PATTERNS OF STRUCTURAL REFLECTION IN THE LARGE-CARDINAL HIERARCHY

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Abstract. We unveil new patterns of Structural Reflection in the large-cardinal hierarchy below the first measurable cardinal. Namely, we give two different characterizations of strongly unfoldable and subtle cardinals in terms of a weak form of the principle of Structural Reflection, and also in terms of weak product structural reflection. Our analysis prompts the introduction of the new notion of $C^{(n)}$-strongly unfoldable cardinal for every natural number n, and we show that these cardinals form a natural hierarchy between strong unfoldable and subtle cardinals analogous to the known hierarchies of $C^{(n)}$-extendible and Σ_n-strong cardinals. These results show that the relatively low region of the large-cardinal hierarchy comprised between the first strongly unfoldable and the first subtle cardinals is completely analogous to the much higher region between the first strong and the first Woodin cardinals, and also to the much further upper region of the hierarchy ranging between the first supercompact and the first Vopěnka cardinals.

1. Introduction

Large cardinals are transfinite cardinal numbers with associated properties that make them very large, so much so that their existence cannot be proved in ZFC. Since the weakest large cardinals, the *weakly inaccessible*, were first defined and studied by Hausdorff over a century ago, in 1908, a plethora of different and much stronger large cardinals have since then been identified in a great variety of contexts and taking many different forms. The book of Kanamori [22] gives a comprehensive overview of the rich world of large cardinals, the world of the "*Higher Infinite*" as the book's title reads. Since the book's first edition, published in 1994 in the wake of the groundbreaking results of Martin-Steel [26] and Woodin [29] establishing the tight connections between large cardinals and the determinacy of sets of reals, the theory of large cardinals has been expanding in multiple directions, yielding solutions to well-known set-theoretic problems – e.g., in the arithmetic of singular cardinals (see [17]) or in the combinatorial properties of small uncountable cardinals (see [15] and [16]) – as well as fertile applications to other areas, from general topology (see [7] and [14]) to algebraic topology and homotopy theory (see [3] and [9]), to abelian groups (see [6] and [11]), etc. The use of large cardinals is most effective in conjunction with the forcing technique (e.g., Prikry-type forcing), and also via the construction and analysis of canonical inner models in which large cardinals exist – the so-called *inner model program.* Indeed, to settle a given statement φ one typically assumes, on the one hand, the existence of some suitable large cardinal

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and, by forcing, produces a model of set theory in which φ holds. On the other hand, one assumes φ and shows that some large cardinal (or even the same kind of large cardinal) exists in a canonical inner model, thereby showing that φ is (equi)consistent with the existence of that large cardinal.

However, in spite of their enormous success, large-cardinal axioms, i.e., the axioms asserting that such and such large cardinals exist, have remained a long-standing mystery from a foundational point of view. Indeed, a precise definition of *large cardinal*, which would encompass all the variety of known large cardinals into a single notion, is still lacking. Moreover, the fact that all known large cardinals line up into a well-ordered hierarchy of consistency strength remains unexplained. Furthermore, no convincing intrinsic justification of their naturalness, or their status as true axioms of set theory has been yet put forward, and their acceptance has so far been supported mainly by their fruitful consequences.

In recent years, a new program of reformulating large cardinals in terms of a general reflection principle called *Structural Reflection* (SR) (see [2]), has succeeded at characterizing well-known large cardinals at several regions of the large-cardinal hierarchy in terms of this principle. The program started, still in a veiled form, with [1] and [3], in which large cardinals in the region spanning from supercompact and extendible cardinals to Vopěnka's Principle are characterized in terms of a strong form of Löwenheim-Skolem type of reflection, equivalent to stratified forms of Vopěnka's Principle restricted to classes of structures of a given definitional complexity. The work continued in [9], in which a similar characterization in terms of SR, this time using products and homomorphisms instead of elementary embeddings, was given for large cardinals in the region between strong and Woodin cardinals. Further work along the same lines was done in the lower regions of the large-cardinal hierarchy, below the first measurable cardinal. In [8] (see also [24] and [2]), reformulations in terms of SR are given for the weakest of large cardinals, namely those between weakly inaccessible and weakly compact cardinals. Furthermore, in [4], a generic form of SR is used to characterize cardinals that lie between almost-remarkable and virtually extendible. Several other similar SR characterizations of large cardinals lying in other regions of the large-cardinal hierarchy are given in [2] – e.g., for globally superstrong cardinals – and even large-cardinal principles such as "$0^\sharp$ *exists*" or "$0^\dagger$ *exists*" are shown to be equivalent to SR for definable classes of structures belonging to canonical inner models. Finally, in [5], the authors gave characterizations of large cardinals in the uppermost regions of the large-cardinal hierarchy, between Vopěnka's Principle and I1-cardinals. Moreover, our analysis of those cardinals in terms of SR allowed to formulate a new hierarchy of large cardinal notions that has the potential to go beyond all known large cardinals not known to be inconsistent with ZFC.

In the present article, we show that the same pattern of SR that holds between a supercompact and a Vopěnka cardinal, and between a strong and a Woodin cardinal, also holds between a strongly unfoldable and a subtle cardinal. These correlations can be informally expressed by the following equations:

$$\frac{\textit{Vopěnka}}{\textit{supercompact}} \quad = \quad \frac{\textit{Woodin}}{\textit{strong}} \quad = \quad \frac{\textit{subtle}}{\textit{strongly unfoldable}}$$

As strongly unfoldable and subtle cardinals lie below the first measurable cardinal and can exist in Gödel's constructible universe L, our results show that the large-cardinal hierarchy is highly homogeneous, repeating the same pattern in the upper, central, and lower regions of the hierarchy. The reformulation of large cardinals in terms of SR does explain the empirical fact that they form a well-ordered hierarchy, and also attests to their naturalness and intrinsic justification as true axioms of set theory (see [2]). Moreover, each new SR characterization, like the ones given in the present article, constitutes a further step towards the ultimate goal of yielding a uniform formulation of all known large cardinals in terms of a single unifying principle. Furthermore, what could be termed as a "side effect" is that every SR reformulation of a known large-cardinal notion suggests and gives rise to new definitions of large cardinals, filling in the gaps in the large-cardinal hierarchy, e.g., $C^{(n)}$-extendible cardinals, which lie between supercompact cardinals and Vopěnka's Principle. Let us describe next more precisely our results and explain how they fit in and contribute to the SR program.

1.1. **Structural Reflection.** Recall that, given a natural number n, $C^{(n)}$ is the Π_n-definable closed unbounded class of ordinals α that are Σ_n-correct in V, that is, V_α is a Σ_n-elementary substructure of V, written $V_\alpha \prec_{\Sigma_n} V$. Thus, $C^{(0)}$ is the class of all ordinal numbers and $C^{(1)}$ is the class of all uncountable cardinals κ such that $V_\kappa = H_\kappa$.

Let us now recall the following different variants of the principle of Structural Reflection:

Definition 1.1. Let $\mathcal{C}$ be a class[1] of structures[2] of the same type and let κ be an infinite cardinal.

(1) (Bagaria-Väänänen, [8]) $\mathrm{SR}_{\mathcal{C}}(\kappa)$ denotes the statement that for every structure B in $\mathcal{C}$, there exists a structure A in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of A into B.
(2) (Bagaria, [1]) $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ denotes the statement that for every structure B in $\mathcal{C}$, there exists a structure A in $\mathcal{C} \cap H_\kappa$ and an elementary embedding of A into B.
(3) (Bagaria, [2]) $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ denotes the statement that for every structure B in $\mathcal{C}$, there exists a structure A in $\mathcal{C} \cap V_\kappa$ and an elementary embedding of A into B.

Clearly, for a given class $\mathcal{C}$ and a cardinal κ, the principle $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ implies both the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ and the principle $\mathrm{VSR}_{\mathcal{C}}(\kappa)$. Also, if κ is an element of $C^{(1)}$, then we have $H_\kappa = V_\kappa$, and therefore the principles $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ are identical. Moreover, for classes $\mathcal{C}$ that are closed under isomorphic images, the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ implies the principle $\mathrm{HSR}_{\mathcal{C}}(\kappa)$. Thus, for a class $\mathcal{C}$ closed under isomorphic images and $\kappa \in C^{(1)}$, the three principles $\mathrm{SR}_{\mathcal{C}}(\kappa)$, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ are the same.

The following theorem, proved in [1, Corollary 4.10] using Magidor's characterization of the first supercompact cardinal as the first cardinal that reflects the V_α (see [25, Theorem 1]), gives a reformulation of supercompact cardinals in terms of SR and it may be considered the first result of the SR program.

Theorem 1.2 ([1])**.** *The following statements are equivalent for every cardinal κ:*

(1) *The cardinal κ is either supercompact or a limit of supercompact cardinals.*
(2) *The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ (equivalently, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$, or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$) holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Corollary 1.3. *The following statements are equivalent for every cardinal κ:*

(1) *The cardinal κ is the least supercompact cardinal.*
(2) *The cardinal κ is the least cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ (equivalently, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$, or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$) holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.* □

[1]We work in ZFC. Therefore, when considering proper classes, we shall always mean classes that are definable, possibly using sets as parameters.

[2]In this paper, the term *structure* refers to structures for countable first-order languages. The cardinality of a structure is defined as the cardinality of its domain.

For classes of structures of higher definitional complexity an analogous equivalence is given in the theorem below, proved in [1, Theorem 4.18], which characterizes extendible and $C^{(n)}$-extendible cardinals in terms of SR. Recall that a cardinal κ is *$C^{(n)}$-extendible* if for every $\lambda \geq \kappa$, there is an ordinal μ and an elementary embedding $j : V_\lambda \longrightarrow V_\mu$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. A cardinal is *extendible* if and only if it is $C^{(1)}$-extendible.

Theorem 1.4 ([1]). *The following statements are equivalent for every cardinal κ:*

(1) *κ is either $C^{(n)}$-extendible or a limit of $C^{(n)}$-extendible cardinals.*
(2) *The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ (equivalently, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$, or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$) holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+2}-formula with parameters in V_κ.*

Corollary 1.5. *The following statements are equivalent for every cardinal κ:*

(1) *κ is the least $C^{(n)}$-extendible cardinal.*
(2) *κ is the least cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ (equivalently, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$, or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$) holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+2}-formula with parameters in V_κ.* □

Remember that *Vopěnka's Principle* is scheme of axioms stating that for every proper class $\mathcal{C}$ of structures of the same type, there exist $A, B \in \mathcal{C}$ with $A \neq B$ and an elementary embedding from A to B. In [1], the first author obtains the following characterization of this principle:

Theorem 1.6 ([1]). *The following schemes of axioms are equivalent over* ZFC*:*

(1) *Vopěnka's Principle.*
(2) *For every natural number n, there exists a $C^{(n)}$-extendible cardinal.*
(3) *For every natural number n, there exists a proper class of $C^{(n)}$-extendible cardinals.*
(4) *For every class $\mathcal{C}$ of structures of the same type, there exists a cardinal κ such that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ (equivalently, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$, or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$) holds.*

The next result, which shall be proved in Section 2 below, uses the *set*-version of this equivalence to provide a canonical characterization of *Vopěnka cardinals*, i.e., inaccessible cardinals δ with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exist $A, B \in \mathcal{C}$ with $A \neq B$ and an elementary embedding from A to B (see, for example, [18] and [27]).

Theorem 1.7. *The following statements are equivalent for every uncountable cardinal δ:*

(1) *The cardinal δ is a Vopěnka cardinal.*
(2) *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ (equivalently, $\mathrm{HSR}_{\mathcal{C}}(\kappa)$, or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$) holds.*

1.2. **Weak Structural Reflection.** We now use weaker forms of SR, defined below, to obtain canonical characterizations of smaller large cardinals, lying below the first measurable cardinal.

Definition 1.8. Let $\mathcal{C}$ be a non-empty class of structures of the same type and let κ be an infinite cardinal.

(1) (Bagaria–Väänänen, [8]) $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ is the statement that for every structure B in $\mathcal{C}$ of cardinality κ, there exists a structure A in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of A into B.
(2) (Bagaria–Väänänen, [8]) Let $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ denote the statement that $\mathcal{C}$ contains a structure of cardinality less than κ.

(3) $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ denotes the conjunction of $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$.
(4) Let $\mathrm{HSR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}^{-}_{\mathcal{C}}(\kappa)$ be the same as $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}_{\mathcal{C}}(\kappa)$, respectively (as given in Definition 1.1), but restricted to structures B of cardinality κ.

For a given class $\mathcal{C}$ and a cardinal κ, the principle $\mathrm{HSR}^{-}_{\mathcal{C}}(\kappa)$ clearly implies both $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}^{-}_{\mathcal{C}}(\kappa)$; and if κ is an element of $C^{(1)}$, then the principles $\mathrm{HSR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}^{-}_{\mathcal{C}}(\kappa)$ are equivalent. Also, if $\mathcal{C}$ is closed under isomorphic images, then the principle $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ implies the principle $\mathrm{HSR}_{\mathcal{C}}(\kappa)$. Hence, for $\mathcal{C}$ closed under isomorphic images and $\kappa \in C^{(1)}$, the principles $\mathrm{HSR}^{-}_{\mathcal{C}}(\kappa)$, $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}^{-}_{\mathcal{C}}(\kappa)$ are the same.

Next, we recall the large-cardinal notion of strong unfoldability, introduced by Villaveces in model-theoretic investigations of models of set theory.

Definition 1.9 (Villaveces, [28]). An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^{-}-model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$ and $j(\kappa) \geq \lambda$.

