $\dot{c}$ and a unary function symbol $\dot{f}$. Define $\mathcal{C}$ to be the class of all $\mathcal{L}$-structures of the form $\langle \nu, \in, E, \alpha, f\rangle$ with the property that ν is an ordinal and $f : \langle \nu, \in\rangle \to \langle \mathrm{ran}(f), E\rangle$ is an order-isomorphism. Then it is easy to see that $\mathcal{C}$ is definable by a Σ_0-formula without parameters.

Now, fix a real a and a cardinal $\lambda > \kappa$ such that ESR$_{\mathcal{C}}(\kappa,\lambda)$ holds. Pick a bijection $b : \mathrm{L}_\lambda[a] \to \lambda$ and set

$$E \;=\; \{\langle b(x), b(y)\rangle \mid x \in y \in \mathrm{L}_\lambda[a]\}.$$

Then

$$B \;=\; \langle \lambda, \in, E, b(\kappa), b \restriction \lambda\rangle$$

is an $\mathcal{L}$-structure of rank λ in $\mathcal{C}$. By our assumption, there exists a binary relation R on κ, a function $f : \kappa \to \kappa$ and $\alpha < \kappa$ such that

$$A \;=\; \langle \kappa, \in, R, \alpha, f\rangle$$

is a structure in $\mathcal{C}$ with the property that there exists an elementary embedding i of A into B. Since our construction ensures that $\langle \lambda, E\rangle$ is well-founded and $\langle \kappa, R\rangle$ embeds into $\langle \lambda, E\rangle$, it follows that $\langle \kappa, R\rangle$ is well-founded too. Moreover, elementarity implies that $\langle \kappa, R\rangle$ is extensional. Let $\pi : \langle \kappa, R\rangle \to \langle M, \in\rangle$ denote the corresponding transitive collapse and set

$$j \;=\; b^{-1} \circ i \circ \pi^{-1} : M \to \mathrm{L}_\lambda[a].$$

Then j is an elementary embedding of transitive structures.

Now, note that elementarity implies that $\mathrm{ran}(f) = \pi^{-1}[M \cap \mathrm{Ord}]$ and

$$\pi \restriction \mathrm{ran}(f) : \langle \mathrm{ran}(f), R\rangle \to \langle M \cap \mathrm{Ord}, \in\rangle$$

is an order-isomorphism. But this shows that

$$\pi \circ f : \langle \kappa, \in\rangle \to \langle M \cap \mathrm{Ord}, \in\rangle$$

is also an order-isomorphism and hence we can conclude that $M \cap \mathrm{Ord} = \kappa$. In particular, elementarity implies that $M = \mathrm{L}_\kappa[a]$.

Finally, since $j(\pi(\alpha)) = \kappa > \pi(\alpha)$, we know that $j : \mathrm{L}_\kappa[a] \to \mathrm{L}_\lambda[a]$ is a non-trivial elementary embedding. But then $|\mathrm{crit}(j)| < \kappa$ and the proof of [Jec02, Theorem 18.27] shows that $a^\#$ exists. □

The next result provides an upper bound for the consistency strength of the assumption of Lemma 3.2. In particular, it shows that this assumption does not imply the existence of an inner model with a measurable cardinal. Its proof is based on arguments contained in the proof of [VW01, Theorem 2.3].

Lemma 3.3. *If δ is a Ramsey cardinal, then the set of inaccessible cardinals $\kappa < \delta$ with the property that $\Sigma_1(V_\kappa)$-ESR(κ) holds in V_δ is unbounded in δ.*

Proof. Fix $\xi < \delta$ and pick $A \subseteq \delta$ such that $V_\delta = \mathrm{L}_\delta[A]$. By our assumption, there exists a good set I of indiscernibles for the structure $\langle \mathrm{L}_\delta[A], \in, A\rangle$ (see [DJK81, Section 1]) that is unbounded in δ and satisfies $\min(I) > \xi$. Set

$$X \;=\; \mathrm{Hull}_{\langle \mathrm{L}_\delta[A],\in,A\rangle}(\min(I) \cup (I \setminus \{\min(I)\}))$$

and let $\pi : X \to M$ denote the corresponding transitive collapse. Since I is unbounded in δ, we know that $M \cap \mathrm{Ord} = \delta$. Moreover, indiscernibility ensures that $\min(I) \notin X$ and hence $\pi^{-1} : M \to V_\delta$ is a non-trivial elementary embedding with critical point $\min(I)$. Set $\kappa = \pi^{-1}(\min(I)) \in M$ and $\lambda = \pi^{-1}(\kappa) \in M$. Then κ and λ are both inaccessible cardinals greater than ξ.

Claim. *In V_δ, the principle $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_1-formula with parameters in V_κ.*

Proof of the Claim. Assume, towards a contradiction, that there is a Σ_1-formula $\varphi(v_0, v_1)$ with the property that for some $z \in V_\kappa$, the class $\mathcal{C} = \{A \in V_\delta \mid \varphi(A, z)\}$ consists of structures of the same type and there exists $B \in \mathcal{C}$ of rank λ such that for all $A \in \mathcal{C}$ of rank κ, there is no elementary embedding of A into B. Using elementarity, we now know that, in M, there exist $z_0 \in V_{\min(I)}$ and $B_0 \in V_{\kappa+1} \setminus V_\kappa$ with the property that the class $\mathcal{C}_0 = \{A \in M \mid \varphi(A, z_0)\}$ consists of structures of the same type, $B_0 \in \mathcal{C}_0$ and for all $A \in \mathcal{C}_0$ of rank $\min(I)$, there is no elementary embedding of A into B_0. Since we have

$$\pi^{-1} \restriction V^M_{\min(I)} \;=\; \mathrm{id}_{V^M_{\min(I)}},$$

the elementarity of π^{-1} implies that $\mathcal{C}_* = \{A \in V_\delta \mid \varphi(A, z_0)\}$ consists of structures of the same type, $\pi^{-1}(B_0) \in \mathcal{C}_*$ and for all $A \in \mathcal{C}_*$ of rank κ, there is no elementary embedding of A into $\pi^{-1}(B_0)$. But this yields a contradiction, because the upwards absoluteness of Σ_1-formulas implies that B_0 is a structure in $\mathcal{C}_*$ of rank κ and π^{-1} induces an elementary embedding of B_0 into $\pi^{-1}(B_0)$. □

The above claim completes the proof of the lemma. □

4. The Π_n-case for isomorphism-closed classes

We now show that the structural reflection principles introduced in Definition 1.1 become very strong when they hold for more complex classes of structures. In particular, the validity of these principles for Π_1-definable classes $\mathcal{C}$ of structures closed under isomorphic copies at singular cardinals already implies non-trivial fragments of supercompactness, and the corresponding principles for a regular cardinal will turn out to imply the existence of many almost huge cardinals.

Recall that a cardinal κ is *λ-supercompact* if there is a transitive class M closed under λ-sequences and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Lemma 4.1. *Let $\kappa < \lambda$ be infinite cardinals such that κ is singular and $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$. If the interval $(\mathrm{cof}(\kappa), \kappa)$ contains a $\beth_\lambda$-supercompact cardinal μ, then $\Pi_1(V_\mu)^{ic}$-ESR(κ, λ) holds.*

Proof. Fix a Π_1-formula $\varphi(v_0, v_1)$ and an element z of V_μ such that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ is a class of structures of the same type that is closed under isomorphic copies. Pick a structure B in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$ and whose domain is a subset of $\beth_\lambda$. Now, fix an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \mu$, $j(\mu) > \beth_\lambda$ and ${}^{\beth_\lambda}M \subseteq M$. Then the closure properties of M ensure that B is an element of M and the elementary embedding of B into $j(B)$ induced by j is also contained in M.

Moreover, Π_1-downwards absoluteness for transitive classes implies that $\varphi(B, z)$ holds in M. Thus, M satisfies that there exists a structure A whose cardinality is contained in the interval $[\mathrm{cof}(\kappa), j(\mu))$ such that $\varphi(A, z)$ holds, and there exists an elementary embedding $e : A \to j(B)$, as this is witnessed by B and $j \restriction B : B \to j(B)$. Since $j(z) = z$ and $j(\mathrm{cof}(\kappa)) = \mathrm{cof}(\kappa)$, the elementarity of j yields an elementary embedding of a structure A of the given type into B such that $\varphi(A, z)$ holds and the cardinality of A is contained in the interval $[\mathrm{cof}(\kappa), \mu)$. This shows that there is an elementary embedding of a structure in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\kappa), \beth_\kappa]$ into B. By Proposition 2.1, this proves the lemma. □

Corollary 4.2. *If $\kappa < \lambda$ are cardinals such that $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$ and κ is a singular limit of $\beth_\lambda$-supercompact cardinals, then $\Pi_1(V_\kappa)^{ic}$-$\mathrm{ESR}(\kappa, \lambda)$ holds. Hence, if κ is a singular limit of supercompact cardinals, then $\Pi_1(V_\kappa)^{ic}$-$\mathrm{ESR}(\kappa, \lambda)$ holds for a proper class of cardinals λ.* □

We continue by showing that, for all $n > 1$, analogous statements hold for Π_n-definable classes and $[\beth_\kappa, \beth_\lambda + 1)$-$C^{(n-1)}$-extendible cardinals.

Lemma 4.3. *Let $n > 0$ be a natural number and let $\kappa < \lambda$ be infinite cardinals such that κ is singular and $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$ holds. If the interval $(\mathrm{cof}(\kappa), \kappa)$ contains a cardinal δ that is $[\mu, \beth_\lambda + 1)$-$C^{(n)}$-extendible for some ordinal $\delta \leq \mu \leq \beth_\kappa$, then $\Pi_{n+1}(V_\delta)^{ic}$-$\mathrm{ESR}(\kappa, \lambda)$ holds.*

Proof. Fix a Π_{n+1}-formula $\varphi(v_0, v_1)$ and an element z of V_δ such that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ is a class of structures of the same type that is closed under isomorphic copies. Pick a structure B in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$ and whose domain is a subset of $\beth_\lambda$. Let $\nu \in C^{(n+1)} \cap [\mu, \beth_\lambda + 1)$ and let $j : V_{\beth_\lambda + 1} \to V_\eta$ be an elementary embedding with $\mathrm{crit}(j) = \delta$, $j(\mu) \geq \beth_\lambda + 1$, and $j(\nu) \in C^{(n)}$. Then the fact that $\beth_\lambda < j(\mu) \leq j(\nu)$ implies that B is an element of $V_{j(\nu)}$ and the elementary embedding of B into $j(B)$ induced by j is contained in V_η. Moreover, since $j(\nu) \in C^{(n)}$, by Π_{n+1}-downwards absoluteness for $V_{j(\nu)}$, we have that $\varphi(B, z)$ holds in $V_{j(\nu)}$. Thus, V_η satisfies that there exists a structure A whose cardinality is contained in the interval $[\mathrm{cof}(\kappa), j(\mu))$ such that $\varphi(A, z)$ holds in $V_{j(\nu)}$, and there exists an elementary embedding $e : A \to j(B)$, as this is witnessed by B and $j \restriction B : B \to j(B)$. Since $j(z) = z$ and $j(\mathrm{cof}(\kappa)) = \mathrm{cof}(\kappa)$, the elementarity of j yields an elementary embedding of a structure A of the given type into B such that $\varphi(A, z)$ holds in V_ν and the cardinality of A is contained in the interval $[\mathrm{cof}(\kappa), \mu)$. But since $\nu \in C^{(n+1)}$, we know that $\varphi(A, z)$ also holds in V. This shows that there is an elementary embedding of a structure in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\kappa), \beth_\kappa]$ into B. By Proposition 2.1, this proves the lemma. □

The following observation shows that $C^{(n)}$- extendible cardinals provide natural examples of $[\mu, \infty)$-$C^{(n)}$-extendible cardinals.

Proposition 4.4.

(i) *If κ is a λ-$C^{(n)}$-extendible cardinal and $C^{(n+1)} \cap [\kappa, \lambda) \neq \varnothing$, then κ is $[\kappa, \lambda)$-$C^{(n)}$-extendible.*

(ii) *Every $C^{(n)}$-extendible cardinal κ is $[\mu, \infty)$-$C^{(n)}$-extendible, for every $\mu \geq \kappa$.*

Proof. (i): Assume κ is λ-$C^{(n)}$-extendible and $\nu \in C^{(n+1)} \cap [\kappa, \lambda)$. Since $\kappa \in C^{(n)}$, and since every true Σ_{n+1} statement, with parameters in V_κ is true in V_ν, the assumption that κ is λ-$C^{(n)}$-extendible easily yields that $\kappa \in C^{(n+1)}$. Then κ itself witnesses the $[\kappa, \lambda)$-$C^{(n)}$-extendibility of κ.

(ii): As shown in [Tsa18], a cardinal κ is $C^{(n)}$-extendible if and only if it is $C^{(n)^+}$-extendible, i.e., for a proper class of $\lambda \in C^{(n)}$ there exists an elementary embedding $j : V_\lambda \to V_\eta$ for some $\eta \in C^{(n)}$, with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. Thus, if κ is $C^{(n)}$-extendible, then it is $[\mu, \lambda)$-$C^{(n)}$-extendible, for every $\mu \geq \kappa$ and every $\lambda \in C^{(n)}$ such that $C^{(n+1)} \cap [\mu, \lambda) \neq \varnothing$. In particular, every $C^{(n)}$-extendible cardinal κ is $[\mu, \infty)$-$C^{(n)}$-extendible for every $\mu \geq \kappa$. □

Corollary 4.5. *Let $n > 0$ be a natural number, let κ be a singular cardinal and let $\lambda > \kappa$ be a cardinal with $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$.*

(i) *If κ is a limit of cardinals δ that are $[\mu, \beth_\lambda + 1)$-$C^{(n)}$-extendible for some $\delta \leq \mu \leq \kappa$, then $\Pi_{n+1}(V_\kappa)^{ic}$-ESR(κ, λ) holds.*
(ii) *If κ is a limit point of $C^{(n+1)}$ and a limit of $(\beth_\lambda + 1)$-$C^{(n)}$-extendible cardinals, then $\Pi_{n+1}(V_\kappa)^{ic}$-ESR(κ, λ) holds.*
(iii) *If κ is a limit of $C^{(n)}$-extendible cardinals, then $\Pi_{n+1}(V_\kappa)^{ic}$-ESR(κ, λ) holds*

Proof. The first statement follows directly from Lemma 4.3. For the second statement, notice that our assumption implies that κ is a limit of $(\beth_\lambda + 1)$-$C^{(n)}$-extendible cardinals δ with the property that $C^{(n+1)} \cap (\delta, \kappa) \neq \varnothing$. By Proposition 4.4, this implies that κ is a limit of cardinals δ that are $[\delta, \beth_\lambda + 1)$-$C^{(n)}$-extendible and therefore we can use the first part to derive the desired conclusion. Finally, since [Bag12, Proposition 3.4] shows that all $C^{(n)}$-extendible cardinals are elements of $C^{(n+2)}$, we can apply the second part of the corollary to prove the third statement. □

We are now ready to show that Vopěnka's Principle can be characterized through principles of exact structural reflection.

Proof of Theorem 1.5. First, assume that (i) holds and fix a natural number $n > 0$. Since [BCMR15, Corollary 6.9] shows that our assumption implies the existence of a proper class of $C^{(n)}$-extendible cardinals, there exists a proper class of cardinals κ of countable cofinality that are limits of $C^{(n)}$-extendible cardinals. The third part of Corollary 4.5 now implies that for every such cardinal κ and every $\lambda > \kappa$, the principle $\Pi_{n+1}(V_\kappa)^{ic}$-ESR(κ, λ) holds. This shows that (iii) holds in this case.

Now, assume that (ii) holds and let $\mathcal{C}_0$ be a proper class of structures of the same type. Let $\mathcal{C}$ denote the class of all structures of the given type that are isomorphic to a structure in $\mathcal{C}_0$ and define $C = \{|A| \mid A \in \mathcal{C}\}$. By our assumptions, there exists a cardinal κ with the property that ESR$_{\mathcal{C}}(\kappa, \lambda)$ holds for all $\lambda > \kappa$. If C is a proper class, then there exists a structure $B \in \mathcal{C}_0$ of cardinality greater than $\beth_\kappa$ and, since the principle ESR$_{\mathcal{C}}(\kappa, |B|)$ holds, we can use Proposition 2.1 to find a structure $A \in \mathcal{C}_0$ of cardinality at most $\beth_\kappa$ and an elementary embedding of A into B. In particular, we know that Vopěnka's Principle for $\mathcal{C}$ holds in this case. In the other case, namely if C is a set, then we can find distinct $A, B \in \mathcal{C}_0$ that are isomorphic and hence Vopěnka's Principle for $\mathcal{C}$ also holds in this case. This allows us to conclude that (i) holds.

This concludes the proof of the theorem, because (iii) obviously implies (ii). □

Similar results hold also for $\Sigma_n(V_\kappa)$-definable classes closed under isomorphic copies, assuming κ is a singular limit of supercompact cardinals, in the case $n = 2$, or a singular limit of $[\kappa, \infty)$-$C^{(n-2)}$-extendible cardinals, in the case $n > 2$.

Corollary 4.6. *Let κ be a singular cardinal and let $\lambda > \kappa$ be a cardinal with* $\mathrm{cof}(\kappa) \leq \mathrm{cof}(\lambda)$.
(i) *If κ is a limit of supercompact cardinals, then $\Sigma_2(V_\kappa)^{ic}$-ESR(κ, λ) holds.*
(ii) *If $n > 0$ and κ is a limit of $[\kappa, \infty)$-$C^{(n)}$-extendible cardinals, then $\Sigma_{n+2}(V_\kappa)^{ic}$-ESR$(\kappa, \lambda)$ holds.*

Proof. (i) Assume that κ is a limit of supercompact cardinals and fix $z \in V_\kappa$. Let $\varphi(v_0, v_1)$ be a Σ_2-formula such that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ is a class of structures of the same type that is closed under isomorphic copies. Pick a structure B in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$. We can use our assumption to find cardinals $\mu < \kappa$ and $\theta \geq \beth_\lambda$ such that $z \in V_\mu$ and there exists a transitive class M containing B and closed under θ-sequences, and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \mu$, $j(\mu) > \theta$ and the property that $\varphi(B, z)$ holds in M. In this situation, we can repeat the proof of Lemma 4.1 to find a structure A in $\mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\kappa), \beth_\kappa]$ and an elementary embedding of A into B.

(ii) Now, assume that κ is a limit of $[\kappa, \infty)$-$C^{(n)}$-extendible cardinals. Fix a Σ_{n+2}-formula $\varphi(v_0, v_1)$ and $z \in V_\kappa$ that define a class $\mathcal{C}$ of structures of the same type closed under isomorphic copies. If we now pick a structure $B \in \mathcal{C}$ whose cardinality is contained in the interval $[\mathrm{cof}(\lambda), \beth_\lambda]$, then there is a cardinal $\beth_\lambda < \theta \in C^{(n+2)}$, a cardinal $\nu \in C^{(n+1)} \cap [\kappa, \theta)$ and an elementary embedding $j : V_\theta \to V_\eta$ for some η such that $z \in V_{\mathrm{crit}(j)}$, $j(\kappa) \geq \theta$ and

$j(\nu) \in C^{(n)}$. Using the fact that $\theta \in C^{(n+2)}$, we can now continue as in the proof of Lemma 4.3 to obtain the desired elementary embedding. □

We shall next prove several results that will allow us to derive high lower bounds for the consistency strength of the principle Π^{ic}_1-ESR. In particular, these results will show that passing from Σ_1-definable to Π_1-definable classes of structures drastically increases the strength of the principle ESR.

Given a set z and a natural number $n > 0$, we let $\mathcal{W}_n(z)$ denote the class of all structures (in the language $\mathcal{L}_\in$ of set theory extended by five constant symbols and two unary function symbols) of the form $\langle D, E, a, b, c, d, e, f, g\rangle$ with the property that the relation E is well-founded and extensional, and, if $\pi : \langle D, E\rangle \to \langle M, \in\rangle$ denotes the corresponding transitive collapse, then the following statements hold:

(i) $\pi(a), \pi(b), \pi(c), \pi(d) \in M \cap \mathrm{Ord}$, and $V_{\pi(c)} \subseteq M$.
(ii) If $\pi(b) > 0$, then $\pi(b) \in C^{(n)}$.
(iii) $\pi(e) = z \in V_{\pi(a)}$.
(iv) M is $\Pi_n(V_{\pi(c)+1})$-correct.
(v) The map $\pi \circ f \circ \pi^{-1}$ induces a bijection between $V_{\pi(c)}$ and $\pi(d)$.
(vi) The map $g \circ \pi^{-1}$ restricts to a bijection between $\pi(d)$ and D.

Then it is easy to see that the class $\mathcal{W}_n(z)$ is closed under isomorphic copies and is definable by a Π_n-formula with parameter z.

Lemma 4.7. *Let $\mu < \lambda$ be cardinals such that $\mathrm{ESR}_{\mathcal{W}_n(z)}(\mu, \lambda)$ holds for some element z of V_μ and some natural number $n > 0$. Given a cardinal $\lambda' \in C^{(n)}$ greater than $\beth_\lambda$, there exist*

- *a cardinal $\kappa \leq \mu$ with $z \in V_\kappa$ and $\mathrm{cof}(\mu) \leq \beth_\kappa$,*
- *a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\beth_\kappa\} \subseteq M$, and*
- *an elementary embedding $j : M \to H_{\lambda'}$ with $\mathrm{crit}(j) < \kappa$, $j(\mathrm{crit}(j)) \leq \mu$, $j(z) = z$, $\mu \in \mathrm{ran}(j)$ and $j(\kappa) = \lambda$.*

In addition, for every $\nu \in C^{(n)} \cap [\mu, \lambda)$, we can find objects satisfying the above statements such that $j(\zeta) = \nu$ holds for some $\zeta \in C^{(n)} \cap \kappa$.

Proof. Pick an elementary substructure X of $H_{\lambda'}$ of cardinality $\beth_\lambda$ with $V_\lambda \cup (\beth_\lambda + 1) \subseteq X$, a map $h_0 : X \to X$ that extends a bijection between V_λ and $\beth_\lambda$, and a map $h_1 : X \to X$ that extends a bijection between $\beth_\lambda$ and X. Fix an ordinal ν such that either $\nu = 0$ or $\nu \in C^{(n)} \cap [\mu, \lambda)$. Since $V_{\lambda+1} \cap X$ is contained in the transitive part of X, it follows that the transitive collapse of X is $\Pi_n(V_{\lambda+1})$-correct and hence

$$\langle X, \in, \mu, \nu, \lambda, \beth_\lambda, z, h_0, h_1\rangle$$

is a structure in $\mathcal{W}_n(z)$ of cardinality $\beth_\lambda$. By Proposition 2.1, our assumptions allow us to find a structure $\langle D, E, a, b, c, d, e, f, g\rangle$ in $\mathcal{W}_n(z)$ whose cardinality is contained in the interval $[\mathrm{cof}(\mu), \beth_\mu]$ and an elementary embedding

$$i : \langle D, E, a, b, c, d, e, f, g\rangle \to \langle X, \in, \mu, \nu, \lambda, \beth_\lambda, z, h_0, h_1\rangle.$$

Let $\pi : \langle D, E\rangle \to \langle M, \in\rangle$ denote the corresponding transitive collapse. Set $\theta = \pi(a)$, $\zeta = \pi(b)$, and $\kappa = \pi(c)$. Then we have $\pi(e) = z \in V_\theta \subseteq V_\kappa \subseteq M$,

$$\beth_\kappa \;=\; |V_\kappa| \;=\; |M| \;\in\; [\mathrm{cof}(\mu), \beth_\mu]$$

and therefore $\mathrm{rnk}(z) < \theta < \kappa \leq \mu$. Moreover, since the ordinal $\pi(d)$ has cardinality $\beth_\kappa$, our setup ensures that $\beth_\kappa \in M$. The definition of the class $\mathcal{W}_n(z)$ also ensures that M is $\Pi_n(V_{\kappa+1})$-correct. Define

$$j \;=\; i \circ \pi^{-1} : M \to H_{\lambda'}.$$

Then j is an elementary embedding satisfying $j(z) = z$, $j(\theta) = \mu \geq \kappa > \theta$, and $j(\kappa) = \lambda$. In particular, we know that $\mathrm{crit}(j) \leq \theta < \kappa$ and $j(\mathrm{crit}(j)) \leq j(\theta) = \mu$. Finally, if we have $\nu \in C^{(n)} \cap [\mu, \lambda)$, then elementarity implies that $\zeta > 0$ and this allows us to conclude that ζ is an element of $C^{(n)} \cap \kappa$ with $j(\zeta) = \nu$. □

The following direct consequence of the *Kunen Inconsistency* (see [Kan03, Corollary 23.14]) will be used in our subsequent arguments:

Proposition 4.8. *Given cardinals $\kappa < \lambda$ and an ordinal $\alpha < \kappa$, if $j : V_\kappa \to V_\lambda$ is a non-trivial elementary embedding with $j(\alpha) = \alpha$, then* $\mathrm{crit}(j) > \alpha$.

Proof. Assume, towards a contradiction, that $\mathrm{crit}(j) < \alpha$ holds. Then $j^n(\mathrm{crit}(j)) < \alpha < \kappa$ holds for all $n < \omega$ and hence we can conclude that $\rho = \sup_{n<\omega} j^n(\mathrm{crit}(j)) \leq \alpha < \kappa$. Since $j(\rho) = \rho$ and $V_{\rho+2} \subseteq V_{\alpha+2} \subseteq V_\kappa$, we know that $j \restriction V_{\rho+2} : V_{\rho+2} \to V_{\rho+2}$ is a non-trivial elementary embedding. This contradicts the *Kunen Inconsistency*. □

The next results show that the assumptions of Corollary 4.2 are close to optimal.

Lemma 4.9. *Let $\mu < \lambda$ be cardinals and let $\alpha < \mu$ be an ordinal.*

(i) *If $\Pi_1(\{\alpha\})^{ic}$-ESR(μ, λ) holds, then the interval $(\alpha, \mu]$ contains a $<\lambda$-supercompact cardinal.*
(ii) *Given a natural number $n > 0$, if $C^{(n+1)} \cap [\mu, \lambda) \neq \varnothing$ and $\Pi_{n+1}(\{\alpha\})^{ic}$-ESR$(\mu, \lambda)$ holds, then the interval $(\alpha, \mu]$ contains a $[\mu, \lambda)$-$C^{(n)}$-extendible cardinal.*

Proof. (i) Pick a cardinal $\lambda' > \beth_\lambda$ with the property that $H_{\lambda'}$ is sufficiently elementary in V. Since $\mathrm{ESR}_{\mathcal{W}_1(\alpha)}(\mu, \lambda)$ holds, an application of Lemma 4.7 shows that there is a cardinal $\alpha < \kappa \leq \mu$, a transitive set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $\mathrm{crit}(j) < \kappa$, $j(\mathrm{crit}(j)) \leq \mu$, $j(\alpha) = \alpha$ and $j(\kappa) = \lambda$. In this situation, Proposition 4.8 shows that $\mathrm{crit}(j) > \alpha$. By [Mag71, Lemma 2], the fact that $V_\kappa \subseteq M$ implies that $\mathrm{crit}(j)$ is $<\kappa$-supercompact. Since all ultrafilters witnessing this property are contained in $V_\kappa \subseteq M$, it follows that $\mathrm{crit}(j)$ is $<\kappa$-supercompact in M and hence $j(\mathrm{crit}(j)) \in (\alpha, \mu]$ is $<\lambda$-supercompact in both $H_{\lambda'}$ and V.

(ii) Pick some $\nu \in C^{(n+1)} \cap [\mu, \lambda)$ and $\lambda' \in C^{(n+1)}$ greater than $\beth_\lambda$. Since $\mathrm{ESR}_{\mathcal{W}_{n+1}(\alpha)}(\mu, \lambda)$ holds, Lemma 4.7 allows us to find a cardinal κ with $\alpha < \kappa \leq \mu$, a cardinal $\zeta \in C^{(n+1)} \cap \kappa$, a cardinal $\theta \leq \zeta$, a $\Pi_{n+1}(V_{\kappa+1})$-correct transitive set M with $V_\kappa \cup \{\kappa\} \subseteq M$, and an elementary embedding $j : M \to H_{\lambda'}$ with $\mathrm{crit}(j) \leq \theta$, $j(\alpha) = \alpha$, $j(\theta) = \mu$, $j(\zeta) = \nu$ and $j(\kappa) = \lambda$. By Proposition 4.8, we have $\mathrm{crit}(j) > \alpha$. Now, notice that λ and $j \restriction V_\kappa$ witness that there exists an ordinal η and an elementary embedding $i : V_\kappa \to V_\eta$ with $\mathrm{crit}(i) \in (\alpha, \theta]$, $i(\theta) \geq \kappa$ and $i(\zeta) \in C^{(n)}$, and this statement can be expressed by a Σ_{n+1}-formula with parameters α, κ, θ and ζ. Moreover, since $\kappa + 1 \subseteq M$ and M is $\Pi_{n+1}(V_{\kappa+1})$-correct, this statement holds in M. By the elementarity of j and the fact that $\lambda' \in C^{(n+1)}$, we now know that, in V, there exists an ordinal η and an elementary embedding $i : V_\lambda \to V_\eta$ with $\mathrm{crit}(i) \in (\alpha, \mu]$, $i(\mu) \geq \lambda$ and $i(\nu) \in C^{(n)}$. This shows that $\mathrm{crit}(i) \in (\alpha, \mu]$ is a $[\mu, \lambda)$-$C^{(n)}$-extendible cardinal. □

Corollary 4.10. *Let κ be a cardinal and let $\alpha < \kappa$ be an ordinal.*

(i) *If $\Pi_1(\{\alpha\})^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ, then the interval $(\alpha, \kappa]$ contains a supercompact cardinal.*
(ii) *For each natural number $n > 0$, if $\Pi_{n+1}(\{\alpha\})^{ic}$-ESR$(\kappa, \lambda)$ holds for a proper class of cardinals λ, then the interval $(\alpha, \kappa]$ contains a $[\kappa, \infty)$-$C^{(n)}$-extendible cardinal.* □

A combination of the above results now yield short proofs of Theorems 1.2 and 1.4 stated in the Introduction.