Every strongly unfoldable cardinal is a totally indescribable element of $C^{(2)}$ (see [13, Theorem 2] and [23, Section 3]). In particular, the first strongly unfoldable cardinal is bigger than the first subtle cardinal (see Definition 1.15 below) and smaller than the first strong cardinal. The following characterization of strongly unfoldable cardinals in terms of weak SR will be proved in Section 3:

Theorem 1.10. *The following statements are equivalent for every cardinal κ:*
(1) *κ is either strongly unfoldable or a limit of supercompact cardinals.*
(2) *The principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
(3) *κ is an element of $C^{(2)}$ and the principle $\mathrm{HSR}^{-}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

The fact that supercompact cardinals are strongly unfoldable now directly yields a characterization of strongly unfoldable cardinals through weak principles of structural reflection:

Corollary 1.11. *The following statements are equivalent for every cardinal κ:*
(1) *κ is the least strongly unfoldable cardinal.*
(2) *κ is the least cardinal with the property that the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
(3) *κ is the least element of the class $C^{(2)}$ with the property that the principle $\mathrm{HSR}^{-}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.* □

For classes of structures of higher definitional complexity, we shall prove analogous equivalences, using the following natural strengthening of strong unfoldability:

Definition 1.12. Given a natural number n, an inaccessible cardinal κ is *$C^{(n)}$-strongly unfoldable* if for every ordinal $\lambda \in C^{(n)}$ greater than κ and every transitive ZF^{-}-model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$.

By definition, a cardinal is strongly unfoldable if and only if it is $C^{(0)}$-strongly unfoldable. A short argument (see Proposition 9.2 below) shows that strong unfoldability also coincides with $C^{(1)}$-strong unfoldability. Moreover, it is also easy to see that all $C^{(n)}$-extendible cardinals are $C^{(n+1)}$-strongly unfoldable (see Proposition 9.3 below). The following result, which shall be proved in Section 9, characterizes $C^{(n)}$-strongly unfoldable cardinals in terms of weak SR. For every natural number $n > 0$, the theorem shows that $C^{(n+1)}$-strongly unfoldable cardinals are related, via weak SR, to strongly unfoldable cardinals, as $C^{(n)}$-extendible cardinals are related, via SR, to supercompact cardinals.

Theorem 1.13. *Given a natural number $n > 1$, the following statements are equivalent for every cardinal κ:*

(1) *κ is either $C^{(n)}$-strongly unfoldable or a limit of $C^{(n-1)}$-extendible cardinals.*
(2) *The principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.*
(3) *κ is an element of $C^{(n+1)}$ and the principle $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.*

By combining this theorem with Corollary 1.11 and the fact that all $C^{(n)}$-extendible cardinals are $C^{(n+1)}$-strongly unfoldable, we now obtain a uniform characterization of the least $C^{(n)}$-strongly unfoldable cardinal through weak principles of structural reflection:

Corollary 1.14. *Given a natural number $n > 0$, the following statements are equivalent for every cardinal κ:*

(1) *κ is the least $C^{(n)}$-strongly unfoldable cardinal.*
(2) *κ is the least cardinal with the property that the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.*
(3) *κ is the least cardinal in $C^{(n+1)}$ with the property that the principle $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.* □

We now want to use weak SR to isolate the canonical variations of Vopěnka's Principle and Vopěnka cardinals for the considered region of the large cardinal hierarchy. This analysis turns out to be closely related to the notion of *subtle cardinals*, introduced by Jensen and Kunen in their work on the validity of strong diamond principles in the constructible universe L. Remember that, given a set A of ordinals, a sequence $\langle E_\alpha \mid \alpha \in A\rangle$ is called an *A-list* if $E_\alpha \subseteq \alpha$ holds for every $\alpha \in A$.

Definition 1.15 (Jensen-Kunen, [21])**.** An infinite cardinal δ is *subtle* if for every δ-list $\langle E_\gamma \mid \gamma < \delta\rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with $E_\beta = E_\gamma \cap \beta$.

Subtle cardinals are strongly inaccessible, and below the first subtle cardinal there are many totally indescribable cardinals. However, the first subtle cardinal is Π^1_1-describable, and therefore not weakly compact. Moreover, if δ is a subtle cardinal, then the set of $\kappa < \delta$ that are strongly unfoldable in V_δ is stationary in δ (see [13, Theorem 3]).

The notion of subtleness has a natural *class*-version ("Ord *is subtle*") that postulates, as a schema, that for every closed unbounded class C of ordinals and every class sequence $\langle E_\gamma \mid \gamma \in \mathrm{Ord}\rangle$, there exist $\beta < \gamma$ in C with $E_\beta = E_\gamma \cap \beta$ (see [10, Section 5]). It turns out that for our purposes, a slight strengthening of this principle is needed. The formulation of this principle is motivated by the trivial observation that the Axiom of Choice allows us to show that a cardinal δ is subtle if and only if for every closed unbounded subset C of δ and every sequence $\langle \mathcal{E}_\gamma \mid \gamma < \delta\rangle$ with $\varnothing \neq \mathcal{E}_\gamma \subseteq \mathcal{P}(\gamma)$ for all $\gamma < \delta$, there exist $\beta < \gamma$ in C and $E \in \mathcal{E}_\gamma$ with $E \cap \beta \in \mathcal{E}_\beta$.

Definition 1.16. Let "Ord *is essentially subtle*" denote the axiom schema stating that for every closed unbounded class C of ordinals and every class function $\mathcal{E}$ on the ordinals with the property that $\varnothing \neq \mathcal{E}(\gamma) \subseteq \mathcal{P}(\gamma)$ for all $\gamma \in \mathrm{Ord}$, there exist $\beta < \gamma$ in C and $E \in \mathcal{E}(\gamma)$ with $E \cap \beta \in \mathcal{E}(\beta)$.

Note that this principle implies that Ord is subtle, and, in the presence of a definable well-ordering of V, both principles are equivalent. Moreover, the assumption that Ord is essentially subtle is obviously downwards absolute to L, and hence both principles have the same consistency strength over ZFC. In contrast, the proof of [10, Theorem 5.6] produces a model of set-theory that witnesses that both principles are not equivalent over ZFC. The next result, proved in Section 9, provides a direct analog of Theorem 1.6 by showing that strongly unfoldable and $C^{(n+1)}$-strongly unfoldable cardinals are related, via weak SR, to the assumption that Ord is essentially subtle, as supercompact and $C^{(n)}$-extendible cardinals are related, via SR, to Vopěnka's Principle.

Theorem 1.17. *The following schemes of axioms are equivalent over* ZFC*:*

(1) Ord *is essentially subtle.*
(2) *For every natural number n, there exists a $C^{(n)}$-strongly unfoldable cardinal.*
(3) *For every natural number n, there exists a proper class of $C^{(n)}$-strongly unfoldable cardinals.*
(4) *For every natural number n and every class $\mathcal{C}$ of structures of the same type, there exists a cardinal $\kappa \in C^{(n)}$ with the property that $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds.*

Finally, we will also show that an analog of Theorem 1.7 holds for this region of the large cardinal hierarchy. The following theorem, which will be proved in Section 4, shows that subtle cardinals are related, via weak SR, to strongly unfoldable and $C^{(n+1)}$-strongly unfoldable cardinals, as Vopěnka cardinals are related, via SR, to supercompact and $C^{(n)}$-extendible cardinals.

Theorem 1.18. *The following statements are equivalent for every uncountable cardinal δ:*

(1) *δ is either subtle or a limit of subtle cardinals.*
(2) *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds.*

Corollary 1.19. *The following statements are equivalent for every uncountable cardinal δ:*

(1) *δ is the least subtle cardinal.*
(2) *δ is the least cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$, there exists a cardinal $\kappa < \delta$ such that the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds.* □

1.3. **Product Structural Reflection.** In analogy with the characterization (given essentially in [9], but outlined below) of strong, Σ_n-strong, and Woodin cardinals, in terms of *product* SR (see Definition 1.21 below), we work towards additional characterizations of strongly unfoldable, $C^{(n)}$-strongly unfoldable, and subtle cardinals in terms of *weak product* SR (see Definition 1.29 below).

Let us recall that for any set S of structures of the same type, the set-theoretic product $\prod S$ is the structure whose universe is the set of all functions f with domain S such that $f(\mathcal{A}) \in A$ for every $\mathcal{A} = \langle A, \ldots \rangle \in S$, and whose interpretation of constant, function and relation symbols is defined point-wise. In addition, note that, if X is a substructure of $\prod S$ and $\mathcal{A} \in S$, then the canonical projection map

$$p : X \longrightarrow \mathcal{A};\ f \mapsto f(\mathcal{A})$$

is a homomorphism.[3]

The following *Product Reflection Principle* (PRP) was introduced in [9, Definition 3.1]:

[3]As defined in [20, Section 1.2].

Definition 1.20 (Bagaria-Wilson, [9])**.** Given a class $\mathcal{C}$ of structures of the same type, the principle $\mathrm{PRP}_{\mathcal{C}}$ asserts that there is a subset S of $\mathcal{C}$ such that for $B \in \mathcal{C}$, there is a homomorphism from $\prod S$ to B.

The principle $\mathrm{PRP}_{\mathcal{C}}$ holds for every class $\mathcal{C}$ that is definable by a Σ_1-formula with parameters, as witnessed by sets of the form $V_\kappa \cap \mathcal{C}$, where $\kappa \in C^{(1)}$ has the property that V_κ contains the parameters used in the definition of $\mathcal{C}$ (see [9, Proposition 3.2]). Also, if κ is either a strong cardinal or a limit of strong cardinals, then $\mathrm{PRP}_{\mathcal{C}}$ holds for every class $\mathcal{C}$ that is defined by a Σ_2-formula with parameters in V_κ, as witnessed by $\mathcal{C} \cap V_\kappa$ (see [9, Proposition 3.3]). Moreover, the proof of [9, Theorem 4.1] shows that if κ is a cardinal such that $\mathcal{C} \cap V_\kappa$ is non-empty and witnesses $\mathrm{PRP}_{\mathcal{C}}$ for every class $\mathcal{C}$ that defined by a Π_1-formula without parameters, then there exists a strong cardinal less than or equal to κ. If the same holds for every class $\mathcal{C}$ that is defined by a Π_1-formula with parameters in V_κ, then κ is either a strong cardinal or a limit of strong cardinals. Hence, the following statements are equivalent for every infinite cardinal κ:

(1) κ is either a strong cardinal or a limit of strong cardinals.
(2) $\mathcal{C} \cap V_\kappa$ witnesses $\mathrm{PRP}_{\mathcal{C}}$ for all classes $\mathcal{C}$ of structures of the same type that are definable by a Π_1-formula (equivalently, by a Σ_2-formula) with parameters in V_κ.

In view of this equivalence, we may reformulate the principle PRP as a principle of Structural Reflection for products, as follows:

Definition 1.21. Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ denote the statement that $\mathcal{C} \cap V_\kappa \neq \varnothing$ and for every $B \in \mathcal{C}$, there exists a homomorphism from $\prod(\mathcal{C} \cap V_\kappa)$ to B.

The above equivalence can now be stated in the following way:

Theorem 1.22 ([9])**.** *The following statements are equivalent for every cardinal κ:*

(1) *κ is either a strong cardinal or a limit of strong cardinals.*
(2) *The principle $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Corollary 1.23. *The following statements are equivalent for every cardinal κ:*

(1) *κ is the least strong cardinal.*
(2) *κ is the least cardinal with the property that $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

More generally, recall the following strengthening of strongness introduced in [9, Definition 5.1]:

Definition 1.24 (Bagaria-Wilson, [9])**.** Let $n > 0$ be a natural number.

(1) Given an ordinal λ, a cardinal κ is λ-Σ_n*-strong* if for every class A that is definable by a Σ_n-formula without parameters, there is a transitive class M with $V_\lambda \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \lambda$ and $A \cap V_\lambda \subseteq j(A)$.
(2) A cardinal κ is Σ_n*-strong* if it is λ-Σ_n-strong for every ordinal λ.

Every strong cardinal is Σ_2-strong (see [9, Proposition 5.2]). Arguments contained in the proofs of [9, Claim 5.12] and [9, Theorem 5.13] now yield the following equivalence:

Theorem 1.25 ([9])**.** *Given a natural number $n > 1$, the following statements are equivalent for every cardinal κ:*

(1) *κ is either a Σ_n-strong cardinal or a limit of Σ_n-strong cardinals.*

(2) *The principle* $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *of structures of the same type that is definable by a* Σ_n*-formula with parameters in* V_κ.

Corollary 1.26. *Given a natural number* $n > 1$, *the following statements are equivalent for every cardinal* κ:

(1) κ *is the least* Σ_n*-strong cardinal.*
(2) κ *is the least cardinal with the property that* $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *of structures of the same type that is definable by a* Σ_n*-formula with parameters in* V_κ.

The results of [9] also show that the canonical variation of Vopěnka's Principle corresponding to the above large cardinal notions is the assumption that "Ord *is Woodin*". This axiom schema asserts that for every class A, there exists some cardinal κ which is A-strong, i.e., for every λ, there is an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \gamma$, $V_\gamma \subseteq M$ and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Theorem 1.27 ([9])**.** *The following statements are equivalent:*

(1) Ord *is Woodin.*
(2) *For every class* $\mathcal{C}$ *of structures of the same type there exists a cardinal* κ *with the property that the principle* $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ *holds.*
(3) *For every natural number* n, *there exists a* Σ_n*-strong cardinal.*
(4) *For every natural number* n, *there exists a proper class of* Σ_n*-strong cardinals.*

Using the characterization of a cardinal δ being a Woodin cardinal in terms of the existence of cardinals $\kappa < \delta$ that are A-strong, for every subset A of V_δ (see [22, Theorem 26.14]), we will extend the argument given in [9, Theorem 5.13] to prove a version of Theorem 1.7 for Woodinness and obtain the following characterization of Woodin cardinals in terms of PSR in Section 7:

Theorem 1.28. *The following statements are equivalent for every uncountable cardinal* δ:

(1) δ *is a Woodin cardinal.*
(2) *For every set* $\mathcal{C}$ *of structures of the same type with* $\mathcal{C} \subseteq V_\delta$, *there exists a cardinal* $\kappa < \delta$ *with the property that the principle* $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ *holds.*

1.4. **Weak Product Structural Reflection.** We now introduce a weakening of PSR and show that this principle produces analogous characterizations of strongly unfoldable, $C^{(n)}$-strongly unfoldable, and subtle cardinals.