Proof of Theorem 1.2. Let κ be a singular cardinal. If κ is a limit of supercompact cardinals, then Corollary 4.6 shows that $\Sigma_2(V_\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ. In the other direction, if $\Pi_1(\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ, then we can apply Corollary 4.10 to show that κ is a limit of supercompact cardinals. □

Proof of Theorem 1.4. Let κ be a singular cardinal and let $n > 0$ be a natural number. If κ is a limit of $[\kappa, \infty)$-$C^{(n)}$-extendible cardinals, then Corollary 4.6 shows that $\Sigma_{n+2}(V_\kappa)^{ic}$-ESR$(\kappa, \lambda)$ holds for a proper class of cardinals λ. For the other direction, if $\Pi_{n+1}(V_\kappa)^{ic}$-ESR(κ, λ) holds for a proper class of cardinals λ, then Corollary 4.10 allows us to conclude that κ is a limit of $[\kappa, \infty)$-$C^{(n)}$-extendible cardinals. □

We will eventually show that the strength of the principle Π^{ic}_1-ESR(κ) further increases significantly if κ is a regular cardinal. More specifically, we will show (see Corollary 8.2 below and Theorem 1.7 stated in the Introduction) that this assumption implies the existence of an almost huge cardinal. The next lemma is the starting point of this analysis. It will allow

us to show that the least regular cardinal μ satisfying Π^{ic}_n-ESR(μ) coincides with the least cardinal ν satisfying $\Pi_n(V_\nu)$-ESR(ν).

Lemma 4.11. *Given a natural number $n > 0$, assume that*

- *$\kappa < \lambda < \lambda'$ are cardinals with $\beth_\lambda < \lambda' \in C^{(n)}$,*
- *M is a transitive, $\Pi_n(V_{\kappa+1})$-correct set with $V_\kappa \cup \{\kappa\} \subseteq M$, and*
- *$j : M \to H_{\lambda'}$ is an elementary embedding with $j(\kappa) = \lambda$.*

Then the following statements hold:

(i) *If $\mu \in \mathrm{ran}(j) \cap (\mathrm{crit}(j), \kappa]$ and $z \in M$ with $j(z) = z$, then $j(\mu) > \mu$ and $\Pi_n(\{z\})$-ESR$(\mu, j(\mu))$ holds.*
(ii) *If $\mu = j(\mathrm{crit}(j))$, then $\mu \leq \kappa$ implies that $\Pi_n(V_\mu)$-ESR$(\mu, j(\mu))$ holds.*

Proof. (i) Pick $\theta \in M$ with $j(\theta) = \mu$ and set $\nu = j(\mu) \leq \lambda$. Since $j(\kappa) = \lambda > \kappa \geq \mu > \mathrm{crit}(j)$, we know that $\mathrm{crit}(j) \leq \theta$ and this allows us to apply Proposition 4.8 to show that $\theta < \mu < \nu$. Fix a Π_n-formula $\varphi(v_0, v_1)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of structures of the same type. Assume, towards a contradiction, that there exists $B \in \mathcal{C}$ of rank ν with the property that for all $A \in \mathcal{C}$ of rank μ, there exists no elementary embedding of A into B. Since $\beth_\nu \leq \beth_\lambda < \lambda'$, we know that $B \in H_{\lambda'}$ and the fact that $\lambda' \in C^{(n)}$ implies that $\varphi(B, z)$ holds in $H_{\lambda'}$ and for every structure A of the given type and rank μ, either $\varphi(A, z)$ fails in $H_{\lambda'}$ or $H_{\lambda'}$ contains no elementary embedding of A into B. The elementarity of j then yields a structure $B_0 \in M$ of the given type and rank μ such that $\varphi(B_0, z)$ holds in M and for every structure A of the given type and rank θ, either $\varphi(A, z)$ fails in M or M contains no elementary embedding of A into B_0. Our setup then ensures that B_0 is an element of $\mathcal{C}$ of rank μ and the embedding j gives rise to an elementary embedding i of B_0 into $j(B_0)$. But this yields a contradiction to the elementarity of j, because B_0 and i are both contained in $H_{\lambda'}$, and the fact that M is correct about the sentence $\varphi(B_0, z)$, and $\lambda' \in C^{(n)}$, implies that $\varphi(B_0, u)$ holds in $H_{\lambda'}$.

(ii) Set $\nu = j(\mu) > \mu$ and fix a Π_n-formula $\varphi(v_0, v_1)$. Assume, towards a contradiction, that there exists $z \in V_\mu$ with the property that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ is a class of structures of the same type and there exists $B \in \mathcal{C}$ of rank ν such that for all $A \in \mathcal{C}$ of rank μ, there is no elementary embedding of A into B. Then the fact that $\nu \leq \lambda \leq \beth_\lambda < \lambda' \in C^{(n)}$ implies that this statement also holds in $H_{\lambda'}$, and hence, in M, there exists $z_0 \in V_{\mathrm{crit}(j)}$ with the property that $\mathcal{C}_0 = \{A \mid \varphi(A, z_0)\}$ is a class of structures of the same type and there exists $B \in \mathcal{C}_0$ of rank μ such that for all $A \in \mathcal{C}_0$ of rank $\mathrm{crit}(j)$, there is no elementary embedding of A into B. Since M is $\Pi_n(V_{\kappa+1})$-correct, this is true in V. So, since $j(z_0) = z_0$ and $\mu \in (\mathrm{crit}(j), \kappa]$, we can now proceed as in the proof of (i) to derive a contradiction. □

Lemma 4.12. *Given a natural number $n > 0$, let μ be a regular cardinal with the property that Π^{ic}_n-ESR(μ, λ) holds for some cardinal $\lambda > \mu$. Then there exists an inaccessible cardinal $\delta \leq \mu$ with the property that $\Pi_n(V_\delta)$-ESR(δ, ρ) holds for an inaccessible cardinal ρ with $\delta < \rho \leq \lambda$.*

Proof. Pick $\lambda' \in C^{(n)}$ greater than $\beth_\lambda$ and use Lemma 4.7 to find a cardinal $\kappa \leq \mu$ with $\mu = \mathrm{cof}(\mu) \leq \beth_\kappa$, a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\beth_\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $\mathrm{crit}(j) < \kappa$, $j(\mathrm{crit}(j)) \leq \mu$, $\mu \in \mathrm{ran}(j)$ and $j(\kappa) = \lambda$. Since $V_\kappa \subseteq M$, we have that $\mathrm{crit}(j)$ is an inaccessible cardinal.

Claim. $j(\mathrm{crit}(j)) \leq \kappa$.

Proof of the Claim. Let ε be minimal with $\beth_\varepsilon \geq \mu$. Note that $\varepsilon \leq \kappa$. Since μ is an element of $\mathrm{ran}(j)$, and ε is definable from μ, we can find $\zeta \in M$ with $j(\zeta) = \varepsilon$. Moreover, since $\mathrm{crit}(j)$ is an inaccessible cardinal smaller than $\beth_\varepsilon$, we know that $\zeta \geq \mathrm{crit}(j)$. But this allows us to conclude that $j(\mathrm{crit}(j)) \leq j(\zeta) = \varepsilon \leq \kappa$. □

Define $\delta = j(\mathrm{crit}(j)) \in M$ and $\rho = j(\delta)$. Then elementarity implies that δ and ρ are also inaccessible cardinals. Moreover, an application of the second part of Lemma 4.11 directly shows that $\Pi_n(V_\delta)$-ESR(δ, ρ) holds. □

5. The Π_n-case for arbitrary classes

In order to study principles of exact structural reflection for Π_n-definable classes of structures that are not necessarily closed under isomorphic copies, we analyse connections between the validity of these principles and the existence of weakly n-exact cardinals. The following variation of Definition 1.6 will allow us to state our results more precisely:

Definition 5.1. *Given cardinals $\kappa < \lambda$, a set $z \in V_\kappa$ and a natural number $n > 0$, we say that κ is* weakly n-exact for λ and z *if for every $A \in V_{\lambda+1}$, there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(z) = z$ and $A \in \mathrm{ran}(j)$.*

Proposition 5.2. *If a cardinal κ is weakly parametrically n-exact for some cardinal λ, then κ is weakly n-exact for λ and all $z \in V_\kappa$.*

Proof. Given $z \in V_\kappa$ and $A \in V_{\lambda+1}$, our assumptions yield a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(\mathrm{crit}(j)) = \kappa$ and $A, z \in \mathrm{ran}(j)$. Pick $z_0 \in M$ with $j(z_0) = z$. Since we have $j(\mathrm{rnk}(z_0)) = \mathrm{rnk}(z) < \kappa = j(\mathrm{crit}(j))$, it follows that $z_0 \in M \cap V_{\mathrm{crit}(j)}$ and hence $z_0 = j(z_0) = z$. □

Proposition 5.3. *If κ is weakly n-exact for λ and z, then $\Pi_n(\{z\})$-ESR(κ, λ) holds. In particular, if κ is weakly parametrically n-exact for λ, then $\Pi_n(V_\kappa)$-ESR(κ, λ) holds.*

Proof. Fix a Π_n-formula $\varphi(v_0, v_1)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of structures of the same type and $B \in \mathcal{C}$ of rank λ. By our assumptions, there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(z) = z$ and $B \in \mathrm{ran}(j)$. Pick $A \in M$ with $j(A) = B$. Since all Π_n-formulas with parameters in $H_{\lambda'}$ are downwards absolute from V to $H_{\lambda'}$, we know that $\varphi(B, z)$ holds in $H_{\lambda'}$ and the structure B has rank λ in $H_{\lambda'}$. But this means that, in M, the statement $\varphi(A, z)$ holds and A has rank κ. The fact that M is correct about the statement $\varphi(A, z)$ allows us to conclude that A is a structure in $\mathcal{C}$ of rank κ and the embedding j induces an elementary embedding of A into B. The second part of the proposition follows directly from Proposition 5.2. □

For each natural number $n > 0$ and every set z, let $\mathcal{E}_n(z)$ denote the class of structures $\langle D, E, a, b, c\rangle$ (in the language of set theory with three additional constant symbols) with the property that $\mathrm{rnk}(D) \subseteq D$, E is a well-founded and extensional relation on D and, if $\pi : \langle D, E\rangle \to \langle M, \in\rangle$ is the corresponding transitive collapse, then $V_{\mathrm{rnk}(D)} \cup \{\mathrm{rnk}(D)\} \subseteq M$, M is $\Pi_n(V_{\mathrm{rnk}(D)+1})$-correct, $\pi(b) = \mathrm{rnk}(D)$, $\pi(c) = z$ and $\pi^{-1} \restriction \mathrm{rnk}(D) = \mathrm{id}_{\mathrm{rnk}(D)}$. Note that the class $\mathcal{E}_n(z)$ is definable by a Π_n-formula with parameter z.

Lemma 5.4. *Let $\kappa < \lambda$ be cardinals, let $z \in V_\kappa$ and let $n > 0$ be a natural number with the property that $\mathrm{ESR}_{\mathcal{E}_n(z)}(\kappa, \lambda)$ holds. Given $B \in V_{\lambda+1}$ and a cardinal $\lambda' \in C^{(n)}$ greater than λ,[5] there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(z) = z$ and $B \in \mathrm{ran}(j)$. In particular, the cardinal κ is weakly n-exact for λ and z.*

Proof. Let X be an elementary submodel of $H_{\lambda'}$ of cardinality $\beth_\lambda$ with $V_\lambda \cup \{\lambda, B\} \subseteq Y$. Pick a bijection $f : X \to V_\lambda$ with $f \restriction \lambda = \mathrm{id}_\lambda$ and let R be the induced binary relation on V_λ. Since the transitive collapse of $\langle V_\lambda, R\rangle$ is the composition of f^{-1} and the transitive collapse of $\langle X, \in\rangle$ and all Σ_n-formulas using parameters from the transitive part of X are absolute between V and the transitive collapse of X, it follows that the structure $\langle V_\lambda, R, f(B), f(\lambda), f(z)\rangle$ is an element of $\mathcal{E}_n(z)$ of rank λ.

By our assumptions, we can find a structure $\langle D, E, a, b, c\rangle$ of rank κ in $\mathcal{E}_n(z)$ such that there exists an elementary embedding

$$i : \langle D, E, a, b, c\rangle \to \langle V_\lambda, R, f(B), f(\lambda), f(z)\rangle.$$

[5]Since the class $C^{(1)}$ consists of all cardinals μ with the property that $H_\mu = V_\mu$, the given assumption ensures that λ' is greater than $\beth_\lambda$.

Let $\pi : \langle D, E\rangle \to \langle M, \in\rangle$ denote the corresponding transitive collapse. Set $A = \pi(a)$ and

$$j \;=\; f^{-1} \circ i \circ \pi^{-1} : M \to H_{\lambda'}.$$

Then M is a transitive set with $V_\kappa \cup \{\kappa\} \subseteq M$, $A \in M \cap V_{\kappa+1}$ with the property that M is $\Pi_n(V_{\kappa+1})$-correct and j is an elementary embedding with $j(\kappa) = \lambda$, $j(A) = B$ and $j(z) = z$. □

A combination of Proposition 5.3 and Lemma 5.4 now yields the following equivalence:

Corollary 5.5. *The following statements are equivalent for all natural numbers $n > 0$, all cardinals $\kappa < \lambda$ and all $z \in V_\kappa$:*

(i) $\Pi_n(\{z\})$-ESR(κ, λ).
(ii) *κ is weakly n-exact for λ and z.*
(iii) *For all $A \in V_{\lambda+1}$ and $\lambda < \lambda' \in C^{(n)}$, there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ satisfying $j(\kappa) = \lambda$, $j(z) = z$ and $A \in \mathrm{ran}(j)$.* □

The techniques developed above now allow us to show that exact structural reflection for Π_n-definable classes implies exact structural reflection for Σ_n-definable classes.

Proposition 5.6. *Let $\kappa < \lambda$ be cardinals, let $z \in V_\kappa$ and let $n > 0$ be a natural number. Then $\Pi_n(\{z\})$-ESR(κ, λ) implies $\Sigma_n(\{z\})$-ESR(κ, λ).*

Proof. Fix a Σ_n-formula $\varphi(v_0, v_1)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of structures of the same type and a structure B in $\mathcal{C}$ of rank λ. Using Corollary 5.5, we can find a cardinal $\lambda < \lambda' \in C^{(n)}$, a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(z) = z$ and $B \in \mathrm{ran}(j)$. Pick $B_0 \in M$ with $j(B_0) = B$. Since $\lambda' \in C^{(n)}$, we know that $\varphi(B, z)$ holds in $H_{\lambda'}$ and hence $\varphi(B_0, z)$ holds in M. By the $\Pi_n(V_{\kappa+1})$-correctness of M, this shows that B_0 is an element of $\mathcal{C}$ of rank κ. Moreover, the map j induces an elementary embedding of B_0 into B. □

Arguments contained in the proofs of the above results also allow us to prove the following parametrical version of Corollary 5.5 that will be needed later on.

Lemma 5.7. *Let $n > 0$ be a natural number, and let κ be weakly parametrically n-exact for some cardinal $\lambda > \kappa$. If $\lambda < \lambda' \in C^{(n)}$ and $B \in V_{\lambda+1}$, then there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ satisfying $j(\mathrm{crit}(j)) = \kappa$, $j(\kappa) = \lambda$ and $B \in \mathrm{ran}(j)$.*

Proof. Pick an elementary submodel X of $H_{\lambda'}$ of cardinality λ with $V_\lambda \cup \{\lambda, B\} \subseteq X$ and a bijection $f : X \to V_\lambda$ with $f \restriction \lambda = \mathrm{id}_\lambda$. Let $\langle V_\lambda, R, f(B), f(\lambda), \varnothing\rangle$ denote the corresponding structure in $\mathcal{E}_n(\varnothing)$ of rank λ constructed in the proof of Lemma 5.4. By our assumptions, we can find a transitive, $\Pi_n(V_{\kappa+1})$-correct set N with $V_\kappa \cup \{\kappa\} \subseteq N$, a cardinal $\beth_\lambda < \eta \in C^{(n-1)}$ and an elementary embedding $i : N \to H_\eta$ with $i(\mathrm{crit}(i)) = \kappa$, $i(\kappa) = \lambda$ and $R, f(B), f(\lambda) \in \mathrm{ran}(i)$. Pick $R_0, a_0, b_0 \in N$ with $i(R_0) = R$, $i(a_0) = f(B)$ and $i(b_0) = f(\lambda)$. Then the elementarity of i, the $\Pi_n(V_{\kappa+1})$-correctness of N and the fact that V_λ is contained in H_η ensure that $\langle V_\kappa, R_0, a_0, b_0, \varnothing\rangle$ is a structure in $\mathcal{E}_n(\varnothing)$ of rank κ and

$$i \restriction V_\kappa : \langle V_\kappa, R_0, a_0, b_0, \varnothing\rangle \to \langle V_\lambda, R, f(B), f(\lambda), \varnothing\rangle$$

is an elementary embedding. Let $\pi : \langle V_\kappa, R_0\rangle \to \langle M, \in\rangle$ denote the corresponding transitive collapse. Then M is a $\Pi_n(V_{\kappa+1})$-correct set with $V_\kappa \cup \{\kappa\} \subseteq M$, $\pi^{-1} \restriction \kappa = \mathrm{id}_\kappa$ and $\pi(b_0) = \kappa$. If we now define

$$j \;=\; f^{-1} \circ i \circ \pi^{-1} : M \to H_{\lambda'},$$

then j is a non-trivial elementary embedding between transitive structures with $j \restriction \mathrm{crit}(i) = \mathrm{id}_{\mathrm{crit}(i)}$, $j(\mathrm{crit}(i)) = \kappa > \mathrm{crit}(i) = \mathrm{crit}(j)$, $j(\kappa) = \lambda$ and $B \in \mathrm{ran}(j)$. □

In the remainder of this section, we prove that for all natural numbers $n > 0$, exact structural reflection for Π_n-definable classes is strictly stronger than exact structural reflection for Σ_n-definable classes. This will follow from Corollary 5.5 and the next Lemma.

Lemma 5.8. *Let $n > 0$ be a natural number and let κ be weakly parametrically n-exact for some cardinal $\lambda > \kappa$. Then the set of cardinals $\mu < \kappa$ such that $\Sigma_n(V_\mu)$-ESR(μ, κ) holds is stationary in κ.*

Proof. Fix a closed unbounded subset K of κ. By Lemma 5.7, there exists a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, cardinals $\kappa < \lambda < \lambda' \in C^{(n)}$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\mathrm{crit}(j)) = \kappa$, $j(\kappa) = \lambda$ and $K \in \mathrm{ran}(j)$. Then $\mathrm{crit}(j) \in K$. Assume, towards a contradiction, that $\Sigma_n(V_{\mathrm{crit}(j)})$-ESR$(\mathrm{crit}(j), \kappa)$ fails. Then there exists a Σ_n-formula $\varphi(v_0, v_1)$, $z \in V_{\mathrm{crit}(j)}$ and $B \in V_{\kappa+1} \setminus V_\kappa$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of structures of the same type, $\varphi(B, z)$ holds and for all $A \in \mathcal{C}$ of rank $\mathrm{crit}(j)$, there is no elementary embedding of A into B. Since κ is inaccessible and therefore every elementary embedding of a structure of rank less than κ into a structure of rank κ is an element of V_κ, there is a Σ_n-formula with parameters in $V_\kappa \cup \{V_\kappa\}$ that expresses the statement that there exists a structure $B \in \mathcal{C}$ of rank κ with the property that for every $A \in \mathcal{C}$ of rank $\mathrm{crit}(j)$, there is no elementary embedding of A into B. Since M is $\Pi_n(V_{\kappa+1})$-correct and $V_\kappa \cup \{V_\kappa\} \subseteq M$, this yields $B_0 \in \mathcal{C} \cap M$ of rank κ with the property that, in M, for all $A \in V_{\mathrm{crit}(j)+1} \setminus V_{\mathrm{crit}(j)}$ such that $\varphi(A, z)$ holds, there is no elementary embedding of A into B_0. Then, in $H_{\lambda'}$, for all $A \in V_{\kappa+1} \setminus V_\kappa$ such that $\varphi(A, z)$ holds, there is no elementary embedding of A into $j(B_0)$. Since $\kappa < \lambda < \lambda' \in C^{(n)}$, it now follows that $\varphi(B_0, z)$ holds in $H_{\lambda'}$ and the map $j \restriction B_0 : B_0 \to j(B_0)$ is an element of $H_{\lambda'}$. But this yields a contradiction, because $j \restriction B_0$ is an elementary embedding of B_0 into $j(B_0)$ in $V_{\lambda'}$. □

6. The Σ_{n+1}-case

Analogously to the theory developed in the previous section, we now analyse the relationship between the principle Σ_{n+1}-ESR and n-exact cardinals. First, let us consider the following variation of Definition 1.8.

Definition 6.1. *Given cardinals $\kappa < \lambda$, a set $z \in V_\kappa$ and $n < \omega$, the cardinal κ is n-*exact for λ and z *if for every $A \in V_{\lambda+1}$, there exists a cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\kappa$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(z) = z$ and $A \in \mathrm{ran}(j)$.*

Proposition 6.2. *If a cardinal κ is parametrically n-exact for some cardinal λ (see Definition 1.8), then κ is n-exact for λ and all $z \in V_\kappa$.* □

Proposition 6.3. *If κ is n-exact for λ and z, then $\Sigma_{n+1}(\{z\})$-ESR(κ, λ) holds. In particular, if κ is parametrically n-exact for λ, then $\Sigma_{n+1}(V_\kappa)$-ESR(κ, λ) holds.*

Proof. Pick a Σ_{n+1}-formula $\varphi(v_0, v_1)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of structures of the same type and fix $B \in \mathcal{C}$ of rank λ. By our assumptions, there exists a cardinal $\beth_\kappa < \kappa' \in C^{(n)}$, a cardinal $\lambda < \lambda' \in C^{(n+1)}$, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(z) = z$ and $B \in \mathrm{ran}(j)$. Pick $A \in X$ with $j(A) = B$. Then our setup ensures that $\varphi(A, z)$ holds and hence A is a structure in $\mathcal{C}$ of rank κ. Moreover, the map j induces an elementary embedding of A into B. These computations yield the first part of the proposition. The second part follows directly from a combination of the first part and Proposition 6.2. □

For each natural number $n > 0$ and every set z, we let $\mathcal{D}_n(z)$ denote the class of structures $\langle D, E, a, b, c\rangle$ (in the language of set theory with three additional constant symbols) with the property that for some cardinal $\theta \in C^{(n)}$ greater than $\beth_{\mathrm{rnk}(D)}$, there exists an elementary submodel X of H_θ with $V_{\mathrm{rnk}(D)} \cup \{\mathrm{rnk}(D)\} \subseteq X$ and an isomorphism $\pi : \langle D, E\rangle \to \langle X, \in\rangle$ with $\pi(b) = z$ and $\pi(c) = \mathrm{rnk}(D)$. Note that the class $\mathcal{D}_n(z)$ is definable by a Σ_{n+1}-formula with parameter z.

Lemma 6.4. *Let $\kappa < \lambda$ be cardinals, let $z \in V_\kappa$ and let $n > 0$ be a natural number with the property that $\mathrm{ESR}_{\mathcal{D}_n(z)}(\kappa, \lambda)$ holds. Then κ is n-exact for λ and z.*

Proof. Fix $A \in V_{\lambda+1}$, $\lambda' \in C^{(n+1)}$ greater than λ and an elementary submodel Y of $H_{\lambda'}$ of cardinality $\beth_\lambda$ with $V_\lambda \cup \{A, \lambda\} \subseteq Y$. Pick a bijection $f : Y \to V_\lambda$ and let R be the binary relation on V_λ induced by f and $\in$. Then λ' and f^{-1} witness that $\langle V_\lambda, R, f(A), f(z), f(\lambda)\rangle$

is a structure of rank λ in $\mathcal{D}_n(z)$. By our assumptions, there exists a structure $\langle D, E, a, b, c\rangle$ of rank κ in $\mathcal{D}_n(z)$ and an elementary embedding

$$i : \langle D, E, a, b, c\rangle \to \langle V_\lambda, R, f(A), f(z), f(\lambda)\rangle.$$

Pick a cardinal $\kappa' \in C^{(n)}$, an elementary submodel X of $H_{\kappa'}$ and an isomorphism $\pi : \langle D, E\rangle \to \langle X, \in\rangle$ witnessing that $\langle D, E, a, b, c\rangle$ is an element of $\mathcal{D}_n(z)$. Then $\pi(b) = z$, $\pi(c) = \kappa$, $\beth_\kappa < \kappa'$ and $V_\kappa \cup \{\kappa\} \subseteq X$. Define

$$j \;=\; f^{-1} \circ i \circ \pi^{-1} : X \to H_{\lambda'}.$$

Then j is an elementary embedding with $j(\kappa) = \lambda$, $j(z) = z$ and $A = j(\pi(a)) \in \mathrm{ran}(j)$. □

Corollary 6.5. *Let $\kappa < \lambda$ be cardinals, let $z \in V_\kappa$ and let $n > 0$ be a natural number. If κ is weakly $(n+1)$-exact for λ and z, then κ is n-exact for λ and z.*

Proof. By Corollary 5.5, our assumption implies that the principle $\Pi_{n+1}(\{z\})$-ESR(κ, λ) holds and this allows us to use Corollary 5.6 to show that $\Sigma_{n+1}(\{z\})$-ESR(κ, λ) also holds true. Using Lemma 6.4, we can conclude that κ is n-exact for λ and z. □

7. Proofs of the main theorems

We shall now give a proof of Theorems 1.7 and 1.9.

Proof of Theorem 1.7. Fix a cardinal κ and a natural number $n > 0$.

The next two claims, together with Proposition 5.3 and Corollary 5.5, will allow us to conclude that all statements (i)-(v) listed in the theorem are equivalent.

Claim. *If κ is the least regular cardinal with the property that Π^{ic}_n-ESR(κ) holds, then κ is weakly parametrically n-exact for some $\lambda > \kappa$.*

Proof of the Claim. By Lemma 4.12 and the minimality of κ, we know that κ is an inaccessible cardinal with the property that the principle $\Pi_n(V_\kappa)$-ESR(κ, λ) holds for some inaccessible cardinal $\lambda > \kappa$. Assume, towards a contradiction, that κ is not weakly parametrically n-exact for λ. Then there exists $A \in V_{\lambda+1}$ with the property that $j(\mathrm{crit}(j)) \neq \kappa$ holds whenever M is a transitive, $\Pi_n(V_{\kappa+1})$-correct set with $V_\kappa \cup \{\kappa\} \subseteq M$, $\lambda' \in C^{(n-1)}$ is a cardinal greater than λ and $j : M \to H_{\lambda'}$ is an elementary embedding with $j(\kappa) = \lambda$ and $A \in \mathrm{ran}(j)$. An application of Lemma 5.4 now allows us to find a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$, a cardinal λ' with $\lambda < \lambda' \in C^{(n)}$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $A, \kappa \in \mathrm{ran}(j)$. Then $j(\mathrm{crit}(j)) \leq \kappa$ and hence $j(\mathrm{crit}(j)) < \kappa$. Set $\mu = j(\mathrm{crit}(j)) \in M \cap \kappa$ and $\nu = j(\mu)$. We can now apply Lemma 4.11 to conclude that $\Pi_n(V_\mu)$-ESR(μ, ν) holds, contradicting the minimality of κ. □

Claim. *If κ is the least cardinal that is weakly n-exact for some cardinal λ, then κ is regular.*

Proof of the Claim. Assume, towards a contradiction, that κ is singular. Then Proposition 5.3 yields a cardinal $\lambda > \kappa$ with the property that Π_n-ESR(κ, λ) holds. In this situation, we can apply Lemma 5.4 to find a cardinal λ' with $\beth_\lambda < \lambda' \in C^{(n)}$, a transitive, $\Pi_n(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $\kappa \in \mathrm{ran}(j)$. Set $\mu = j(\mathrm{crit}(j))$. Then elementarity implies that μ is regular and, since the fact that $\kappa \in j(\kappa) \cap \mathrm{ran}(j)$ ensures that $\mu \leq \kappa$, we know that $\mu < \kappa$. In this situation, the second part of Lemma 4.11 implies that Π_n-ESR$(\mu, j(\mu))$ holds and this allows us to apply Corollary 5.5 to conclude that μ is weakly n-exact for $j(\mu)$, contradicting the minimality of κ. □

By the first claim above, the cardinal that satisfies (i) of the Theorem is greater than or equal to the cardinal that satisfies (v), which, by Proposition 5.3, is greater than or equal to the cardinal that satisfies (iii). Moreover, the cardinal satisfying (iii) is obviously greater than or equal to the cardinal that satisfies (ii), and an application of Corollary 5.5 then shows that the cardinal satisfying (ii) is greater than or equal to the cardinal that satisfies (iv). Our second claim then shows that the cardinal satisfying (iv) is regular and this allows us to use Corollary 5.5 again to conclude that it is greater than or equal to the cardinal satisfying (i). This shows that all of these cardinals are equal. □

Proof of Theorem 1.9. Fix a cardinal κ and a natural number $n > 0$.

The next two claims, together with Proposition 6.3, will allow us to conclude that all statements (i)-(iv) listed in the theorem are equivalent.