Definition 1.29. Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ denote the statement that $\mathcal{C} \cap V_\kappa \neq \varnothing$ and for every substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality at most κ and every $B \in \mathcal{C}$, there exists a homomorphism from X to B.

Note that, if X is a substructure of $\prod(\mathcal{C} \cap V_\kappa)$, then the identity map $\mathrm{id} : X \longrightarrow \prod(\mathcal{C} \cap V_\kappa)$ is a homomorphism. In particular, for every class $\mathcal{C}$ and every cardinal κ, the principle $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ implies the principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$. In Section 6, we will prove the following theorem that yields another canonical characterization of strong unfoldability:

Theorem 1.30. *The following statements are equivalent for every cardinal* κ:

(1) κ *is either strongly unfoldable or a limit of strong cardinals.*
(2) *The principle* $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *of structures of the same type that is definable by a* Σ_2*-formula with parameters in* V_κ.

The fact that strong cardinals are strongly unfoldable now yields the following corollary:

Corollary 1.31. *The following statements are equivalent for every cardinal* κ:

(1) *κ is the least strongly unfoldable cardinal.*
(2) *κ is the least cardinal with the property that the principle* $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ *holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.* □

The generalization of the last theorem to $C^{(n)}$-strongly unfoldable cardinals is given by the following theorem, which shall be proved in Section 10:

Theorem 1.32. *Given a natural number $n > 1$, the following statements are equivalent for every cardinal κ:*

(1) *κ is either $C^{(n)}$-strongly unfoldable or a limit of Σ_{n+1}-strong cardinals.*
(2) *The principle* $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ *holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.*

The next result shows that the above pattern also holds for the class-version of WPSR. The proof will be given in Section 10.

Theorem 1.33. *The following schemes of axioms are equivalent over* ZFC*:*

(1) Ord *is essentially subtle.*
(2) *For every non-empty class $\mathcal{C}$ of structures of the same type, there exists a cardinal κ such that* $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ *holds.*

Finally, we also have the analog characterization of subtle cardinals in terms of weak product SR. The proof of this result is contained in Section 8.

Theorem 1.34. *The following statements are equivalent for every uncountable cardinal δ:*

(1) *δ is a subtle cardinal.*
(2) *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle* $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ *holds.*

2. Vopěnka cardinals

In this section, we present the proof of Theorem 1.7. Recall that an inaccessible cardinal δ is a *Vopěnka cardinal* if for every set $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$ of structures of the same type, there exist distinct $A, B \in \mathcal{C}$ with an elementary embedding from A to B.

Lemma 2.1. *If δ is a Vopěnka cardinal and $\mathcal{C} \subseteq V_\delta$ is a set of structures of the same type, then there exists a cardinal $\kappa < \delta$ with the property that* $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ *(and hence also* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *and* $\mathrm{VSR}_{\mathcal{C}}(\kappa)$*) holds.*

Proof. Assume that δ is a Vopěnka cardinal and $\mathcal{C} \subseteq V_\delta$ is a set of structures of the same type. By [22, Exercise 24.19], there is a cardinal $\kappa < \delta$ that is *η-extendible for $\mathcal{C}$*, for every $\eta < \delta$, i.e., for every ordinal $\kappa < \eta < \delta$, there is an ordinal ζ and an elementary embedding

$$j : \langle V_\eta, \in, \mathcal{C} \cap V_\eta \rangle \longrightarrow \langle V_\zeta, \in, \mathcal{C} \cap V_\zeta \rangle$$

with $\mathrm{crit}\,(j) = \kappa$ and $j(\kappa) > \eta$. Now, fix a structure B in $\mathcal{C}$ and let $\kappa < \eta < \delta$ be a limit ordinal such that $B \in V_\eta$. Since B is an element of $\mathcal{C} \cap V_{j(\kappa)}$ and j induces an elementary embedding of B into $j(B)$, elementarity yields an element A of $\mathcal{C} \cap V_\kappa$ and an elementary embedding of A into B. Moreover, since κ is inaccessible, we know that B is contained in H_κ. These computations show that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds. □

Lemma 2.2. *If δ is a singular cardinal, then there exists a class $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$ and the property that the principles* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *and* $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ *fail for every cardinal $\kappa < \delta$.*

Proof. First, assume that $\operatorname{cof}(\delta) = \omega$ and fix a strictly increasing sequence $\langle \delta_n \mid n < \omega \rangle$ of infinite cardinals that is cofinal in δ. Let $\mathcal{L}$ denote the first-order language that extends the language $\mathcal{L}_\in$ of set theory by a constant symbol. For every $n < \omega$, we let A_n denote the $\mathcal{L}$-structure $\langle \delta_n, \in, n \rangle$. Define $\mathcal{C} = \{A_n \mid n < \omega\} \subseteq V_\delta$. Note that for all $m < n < \omega$, there is no elementary embedding of A_m into A_n, because such an embedding would map m to n and this would contradict the elementarity of the given map. But this directly shows that $\mathrm{SR}_{\mathcal{C}}(\kappa)$, and also $\mathrm{VSR}_{\mathcal{C}}(\kappa)$, fail for every cardinal $\kappa < \delta$.

Now, assume that $\operatorname{cof}(\delta)$ is uncountable and fix a strictly increasing sequence $\langle \delta_\xi \mid \xi < \operatorname{cof}(\delta) \rangle$ of cardinals greater than $\operatorname{cof}(\delta)$ that is cofinal in δ. Define $\mathcal{L}$ to be the first-order language that extends $\mathcal{L}_\in$ by two constant symbols. Given $\xi < \operatorname{cof}(\delta)$, define A^ξ to be the $\mathcal{L}$-structure $\langle V_{\delta_\xi}, \in, \operatorname{cof}(\delta), \xi \rangle$. Define $\mathcal{C} = \{A^\xi \mid \xi < \operatorname{cof}(\delta)\} \subseteq V_\delta$.

Claim. *If $\xi, \zeta < \operatorname{cof}(\delta)$ have the property that there exists an elementary embedding from A^ξ to A^ζ, then $\xi = \zeta$.*

Proof of the Claim. Let $j : V_{\delta_\xi} \longrightarrow V_{\delta_\zeta}$ be an elementary embedding with $j(\operatorname{cof}(\delta)) = \operatorname{cof}(\delta)$ and $j(\xi) = \zeta$. Then $j(V_{\operatorname{cof}(\delta)}) = V_{\operatorname{cof}(\delta)}$ and the map $j \restriction V_{\operatorname{cof}(\delta)} : V_{\operatorname{cof}(\delta)} \longrightarrow V_{\operatorname{cof}(\delta)}$ is an elementary embedding. The fact that $\operatorname{cof}(\delta)$ is uncountable then implies that j has unboundedly many fixed points below $\operatorname{cof}(\delta)$ and this allows us to use the *Kunen Inconsistency* to conclude that $j \restriction V_{\operatorname{cof}(\delta)} = \mathrm{id}_{V_{\operatorname{cof}(\delta)}}$. In particular, we know that $\xi = j(\xi) = \zeta$. □

The above claim directly shows that the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$, and also $\mathrm{VSR}_{\mathcal{C}}(\kappa)$, fail for every cardinal $\kappa < \delta$. □

Proposition 2.3. *If δ is an uncountable cardinal that is not a limit of inaccessible cardinals, then there exists a class $\mathcal{C}$ of structures of the same type, which is Π_1-definable in V_δ with an ordinal as a parameter, and such that the principles $\mathrm{SR}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ fail for every cardinal $\kappa < \delta$.*

Proof. Fix an ordinal $\lambda < \delta$ with the property that there are no inaccessible cardinals between λ and δ. Let $\mathcal{L}$ denote the first-order language that extending $\mathcal{L}_\in$ by a constant symbol and define $\mathcal{C}$ to be the set of all $\mathcal{L}$-structures of the form $\langle V_{\gamma+2}, \in, \lambda \rangle$ with $\lambda < \gamma < \delta$. Then $\varnothing \neq \mathcal{C} \subseteq V_\delta$ and $\mathcal{C}$ is definable in V_δ by a Π_1-formula with parameter λ. Assume, towards a contradiction, that there is a cardinal $\kappa < \delta$ with the property that either $\mathrm{SR}_{\mathcal{C}}(\kappa)$ or $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ holds. Then there is an ordinal $\lambda < \gamma < \kappa$ with the property that there exists an elementary embedding $j : V_{\gamma+2} \longrightarrow V_{\kappa+2}$ with $j(\lambda) = \lambda$. Since elementarity ensures that $j(\gamma) = \kappa$, we know that j is non-trivial. Moreover, we have $j(V_{\lambda+2}) = V_{\lambda+2}$ and the *Kunen Inconsistency* ensure that the elementary embedding $j \restriction V_{\lambda+2} : V_{\lambda+2} \longrightarrow V_{\lambda+2}$ is trivial. This allows us to conclude that $\operatorname{crit}(j)$ is an inaccessible cardinal greater than λ, a contradiction. □

Proof of Theorem 1.7. The implication (1) ⇒ (2) is given by Lemma 2.1. In order to prove the implication (2) ⇒ (1), first notice that of Lemma 2.2 and Proposition 2.3 shows that (2) implies that δ is inaccessible. The implication (2) ⇒ (1) then directly follows from the simple observation that for every set $\mathcal{C} \in V_{\delta+1} \setminus V_\delta$ of structures of the same type and every cardinal $\kappa < \delta$, both $\mathrm{SR}_{\mathcal{C}}(\kappa)$ and $\mathrm{VSR}_{\mathcal{C}}(\kappa)$ imply that there exist distinct $A, B \in \mathcal{C}$ with an elementary embedding from A to B. □

3. Strongly unfoldable reflection

We shall next demonstrate the connection between strong unfoldability and the validity of the principle WSR for Σ_2-definable classes by proving Theorem 1.10. The starting point is the following characterization of the elements of the class $C^{(n)}$-cardinals through the principle SR^{--}.

Lemma 3.1. *Given a natural number $n > 1$, the following statements are equivalent for every cardinal κ:*

1. *The cardinal κ is an element of $C^{(n)}$.*
2. *The principle $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ holds for every non-empty class $\mathcal{C}$ of structures of the same type that is definable by a Σ_n-formula with parameters in V_κ.*
3. *Every non-empty class of structures of the same type that is definable by a Σ_n-formula with parameters in H_κ contains an element of V_κ.*
4. *Every non-empty class of structures of the same type that is definable by a Σ_n-formula with parameters in H_κ contains a structure of cardinality less than κ.*
5. *Every non-empty class of structures of the same type that is definable by a Σ_n-formula with parameters in V_κ contains an element of H_κ.*
6. *Every non-empty class of structures of the same type that is definable by a Σ_n-formula with parameters in V_κ contains an element of V_κ.*

Proof. First, assume that (1) holds. Fix a Σ_n-formula $\varphi(v_0, v_1)$, and $a \in V_\kappa$ with the property that the class $\mathcal{C} = \{A \mid \varphi(A, a)\}$ is non-empty and consists of structures of the same type. Then the Σ_n-statement $\exists v_0 \varphi(v_0, a)$ holds in V and our assumptions imply that it also holds in V_κ. Hence, there is $A \in V_\kappa$ with the property that $\varphi(A, a)$ holds in V_κ. Our assumption now implies that $\varphi(A, a)$ holds in V. But our assumption also ensures that $H_\kappa = V_\kappa$ and hence we know that $\mathcal{C} \cap H_\kappa \neq \emptyset$. This shows that (2), and also (5) and (6), hold.

Next, assume that either (3), or (4), or (6) holds, and we shall prove (1).

Claim. *If ρ is an ordinal with the property that the set $\{\rho\}$ is definable by a Σ_n-formula with parameters in H_κ, then $\rho < \kappa$.*

Proof of the Claim. Let $\mathcal{L}$ denote the trivial first-order language and let B be the unique $\mathcal{L}$-structure with domain ρ. Then the class $\mathcal{C} = \{B\}$ is definable by a Σ_n-formula with parameters in H_κ and hence each of our assumptions allows us to conclude that $\rho < \kappa$. □

The above claim implies that κ is a limit point of $C^{(n-1)}$, because for every $\alpha < \kappa$, the least cardinal in $C^{(n-1)}$ greater than α is definable by a Σ_n-formula with parameter α. To show (1), fix a Σ_n-formula $\exists x \psi(x, y)$, with ψ being Π_{n-1}, and some $a \in V_\kappa$ with the property that $\exists x \psi(x, a)$ holds. Let ρ denote the least element of $C^{(n-1)}$ such that $a \in V_\rho$ and for some $b \in V_\rho$, $\psi(b, a)$ holds in V_ρ. Then the set $\{\rho\}$ is definable by a Σ_n-formula with parameter $a \in H_\kappa$ and the previous claim shows that $\rho < \kappa$. Since $\kappa \in C^{(n-1)}$ and Π_{n-1}-formulas are upwards absolute between V_ρ and V_κ, we can now conclude that $\psi(b, a)$ holds in V_κ, for some b. This shows that (1) holds.