Claim. *If κ is the least cardinal with the property that Σ_{n+1}-ESR(κ) holds, then κ is parametrically n-exact for some $\lambda > \kappa$.*

Proof of the Claim. Fix $D \in V_{\lambda+1}$. Using Lemma 6.4, we find a cardinal $\beth_\kappa < \kappa' \in C^{(n)}$, cardinals $\kappa < \lambda < \lambda' \in C^{(n+1)}$, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to V_{\lambda'}$ with $j(\kappa) = \lambda$ and $D, \kappa \in \mathrm{ran}(j)$. Set $\mu = j(\mathrm{crit}(j)) \leq \kappa \in X$ and $\nu = j(\mu)$. Assume, towards a contradiction, that Σ_{n+1}-ESR(μ, ν) fails. Then we can find a Σ_{n+1}-formula φ with the property that the class $\mathcal{C} = \{A \mid \varphi(A)\}$ consists of structures of the same type and $B \in \mathcal{C}$ of rank ν with the property that for all $A \in \mathcal{C}$ of rank μ, there is no elementary embedding of A into B. Since $\lambda' \in C^{(n+1)}$, this statement also holds in $H_{\lambda'}$ and therefore the elementarity of j allows us to find $B_0 \in X$ of rank μ with the property that $\varphi(j(B_0))$ holds in $H_{\lambda'}$ and for all $A \in V_{\mu+1} \setminus V_\mu$ such that $\varphi(A)$ holds, there is no elementary embedding of A into $j(B_0)$ in $H_{\lambda'}$. But this yields a contradiction, because our setup ensures that $\varphi(B_0)$ holds in V and the fact that $V_\kappa \subseteq X$ implies that j induces an elementary embedding of B_0 into $j(B_0)$ that is an element of $H_{\lambda'}$. In this situation, the minimality of κ implies that $\kappa = \mu$ and $\nu = \lambda$. In particular, we can conclude that κ is parametrically n-exact for λ. □

Since Proposition 6.3 shows that $\Sigma_{n+1}(V_\kappa)$-ESR(κ) holds whenever κ is parametrically n-exact for some cardinal $\lambda > \kappa$ and therefore all statements listed in the theorem imply that Σ_{n+1}-ESR(κ) holds, the above claim allows us to conclude that all of the listed statements are equivalent. □

In the remainder of this section, we show that, for all $n > 0$, exact structural reflection for Σ_{n+1}-definable classes is strictly stronger than exact structural reflection for Π_n-definable classes. In combination with the equivalences provided by Theorems 1.7 and 1.9, the following lemma shows that, in general, the principle $\Pi_n(V_\kappa)$-ESR(κ) does not imply the principle Σ_{n+1}-ESR(κ). This statement should be compared with the results of [Bag12, Section 4], showing that the validity of the principle SR for Π_n-definable classes of structures is equivalent to the validity of this principle for Σ_{n+1}-definable classes.

Lemma 7.1. *Let $n > 0$ be a natural number and let κ be parametrically n-exact for some cardinal $\lambda > \kappa$. Then the set of cardinals $\mu < \kappa$ with the property that $\Pi_n(V_\mu)$-ESR(μ, κ) holds is stationary in κ. In particular, there exists a cardinal $\mu < \kappa$ that is weakly parametrically n-exact for some cardinal $\nu > \mu$.*

Proof. Fix a closed unbounded subset K of κ. Using our assumptions, we can find a cardinal $\kappa < \kappa' \in C^{(n)}$, a cardinal $\lambda < \lambda' \in C^{(n+1)}$, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$, $j(\mathrm{crit}(j)) = \kappa$ and $K \in \mathrm{ran}(j)$. Then $\mathrm{crit}(j)$ is an element of K.

Now, assume towards a contradiction, that there is a Π_n-formula $\varphi(v_0, v_1)$ and $z \in V_{\mathrm{crit}(j)}$ such that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of structures of the same type and there exists $B \in \mathcal{C}$ of rank κ such that for all $A \in \mathcal{C}$ of rank $\mathrm{crit}(j)$, there is no elementary embedding from A into B. Since $\kappa < \kappa' \in C^{(n)}$, these statements hold in $H_{\kappa'}$ and we can find $B \in \mathcal{C} \cap X$ of rank κ with the property that for every $A \in \mathcal{C}$ of rank $\mathrm{crit}(j)$, there is no elementary embedding of A into B. In this situation, since $\lambda < \lambda' \in C^{(n+1)}$, elementarity implies that $j(B)$ is an element of $\mathcal{C}$ and for every $A \in \mathcal{C}$ of rank κ, there is no elementary embedding of A into $j(B)$. But this yields a contradiction, because the fact that V_κ is a subset of X implies that j induces an elementary embedding of B into $j(B)$.

The above computations yield the first part of the lemma. The second part follows directly from a combination of the first part with Theorem 1.7. □

We end this section by proving the following parametrical version of Corollary 6.5:

Proposition 7.2. *Given a natural number $n > 0$, if a cardinal κ is weakly parametrically $(n+1)$-exact for some cardinal λ, then κ is also parametrically n-exact for λ.*

Proof. Fix $A \in V_{\lambda+1}$, a cardinal $\lambda < \lambda' \in C^{(n+1)}$, an elementary submodel Y of $H_{\lambda'}$ of cardinality λ with $V_\lambda \cup \{A, \lambda\} \subseteq Y$ and a bijection $b : \lambda \to Y$ with $b(0) = \lambda$ and $b(\omega \cdot (1 + \gamma)) = \gamma$ for all $\gamma < \lambda$. Set $R = \{\langle \alpha, \beta\rangle \mid b(\alpha) \in b(\beta)\} \in V_{\lambda+1}$. By Lemma 5.7, there exists a transitive, $\Pi_{n+1}(V_{\kappa+1})$-correct set M with $V_\kappa \cup \{\kappa\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ with $j(\mathrm{crit}(j)) = \kappa$, $j(\kappa) = \lambda$ and $A, R \in \mathrm{ran}(j)$. Pick $A_0, R_0 \in M$ with $j(A_0) = A$ and $j(R_0) = R$. Now, note that, since $n > 0$, the statement "*There exists a cardinal η with $\lambda < \eta \in C^{(n)}$ and an elementary embedding $k : \langle \lambda, R\rangle \to \langle H_\eta, \in\rangle$ with $V_\lambda \cup \{A, \lambda\} \subseteq \mathrm{ran}(k)$, $k(0) = \lambda$ and $k(\omega \cdot (1+\gamma)) = \gamma$ for all $\gamma < \lambda$*" can be expressed by Σ_{n+1}-formula with parameters λ and A, and therefore this statement also holds in $H_{\lambda'}$. Using the elementarity of j and the Π_{n+1}-correctness of M, we can find a cardinal κ' with $\kappa < \kappa' \in C^{(n)}$ and an elementary embedding $i : \langle \kappa, R_0\rangle \to \langle H_{\kappa'}, \in\rangle$ with $V_\kappa \cup \{A_0, \kappa\} \subseteq \mathrm{ran}(i)$, $i(0) = \kappa$ and $i(\omega \cdot (1 + \alpha)) = \alpha$ for all $\alpha < \kappa$. Set $X = \mathrm{ran}(i)$ and

$$j_0 \;=\; b \circ j \circ i^{-1} : X \to H_{\lambda'}.$$

Then X is an elementary submodel of $H_{\kappa'}$ with $V_\kappa \cup \{A_0, \kappa\} \subseteq X$ and, since $i(\mathrm{crit}(j)) = \mathrm{crit}(j)$, $b(\kappa) = \kappa$ and $j \restriction \kappa : \langle \kappa, R_0\rangle \to \langle \lambda, R\rangle$ is an elementary embedding, the map j_0 is an elementary embedding with $j_0 \restriction \mathrm{crit}(j) = \mathrm{id}_{\mathrm{crit}(j)}$, $j_0(\mathrm{crit}(j)) = \kappa$, $j_0(\kappa) = \lambda$ and $A \in \mathrm{ran}(j_0)$. □

8. The strength of exact cardinals

In this section, we measure the strength of the principles of exact structural reflection introduced above by positioning exact and weakly exact cardinals in the hierarchy of large cardinals. We start by deriving lower bounds for their consistency strength by showing that the existence of such cardinals implies the existence of many *almost huge* cardinals below them.

Recall that a cardinal κ is *almost huge* if there exists a transitive class M and a non-trivial elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and ${}^{<j(\kappa)}M \subseteq M$. We then say that a cardinal κ is almost huge with *target* λ if there exists an embedding j witnessing the hugeness of κ with $j(\kappa) = \lambda$.

The following standard argument will allow us to prove these implications:

Lemma 8.1. *Let $\kappa < \lambda$ be cardinals with the property that there exists a non-trivial elementary embedding $j : V_\kappa \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$. Then $\mathrm{crit}(j)$ is almost huge with target κ.*

Proof. Set $\mu = \mathrm{crit}(j)$. Given $\mu \leq \gamma < \kappa$, define

$$U_\gamma \;=\; \{A \subseteq \mathcal{P}_\mu(\gamma) \mid j[\gamma] \in j(A)\}.$$

Then it is easy to see that for every $\mu \leq \gamma < \kappa$, the collection U_γ is a normal ultrafilter over $\mathcal{P}_\mu(\gamma)$. Moreover, this definition directly ensures that

$$U_\gamma \;=\; \{\{a \cap \gamma \mid a \in A\} \mid A \in U_\delta\}$$

holds for all $\mu \leq \gamma \leq \delta < \kappa$.

Now, given $\mu \leq \gamma < \kappa$, we let $i_\gamma : V \to M_\gamma$ denote the ultrapower embedding induced by U_γ. In addition, for all $\mu \leq \gamma \leq \delta < \kappa$, we let $k_{\gamma,\delta} : M_\gamma \to M_\delta$ denote the canonical embedding satisfying $i_\delta = k_{\gamma,\delta} \circ i_\gamma$ (see [Kan03, p. 333]).

Claim. *If $\mu \leq \gamma < \kappa$ and $\gamma \leq \alpha < i_\gamma(\mu)$, then there exists $\gamma \leq \delta < \kappa$ with $k_{\gamma,\delta}(\alpha) = \delta$.*

Proof of the Claim. Pick a function $f : \mathcal{P}_\mu(\gamma) \to \mu$ with $[f]_{U_\gamma} = \alpha$ and define

$$\delta \;=\; j(f)(j[\gamma]) \;<\; \kappa.$$

Since normality allows us to conclude that $[a \mapsto \mathrm{ot}(a)]_{U_\gamma} = \gamma$, we know that

$$\{a \in \mathcal{P}_\mu(\gamma) \mid \mathrm{ot}(a) \leq f(a)\} \;\in\; U_\gamma$$

and hence

$$\gamma \;=\; \mathrm{ot}(j[\gamma]) \;\leq\; j(f)(j[\gamma]) \;=\; \delta.$$

Moreover, we have

$$j(f)(j(\gamma) \cap j[\delta]) \;=\; j(f)(j[\gamma]) \;=\; \delta \;=\; \mathrm{ot}(j[\delta])$$

and hence $\{a \in \mathcal{P}_\mu(\gamma) \mid f(a \cap \gamma) = \mathrm{ot}(a)\} \in U_\delta$. But then

$$k_{\gamma,\delta}(\alpha) \;=\; k_{\gamma,\delta}([f]_{U_\gamma}) \;=\; [a \mapsto f(a \cap \gamma)]_{U_\delta} \;=\; [a \mapsto \mathrm{ot}(a)]_{U_\delta} \;=\; \delta. \qquad \square$$

By [Kan03, Theorem 24.11], this shows that μ is almost huge with target κ. $\square$

Corollary 8.2. *Let $\kappa < \lambda$ be cardinals with the property that κ is either parametrically 0-exact for λ or weakly parametrically 1-exact for λ. Then the set of cardinals $\mu < \kappa$ with the property that μ is almost huge with target κ is stationary in κ.*

Proof. Let C be a closed unbounded subset of κ. By definition, both of the listed assumption imply the existence of a non-trivial elementary embedding $j : V_\kappa \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$ and $C \in \mathrm{ran}(j)$. Then $\mathrm{crit}(j)$ is an element of C and Lemma 8.1 shows that $\mathrm{crit}(j)$ is almost huge with target κ. $\square$

Corollary 8.3. *Let κ be a cardinal that is parametrically 0-exact for some cardinal $\lambda > \kappa$. Then κ is almost huge with target λ.*

Proof. By definition, there exist a cardinal $\kappa' > \kappa$, a cardinal $\lambda < \lambda' \in C^{(1)}$, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$ and $j(\mathrm{crit}(j)) = \kappa$. Then Lemma 8.1 implies that $\mathrm{crit}(j)$ is almost huge with target κ. Since the system of filters witnessing this statement is contained in $H_{\kappa'}$, the model X also contains such a system. But then the elementarity of j implies that, in $H_{\lambda'}$, there is a system of ultrafilters witnessing that κ is almost huge with target λ. Since $\lambda < \lambda' \in C^{(1)}$, this statement also holds in V. $\square$

Recall that a cardinal κ is *huge* if there exists a transitive class M and a non-trivial elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and ${}^{j(\kappa)}M \subseteq M$. We then say that a cardinal κ is huge with *target* λ if there exists an embedding j witnessing the hugeness of κ with $j(\kappa) = \lambda$. It is well-known that κ is huge with target λ if and only if there exists a κ-complete normal ultrafilter $\mathcal{U}$ over $\mathcal{P}(\lambda)$ such that $\{x \in \mathcal{P}(\lambda) \mid \mathrm{otp}\,(x) = \kappa\} \in \mathcal{U}$ (see [Kan03, Theorem 24.8]).

Proposition 8.4. *If κ is huge with target λ, then κ is weakly parametrically 1-exact for λ.*

Proof. Let M be an inner model with ${}^\lambda M \subseteq M$ and let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) = \lambda$. Fix $A \in V_{\lambda+1}$ and let N be an elementary submodel of H_{λ^+} of cardinality λ with $V_\lambda \cup \{A, \lambda\} \subseteq N$. We then have $N \in M$ and, since $H_{\lambda^+} = H^M_{\lambda^+}$, Σ_1-absoluteness implies that N is $\Pi_1(V_{\lambda+1})$-correct in M. Set $j_0 = j \restriction N : N \to H^M_{j(\lambda)^+}$. Then j_0 is an elementary embedding that is an element of M. Thus, in M, there exists a transitive, $\Pi_1(V_{j(\kappa)+1})$-correct set K with $V_{j(\kappa)} \cup \{j(\kappa)\} \subseteq K$ (namely N) and an elementary embedding $k : K \to H_{j(\lambda^+)}$ with $k(\mathrm{crit}(k)) = j(\kappa)$, $k(j(\kappa)) = j(\lambda)$ and $j(A) \in \mathrm{ran}(k)$ (namely j_0) Hence, the elementarity of j implies that, in V, there exists a transitive, $\Pi_1(V_{\kappa+1})$-correct set K with $V_\kappa \cup \{\kappa\} \subseteq K$ and an elementary embedding $k : K \to H_{\lambda^+}$ with $k(\mathrm{crit}(k)) = \kappa$, $k(\kappa) = \lambda$ and $A \in \mathrm{ran}(k)$. $\square$

This result also allows us to show that the consistency strength of huge cardinals is strictly larger than the consistency strength of weakly 1-exact cardinals.

Corollary 8.5. *If κ is huge with target λ, then there is an inaccessible cardinal $\rho < \kappa$ such that $\Pi_1(V_\rho)$-ESR(ρ, κ) holds in V_λ.*

Proof. Let $j : V \to N$ be an elementary embedding with critical point κ such that $j(\kappa) = \lambda$ and N is closed under λ-sequences. Note that $H_{\lambda^+} \in V^N_{j(\lambda)}$. Fix a Π_1-formula $\varphi(v_0, v_1)$ and $z \in V_\kappa$ with the property that, in $V^N_{j(\lambda)}$, the class $\{A \mid \varphi(A, z)\}$ consists of structures of the same type. Pick a structure B of rank λ with the property that $\varphi(B, z)$ holds in $V^N_{j(\lambda)}$. Since $H_{\lambda^+} \subseteq V^N_{j(\lambda)}$, Σ_1-absoluteness implies that $\varphi(B, z)$ also holds in V. By our assumptions, we can now apply Proposition 8.4 to find a structure A of rank κ with the property that $\varphi(A, z)$ holds in V and an elementary embedding i of A into B. But then A is contained in $V^N_{j(\lambda)}$, $\varphi(A, z)$ holds in $V^N_{j(\lambda)}$ and the map i is an element of $V^N_{j(\lambda)}$.

These computations show that the principle $\Pi_1(V_\kappa)$-ESR(κ, λ) holds in $V^N_{j(\lambda)}$. Using the elementarity of j, we can now conclude that, in V_λ, there is an inaccessible cardinal $\rho < \kappa$ with the property that $\Pi_1(V_\rho)$-ESR(ρ, κ) holds . □

We now show that the implication given by Proposition 8.4 is optimal. Note that, by Corollary 6.5, the next result also shows that the least huge cardinal κ is not weakly 2-exact for some cardinal $\lambda > \kappa$.

Proposition 8.6. *If κ is the least huge cardinal, then κ is not 1-exact for any cardinal $\lambda > \kappa$.*

Proof. Assume, towards a contradiction, that there exists a cardinal $\kappa < \kappa' \in C^{(1)}$, cardinals $\lambda < \lambda' \in C^{(2)}$, an elementary submodel X of $H_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\kappa) = \lambda$. Note that the statement "*There exists a huge cardinal smaller than* λ" can be formulated by a Σ_2-formula with parameter λ and, since $\lambda < \lambda' \in C^{(2)}$, this statement holds in $H_{\lambda'}$. But then the elementarity of j and the fact that $\kappa' \in C^{(1)}$ allow us to conclude that there exists a huge cardinal smaller than κ, a contradiction. □

The next proposition gives a consistency upper bound for the existence of a parametrically exact cardinal. Recall that an *$I1$-embedding* is a non-trivial elementary embedding $j : V_{\delta+1} \to V_{\delta+1}$ for some limit ordinal δ. Also, an *$I2$-embedding* is an elementary embedding $j : V \to M$ for some transitive class M such that $V_\delta \subseteq M$ for some limit ordinal $\delta > \mathrm{crit}(j)$ satisfying $j(\delta) = \delta$. Finally, an *$I3$-embedding* is a non-trivial elementary embedding $j : V_\delta \to V_\delta$, for some limit ordinal δ (see [Kan03, §24]). Note that, if $j : V_\delta \to V_\delta$ is an $I3$-embedding with critical point κ, then V_δ is a model of ZFC and the sequence $\langle j^m(\kappa) \mid m < \omega\rangle$ is cofinal in δ.

Proposition 8.7. *Assume that κ is the critical point of an $I3$-embedding $j : V_\delta \to V_\delta$. If $l, m, n < \omega$, then, in V_δ, the cardinal $j^l(\kappa)$ is parametrically n-exact for $j^{l+m+1}(\kappa)$.*

Proof. Given $0 < m < \omega$, set $\kappa_m = j^m(\kappa)$. Then, in V_δ, every κ_m is inaccessible and belongs to $C^{(n)}$, for all $n < \omega$. Pick $0 < m < \omega$, set $\lambda = \kappa_m$ and fix $A \in V_{\lambda+1}$. Then, in V_δ, the map $j^m \restriction H_{\kappa_{m+1}} : H_{\kappa_{m+1}} \to H_{\kappa_{2m+1}}$ witnesses that there exists an elementary embedding $i : H_{j^m(\kappa_1)} \to H_{j^m(\kappa_{m+1})}$ with $j^m(A) \in \mathrm{ran}(i)$, $i(j^m(\kappa)) = j^m(\lambda)$ and $i(\mathrm{crit}(i)) = j^m(\kappa)$. But then the elementarity of $j^m : V_\delta \to V_\delta$ implies that, in V_δ, there exists an elementary embedding $i : H_{\kappa_1} \to H_{\kappa_{m+1}}$ with $A \in \mathrm{ran}(i)$, $i(\kappa) = \lambda$ and $i(\mathrm{crit}(i)) = \kappa$. Since $\kappa < \kappa_1 \in (C^{(n)})^{V_\delta}$ and $\lambda < \kappa_{m+1} \in (C^{(n)})^{V_\delta}$ for all $n < \omega$, these computations show that, in V_δ, the cardinal κ is parametrically n-exact for λ for all $n < \omega$. By elementarity of the iterated embedding j^l, this yields the statement of the proposition. □

In the following, we will derive a much lower upper bound for the consistency strength of the existence of a cardinal κ that is weakly parametrically n-exact for some cardinal λ for all $n < \omega$.

Definition 8.8. *Given a natural number $n > 0$, a cardinal κ is* n-superstrong *if there exists a transitive class M and an elementary embedding $j : V \to M$ with* $\mathrm{crit}(j) = \kappa$ *and $V_{j^n(\kappa)} \subseteq M$. If, moreover, ${}^{j(\kappa)}V_{j^n(\kappa)} \subseteq M$, then we say that κ is* hugely n-superstrong.

Notice that, given an elementary embedding $j : V \to M$ and a natural number $n > 1$, the embedding j witnesses that κ is hugely n-superstrong if and only if it witnesses that κ is n-superstrong and $\mathrm{cof}(j^n(\kappa)) > j(\kappa)$. Also note that every huge cardinal is hugely 1-superstrong and, for $n > 1$, every almost n-huge cardinal[6] is hugely n-superstrong.

Proposition 8.9. *If κ is a hugely 2-superstrong cardinal, then there exists an inaccessible cardinal $\lambda > \kappa$ and a cardinal $\rho > \lambda$ such that V_ρ is a model of* ZFC and, in V_ρ, the cardinal κ is weakly parametrically n-exact for λ for all natural numbers $n > 0$.

[6]Recall that κ is *almost n-huge* if there exists an elementary embedding $j : V \to M$, with $\mathrm{crit}(j) = \kappa$, and with M transitive and closed under $<j^n(\kappa)$-sequences.

Proof. Let $j : V \to M$ with be an elementary embedding with M transitive, $\mathrm{crit}(j) = \kappa$, $V_{j^2(\kappa)} \subseteq M$, and ${}^{j(\kappa)}V_{j^2(\kappa)} \subseteq M$. Set $\lambda = j(\kappa)$ and $\rho = j^2(\kappa)$. Then our assumptions ensure that λ is an inaccessible cardinal and ρ is a cardinal with the property that V_ρ is a model of ZFC. Notice that, since $V_\kappa \preceq V_{j(\kappa)} = V_\lambda$, elementarity implies that $V_\lambda = V_{j(\kappa)} \preceq V_{j(\lambda)} = V_\rho$ and therefore also $V_\rho = V_{j(\lambda)} \preceq V^M_{j(\rho)}$.

Now, fix $A \in V_{\lambda+1}$ and, in M, pick an elementary submodel X of V_ρ of cardinality λ with $V_\lambda \cup \{A, \lambda\} \subseteq X$. Let $\pi : X \to N$ denote the corresponding transitive collapse. Then $V_\lambda \cup \{A, \lambda\} \subseteq N$. Moreover, since $\pi \restriction (V_{\lambda+1} \cap X) = \mathrm{id}_{V_{\lambda+1}\cap X}$, it follows that N is $\Pi_n(V_{\lambda+1})$-correct in V_ρ for all $n < \omega$. But since $V_\rho \preceq V^M_{j(\rho)}$, the set N is also $\Pi_n(V_{\lambda+1})$-correct in $V^M_{j(\rho)}$ for all $n < \omega$. Finally, pick a bijection $b : X \to \lambda$ with $b(\lambda) = 0$ and $b(\gamma) = \omega \cdot (1 + \gamma)$ for all $\gamma < \lambda$. Set

$$E \;=\; \{\langle b(x_0), b(x_1)\rangle \mid x_0, x_1 \in X,\; x_0 \in x_1\} \;\in\; M.$$

Then the map $j \restriction \lambda : \langle \lambda, E\rangle \to \langle \rho, j(E)\rangle$ is an elementary embedding of $\mathcal{L}_\in$-structures and, since it is a subset of V_ρ of cardinality λ, the closure properties of M ensure that this map is an element of M.

We now have that, in M, the map

$$i \;=\; j(b^{-1}) \circ (j \restriction \lambda) \circ b \circ \pi^{-1} : N \longrightarrow V^M_{j(\rho)}$$

is an elementary embedding with $i \restriction \kappa = \mathrm{id}_\kappa$, $i(\kappa) = j(\kappa) = \lambda$, $i(\lambda) = j(\lambda) = \rho$ and $j(A) \in \mathrm{ran}(i)$. Finally, we know that $N \in V_\rho$, because N is a subset of V_ρ of cardinality λ in M and ρ is inaccessible in M. Since the closure properties of M imply that ρ is a limit cardinal of cofinality greater than λ in V, we can find an M-cardinal $\rho < \eta < j(\rho)$ with $j(N) \subseteq H^M_\eta \prec V^M_{j(\rho)}$.

Fix $0 < n < \omega$. The above computations now show that, in $V^M_{j(\rho)}$, there exists a cardinal $\rho < \eta \in C^{(n)}$, a transitive $\Pi_n(V_{\lambda+1})$-correct set N with $V_\lambda \cup \{\lambda\} \subseteq N$ and a non-trivial elementary embedding $i : N \to H_\eta$ with $i(\mathrm{crit}(i)) = \lambda$, $i(\lambda) = \rho$ and $j(A) \in \mathrm{ran}(i)$. In this situation, the elementarity of j implies that, in V_ρ, there exists a cardinal $\lambda < \lambda' \in C^{(n)}$, a transitive $\Pi_n(V_{\kappa+1})$-correct set N_0 with $V_\kappa \cup \{\kappa\} \subseteq N_0$ and a non-trivial elementary embedding $i_0 : N_0 \to H_{\lambda'}$ with $i_0(\mathrm{crit}(i_0)) = \kappa$, $i_0(\kappa) = \lambda$ and $A \in \mathrm{ran}(i_0)$. □

9. Beyond huge reflection

In this section, we introduce a generalization of the principle $\mathrm{ESR}_\mathcal{C}(\kappa, \lambda)$ to sequences of cardinals in order to obtain principles of structural reflection that imply the existence of even stronger large cardinals. The following definition is motivated by the formulation of *Chang's Conjecture*.

Definition 9.1. *Let $0 < \eta \leq \omega$ and let $\mathcal{L}$ be a first-order language containing unary predicate symbols $\vec{P} = \langle \dot{P}_i \mid i < \eta\rangle$.*

(i) *Given a sequence $\vec{\mu} = \langle \mu_i \mid i < \eta\rangle$ of cardinals with supremum μ, an $\mathcal{L}$-structure A has* type $\vec{\mu}$ (with respect to $\vec{P}$) *if the universe of A has rank μ and $\mathrm{rnk}(\dot{P}_i^A) = \mu_i$ for all $i < \eta$.*

(ii) *Given a class $\mathcal{C}$ of $\mathcal{L}$-structures and a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta\rangle$ of cardinals, we let $\mathrm{ESR}_\mathcal{C}(\vec{\lambda})$ denote the statement that for every structure B in $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < \eta\rangle$, there exists an elementary embedding of a structure A in $\mathcal{C}$ of type $\langle \lambda_i \mid i < \eta\rangle$ into B.*

(iii) *Given a definability class Γ and a class P, we let $\Gamma(P)$-$\mathrm{ESR}(\vec{\lambda})$ denote the statement that $\mathrm{ESR}_\mathcal{C}(\vec{\lambda})$ holds for every class $\mathcal{C}$ of structures of the same type that is Γ-definable with parameters in P.*

In order to determine the large cardinal strength of the above principles, we consider the following *sequential* versions of n-exact and weakly n-exact cardinals:

Definition 9.2. *Let $0 < \eta \leq \omega$ and let $\vec{\lambda} = \langle \lambda_m \mid m < \eta\rangle$ be a strictly increasing sequence of cardinals with supremum λ.*

(i) *Given $n < \omega$, a cardinal $\kappa < \lambda_0$ is* n-exact for $\vec{\lambda}$ *if for every $A \in V_{\lambda+1}$, there exists a cardinal ρ,*[7] *a cardinal $\kappa' \in C^{(n)}$ greater than $\beth_\rho$, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an elementary submodel X of $H_{\kappa'}$ with $V_\rho \cup \{\rho\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $A \in \mathrm{ran}(j)$, $j(\rho) = \lambda$, $j(\kappa) = \lambda_0$ and $j(\lambda_{m-1}) = \lambda_m$ for all $0 < m < \eta$. If we further require that $j(\mathrm{crit}(j)) = \kappa$, then we say that κ* is parametrically n-exact for $\vec{\lambda}$.

(ii) *Given $0 < n < \omega$, a cardinal $\kappa < \lambda_0$ is* weakly n-exact for $\vec{\lambda}$ *if for every $A \in V_{\lambda+1}$, there exists a cardinal ρ, a transitive, $\Pi_n(V_{\rho+1})$-correct set M with $V_\rho \cup \{\rho\} \subseteq M$, a cardinal $\lambda' \in C^{(n-1)}$ greater than $\beth_\lambda$ and an elementary embedding $j : M \to H_{\lambda'}$ with $A \in \mathrm{ran}(j)$, $j(\rho) = \lambda$, $j(\kappa) = \lambda_0$ and $j(\lambda_{m-1}) = \lambda_m$ for all $0 < m < \eta$. If we further require that $j(\mathrm{crit}(j)) = \kappa$, then we say that κ is* weakly parametrically n-exact for $\vec{\lambda}$.

Note that, if $n > 0$ is a natural number and $\langle \lambda_i \mid i \leq n \rangle$ is a strictly increasing sequence of cardinals such that λ_0 is weakly 1-exact for $\langle \lambda_{i+1} \mid i < n \rangle$, then the instance

$$(\lambda_n, \ldots, \lambda_1, \lambda_0) \twoheadrightarrow (\lambda_{n-1}, \ldots, \lambda_0, {<}\lambda_0)$$

of Chang's Conjecture (see, for example, [For10, p. 914]) holds true. Analogous implications hold true for sequences of cardinals of length ω.

We show next that the large cardinal notions introduced above are located in the uppermost regions of the large cardinal hierarchy. Recall that, given a natural number $n > 0$, a cardinal κ is *n-huge* if there exists a transitive class M and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and ${}^{j^n(\kappa)}M \subseteq M$.

Proposition 9.3. *Let $n > 0$ be a natural number.*

(i) *If κ is an n-huge cardinal, witnessed by an elementary embedding $j : V \to M$, then κ is weakly parametrically 1-exact for the sequence $\langle j^{m+1}(\kappa) \mid m < n \rangle$.*

(ii) *If κ is a cardinal and $\vec{\lambda} = \langle \lambda_m \mid m \leq n \rangle$ is a sequence of cardinals such that κ is either weakly 1-exact for $\vec{\lambda}$ or 0-exact for $\vec{\lambda}$, then some cardinal less than κ is n-huge.*

Proof. (i) Set $\rho = j^{n-1}(\kappa)$ and $\lambda = j^n(\kappa)$. Fix $A \in V_{\lambda+1}$. Let N be an elementary submodel of H_{λ^+} of size λ with $V_\lambda \cup \{\lambda, A\} \subseteq N$. Then N is an element of M and N is $\Pi_1(V_{\lambda+1})$-correct in M. Since the map $j \restriction N : N \to H^M_{j(\lambda)^+}$ is also contained in M, elementarity allows us to conclude that, in V, there exists a transitive, $\Pi_1(V_{\rho+1})$-correct set K with $V_\rho \cup \{\rho\} \subseteq K$ and an elementary embedding $k : K \to H_{\lambda^+}$ with $A \in \mathrm{ran}(k)$, $k(\rho) = \lambda$, $k(\mathrm{crit}(k)) = \kappa$ and $k(j^m(\kappa)) = j^{m+1}(\kappa)$ for all $m < n$.