The above implications now yield the statement of the lemma, because (2) directly implies (4), and (5) obviously implies all of (2), (3), and (6). □

Let us also observe that Clause (1) of the above lemma is also equivalent to each of (2), (3), (4), (5), (6), restricted to Σ_n-definable classes $\mathcal{C}$ that are closed under isomorphic images. The reason is that the above claim also holds under this restriction by taking $\mathcal{C}$ in the proof to be the class of all $\mathcal{L}$-structures of cardinality ρ, which is closed under isomorphic images, instead of the singleton $\{B\}$. Using this fact, the following proposition shows that, when restricted to classes $\mathcal{C}$ that are

Σ_n-definable with $n > 1$ and closed under isomorphic images, the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ is equivalent to $\mathrm{VSR}^-_{\mathcal{C}}(\kappa)$ (see Definition 1.8).

We are now ready to prove the equivalences stated in Theorem 1.10. The following arguments rely on Magidor's characterization of supercompactness in [25] and the analysis of strong unfoldability provided by the results of [23] and [24].

Proof of Theorem 1.10. First, note that Lemma 3.1 directly yields the implication (3) $\Rightarrow$ (2).

Next, assume that κ is a strongly unfoldable cardinal. Then κ is an element of $C^{(2)}$ (see [23, Section 3] or Proposition 9.1 below). Fix a class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ and a structure B in $\mathcal{C}$ of cardinality κ. Pick a Σ_2-formula $\varphi(v_0, v_1)$ and an element z of H_κ with $\mathcal{C} = \{A \mid \varphi(A, z)\}$, and let $\theta > \kappa$ be a cardinal with the property that $B \in H_\theta$ and $\varphi(B, z)$ holds in H_θ. Using [23, Theorem 1.3 and Lemma 2.1] (see also Theorem 9.4 below), we can find cardinals $\bar{\kappa} < \bar{\theta} < \kappa$, an elementary submodel X of $H_{\bar{\theta}}$ with $(\bar{\kappa}+1) \cup {}^{<\bar{\kappa}}X \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, and $B, z \in \mathrm{ran}(j)$. We then have $j \restriction (H_{\bar{\kappa}} \cap X) = id_{H_{\bar{\kappa}} \cap X}$, and therefore $z \in H_{\bar{\kappa}}$ and $j(z) = z$. Pick $A \in X$ with $j(A) = B$. Then elementarity and Σ_1-absoluteness implies that $\varphi(A, z)$ holds and hence A is an element of $\mathcal{C}$. Since A belongs to X, and therefore to $H_{\bar{\theta}}$, it also belongs to H_κ. Moreover, since $\bar{\kappa}$ is a subset of X, and A has cardinality $\bar{\kappa}$ in X, the restriction of j to A yields an elementary embedding of A into B. This shows that $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds in this case.

Now, assume that κ is a limit of supercompact cardinals. Then κ is an element of $C^{(2)}$. Moreover, Theorem 1.2 shows that $\mathrm{HSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ. In combination with the above computations, this yields the implication (1) $\Rightarrow$ (3).

Finally, assume that κ is a cardinal that is not strongly unfoldable and the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ. By Lemma 3.1, we know that κ is an element of $C^{(2)}$. In particular, κ is a limit cardinal and the set V_κ has cardinality κ.

Claim. *If $\theta > \kappa$ is a cardinal, $y \in V_\kappa$ and $z \in H_\theta$, then there are cardinals $\bar{\kappa} < \bar{\theta} < \kappa$ with $y \in V_{\bar{\kappa}}$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j(\bar{\kappa}) = \kappa$, $j(y) = y$ and $z \in \mathrm{ran}(j)$.*

Proof of the Claim. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}_\in$ by three constant symbols and let $\mathcal{C}$ denote the class of all $\mathcal{L}$-models of the form $\langle M, \in, \mu, a, y\rangle$ such that μ is a cardinal in $C^{(1)}$, $y \in V_\mu$ and there exists a cardinal $\nu > \mu$ and an elementary submodel X of H_ν with $V_\mu \cup \{\mu\} \subseteq X$ and the property that M is the transitive collapse of X. It is then easy to see that the class $\mathcal{C}$ is definable by a Σ_2-formula with parameter y. Now, let Y be an elementary submodel of H_θ of cardinality κ with $V_\kappa \cup \{\kappa, z\}$ and let $\tau : Y \longrightarrow N$ denote the corresponding transitive collapse. Then θ and Y witness that $B = \langle N, \in, \kappa, \tau(z), y\rangle$ is an element of $\mathcal{C}$ of cardinality κ. Our assumptions then yield cardinals $\bar{\kappa} < \bar{\theta}$ with $\bar{\kappa} \in C^{(1)} \cap \kappa$, an elementary submodel X of $H_{\bar{\theta}}$ of cardinality less than κ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $i : M \longrightarrow N$ with $i(\bar{\kappa}) = \kappa$, $i(y) = y$ and $\tau(z) \in \mathrm{ran}(i)$. Since $\kappa \in C^{(2)}$ and $M \in V_\kappa$, we may assume that $\bar{\theta} < \kappa$. Now let $\pi : X \longrightarrow M$ denote the transitive collapse, and define

$$j \ = \ \tau^{-1} \circ i \circ \pi : X \longrightarrow H_\theta.$$

Then j is an elementary embedding with $j(\bar{\kappa}) = \kappa$, $j(y) = y$ and $z \in \mathrm{ran}(j)$. □

Claim. *If $\theta > \kappa$ is a cardinal, $y \in V_\kappa$ and $z \in H_\theta$, then there are cardinals $\bar{\kappa} < \bar{\theta} < \kappa$ with $y \in V_{\bar{\kappa}}$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j(\bar{\kappa}) = \kappa$, $j(y) = y$, $z \in \operatorname{ran}(j)$ and $j \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$.*

Proof of the Claim. Since κ is not strongly unfoldable, we can apply [23, Lemma 2.1] (see also Theorem 9.4 3 below and the Remark 9.5 that follows) to find a cardinal $\vartheta > \theta$ and $z' \in V_\vartheta$ such that for all cardinals $\bar{\kappa} < \bar{\vartheta}$ and all elementary submodels X of $H_{\bar{\vartheta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, there is no elementary embedding $j : X \longrightarrow H_\vartheta$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, and $z, z', \theta \in \operatorname{ran}(j)$. An application of our previous claim now yields cardinals $\bar{\kappa} < \bar{\vartheta} < \kappa$ with $y \in V_{\bar{\kappa}}$, an elementary submodel Y of $H_{\bar{\vartheta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq Y$ and an elementary embedding $i : Y \longrightarrow H_\vartheta$ with $i(\bar{\kappa}) = \kappa$, $i(y) = y$ and $z, z', \theta \in \operatorname{ran}(i)$. Therefore, we must have $i \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$. Pick $\bar{\theta} \in Y$ with $i(\bar{\theta}) = \theta$. Then elementarity implies that $\bar{\theta}$ is a cardinal. Set $X = Y \cap H_{\bar{\theta}}$ and $j = i \restriction X$. In this situation, we can conclude that $\bar{\kappa} < \bar{\theta} < \kappa$, X is an elementary submodel of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and $j : X \longrightarrow H_\theta$ is an elementary embedding with $j(\bar{\kappa}) = \kappa$, $j(y) = y$, $z \in \operatorname{ran}(j)$ and $j \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$. □

Claim. *There are unboundedly many cardinals below κ that are α-supercompact for every $\alpha < \kappa$.*

Proof of the Claim. Fix an uncountable regular cardinal $\rho < \kappa$ and a cardinal θ in $C^{(1)}$ above κ. By our previous claim, we can find cardinals $\rho < \bar{\kappa} < \bar{\theta} < \kappa$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $i : X \longrightarrow H_\theta$ with $i(\bar{\kappa}) = \kappa$, $j(\rho) = \rho$ and $i \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$. Set $j = i \restriction V_{\bar{\kappa}} : V_{\bar{\kappa}} \longrightarrow V_\kappa$. Our setup then ensures that j is a non-trivial elementary embedding. Since the *Kunen Inconsistency* implies that $i \restriction V_\rho = \mathrm{id}_\rho$, we know that $\operatorname{crit}(j) > \rho$. Moreover, [25, Lemma 2] directly shows that $\operatorname{crit}(j)$ is α-supercompact for all $\alpha < \bar{\kappa}$. By elementarity, we know that $\bar{\theta}$ is an element of $C^{(1)}$ and this shows that, in X, the cardinal $\operatorname{crit}(j)$ is α-supercompact for all $\alpha < \bar{\kappa}$. But this allows us to conclude that $j(\operatorname{crit}(j))$ is a cardinal in the interval (ρ, κ) that is α-supercompact for all $\alpha < \kappa$. □

Claim. *Every cardinal below κ that is α-supercompact for every $\alpha < \kappa$ is supercompact.*

Proof of the Claim. Let $\mu < \kappa$ be a cardinal that is α-supercompact for all $\alpha < \kappa$, let $\lambda > \kappa$ be an ordinal and let $\theta > \lambda$ be an element of $C^{(1)}$. By our first claim, there exist cardinals $\mu < \bar{\kappa} < \bar{\theta} < \kappa$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j(\bar{\kappa}) = \kappa$, $j(\mu) = \mu$ and $\lambda \in \operatorname{ran}(j)$. Pick $\bar{\lambda} \in X$ with $j(\bar{\lambda}) = \lambda$. Then $\mu < \bar{\lambda} < \kappa$ and μ is $\bar{\lambda}$-supercompact. Since elementarity ensures that $\bar{\theta}$ is an element of $C^{(1)}$, we now know that μ is $\bar{\lambda}$-supercompact in X and this shows that μ is λ-supercompact. □

The combination of the above claims now shows that κ is a limit of supercompact cardinals in this case. In particular, these arguments prove the implication (2) ⇒ (1). □

In addition to Corollary 1.11, Theorem 1.10 can directly be used to derive several interesting equivalences. For example, it shows that for cardinals that are not strongly unfoldable, the validity of the principle SR for Σ_2-definable classes is equivalent to the validity of the principle WSR, and also to HSR^-, for these classes. In particular, this equivalence holds for all singular cardinals.

Corollary 3.2. *The following statements are equivalent for every cardinal κ that is not strongly unfoldable:*

1. *The cardinal κ is a limit of supercompact cardinals.*
2. *The principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
3. *The principle $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.* □

4. Subtle reflection

We will next give a proof of Theorem 1.18. As a side product of the developed techniques, we will obtain a combinatorial characterization of cardinals that are either subtle or limits of subtle cardinals (see Corollary 4.5 below). Recall that, given an ordinal δ, a sequence $\langle E_\gamma \mid \gamma < \delta \rangle$ is a *δ-list* if $E_\gamma \subseteq \gamma$ holds for every $\gamma < \delta$.

Lemma 4.1. *Let δ be an element of $C^{(1)}$ with the property that for every δ-list $\langle E_\gamma \mid \gamma < \delta \rangle$ and every $\rho < \delta$, there exist cardinals $\rho < \mu < \nu < \delta$ with $E_\mu \subseteq E_\nu$. If $\mathcal{C}$ is a non-empty set of structures of the same type with $\mathcal{C} \subseteq V_\delta$, then there exists a cardinal $\kappa < \delta$ with the property that $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds.*

Proof. Let $\mathcal{L}$ denote the signature of $\mathcal{C}$ and let ρ denote the minimal cardinality of structures in $\mathcal{C}$. Since $\delta \in C^{(1)}$, we know that $\rho < \delta$. In addition, let $\mathsf{Sat}_{\mathcal{L}}$ denote the formalized satisfaction relation for $\mathcal{L}$ and let $\mathsf{Fml}_{\mathcal{L}}$ denote the set of Gödel numbers of formalized $\mathcal{L}$-formulas. Assume, towards a contradiction, that the principle $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ fails for every cardinal $\rho < \kappa < \delta$. Given a cardinal $\rho < \kappa < \delta$, we can now fix a structure $A_\kappa \in \mathcal{C}$ of cardinality κ with the property that there is no elementary embedding of a structure in $\mathcal{C}$ of cardinality less than κ into A_κ. Let b_κ be a bijection between κ and the domain of A_κ. Let $\langle E_\gamma \mid \gamma < \delta \rangle$ be a δ-list with the property that for every cardinal $\rho < \kappa < \delta$, the set E_κ consists of all ordinals of the form[4]

$$\prec \ell, \alpha_0, \ldots, \alpha_{k-1} \succ$$

for some $\ell \in \mathsf{Fml}_{\mathcal{L}}$ that codes a formula with k free variables and $\alpha_0, \ldots, \alpha_{k-1} < \kappa$ with

$$\mathsf{Sat}_{\mathcal{L}}(A_\kappa, \ell, \langle b_\kappa(\alpha_0), \ldots, b_\kappa(\alpha_{k-1}) \rangle).$$

By our assumptions, there exist cardinals $\rho < \mu < \nu < \delta$ with the property that $E_\mu \subseteq E_\nu$.

Claim. *The map $b_\nu \circ b_\mu^{-1}$ is an elementary embedding of A_μ into A_ν.*

Proof of the Claim. Fix an $\mathcal{L}$-formula $\varphi(v_0, \ldots, v_{k-1})$ and $\alpha_0, \ldots, \alpha_{k-1} < \mu$ such that

$$A_\mu \models \varphi(b_\mu(\alpha_0), \ldots, b_\mu(\alpha_{k-1})).$$

If ℓ_φ is the canonical element of $\mathsf{Fml}_{\mathcal{L}}$ corresponding to φ, then we have

$$\prec \ell_\varphi, \alpha_0, \ldots, \alpha_{k-1} \succ \; \in \; E_\mu \; \subseteq \; E_\nu$$

and this implies that

$$A_\nu \models \varphi(b_\nu(\alpha_0), \ldots, b_\nu(\alpha_{k-1})).$$

By also considering negated formulas, the derived implication yields the statement of the claim. □

Since the above claim yields a contradiction, we now know that the principle $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ holds for some cardinal $\rho < \kappa < \delta$. Moreover, our setup also ensures that $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ holds. □

The above lemma directly yields a proof of the forward implication of Theorem 1.18.