(ii) Set $\lambda = \lambda_n$ and $\rho = \lambda_{n-1}$. Both of our assumptions then yield a cardinal $\lambda' > \lambda$, a set X with $V_\rho \cup \{\rho\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $\kappa \in \mathrm{ran}(j)$, $j(\kappa) = \lambda_0$ and $j(\lambda_{m-1}) = \lambda_m$ for all $0 < m \leq n$. Set $\mu = \mathrm{crit}(j) < \kappa$. An easy induction then shows that $j^{m+2}(\mu) \leq \lambda_m$ holds for all $m \leq n$. In particular, we have $j^n(\mu) < \rho$ and therefore $i = j \restriction V_\rho : V_\rho \to V_\lambda$ is an elementary embedding with $\mathrm{crit}(i) = \mu$ and $i^n(\mu) < \rho$. Using results of Kanamori (see [Kan03, Theorem 24.8]), we can now conclude that μ is n-huge. □

Proposition 9.4. *Let $\vec{\lambda} = \langle \lambda_m \mid m < \omega \rangle$ be a strictly increasing sequence of cardinals with supremum λ and let $\kappa < \lambda_0$ be a cardinal.*

(i) *If κ is either weakly 1-exact for $\vec{\lambda}$ or 0-exact for $\vec{\lambda}$, then there exists an $I3$-embedding $j : V_\lambda \to V_\lambda$.*

(ii) *If κ is either weakly parametrically 1-exact for $\vec{\lambda}$ or parametrically 0-exact for $\vec{\lambda}$, then the set of critical points of $I3$-embeddings is stationary in κ.*

Proof. (i) Both of our assumptions ensure that there exists a cardinal $\lambda' > \lambda$, a set X with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $j(\lambda) = \lambda$ and $j(\lambda_{m-1}) = \lambda_m$ for all $0 < m < \omega$. Then $j \restriction V_\lambda$ is an $I3$-embedding.

[7] Note that, in both parts of this definition, the listed requirements ensure that there is a unique cardinal ρ with these properties. If $\eta = 1$, then $\lambda = \lambda_0$ and $\rho = \kappa$. Next, if $1 < \eta < \omega$, then $\lambda = \lambda_{\eta-1}$ and $\rho = \lambda_{\eta-2}$. Finally, if $\eta = \omega$, then $\lambda = \rho$.

(ii) Fix a closed unbounded subset C of κ. By our assumptions, there exists a cardinal $\lambda' > \lambda$, a set X with $V_\lambda \cup \{\lambda\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $C \in \mathrm{ran}(j)$, $j(\mathrm{crit}(j)) = \kappa$, $j(\lambda) = \lambda$ and $j(\lambda_{m-1}) = \lambda_m$ for all $0 < m < \omega$. Then $\mathrm{crit}(j) \in C$ and the map $j \restriction V_\lambda : V_\lambda \to V_\lambda$ is an $I3$-embedding with critical point $\mathrm{crit}(j)$. □

Proposition 9.5. *If κ is the critical point of an I1-embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ and $k > 0$ is a natural number, then κ is weakly parametrically 1-exact for $\langle j^{k(m+1)}(\kappa) \mid m < \omega\rangle$.*

Proof. By standard coding arguments, our assumptions yields an elementary embedding $i : H_{\lambda^+} \to H_{\lambda^+}$ with $i \restriction V_\lambda = j \restriction V_\lambda$ and therefore $i \restriction V_\kappa = \mathrm{id}_{V_\kappa}$. Fix $A \in V_{\lambda+1}$ and pick an elementary submodel N of H_{λ^+} of cardinality λ with $\lambda \cup \{A, \lambda\} \subseteq N$. Then Σ_1-absoluteness implies that N is $\Pi_1(V_{\lambda+1})$-correct. In this situation, the set N and the map $i^k \restriction N : N \to i^k(N)$ witness that, in H_{λ^+}, there exists a transitive, $\Pi_1(V_{\lambda+1})$-correct set M with $V_\lambda \cup \{\lambda\} \subseteq M$ and an elementary embedding $l : M \to i^k(N)$ with $i^k(A) \in \mathrm{ran}(l)$, $l(\mathrm{crit}(l)) = i^k(\kappa)$, $l(i^k(\kappa)) = i^k(j^k(\kappa))$ and $l(i^k(j^{km}(\kappa))) = i^k(j^{k(m+1)}(\kappa))$ for all $0 < m < \omega$. Using the elementarity of $i^k : V_{\lambda+1} \to V_{\lambda+1}$, Σ_1-absoluteness and the fact that N is an elementary submodel of H_{λ^+}, we can now conclude that there exists a transitive, $\Pi_1(V_{\lambda+1})$-correct set M with $V_\lambda \cup \{\lambda\} \subseteq M$ and an elementary embedding $l : M \to H_{\lambda^+}$ with $A \in \mathrm{ran}(l)$, $l(\mathrm{crit}(l)) = \kappa$, $l(\kappa) = j^k(\kappa)$ and $l(j^{km}(\kappa)) = j^{k(m+1)}(\kappa)$ for all $0 < m < \omega$. □

In the remainder of this section, we show how the validity of the principle $\mathrm{ESR}(\vec{\lambda})$ is connected to the existence of cardinals that are exact or weakly exact for certain sequences of cardinals.

Lemma 9.6. *The following statements are equivalent for every natural number $n > 0$, every ordinal $0 < \eta \leq \omega$, every strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta\rangle$ of uncountable cardinals with supremum λ, and all $z \in V_{\lambda_0}$:*

(i) $\Pi_n(\{z\})$-$\mathrm{ESR}_{\mathcal{C}}(\vec{\lambda})$.

(ii) *For every $A \in V_{\lambda+1}$, there exists a cardinal ρ, a transitive, $\Pi_n(V_{\rho+1})$-correct set M with $V_\rho \cup \{\rho\} \subseteq M$, a cardinal $\lambda < \lambda' \in C^{(n-1)}$ and an elementary embedding $j : M \to H_{\lambda'}$ such that $A \in \mathrm{ran}(j)$, $j(z) = z$, $j(\rho) = \lambda$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$.*

(iii) *For all cardinals $\lambda < \lambda' \in C^{(n)}$ and every $A \in V_{\lambda+1}$, there exists a cardinal ρ, a transitive, $\Pi_n(V_{\rho+1})$-correct set M with $V_\rho \cup \{\rho\} \subseteq M$ and an elementary embedding $j : M \to H_{\lambda'}$ such that $A \in \mathrm{ran}(j)$, $j(z) = z$, $j(\rho) = \lambda$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$.*

Proof. First, assume that (ii) holds. Fix a Π_n-formula $\varphi(v_0, v_1)$ with the property that $\mathcal{C} = \{A \mid \varphi(A, z)\}$ is a suitable class of structures and pick a structure B in $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < \eta\rangle$. Since $B \in V_{\lambda+1}$, we can find a cardinal ρ, a transitive, $\Pi_n(V_{\rho+1})$-correct set M with $V_\lambda \cup \{\lambda\} \subseteq M$, a cardinal $\lambda < \lambda' \in C^{(n-1)}$ and an elementary embedding $j : M \to H_{\lambda'}$ such that $A, \vec{\lambda} \in \mathrm{ran}(j)$, $j(z) = z$, $j(\rho) = \lambda$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$. Elementarity then implies that ρ is the supremum of the sequence $\langle \lambda_i \mid i < \eta\rangle$. Moreover, the fact that $\lambda' \in C^{(n-1)}$ implies that Π_n-statements are downwards absolute from V to $H_{\lambda'}$, and this allows us to conclude that $\varphi(B, z)$ holds in $H_{\lambda'}$. Pick $A \in M \cap V_{\rho+1}$ with $j(A) = B$. In this situation, the elementarity of j and the fact that $j(\lambda_i) = \lambda_{i+1}$ holds for all $i < \eta$ cause $\varphi(A, z)$ to hold in M and, by the $\Pi_n(V_{\rho+1})$-correctness of M, this shows that A is a structure of type $\langle \lambda_i \mid i < \eta\rangle$ in $\mathcal{C}$. Finally, since A has rank ρ and V_ρ is a subset of M, the map j induces an elementary embedding of A into B. This shows that (i) holds in this case.

Next, assume that (i) holds and we shall prove (iii). So fix a cardinal $\lambda < \lambda' \in C^{(n)}$ and $A \in V_{\lambda+1}$. Define $\mathcal{L}$ to be the first-order language extending $\mathcal{L}_\in$ by predicate symbols $\vec{P} = \langle \dot{P}_i \mid i < \eta\rangle$ and constant symbols $\dot{A}$, $\dot{\lambda}$, $\dot{z}$ and $\langle \dot{\lambda}_i \mid i < \eta\rangle$. Define $\mathcal{C}$ to be the class of all $\mathcal{L}$-structures $\langle D, E, \vec{P}, a, b, c, \vec{d}\rangle$ with the property that E is a well-founded and extensional relation on D and, if $\vec{P} = \langle P_i \mid i < \eta\rangle$, $\vec{d} = \langle d_i \mid i < \eta\rangle$ and $\pi : \langle D, E\rangle \to \langle M, \in\rangle$ is the induced transitive collapse, then the following statements hold:

- $\mathrm{rnk}(D)$ is a cardinal and $V_{\mathrm{rnk}(D)} \cup \{\mathrm{rnk}(D)\} \subseteq M$.

- M is $\Pi_n(V_{\mathrm{rnk}(D)+1})$-correct.
- $\pi(b) = \mathrm{rnk}(D)$, $\pi(c) = z$ and $\pi(d_i) = \mathrm{rnk}(P_i)$ for all $i < \eta$.
- $\langle \mathrm{rnk}(P_i) \mid i < \eta \rangle$ is a strictly increasing sequence of cardinals with supremum $\mathrm{rnk}(D)$.

Then $\mathcal{C}$ is definable by a Π_n-formula with parameter z.

Now, let X be an elementary substructure of $V_{\lambda'}$ of cardinality $\beth_\lambda$ with $V_\lambda \cup \{A, \lambda\} \subseteq X$. Pick a bijection $f : X \to V_\lambda$ and let R denote the induced binary relation on V_λ. These choices ensure that the transitive collapse of $\langle V_\lambda, R\rangle$ is $\Pi_n(V_{\lambda+1})$-correct and this allows us to conclude that

$$\langle V_\lambda, R, \langle \lambda_{i+1} \mid i < \eta\rangle, f(A), f(\lambda), f(z), \langle f(\lambda_{i+1}) \mid i < \eta\rangle\rangle$$

is a structure in $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < \eta\rangle$. By our assumptions, there exists an elementary embedding i of a structure

$$\langle D, E, \langle P_i \mid i < \eta\rangle, a, b, c, \langle d_i \mid i < \eta\rangle\rangle$$

of type $\langle \lambda_i \mid i < \eta\rangle$ in $\mathcal{C}$ into the above structure. Let $\pi : \langle D, E\rangle \to \langle M, \in\rangle$ denote the corresponding transitive collapse and set

$$j \;=\; f^{-1} \circ i \circ \pi^{-1} : M \to H_{\lambda'}.$$

Set $\rho = \pi(b) = \mathrm{rnk}(D) = \sup_{i<\eta} \lambda_i$. Then M is a $\Pi_n(V_{\rho+1})$-correct set with $V_\rho \cup \{\rho\} \subseteq M$ and j is an elementary embedding with $j(\rho) = \lambda$, $j(z) = z$ and $A \in \mathrm{ran}(j)$. Moreover, given $i < \eta$, we now have $\lambda_i = \mathrm{rnk}(P_i) = \pi(d_i)$ and this allows us to conclude that

$$j(\lambda_i) \;=\; (f^{-1} \circ i)(d_i) \;=\; \lambda_{i+1}.$$

This shows that (iii) holds in this case.

Since (iii) obviously implies (ii), this concludes the proof of the lemma. □

Corollary 9.7. *Let $0 < n < \omega$, let $0 < \eta \leq \omega$ and let $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta\rangle$ be a strictly increasing sequence of cardinals.*

(i) *The cardinal λ_0 is weakly n-exact for $\langle \lambda_{i+1} \mid i < \eta\rangle$ if and only if Π_n-ESR$(\vec{\lambda})$ holds.*
(ii) *If λ_0 is weakly parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta\rangle$, then $\Pi_n(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds.* □

In the case of sequences of finite length, we can now generalize Theorem 1.7 to principles of the form Π_n-ESR$(\vec{\lambda})$.

Theorem 9.8. *The following statements are equivalent for every cardinal κ and all natural numbers $n, \eta > 0$:*

(i) *κ is the least cardinal such that there exists a strictly increasing sequence $\bar{\lambda} = \langle \lambda_i \mid i < \eta + 1\rangle$ of cardinals with $\lambda_0 = \kappa$ and the property that Π_n-ESR$(\vec{\lambda})$ holds.*
(ii) *κ is the least cardinal such that there exists a strictly increasing sequence $\bar{\lambda} = \langle \lambda_i \mid i < \eta + 1\rangle$ of cardinals with $\lambda_0 = \kappa$ and the property that $\Pi_n(V_\kappa)$-ESR$(\vec{\lambda})$ holds.*
(iii) *κ is the least cardinal that is weakly n-exact for some strictly increasing sequence of cardinals greater than κ of length η.*
(iv) *κ is the least cardinal that is weakly parametrically n-exact for some strictly increasing sequence of cardinals greater than κ of length η.*

Proof. Let κ be the least cardinal such that there exists a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \eta + 1\rangle$ of cardinals with $\lambda_0 = \kappa$ and the property that Π_n-ESR$(\vec{\lambda})$ holds. Set $\lambda = \lambda_\eta$ and $\rho = \lambda_{\eta-1}$. Pick $\lambda' > \lambda$ with the property that $V_{\lambda'}$ is sufficiently elementary in V.