Corollary 4.2. *If δ is either a subtle cardinal or a limit of subtle cardinals and $\mathcal{C}$ is a non-empty class of structures of the same type with $\mathcal{C} \subseteq V_\delta$, then there exists a cardinal $\kappa < \delta$ with the property that $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds.*

[4] Here, we let $\prec \cdot, \ldots, \cdot \succ : \mathrm{Ord}^n \longrightarrow \mathrm{Ord}$ denote the *iterated Gödel pairing function.*

Proof. Let $\langle E_\gamma \mid \gamma < \delta\rangle$ be a δ-list and let ρ be an ordinal smaller than δ. If δ is a subtle cardinal, then we can consider the closed unbounded subset of all cardinals in the interval (ρ, δ) and find elements $\mu < \nu$ of this set with the property that $E_\mu = E_\nu \cap \mu$. In the other case, if δ is a limit of subtle cardinals, then we can fix a subtle cardinal δ_0 with $\rho < \delta_0 < \delta$ and repeat the above argument with the δ_0-list $\langle E_\gamma \mid \gamma < \delta_0\rangle$ to find cardinals $\mu < \nu$ with $\rho < \mu < \nu < \delta_0$ and $E_\mu = E_\nu \cap \mu$. Since our assumptions imply that δ is an element of $C^{(1)}$, these computations allow us to apply Lemma 4.1 to derive the desired conclusion. □

Lemma 4.3. *Let ρ be an ordinal with the property that there exists a cardinal ε greater than ρ such that for every $A \subseteq V_\varepsilon$ and every sufficiently large cardinal θ, there exist cardinals $\rho < \bar{\kappa} < \kappa < \varepsilon$, an elementary submodel X of H_θ with $\bar{\kappa} \cup \{\bar{\kappa}, A\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $j(A) = A$. If δ is the least cardinal greater than ρ with this property, then δ is a subtle cardinal.*

Proof. We start our proof with the following claim.

Claim. *δ is a limit cardinal.*

Proof of the Claim. First, note that our assumptions imply that δ is an uncountable cardinal. Next, assume towards a contradiction that $\delta = \gamma^+$ for some infinite cardinal γ. Let A be a subset of V_γ and let θ be a cardinal that is sufficiently large with respect to δ and the above property. Then we can find cardinals $\rho < \bar{\kappa} < \kappa < \delta$, an elementary submodel X of H_θ with $\bar{\kappa} \cup \{\gamma, \delta, \bar{\kappa}, A\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(\gamma) = \gamma$, $j(\delta) = \delta$ and $j(A) = A$. Since $\kappa \leq \gamma$, $j(\gamma) = \gamma$ and $j(\bar{\kappa}) = \kappa$, we know that $\kappa < \gamma$. But this shows that γ also possesses the relevant property, contradicting the minimality of δ. □

Let $\vec{E} = \langle E_\gamma \mid \gamma < \delta\rangle$ be a δ-list and let C be a closed unbounded subset of δ that consists of cardinals. Now, pick a well-ordering $\lhd$ of V_δ and a cardinal $\theta > \beth_\delta$ such that θ is sufficiently large and H_θ is sufficiently elementary in V. In this situation, our assumptions yields cardinals $\rho < \bar{\kappa} < \kappa < \delta$, an elementary submodel X of H_θ with $\bar{\kappa} \cup \{\bar{\kappa}, C, \vec{E}, \lhd\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(C) = C$, $j(\vec{E}) = \vec{E}$ and $j(\lhd) = \lhd$.

Claim. *$\bar{\kappa} \in C$.*

Proof of the Claim. Assume, towards a contradiction, that $\bar{\kappa}$ is not an element of C. Define

$$\gamma \ = \ \min(C \setminus \bar{\kappa}) \ \in \ X \cap (\bar{\kappa}, \delta).$$

If $C \cap \bar{\kappa} = \emptyset$, then $\gamma = \min(C)$ and the ordinal γ is definable in H_θ by a formula with parameter C. In the other case, if $C \cap \bar{\kappa} \neq \emptyset$, then $\sup(C \cap \bar{\kappa}) < \bar{\kappa}$, $\gamma = \min(C \setminus (\sup(C \cap \bar{\kappa}) + 1))$ and therefore the ordinal γ is definable in H_θ by a formula with parameters in $\bar{\kappa} \cup \{C\}$. In both cases, the ordinal γ is definable in H_θ by a formula whose parameters are elements of X and that are fixed by the embedding j. This shows that γ is a cardinal greater than $\bar{\kappa}$ that is fixed by j. Since $j(\bar{\kappa}) = \kappa$, this also shows that γ is bigger than κ.

Using the minimality of δ, we can now find a cardinal $\vartheta > \beth_\gamma$ and a subset A of V_γ, with the property that for all cardinals $\rho < \bar{\mu} < \mu < \gamma$ and every elementary submodel Y of H_ϑ with $\bar{\mu} \cup \{\bar{\mu}, A\} \subseteq Y$, there is no elementary embedding $i : Y \longrightarrow H_\vartheta$ with $i \restriction \bar{\mu} = \mathrm{id}_{\bar{\mu}}$, $i(\bar{\mu}) = \mu$ and $j(A) = A$. Let ϑ denote the least cardinal greater than $\beth_\gamma$ such that there exists a subset of V_γ with these properties and let A denote the $\lhd$-least subset of V_γ witnessing this statement with respect to γ. Since H_θ was chosen sufficiently elementary in V, it follows that ϑ and A are elements of H_θ and both sets are definable in H_θ from the parameters γ and $\lhd$. But this implies that $\vartheta, A \in X$,

$j(\vartheta) = \vartheta$ and $j(A) = A$. Set $Y = H_\vartheta \cap X$ and $i = j \restriction Y : Y \longrightarrow H_\vartheta$. Since our assumptions on θ ensure that $H_\vartheta \in X$, it follows that Y is an elementary submodel of H_ϑ with $\bar\kappa \cup \{\bar\kappa, A\} \subseteq Y$ and i is an elementary embedding with $i \restriction \bar\kappa = \mathrm{id}_{\bar\kappa}$, $i(\bar\kappa) = \kappa$ and $j(A) = A$, contradicting the properties of ϑ and A. □

By elementarity, the above claim directly implies that κ is an element of C. Moreover, our setup ensures that $E_{\bar\kappa} = j(E_{\bar\kappa}) \cap \bar\kappa = E_\kappa \cap \bar\kappa$.

The above computations show that δ is a limit cardinal with the property that for every δ-list $\langle E_\gamma \mid \gamma < \delta \rangle$ and every closed unbounded set C in δ that consists of cardinals, there are $\mu < \nu$ in C with $E_\mu = E_\nu \cap \nu$. This directly implies that δ has uncountable cofinality and we can conclude that for every δ-list $\langle E_\gamma \mid \gamma < \delta \rangle$ and every closed unbounded set C in δ, there are $\mu < \nu$ in C with $E_\mu = E_\nu \cap \nu$. □

Proposition 4.4. *If δ is an uncountable cardinal that is not an element of $C^{(1)}$, then there exists a class $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$ and the property that the principle $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ fails for every cardinal $\kappa \leq \delta$.*

Proof. Our assumptions on δ directly yield an ordinal $\beta < \delta$ with the property that the set V_β has cardinality greater than δ. Let $\mathcal{C}$ denote the set of all $\mathcal{L}_\in$-structures of the form $\langle V_\gamma, \in \rangle$ with $\beta \leq \gamma < \delta$. Then $\varnothing \neq \mathcal{C} \subseteq V_\delta$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ fails for every cardinal $\kappa \leq \delta$, because $\mathcal{C}$ contains no structures of cardinality less than or equal to δ. □

Proof of Theorem 1.18. Let δ be an uncountable cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ such that the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds. Then Proposition 4.4 shows that δ is an element of $C^{(1)}$. Assume, towards a contradiction, that κ is neither a subtle cardinal nor a limit of subtle cardinals. Pick an uncountable regular cardinal $\rho < \delta$ with the property that the interval $(\rho, \delta]$ contains no subtle cardinals.

Claim. *If $A \subseteq V_\delta$ and $\theta > \beth_\delta$ is a cardinal, then there exist cardinals $\rho < \bar\nu < \nu < \delta$, an elementary submodel X of H_θ with $\bar\nu \cup \{\bar\nu, A\} \subseteq X$ and an elementary embedding $j : X \longrightarrow H_\theta$ with $j \restriction \bar\nu = \mathrm{id}_{\bar\nu}$, $j(\bar\nu) = \nu$ and $j(A) = A$.*

Proof of the Claim. Let $\mathcal{L}$ denote the first order-language that extends $\mathcal{L}_\in$ by three constant symbols and let $\mathcal{C}$ denote the set of all $\mathcal{L}$-structures of the form $\langle M, \in, \rho, \mu, B \rangle$ such that μ is a cardinal strictly between $\beth_\rho$ and δ, and there exists an elementary submodel X of H_θ of cardinality μ with the property that $V_\rho \cup \mu \cup \{\mu, A\} \subseteq X$, M is the transitive collapse of X and $\pi(A) = B$, where $\pi : X \longrightarrow M$ denotes the corresponding collapsing map. We then have $\mathcal{C} \subseteq V_\delta$ and, by our assumptions on δ, there exists a cardinal $\kappa < \delta$ with the property that $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds. Since every structure in $\mathcal{C}$ has cardinality greater than $\beth_\rho$, we know that $\kappa > \rho$.

Now, pick an elementary submodel Y of H_θ of cardinality κ with $V_\rho \cup \kappa \cup \{\kappa, A\} \subseteq Y$ and let $\pi : Y \longrightarrow N$ denote the corresponding transitive collapse. Then $\langle N, \in, \rho, \kappa, \pi(A) \rangle$ is a structure in $\mathcal{C}$ of cardinality κ. By our assumptions, we can now find a cardinal $\bar\kappa$ strictly between $\beth_\rho$ and κ, an elementary submodel X of H_θ of cardinality $\bar\kappa$ with $V_\rho \cup \bar\kappa \cup \{\bar\kappa, A\} \subseteq X$ and an elementary embedding i of $\langle M, \in, \rho, \bar\kappa, \bar\pi(A) \rangle$ into $\langle N, \in, \rho, \kappa, \pi(A) \rangle$, where $\bar\pi : X \longrightarrow M$ denotes the corresponding transitive collapse. We define

$$j \ = \ \pi^{-1} \circ i \circ \bar\pi : X \longrightarrow H_\theta.$$

Then j is an elementary embedding with $j(\rho) = \rho$, $j(\bar\kappa) = \kappa$ and $j(A) = A$. Moreover, since ρ is an uncountable regular cardinal, $V_\rho \subseteq X$ and $j(V_\rho) = V_\rho$, the *Kunen Inconsistency* implies that

$j \upharpoonright V_\rho = \mathrm{id}_\rho$. Let $\bar{\nu}$ denote the least ordinal in X that is moved by j. Then $\bar{\nu}$ is a cardinal with $\rho < \bar{\nu} \leq \bar{\kappa}$. Set $\nu = j(\bar{\nu})$. We then know that ν is a cardinal with $\bar{\nu} < \nu \leq \kappa < \delta$. □

By Lemma 4.3, the above claim shows that the interval $(\rho, \delta]$ contains a subtle cardinal, contradicting our assumptions.

The above computations show that (2) in Theorem 1.18 implies (1) of the theorem. In the other direction, Corollary 4.2 states that (1) also implies (2). □

A combination of Theorem 1.18 and Lemma 4.1 yields the following corollary:

Corollary 4.5. *The following statements are equivalent for every cardinal δ in $C^{(1)}$:*

1. *δ is either subtle or a limit of subtle cardinals.*
2. *For every δ-list $\langle E_\gamma \mid \gamma < \delta \rangle$ and every $\rho < \delta$, there exist cardinals $\rho < \mu < \nu < \delta$ with $E_\mu \subseteq E_\nu$.* □

5. Extenders deriving from Product Reflection

The Product Reflection Principle (PRP) yields the existence of strong cardinals, via extenders derived from the principle holding for a single Π_1-definable class $\mathcal{C}$ of structures (see [9, Theorem 4.1]). In this section, we shall derive extenders in a more general context, including weak forms of Product Structural Reflection (WPSR) restricted to Π_1-definable classes of structures, which shall then be used in the proofs of Theorems 1.30 and 1.34, given below in Sections 6 and 8, respectively.

In the following, let $\mathcal{L}'$ denote a first-order language that extends the language of set theory by constants $\dot{\kappa}$ and $\dot{u}$, plus, possibly, finitely-many other predicate, constant, and function symbols. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}'$ by an $(n+1)$-ary predicate symbol $\dot{T}_\varphi$ for every $\mathcal{L}'$-formula $\varphi(v_0, \ldots, v_n)$ with $(n+1)$-many free variables. Define $\mathcal{S}_\mathcal{L}$ to be the class of all $\mathcal{L}$-structures A such that there exists a cardinal κ_A in $C^{(1)}$ and a limit ordinal $\theta_A > \kappa_A$ such that:

1. The domain of A is V_{θ_A+1}.
2. $\in^A = \in \upharpoonright V_{\theta_A+1}$, $\dot{\kappa}^A = \kappa_A$ and $\dot{u}^A = \theta_A$.
3. If $\varphi(v_0, \ldots, v_n)$ is an $\mathcal{L}'$-formula, then
$$\dot{T}^A_\varphi \ = \ \{\langle x_0, \ldots, x_n \rangle \in V^{n+1}_{\theta_A+1} \mid A \models \varphi(x_0, \ldots, x_n)\}.$$

Note that $\mathcal{S}_\mathcal{L}$ is definable by a Π_1-formula without parameters.

For the rest of the section, we assume that

- $\mathcal{C}$ be a subclass of $\mathcal{S}_\mathcal{L}$, and
- ζ is an ordinal with $\mathcal{C} \cap V_\zeta \neq \emptyset$.

We define
$$\kappa \ = \ \sup\{\kappa_A \mid A \in \mathcal{C} \cap V_\zeta\} \ \leq \ \zeta.$$
For every set x, we let f_x denote the unique function with domain $\mathcal{C} \cap V_\zeta$ such that that $f_x(A) = x$ for all A with $x \in V_{\kappa_A}$, and $f_x(A) = \dot{u}^A$ otherwise. In addition, let f^x denote the unique function with domain $\mathcal{C} \cap V_\zeta$ and such that $f^x(A) = x \cap V_{\kappa_A}$, for all A in the domain. For the remainder of this section, we also assume that

- X is a substructure of $\prod(\mathcal{C} \cap V_\zeta)$ with $f_x \in X$, for all $x \in V_\kappa$,
- B is an element of $\mathcal{S}_\mathcal{L}$ with $\kappa_B \geq \kappa$, and
- $h : X \longrightarrow B$ is a homomorphism.

Lemma 5.1. *The following hold:*

(1) *If* $\varphi(v_0,\ldots,v_{n-1})$ *is an* $\mathcal{L}'$*-formula and* $g_0,\ldots,g_{n-1}\in X$ *with*
$$A\models\varphi(g_0(A),\ldots,g_{n-1}(A))$$
for all $A\in\mathcal{C}\cap V_\zeta$, *then*
$$B\models\varphi(h(g_0),\ldots,h(g_{n-1})).$$
(2) *If* $x\in V_\kappa$ *with* $h(f_x)\neq\dot{u}^B$, *then* $h(f_x)\in V_{\kappa_B}$.
(3) *If* $\alpha<\kappa$ *with* $h(f_\alpha)\neq\dot{u}^B$, *then* $h(f_\alpha)<\kappa_B$.
(4) *If* $E_0,E_1\subseteq V_\zeta$ *with* $f^{E_0},f^{E_1}\in X$, *then* $f^{E_0\cap E_1}\in X$ *implies that* $h(f^{E_0})\cap h(f^{E_1})=h(f^{E_0\cap E_1})$ *and* $f^{E_0\cup E_1}\in X$ *implies that* $h(f^{E_0})\cup h(f^{E_1})=h(f^{E_0\cup E_1})$. *Moreover, if* $f^{E_0\setminus E_1}\in X$, *then* $h(F^{E_0})\setminus h(f^{E_1})=h(f^{E_0\setminus E_1})$.
(5) *If* $E_0\subseteq E_1\subseteq V_\zeta$ *with* $f^{E_0},f^{E_1}\in X$, *then* $h(f^{E_0})\subseteq h(f^{E_1})$.

Proof. (1) Fix an $\mathcal{L}'$-formula $\varphi(v_0,\ldots,v_{n-1})$ and $g_0,\ldots,g_{n-1}\in X$ with $A\models\varphi(g_0(A),\ldots,g_{n-1}(A))$ for all $A\in\mathcal{C}\cap V_\zeta$. Then $A\models\dot{T}_\varphi(g_0(A),\ldots,g_{n-1}(A))$ holds for all $A\in\mathcal{C}\cap V_\zeta$. By the definition of the product structure, this implies that $\prod(\mathcal{C}\cap V_\zeta)\models\dot{T}_\varphi(g_0,\ldots,g_{n-1})$ and hence $X\models\dot{T}_\varphi(g_0,\ldots,g_{n-1})$. Since h is a homomorphism, this shows that $B\models\dot{T}_\varphi(h(g_0),\ldots,h(g_{n-1}))$ holds, and therefore $B\models\varphi(h(g_0),\ldots,h(g_{n-1}))$.

(2) Pick $x\in V_\kappa$ with $h(f_x)\neq\dot{u}^B$. Given $A\in\mathcal{C}\cap V_\zeta$, we have
$$A\models\text{“}f_x(A)\neq\dot{u}\longrightarrow f_x(A)\textit{ is an element of }V_{\dot{\kappa}}\text{”}.$$
Using (1), the fact that $h(f_x)\neq\dot{u}^B$ now directly implies that $h(f_x)$ is an element of V_{κ_B}.

(3) This implication follows directly from the combination of (1) and (2).

(4) Since the given definitions ensure that $f^{E_0}(A)\cap f^{E_1}(A)=f^{E_0\cap E_1}(A)$, $f^{E_0}(A)\cup f^{E_1}(A)=f^{E_0\cup E_1}(A)$, and $f^{E_0}(A)\setminus f^{E_1}(A)=f^{E_0\setminus E_1}(A)$, for all $A\in\mathcal{C}\cap V_\zeta$, the desired implications follow immediately from (1).

(5) Since our assumptions imply that $E_0\cap E_1=E_0$, a combination of (1) and (4) shows that $h(f^{E_0})\cap h(f^{E_1})=h(f^{E_0})$ and hence $h(f^{E_0})\subseteq h(f^{E_1})$. □

Lemma 5.2. *Suppose that* μ *is an ordinal less than or equal to* κ *such that* $h(f_\mu)\neq\mu$, $f^\mu\in X$, *and* $h(f_x)=x$ *for all* $x\in V_\mu$. *Then the following hold:*

(1) *If* $E\subseteq V_\mu$ *with* $f^E\in X$, *then* $h(f^E)\cap V_\mu=E$.
(2) *If* $h(f_\mu)=\dot{u}^B$, *then* $h(f^\mu)=\kappa_B$.
(3) *If* $h(f_\mu)\neq\dot{u}^B$, *then* $h(f^\mu)=h(f_\mu)<\kappa_B$.
(4) μ *is a strong limit cardinal.*
(5) *If there exists* $E\subseteq\mu$ *of order-type* $\operatorname{cof}(\mu)$ *with* $f^E\in X$ *and either* $\mu<h(f^\mu)$ *or* $\kappa_A<\kappa_B$ *holds for all* $A\in\mathcal{C}\cap V_\zeta$, *then* μ *is regular and, by (4), it follows that* μ *is an inaccessible cardinal.*
(6) $h(f^\varnothing)=\varnothing$ *and* $h(f^{[\mu]^{<\omega}})=[h(f^\mu)]^{<\omega}$.

Proof. (1) First, pick $x\in E$. For each $A\in\mathcal{C}\cap V_\zeta$, we have
$$A\models\text{“}f_x(A)\neq\dot{u}\longrightarrow f_x(A)\in f^E(A)\text{”}.$$
By Lemma 5.1.(1), the fact that $h(f_x)=x\neq\dot{u}^B$ implies that $x=h(f_x)\in h(f^E)$. This shows that $E\subseteq h(f^E)\cap V_\mu$.

In the other direction, pick $x\in V_\mu\setminus E$. Then
$$A\models\text{“}f_x(A)\in f^E(A)\longrightarrow f_x(A)=\dot{u}\text{”}$$

holds for all $A \in \mathcal{C} \cap V_\zeta$, and since $h(f_x) = x \neq \dot{u}^B$, by Lemma 5.1.(1) we must have that $x = h(f_x) \notin h(f^E)$. This shows $h(f^E) \cap V_\mu \subseteq E$, thus proving the desired equality.

(2) For each $A \in \mathcal{C} \cap V_\zeta$, we have

$$A \models \text{“}f_\mu(A) = \dot{u} \longrightarrow f^\mu(A) = \dot{\kappa}\text{”}.$$

By Lemma 5.1.(1), this yields the desired implication.

(3) For each $A \in \mathcal{C} \cap V_\zeta$, we have

$$A \models \text{“}f_\mu(A) \neq \dot{u} \longrightarrow f^\mu(A) = f_\mu(A) < \dot{\kappa}\text{”}.$$

The desired implication now follows from another application of Lemma 5.1.(1).

(4) First, note that since $h(f_\mu) \neq \mu$, Lemma 5.1.(1) directly implies that $\mu > \omega$. Now assume, towards a contradiction, that there is a cardinal $\rho < \mu$ with $2^\rho \geq \mu$. Given $A \in \mathcal{C} \cap V_\zeta$ with $\rho < \kappa_A$, the fact that $\kappa_A \in C^{(1)}$ implies that $2^\rho < \kappa_A \leq \kappa$. Using Lemma 5.1.(1), we can now show that $h(f_{2^\rho}) = 2^\rho$, $h(f_{V_{2^\rho}}) = V_{2^\rho}$ and $h(f_x) \in V_{2^\rho}$ for all $x \in V_{2^\rho}$. In particular, since $h(f_\mu) \neq \mu$ and $2^\rho \geq \mu$, we must have $\mu < 2^\rho$. Another application of Lemma 5.1.(1) shows that the map

$$j : V_{2^\rho} \longrightarrow V_{2^\rho};\ x \mapsto h(f_x)$$

is an elementary embedding. Since $\mathrm{cof}(2^\rho) > \rho \geq \omega$, the *Kunen Inconsistency* implies that j is the identity on V_{2^ρ} and hence $h(f_\mu) = j(\mu) = \mu$, a contradiction.

(5) Assume, towards a contradiction, that μ is singular and pick $E \subseteq \mu$ of order-type $\mathrm{cof}(\mu)$ with $f^E \in X$. We then have

$$\begin{aligned} A \models \text{“}[f_{\mathrm{cof}(\mu)}(A) \neq \dot{u} \ \wedge\ f_\mu(A) = \dot{u} \ \wedge\ \mathrm{otp}\left(f^E(A)\right) \geq f_{\mathrm{cof}(\mu)}(A)] \\ \longrightarrow\ f^E(A) \textit{ is a cofinal subset of } \dot{\kappa} \textit{ of order-type } f_{\mathrm{cof}(\mu)}(A)\text{”} \end{aligned}$$

for all $A \in \mathcal{C} \cap V_\zeta$. By Lemma 5.1.(1), and the assumption that $h(f_x) = x$ for all $x \in V_\mu$, which in particular yields $h(f_{\mathrm{cof}(\mu)}) = \mathrm{cof}(\mu)$, we have that

$$\begin{aligned} B \models \text{“}[\mathrm{cof}(\mu) \neq \dot{u} \ \wedge\ h(f_\mu) = \dot{u} \ \wedge\ \mathrm{otp}\left(h(f^E)\right) \geq \mathrm{cof}(\mu)] \\ \longrightarrow\ h(f^E) \textit{ is a cofinal subset of } \dot{\kappa} \textit{ of order-type } \mathrm{cof}(\mu)\text{”}. \end{aligned}$$

Claim. $h(f_\mu) \neq \dot{u}^B$.

Proof of the Claim. Assume, towards a contradiction, that $h(f_\mu) = \dot{u}^B$. Since (1) shows that we have $\mathrm{otp}\left(h(f^E)\right) \geq \mathrm{otp}\,(E) = \mathrm{cof}(\mu)$, a combination of the above observation with the fact that $\mathrm{cof}(\mu) < \mu \leq \kappa < \dot{u}^B$ then shows that $\mu = \kappa_B = \sup(E)$. By (2), this also shows that $h(f^\mu) = \mu$ and our assumptions imply that $\kappa_A < \kappa_B$ holds for all $A \in \mathcal{C} \cap V_\zeta$. But then Lemma 5.1.(1) allows us to find $A \in \mathcal{C} \cap V_\zeta$ with $\mathrm{cof}(\mu) < \kappa_A \leq \mu$ and $\kappa_A = \sup(E) = \kappa_B$, contradicting the assumption that $\kappa_{<}\kappa_B$ holds for all $A \in \mathcal{C} \cap V_\zeta$. □

Now using Lemma 5.1.(1) again, the fact that

$$A \models \text{“}f_\mu(A) \neq \dot{u} \longrightarrow \mathrm{otp}\left(f^E(A)\right) = f_{\mathrm{cof}(\mu)}(A) \ \wedge\ f_\mu(A) = \sup(f^E(A))\text{”}$$

holds for all $A \in \mathcal{C} \cap V_\zeta$, implies that $\mathrm{otp}\left(h(f^E)\right) = \mathrm{cof}(\mu)$ and $h(f_\mu) = \sup(h(f^E))$. Since $E \subseteq h(f^E)$, this implies $h(f_\mu) = \sup(E) = \mu$, contradicting our initial assumptions.

(6) Since $f^\varnothing(A) = \varnothing$ holds for all $A \in \mathcal{C} \cap V_\zeta$, Lemma 5.1.(1) directly implies that $h(f^\varnothing) = \varnothing$. Next, note that we have

$$f^{[\mu]^{<\omega}}(A) \ = \ [\mu]^{<\omega} \cap V_{\kappa_A} \ = \ [\mu \cap \kappa_A]^{<\omega} \ = \ [f^\mu(A)]^{<\omega}$$

for all $A \in \mathcal{C} \cap V_\zeta$. By Lemma 5.1.(1), this implies that $h(f^{[\mu]^{<\omega}}) = [h(f^\mu)]^{<\omega}$. □

Lemma 5.3. *Let μ be as in Lemma 5.2 and assume, moreover, that $\mu < h(f^\mu)$ and $f^E \in X$ for all $E \subseteq V_\mu$. Set $\nu = h(f^\mu)$ and define*

$$\mathsf{E}_a \;=\; \{E \subseteq [\mu]^{|a|} \mid a \in h(f^E)\}$$

for all $a \in [\nu]^{<\omega}$. Then the resulting system

$$\mathcal{E} \;=\; \langle \mathsf{E}_a \mid a \in [\nu]^{<\omega}\rangle$$

*is a (μ,ν)-*extender *(as defined in* [22, pp. 354–355]*).*

Proof. We prove the lemma through a series of claims.