Claim. *κ is weakly parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta\rangle$.*

Proof of the Claim. Assume, towards a contradiction, that $A \in V_{\lambda+1}$ witnesses that κ is not weakly parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta\rangle$. Using Lemma 9.6, we can find a transitive, $\Pi_n(V_{\rho+1})$-correct set M with $V_\rho \cup \{\rho\} \subseteq M$ and an elementary embedding $j : M \to V_{\lambda'}$ with $A, \kappa \in \mathrm{ran}(j)$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$. Our setup then ensures that $j(\mathrm{crit}(j)) \neq \kappa$ and, since $\kappa \in \mathrm{ran}(j)$ and $j(\kappa) = \lambda_1 > \kappa$, this implies that $j(\mathrm{crit}(j)) < \kappa$. Given $i < \eta + 2$, set

$\mu_i = j^i(\mathrm{crit}(j))$. Then $\mu_{i+1} \leq \lambda_i$ for all $i < \eta + 1$. Since $\mu_1 < \kappa$, the minimality of κ yields a Π_n-formula $\varphi(v)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A)\}$ consists of structures of the same type and there exists a structure B of type $\langle \mu_{i+2} \mid i < \eta\rangle$ in $\mathcal{C}$ such that for every structure A of type $\langle \mu_{i+1} \mid i < \eta\rangle$ in $\mathcal{C}$, there exists no elementary embedding of A into B. By the choice of λ', these statements are absolute between V and $V_{\lambda'}$. Elementarity now implies that, in M, there exists a structure B_0 of type $\langle \mu_{i+1} \mid i < \eta\rangle$ with the property that $\varphi(B_0)$ holds and for every structure A of type $\langle \mu_i \mid i < \eta\rangle$ such that $\varphi(A)$ holds, there is no elementary embedding of A into B_0. Since M is $\Pi_n(V_{\rho+1})$-correct and $V_{\lambda'}$ is sufficiently elementary in V, it follows that both $\varphi(B_0)$ and $\varphi(j(B_0))$ hold in $V_{\lambda'}$. But we can now use the elementarity of j to derive a contradiction, because $j(B_0)$ is a structure of type $\langle \mu_{i+2} \mid i < \eta\rangle$, B_0 is a structure of type $\langle \mu_{i+1} \mid i < \eta\rangle$ and the map j induces an elementary embedding of B_0 into $j(B_0)$ that is an element of $V_{\lambda'}$. □

With the help of the above claim, we can apply Lemma 9.6 and Corollary 9.7 to conclude that all statements listed in the theorem are equivalent. □

Note that the above proof cannot be directly generalized to sequences of cardinals of length ω, because, if κ is the least cardinal with the property that Π_n-ESR$(\vec{\lambda})$ holds for some strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \omega\rangle$ of cardinals with $\lambda_0 = \kappa$ and supremum λ, we assume that κ is not weakly parametrically n-exact for $\langle \lambda_{i+1} \mid i < \omega\rangle$ and we repeat the above construction to obtain a transitive, $\Pi_n(V_{\lambda+1})$-correct set M with $V_\lambda \cup \{\lambda\} \subseteq M$ and an elementary embedding $j : M \to V_{\lambda'}$ with $j(\mathrm{crit}(j)) < \kappa$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \omega$, then we do not know wheter the sequence $\langle j^{i+1}(\mathrm{crit}(j)) \mid i < \omega\rangle$ is contained in the range of j and this stops us from repeating the above minimality argument.

Analogously to the above results, the statement of Theorem 1.9 can be generalized to the context of this section.

Lemma 9.9. *The following statements are equivalent for every natural number $n > 0$, every ordinal $0 < \eta \leq \omega$, every strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta\rangle$ of uncountable cardinals with supremum λ, and all $z \in V_{\lambda_0}$:*

(i) $\Sigma_{n+1}(\{z\})$-ESR$_{\mathcal{C}}(\vec{\lambda})$.

(ii) *For every $A \in V_{\lambda+1}$, there exists a cardinal ρ, a cardinal $\kappa' \in C^{(n)}$ greater than ρ, a cardinal $\lambda' \in C^{(n+1)}$ greater than λ, an elementary submodel X of $H_{\kappa'}$ with $V_\rho \cup \{\rho\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $A \in \mathrm{ran}(j)$, $j(z) = z$, $j(\rho) = \lambda$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$.*

Proof. Assume that (ii) holds. Pick a Σ_{n+1}-formula $\varphi(v_0, v_1)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, z)\}$ consists of suitable structures and fix a structure B in $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < \eta\rangle$. Since $B \in V_{\lambda+1}$, there exists a cardinal ρ, a cardinal κ' with $\rho < \kappa' \in C^{(n)}$, a cardinal λ' with $\lambda < \lambda' \in C^{(n+1)}$, an elementary submodel X of $V_{\kappa'}$ with $V_\rho \cup \{\rho\} \subseteq X$, and an elementary embedding $j : X \to H_{\lambda'}$ with $B \in \mathrm{ran}(j)$, $j(z) = z$, $j(\rho) = \lambda$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$. Pick $A \in X$ such that $j(A) = B$. Since $\lambda' \in C^{(n+1)}$, we know that $\varphi(B, z)$ holds in $V_{\lambda'}$. The elementarity of j then implies that $\varphi(A, z)$ holds in $V_{\kappa'}$ and, since $\kappa' \in C^{(n)}$, we may then conclude that $A \in \mathcal{C}$. Moreover, since $j(\lambda_i) = \lambda_{i+1}$ holds for all $i < \eta$, elementarity allows us to conclude that A has type $\langle \lambda_i \mid i < \eta\rangle$. Finally, elementarity also implies that A has rank ρ and, since V_ρ is a subset of X, the map j induces an elementary embedding of A into B.

Next, assume that (i) holds and fix $A \in V_{\lambda+1}$. Let $\mathcal{L}$ be the first-order language extending $\mathcal{L}_\in$ by predicate symbols $\vec{P} = \langle \dot{P}_i \mid i < \eta\rangle$ and constant symbols $\dot{A}$, $\dot{\lambda}$, $\dot{z}$ and $\langle \dot{\lambda}_i \mid i < \eta\rangle$. Define $\mathcal{C}$ to be the class of all $\mathcal{L}$-structures $\langle D, E, \vec{P}, a, b, c, \vec{d}\rangle$ for which there exists $\theta \in C^{(n)}$ and an isomorphism τ between $\langle D, E\rangle$ and an elementary substructure X of H_θ with the property that, if $\vec{P} = \langle P_i \mid i < \eta\rangle$ and $\vec{d} = \langle d_i \mid i < \eta\rangle$, then the following statements hold:

- $\mathrm{rnk}(D)$ is a cardinal and $V_{\mathrm{rnk}(D)} \cup \{\mathrm{rnk}(D)\} \subseteq X$.
- $\tau(b) = \mathrm{rnk}(D)$, $\tau(c) = z$ and $\tau(d_i) = \mathrm{rnk}(P_i)$ for all $i < \eta$.
- $\langle \mathrm{rnk}(P_i) \mid i < \eta\rangle$ is a strictly increasing sequence of cardinals with supremum $\mathrm{rnk}(D)$.

Then the class $\mathcal{C}$ is then Σ_{n+1}-definable with parameter z.

Pick $\lambda < \lambda' \in C^{(n+1)}$, an elementary substructure Y of $H_{\lambda'}$ of cardinality $\beth_\lambda$ with $V_\lambda \cup \{A, \lambda\} \subseteq Y$ and a bijection $f : Y \to V_\lambda$. If we now let R denote the binary relation on V_λ induced by f, then the resulting $\mathcal{L}$-structure

$$\langle V_\lambda, R, \langle \lambda_{i+1} \mid i < \omega \rangle, f(A), f(\lambda), f(z), \langle f(\lambda_{i+1}) \mid i < \eta \rangle \rangle$$

is an element of $\mathcal{C}$ of type $\langle \lambda_{i+1} \mid i < \eta \rangle$. Set $\rho = \sup_{i<\eta} \lambda_i$. By our assumptions, there exists an elementary embedding i of a structure $\langle D, E, \vec{P}, a, b, c, \vec{d} \rangle$ of type $\langle \lambda_i \mid i < \eta \rangle$ in $\mathcal{C}$ into the above structure. Pick a cardinal $\kappa' \in C^{(n)}$, an elementary submodel X of $H_{\kappa'}$ and an isomorphism $\tau : \langle D, E \rangle \to \langle X, \in \rangle$ witnessing that the given structure is contained in $\mathcal{C}$. Then ρ is a cardinal with $\tau(b) = \rho = \mathrm{rnk}(D)$ and $V_\rho \cup \{\rho\} \subseteq X$. Moreover, we have $\tau(c) = z$ and $\lambda_i = \mathrm{rnk}(P_i) = \tau(d_i)$ for all $i < \eta$. If we now define

$$j \;=\; f^{-1} \circ i \circ \tau^{-1} : X \to H_{\lambda'},$$

then j is an elementary embedding with $A \in \mathrm{ran}(j)$, $j(z) = z$, $j(\rho) = \lambda$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$. □

Corollary 9.10. *Let $0 < n < \omega$, let $0 < \eta \leq \omega$ and let $\vec{\lambda} = \langle \lambda_i \mid i < 1 + \eta \rangle$ be a strictly increasing sequence of cardinals.*

(i) *The cardinal λ_0 is n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$ if and only if Σ_{n+1}-ESR$(\vec{\lambda})$ holds.*
(ii) *If λ_0 is parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$, then $\Sigma_{n+1}(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds.* □

As above, we can now generalize Theorem 1.9 to the principle Σ_{n+1}-ESR$(\vec{\lambda})$ for finite sequences of cardinals $\vec{\lambda}$.

Theorem 9.11. *The following statements are equivalent for every cardinal κ and all natural numbers $n, \eta > 0$:*

(i) *κ is the least cardinal such that there exists a strictly increasing sequence $\bar{\lambda} = \langle \lambda_i \mid i < \eta + 1 \rangle$ of cardinals with $\lambda_0 = \kappa$ and the property that Σ_{n+1}-ESR$(\vec{\lambda})$ holds.*
(ii) *κ is the least cardinal such that there exists a strictly increasing sequence $\bar{\lambda} = \langle \lambda_i \mid i < \eta + 1 \rangle$ of cardinals with $\lambda_0 = \kappa$ and the property that $\Sigma_{n+1}(V_\kappa)$-ESR$(\vec{\lambda})$ holds.*
(iii) *κ is the least cardinal that is n-exact for some strictly increasing sequence of cardinals greater than κ of length η.*
(iv) *κ is the least cardinal that is parametrically n-exact for some strictly increasing sequence of cardinals greater than κ of length η.*

Proof. Assume that κ is the least cardinal such that Σ_{n+1}-ESR$(\vec{\lambda})$ holds for some strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \eta + 1 \rangle$ of cardinals with $\lambda_0 = \kappa$. Set $\lambda = \lambda_\eta$ and $\rho = \lambda_{\eta-1}$. Pick $\lambda' > \lambda$ with the property that $V_{\lambda'}$ is sufficiently elementary in V.

Claim. *κ is parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$.*

Proof of the Claim. Assume, towards a contradiction, that $A \in V_{\lambda+1}$ witnesses that κ is not parametrically n-exact for $\langle \lambda_{i+1} \mid i < \eta \rangle$. By Lemma 9.9, a cardinal κ' with $\rho < \kappa' \in C^{(n)}$, a cardinal λ' with $\lambda < \lambda' \in C^{(n+1)}$, an elementary submodel X of $H_{\kappa'}$ with $V_\rho \cup \{\rho\} \subseteq X$ and an elementary embedding $j : X \to H_{\lambda'}$ with $A, \kappa \in \mathrm{ran}(j)$ and $j(\lambda_i) = \lambda_{i+1}$ for all $i < \eta$. We then know that $j(\mathrm{crit}(j)) \neq \kappa$ and, since $\kappa \in \mathrm{ran}(j)$ and $j(\kappa) > \kappa$, this allows us to conclude $j(\mathrm{crit}(j)) < \kappa$. Given $i < \eta + 2$, set $\mu_i = j^i(\mathrm{crit}(j))$. Since $\mu_1 < \kappa$, the minimality of κ yields a Σ_{n+1}-formula $\varphi(v)$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A)\}$ consists of structures of the same type and there exists a structure B of type $\langle \mu_{i+2} \mid i < \eta \rangle$ in $\mathcal{C}$ such that for every structure A of type $\langle \mu_{i+1} \mid i < \eta \rangle$ in $\mathcal{C}$, there exists no elementary embedding of A into B. Our set up now ensures that, in X, there exists a structure B_0 of type $\langle \mu_{i+1} \mid i < \eta \rangle$ with the property that $\varphi(B_0)$ holds and for every structure A of type $\langle \mu_i \mid i < \eta \rangle$ such that $\varphi(A)$ holds, there is no elementary embedding of A into B_0. Then both $\varphi(B_0)$ and $\varphi(j(B_0))$ hold in $V_{\lambda'}$, and j induces an elementary embedding of B_0 into $j(B_0)$ that is an element of $V_{\lambda'}$. Since B_0 has type $\langle \mu_{i+1} \mid i < \eta \rangle = j(\langle \mu_i \mid i < \eta \rangle)$ and $j(B_0)$ has type $\langle \mu_{i+2} \mid i < \eta \rangle = j(\langle \mu_{i+1} \mid i < \eta \rangle)$, we can now use the elementarity of j to derive a contradiction. □

A combination of this claim and Corollary 9.10 now yields the desired equivalences. □

10. Open questions and concluding remarks

We close this paper by discussing some questions raised by the above results.

First, recall that a cardinal κ is *superhuge* if there is a proper class of cardinals λ with the property that κ is huge with target λ. In order to study principles of structural reflection related to superhugeness, Proposition 8.4 suggests to study cardinals κ that are weakly 1-exact for a proper class of cardinals λ. By Corollary 5.5, this property is equivalent to the assumption that the principle Π_1-ESR(κ, λ) holds for a proper class of cardinals λ. Therefore, it is natural to ask whether a variation of Theorem 1.7 can be proven for these cardinals.

Question 10.1. *Are the following statements equivalent for every cardinal κ and every natural number $n > 0$?*

(i) *κ is the least cardinal that is weakly parametrically n-exact for a proper class of cardinals λ.*
(ii) *κ is the least cardinal with the property that Π_n-ESR(κ, λ) holds for a proper class of cardinals λ.*

Our next question deals with the exact position of n-exact and weakly n-exact cardinals in the large cardinal hierarchy. By Corollary 8.2 and Proposition 8.9, these notions are properly contained in the interval given by *almost hugeness* and *almost 2-hugeness*. Moreover, Corollary 8.5 shows that hugeness is strictly stronger than weak 1-exactness. Finally, Proposition 8.6 implies that, if κ is the least huge cardinal, then Σ_2-ESR(κ, λ) fails for all $\lambda > \kappa$. These results leave open the precise relationship between hugeness and exactness, and motivate the following question:

Question 10.2. *Does the consistency of the theory ZFC + "there exists a huge cardinal" imply the consistency of the theory* ZFC + "Σ_2-ESR(κ) *holds for some cardinal κ"?*

The results of [For09, Section 2.2.1] might provide tools to derive a negative answer to this question.

We finally discuss some questions left open about the infinite sequential versions of exact structural reflection principles introduced in Section 9. In the light of Proposition 9.5, it is natural to ask whether the consistency of principles of the form Σ_n-ESR$(\vec{\lambda})$ for infinite sequences $\vec{\lambda}$ of cardinals and natural numbers $n > 1$ can be established from some very strong large cardinal assumption (like ZFC + "*There exists an I0-cardinal*"), or whether these principles are outright inconsistent with ZFC.

Question 10.3. *Does* ZFC prove that the principle Σ_2-ESR$(\vec{\lambda})$ fails for every strictly increasing sequence $\vec{\lambda}$ of cardinals of length ω?

However, if we only assume ZF and κ is a Reinhardt cardinal, witnessed by an elementary embedding $j : V \to V$, then for every natural number $n > 0$, the critical sequence $\vec{\lambda} = \langle \lambda_i \mid i < \omega \rangle$, given by $\lambda_i = j^i(\mathrm{crit}(j))$, witnesses that the principle $\Pi_n(V_{\lambda_0})$-ESR$(\vec{\lambda})$ holds.

Finally, as noted in the discussion following Theorem 9.8, our techniques do not allow us to generalize Theorem 1.7 to principles of the form ESR$(\vec{\lambda})$ for infinite sequences $\vec{\lambda}$. This motivates the following question:

Question 10.4. *If κ is the least cardinal with the property that Π_n-ESR$(\vec{\lambda})$ holds for some strictly increasing sequence $\vec{\lambda}$ of cardinals of length ω with minimum κ, is κ weakly parametrically n-exact for some strictly increasing sequence of cardinals greater than κ of length ω?*

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