Claim. *For every $a \in [\nu]^{<\omega}$, the collection E_a is a $<\mu$-complete ultrafilter on $[\mu]^{|a|}$.*

Proof of the Claim. First, note that a combination of clauses (4) and (5) of Lemma 5.1 and Lemma 5.2.(6) shows that E_a is an ultrafilter on $[\mu]^{|a|}$. Now, fix $\rho < \mu$ and a sequence $\langle E_\alpha \mid \alpha < \rho\rangle$ of elements of E_a. Define $G = \bigcap\{E_\alpha \mid \alpha < \rho\}$ and $H = \{\langle x,\alpha\rangle \mid \alpha < \rho,\ x \in E_\alpha\} \subseteq V_\mu$. Given $A \in \mathcal{C} \cap V_\zeta$ with $\rho \leq \kappa_A$, the fact that μ and κ_A are both cardinals implies that

$$x \;\in\; f^{E_\alpha}(A) \;=\; E_\alpha \cap [\kappa_A]^{|a|} \iff \langle x,\alpha\rangle \;\in\; f^H(A) \;=\; H \cap V_{\kappa_A}$$

holds for all $\alpha < \rho$ and this allows us to conclude that

$$f^G(A) \;=\; \{x \in [f^\mu(A)]^{|a|} \mid \langle x,\alpha\rangle \in H \textit{ for all } \alpha < \rho\}.$$

Since $h(f_\alpha) = \alpha$ for all $\alpha \leq \rho$, we can use Lemma 5.1.(1) to show that

$$h(f^{E_\alpha}) \;=\; \{x \in [\nu]^{|a|} \mid \langle x,\alpha\rangle \in h(f^H)\}$$

for all $\alpha < \rho$ and

$$a \;\in\; \bigcap\{h(f^{E_\alpha}) \mid \alpha < \rho\} \;=\; \{x \in [\nu]^{|a|} \mid \langle x,\alpha\rangle \in h(f^H) \textit{ for all } \alpha < \rho\} \;=\; h(f^G).$$

This proves that $\bigcap\{E_\alpha \mid \alpha < \rho\} \in \mathsf{E}_\alpha$. □

Claim. $\mathsf{E}_{\{\mu\}}$ *is not $<\mu^+$-complete.*

Proof of the Claim. Given $\alpha < \mu$, set $E_\alpha = \{\{\beta\} \mid \alpha < \beta < \mu\} \subseteq [\mu]^1$. For each $\alpha < \mu$, Lemma 5.1.(1) then implies that $h(f^{E_\alpha}) = \{\{\beta\} \mid \alpha < \beta < \nu\}$ and this shows that $E_\alpha \in \mathsf{E}_{\{\mu\}}$. But $\bigcap\{E_\alpha \mid \alpha < \mu\} = \varnothing$ and this yields the statement of the claim. □

Claim. *If $\alpha < \mu$, then $\mathsf{E}_{\{\alpha\}} = \{x \subseteq [\mu]^1 \mid \{\alpha\} \in x\}$.*

Proof of the Claim. Fix $x \subseteq [\mu]^1$. If $\{\alpha\} \in x$, then

$$A \models \text{“}f_\alpha(A) \neq \dot{u} \;\longrightarrow\; f_{\{\alpha\}}(A) \in f^x(A)\text{”}$$

holds for all $A \in \mathcal{C} \cap V_\zeta$, and hence Lemma 5.1.(1) implies that $\{\alpha\} = h(f_{\{\alpha\}}) \in h(f^x)$, which yields $x \in \mathsf{E}_{\{\alpha\}}$. In the other direction, if $x \in \mathsf{E}_{\{\alpha\}}$, then $\{\alpha\} \in h(f^x) \cap [\mu]^1$ and Lemma 5.2.(1) implies that $\{\alpha\} \in x$. □

Given $a \subseteq b \in [\nu]^{<\omega}$, we let $\pi_{b,a} : [\mu]^{|b|} \longrightarrow [\mu]^{|a|}$ denote the canonical induced map, i.e. if $\beta_0 < \ldots < \beta_{|b|-1}$ is the monotone enumeration of b and $i_0 < \ldots < i_{|a|-1}$ is the unique strictly increasing sequence of natural numbers less than $|b|$ such that $a = \{\beta_{i_0}, \ldots, \beta_{|a|-1}\}$, then

$$\pi_{b,a}(x) \;=\; \{\alpha_{i_0}, \ldots, \alpha_{|a|-1}\}$$

for all $x \in [\mu]^{|b|}$ with monotone enumeration $\alpha_0 < \ldots < \alpha_{|b|-1}$.

Claim (Coherence)**.** *If $a \subseteq b \in [\nu]^{<\omega}$, then $\mathsf{E}_a = \{E \subseteq [\mu]^{|a|} \mid \pi_{b,a}^{-1}[E] \in \mathsf{E}_b\}$.*

Proof of the Claim. Given $A \in \mathcal{C} \cap V_\zeta$, let $\pi^A_{b,a} : [f^\mu(A)]^{|b|} \longrightarrow [f^\mu(A)]^{|a|}$ denote the induced canonical projection map.

Fix $E \subseteq [\mu]^{|a|}$. If $A \in \mathcal{C} \cap V_\zeta$ is such that $\mu < \kappa_A$, then we have $f^\mu(A) = \mu$, $f^E(A) = E$, $f^{\pi_{b,a}^{-1}[E]}(A) = \pi_{b,a}^{-1}[E]$, $\pi^A_{b,a} = \pi_{b,a}$, and hence

$$x \in f^{\pi_{b,a}^{-1}[E]}(A) \iff \pi^A_{b,a}(x) \in f^E(A)$$

for all $x \in [f^\mu(A)]^{|b|}$. In the other case, namely if $\mu \geq \kappa_A$, then we have $f^\mu(A) = \kappa_A$, $f^E(A) = E \cap V_{\kappa_A}$, $f^{\pi_{b,a}^{-1}[E]}(A) = (\pi_{b,a}^{-1}[E]) \cap V_{\kappa_A}$ and therefore

$$x \in f^{\pi_{b,a}^{-1}[E]}(A) \iff \pi_{b,a}(x) = \pi^A_{b,a}(x) \in E \iff \pi^A_{b,a}(x) \in f^E(A)$$

for all $x \in [f^\mu(A)]^{|b|} \subseteq V_{\kappa_A}$. If we now define $\tilde{\pi}_{b,a} : [\nu]^{|b|} \longrightarrow [\nu]^{|a|}$ to be the induced canonical projection map, then an application of Lemma 5.1.(1) yields $h(f^{\pi_{b,a}^{-1}[E]}) = \tilde{\pi}_{b,a}^{-1}[h(f^E)]$. This equality allows us now to conclude that

$$E \in \mathsf{E}_a \iff \tilde{\pi}_{b,a}(b) = a \in h(f^E) \iff b \in h(f^{\pi_{b,a}^{-1}E}) \iff \pi_{b,a}^{-1}[E] \in \mathsf{E}_b$$

holds for all $E \subseteq [\mu]^{|a|}$. □

Claim (Well-foundedness)**.** *If $\langle a_n \mid n < \omega \rangle$ is a sequence of elements of $[\nu]^{<\omega}$ and $\langle E_n \mid n < \omega \rangle$ is such that $E_n \in \mathsf{E}_{a_n}$ for all $n < \omega$, then there exists a function*

$$d : \bigcup \{a_n \mid n < \omega\} \longrightarrow \mu$$

with $d[a_n] \in E_n$ for all $n < \omega$.

Proof of the Claim. Assume, towards a contradiction, that there exist sequences $\langle a_n \mid n < \omega \rangle$ and $\langle E_n \mid n < \omega \rangle$ witnessing that the claim fails. Then by the previous Claim (Coherence) we may assume that $|a_n| = n$, $a_{n+1} = a_n \cup \{\max(a_{n+1})\}$ and $E_{n+1} \subseteq \pi_{a_{n+1},a_n}^{-1} E_n$ hold for all $n < \omega$. Define T to be the subset of ${}^{<\omega}\mu$ consisting of all strictly increasing sequences t with $\operatorname{ran}(t) \in E_{\operatorname{lh}(t)}$. Our assumptions then imply that T is a subtree of ${}^{<\omega}\mu$. Moreover, it follows that T is well-founded, because if $c : \omega \longrightarrow \mu$ was a cofinal branch through T, then the map

$$d : \bigcup \{a_n \mid n < \omega\} \longrightarrow \mu; \ \max(a_{n+1}) \mapsto c(n)$$

would satisfy $d[a_n] \in E_n$ for all $n < \omega$. For each $A \in \mathcal{C} \cap V_\zeta$,

$$A \models \text{“}f^T(A) \textit{ is a well-founded subtree of } {}^{<\omega}f^\mu(A)\text{”}.$$

Hence, by Lemma 5.1.(1) and the fact $\mathcal{P}(\mu) \subseteq V_{\theta_B}$, we have that $h(f^T)$ is a well-founded subtree of ${}^{<\omega}\nu$.

Define $E = \{\langle x, n \rangle \mid n < \omega, \ x \in E_n\} \subseteq V_\mu$. Given $n < \omega$, we have

$$A \models \text{“}f^{E_n}(A) = \{x \in [f^\mu]^n \mid \langle x, n \rangle \in f^E(A)\}\text{”}$$

for all $A \in \mathcal{C} \cap V_\zeta$, and hence $h(f^{E_n}) = \{x \in [\nu]^n \mid \langle x, n \rangle \in h(f^E)\}$. Moreover, since we also have that

$$A \models \text{“}f^T(A) \textit{ consists of all strictly increasing } t \textit{ in } {}^{<\omega}f^\mu(A) \textit{ with } \langle \operatorname{ran}(t), \operatorname{lh}(t) \rangle \in f^E(A)\text{”},$$

we can conclude that $h(f^T)$ consists of all strictly increasing sequences t in ${}^{<\omega}\nu$ with the property that $\langle \mathrm{ran}(t), \mathrm{lh}(t)\rangle \in h(f^E)$. Now define

$$c : \omega \longrightarrow \nu;\ n \mapsto \max(a_{n+1}).$$

Given $n < \omega$, we have that $\mathrm{ran}(c \restriction n) = a_n \in h(f^{E_n})$, and therefore $\langle \mathrm{ran}(c \restriction n), n\rangle \in h(f^E)$. This shows that $c \restriction n \in h(f^T)$ for all $n < \omega$, contradicting the well-foundedness of the tree $h(f^T)$. □

Claim (Normality)**.** *If $a \in [\nu]^{<\omega}$ and $r : [\mu]^{|a|} \longrightarrow \mu$ is such that*

$$\{x \in [\mu]^{|a|} \mid r(x) < \max(x)\} \ \in \ \mathsf{E}_a,$$

then there exists $a \subseteq b \in [\nu]^{<\omega}$ with $\max(a) = \max(b)$ and

$$\{x \in [\mu]^{|b|} \mid (r \circ \pi_{b,a})(x) \in x\} \ \in \ \mathsf{E}_b.$$

Proof of the Claim. Set $E = \{x \in [\mu]^{|a|} \mid r(x) < \max(x)\}$. Let $A \in \mathcal{C} \cap V_\zeta$. If $\mu < \kappa_A$, then $f^r(A) = r$, and therefore

$$f^E(A) \ = \ E \ = \ \{x \in [f^\mu(A)]^{|a|} \mid x \in \mathrm{dom}(f^r(A)),\ f^r(A)(x) < \max(x)\}.$$

Now assume $\mu \geq \kappa_A$ and let us see that the last equality also holds in this case. If $x \in f^E(A) = E \cap V_{\kappa_A}$, then $x \in [\kappa_A]^{|a|}$ and $r(x) < \max(x) < \kappa_A$, hence $x \in \mathrm{dom}(f^r(A))$ with $f^r(A)(x) = r(x)$. In the other direction, if $x \in [f^\mu(A)]^{|a|} = [\kappa_A]^{|a|}$ with $x \in \mathrm{dom}(f^r(A))$ and $f^r(A)(x) < \max(x)$, then $r(x) = f^r(A)(x)$ and $x \in E$.

An application of Lemma 5.1.(1) now yields

$$h(f^E) \ = \ \{x \in [\nu]^{|a|} \mid x \in \mathrm{dom}(h(f^r)),\ h(f^r)(x) < \max(a)\}.$$

Since $E \in \mathsf{E}_a$, and so $a \in h(f^E)$, the equality above implies that $a \in \mathrm{dom}(h(f^r))$ and $h(f^r)(a) < \max(a)$. Set $b = a \cup \{h(f^r)(a)\}$ and

$$D \ = \ \{x \in [\mu]^{|b|} \mid (r \circ \pi_{b,a})(x) \in x\}.$$

Letting $\tilde{\pi}_{b,a} : [\nu]^{|b|} \longrightarrow [\nu]^{|a|}$ be the induced canonical projection, a variation of the above argument now shows that

$$h(f^D) \ = \ \{x \in [\nu]^{|b|} \mid \tilde{\pi}_{b,a}(x) \in \mathrm{dom}(h(f^r)),\ (h(f^r) \circ \tilde{\pi}_{b,a})(x) \in x\}.$$

Since $\tilde{\pi}_{b,a}(b) = a$ and $h(f^r)(a) \in b$, we can conclude that $b \in h(f^D)$ and hence $D \in \mathsf{E}_b$. □

This completes the proof of the lemma. □

Lemma 5.4. *Under the assumptions of Lemma 5.3, the following hold:*

(1) *The cardinal μ is inaccessible and ν is an element of $C^{(1)}$.*
(2) *Let $\mathcal{E} = \langle \mathsf{E}_a \mid a \in [\nu]^{<\omega}\rangle$ be the (μ, ν)-*extender *given by Lemma 5.3, let*

$$\langle\langle M_a \mid a \in [\nu]^{<\omega}\rangle,\ \langle j_a : V \longrightarrow M_a \mid a \in [\nu]^{<\omega}\rangle\rangle$$

denote the induced system of ultrapowers of V and ultrapower embeddings (see [22, p. 355]*), let*

$$\langle i_{a,b} : M_a \longrightarrow M_b \mid a \subseteq b \in [\nu]^{<\omega}\rangle$$

denote the system of elementary embeddings induced by the projections $\pi_{b,a} : [\mu]^{|b|} \longrightarrow [\mu]^{|a|}$ (see [22, p. 354]*), let*

$$\langle M_{\mathcal{E}},\ \langle k_a : M_a \longrightarrow M_{\mathcal{E}} \mid a \in [\nu]^{<\omega}\rangle\rangle$$

denote the direct limit of the directed system

$$\langle\langle M_a \mid a \in [\nu]^{<\omega}\rangle,\ \langle i_{a,b} : M_a \longrightarrow M_b \mid a \subseteq b \in [\nu]^{<\omega}\rangle\rangle$$

and let $j_{\mathcal{E}} : V \longrightarrow M_{\mathcal{E}}$ denote the induced embedding. Then $\mathrm{crit}\,(j_{\mathcal{E}}) = \mu$ *and $j_{\mathcal{E}}(\mu) \geq \nu$. Moreover, letting $g : [\mu]^1 \longrightarrow V_\mu$ be a bijection with the property that $g[[\rho]^1] = V_\rho$ holds for every $\rho \in C^{(1)} \cap \mu$, we have that $j_{\mathcal{E}}(g) \restriction [\nu]^1 = h(f^g)$ and $h(f^g)$ is a bijection between $[\nu]^1$ and V_ν, hence $V_\nu \subseteq M_{\mathcal{E}}$.*

Proof. (1) Since we assumed that $\mu < h(f^\mu)$, Lemma 5.2.(5) directly implies that μ is inaccessible. Given $A \in \mathcal{C} \cap V_\zeta$, we have $f^\mu(A) \in \{\mu, \kappa_A\} \subseteq C^{(1)}$ and this implies that $V_{f^\mu(A)} \preceq_{\Sigma_1} V_{\dot{\kappa}^A}$. Using Lemma 5.1.(1), we now know that $V_\nu \preceq_{\Sigma_1} V_{\kappa_B}$ and, since $\kappa_B \in C^{(1)}$, we can conclude that $\nu \in C^{(1)}$.

(2) By [22, Lemma 26.2.(b)], we have that $\mathrm{crit}\,(j_{\mathcal{E}}) = \mu$ and $j_{\mathcal{E}}(\mu) \geq \nu$. Next, for every $A \in \mathcal{C} \cap V_\zeta$ we have that

$$A \models \text{“}f^g(A) \textit{ is a bijection between } [f^\mu(A)]^1 \textit{ and } V_{f^\mu(A)}\text{”}.$$

Hence, by Lemma 5.1.(1), we know that $h(f^g)$ is a bijection between $[\nu]^1$ and V_ν. Also, elementarity ensures that $j_{\mathcal{E}}(g)$ is a bijection between $[j_{\mathcal{E}}(\mu)]^1$ and $V^{M_{\mathcal{E}}}_{j_{\mathcal{E}}(\mu)}$ with the property that $j_{\mathcal{E}}(g)[[\rho]^1] = V^{M_{\mathcal{E}}}_\rho$ holds for all ρ in $(C^{(1)})^{M_{\mathcal{E}}}$ less than or equal to $j_{\mathcal{E}}(\mu)$. Now, since (1) shows that $\nu \in C^{(1)}$ and being in $C^{(1)}$ is a Π_1-property, we know that $\nu \in (C^{(1)})^{M_{\mathcal{E}}}$. Thus, $j_{\mathcal{E}}(g)[[\nu]^1] = V^{M_{\mathcal{E}}}_\nu$, and the map

$$\iota \ = \ j_{\mathcal{E}}(g) \circ h(f^g)^{-1} : V_\nu \longrightarrow V^{M_{\mathcal{E}}}_\nu$$

is a bijection.

Claim. $\iota = \mathrm{id}_{V_\nu}$.

Proof of the Claim. Since V_ν and $V^{M_{\mathcal{E}}}_\nu$ are transitive, it is sufficient to show that ι is an $\in$-homomorphism. Thus, let $x_0, x_1 \in V_\nu$. As $h(f^g) : [\nu]^1 \longrightarrow V_\nu$ is a bijection, let $a_0, a_1 \in [\nu]^1$ be the preimages under $h(f^g)$ of x_0 and x_1, respectively. Set $a = a_0 \cup a_1$ and

$$E \ = \ \{x \in [\mu]^{|a|} \mid g(\pi_{a,a_0}(x)) \in g(\pi_{a,a_1}(x))\}.$$

Given $A \in \mathcal{C} \cap V_\zeta$, the fact that $g[[f^\mu(A)]^1] = V_{f^\mu(A)}$ implies that $f^g(A) = g \restriction [f^\mu(A)]^1$ and hence

$$f^E(A) \ = \ \{x \in [f^\mu(A)]^{|a|} \mid f^g(A)(\pi_{a,a_0}(x)) \in f^g(A)(\pi_{a,a_1}(x))\}.$$

By Lemma 5.1.(1), this shows that

$$h(f^E) \ = \{x \in [\nu]^{|a|} \mid h(f^g)(\pi_{a,a_0}(x)) \in h(f^g)(\pi_{a,a_1}(x))\}.$$

Thus, we have the following equivalences:

$$x_0 \in x_1 \iff h(f^g)(a_0) \in h(f^g)(a_1) \iff a \in h(f^E) \iff E \in \mathsf{E}_a$$

and the latter, by the definition of E and the ultrapower map $j_a : V \longrightarrow M_a$, is equivalent to

$$j_a(g)([\pi_{a,a_0}]_{\mathsf{E}_a}) \in j_a(g)([\pi_{a,a_1}]_{\mathsf{E}_a}).$$

By applying the map k_a to the last displayed sentence, and using the fact that $j_{\mathcal{E}} = k_a \circ j_a$, $k_{a_0} = k_a \circ i_{a_0,a}$, and $i_{a_0,a}([\mathrm{id}_{|a_0|}]_{\mathsf{E}_{a_0}}) = [\mathrm{id}_{|a_0|} \circ \pi_{a_0,a}]_{\mathsf{E}_a} = [\pi_{a_0,a}]_{\mathsf{E}_a}$, and similarly for a_1, we have that

$$j_{\mathcal{E}}(g)(k_{a_0}([\mathrm{id}_{|a_0|}]_{\mathsf{E}_{a_0}})) \in j_{\mathcal{E}}(g)(k_{a_1}([\mathrm{id}_{|a_1|}]_{\mathsf{E}_{a_1}})).$$

Now, as in [22, Lemma 26.2.(a)], $k_{a_0}([\mathrm{id}_{|a_0|}]_{\mathsf{E}_{a_0}}) = a_0$, and similarly for a_1. Thus the last displayed sentence is equivalent to the first term of the following chain of equivalences

$$j_{\mathcal{E}}(g)(a_0) \in j_{\mathcal{E}}(g)(a_1) \iff \iota(h(f^g)(a_0)) \in \iota(h(f^g)(a_1)) \iff \iota(x_0) \in \iota(x_1).$$

We have thus shown that $x_0 \in x_1$ if and only if $\iota(x_0) \in \iota(x_1)$, which proves the Claim. □

The above claim shows that $V_\nu = \mathrm{range}(\iota) \subseteq M_{\mathcal{E}}$, and thus $V_\nu \subseteq M_{\mathcal{E}}$. □

We shall end this section with the following lemma, which will be used in the proofs of Theorems 1.30 and 1.34, given in Sections 6 and 8 below.

Lemma 5.5. *If we define*

$$\lambda \; = \; \min\{\mathrm{rnk}(x) \mid f_x \in X, \; h(f_x) = \dot{u}^B\},$$

then the following statements hold:

(1) *λ is a limit ordinal with $\lambda \leq \kappa$ and $h(f_\lambda) = \dot{u}^B$.*
(2) *If $x \in V_\lambda$ with $h(f_x) \neq x$, then there exists $\alpha \leq \mathrm{rnk}(x)$ with $\alpha < h(f_\alpha)$.*
(3) *If we define*

$$\chi \; = \; \sup\{h(f_\alpha) \mid \alpha < \lambda\},$$

then $\chi \leq \kappa_B$, $\mathrm{rnk}(h(f_x)) < \chi$ for all $x \in V_\lambda$ and the map

$$j : V_\lambda \longrightarrow V_\chi ; \; x \mapsto h(f_x)$$

is a Σ_1-elementary embedding.

Proof. (1) First, note that we have $f_\kappa(A) = \dot{u}^A$ for all $A \in \mathcal{C} \cap V_\zeta$ and this implies that $f_\kappa \in X$ with $h(f_\kappa) = \dot{u}^B$. In particular, we know that $\lambda \leq \kappa$.

Next, assume, towards a contradiction, that $h(f_\lambda) \neq \dot{u}^B$. Then the above computations show that $\lambda < \kappa$. Pick a set $x \in V_\kappa$ with $\mathrm{rnk}(x) = \lambda$ and $h(f_x) = \dot{u}^B$. An application of Lemma 5.1.(1) now yields an element A of $\mathcal{C} \cap V_\zeta$ with $f_\lambda(A) \neq \dot{u}^A$ and $f_x(A) = \dot{u}^A$. But then $\lambda < \kappa_A$ and $x \notin V_{\kappa_A}$, a contradiction.

Now, assume, towards a contradiction, that there is an ordinal α with $\lambda = \alpha+1$. Then $h(f_\alpha) \neq \dot{u}^B$ and $\lambda < \kappa_A$ holds for all $A \in \mathcal{C} \cap V_\zeta$ with $\alpha < \kappa_A$. By Lemma 5.1.(1), this implies that $h(f_\lambda) \neq \dot{u}^B$, a contradiction.

(2) Assume that there exists $x \in V_\lambda$ with $h(f_x) \neq x$. Let $y \in V_\lambda$ be rank-minimal with $h(f_y) \neq y$. Set $\alpha = \mathrm{rnk}(y) \leq \mathrm{rnk}(x) < \lambda \leq \kappa$. Then $h(f_\alpha) \neq \dot{u}^B$, and Lemma 5.1.(3) shows that $h(f_\alpha) < \kappa_B$. Note that, since $y \in V_\lambda$, $h(f_y) \neq \dot{u}^B$. So, since we have

$$A \models \text{“}f_y(A) \neq \dot{u}^A \;\longrightarrow\; f_\alpha(A) = \mathrm{rnk}(f_y(A))\text{”}$$

for all $A \in \mathcal{C} \cap V_\zeta$, this implies that $h(f_\alpha) = \mathrm{rnk}(h(f_y))$.

Assume, towards a contradiction, that $h(f_\alpha) \leq \alpha$. Given $z \in y$, we then have

$$A \models \text{“}f_y(A) \neq \dot{u}^A \longrightarrow f_z(A) \in f_y(A)\text{”}$$

for all $A \in \mathcal{C} \cap V_\zeta$, and this shows that $h(f_z) \in h(f_y)$. By the minimality of y, this implies that $y \subseteq h(f_y)$ and hence there exists $w \in h(f_y) \setminus y$. We now know that

$$\mathrm{rnk}(w) \; < \; \mathrm{rnk}(h(f_y)) \; = \; h(f_\alpha) \; \leq \; \alpha \; = \; \mathrm{rnk}(y)$$

and therefore the minimality of y implies that $h(f_w) = w \in h(f_y)$. Hence there exists A in $\mathcal{C} \cap V_\zeta$ with $f_y(A) \neq \dot{u}^A$ and $f_w(A) \in f_y(A)$. But then $\mathrm{rnk}(w) < \mathrm{rnk}(y) < \kappa_A$ and hence we can conclude that $w = f_w(A) \in f_y(A) = y$, a contradiction.

(3) First, note that Lemma 5.1.(3) directly implies that $\chi \leq \kappa_B$. Since Lemma 5.1.(1) shows that $\mathrm{rnk}(h(f_x)) = h(f_{\mathrm{rnk}(x)}) < \chi$ holds for all $x \in V_\lambda$, we know that the function $j : V_\lambda \longrightarrow V_\chi$ is well-defined.

In the following, fix a Σ_1-formula $\varphi(v)$ and an element x of V_λ. First, assume that $\varphi(x)$ holds in V_λ. By (1), we can find an ordinal $\alpha < \lambda$ with the property that $x \in V_\alpha$ and $\varphi(x)$ holds in V_α. By Σ_1-upwards absoluteness, this shows that we have

$$A \models \text{“}f_\alpha(A) \neq \dot{u}^A \longrightarrow V_{f_\alpha(A)} \models \varphi(f_x(A))\text{”}$$

for all $A \in \mathcal{C} \cap V_\zeta$. Using Lemma 5.1.(1), we now know that $\varphi(j(x))$ holds in $V_{j(\alpha)}$ and therefore Σ_1-upwards absoluteness implies that $\varphi(j(x))$ holds in V_χ. In the other direction, assume that $\varphi(j(x))$ holds in V_χ. Since our computations already show that $j(\alpha) < j(\beta)$ holds for all $\alpha < \beta < \lambda$, we know that $\chi = \text{lub}\{h(f_\alpha) \mid \alpha < \lambda\}$ and hence we can find $\alpha < \lambda$ such that $\varphi(j(x))$ holds in $V_{j(\alpha)}$. Another application of Lemma 5.1.(1) then shows that $\varphi(x)$ holds in V_α and Σ_1-upwards absoluteness allows us to conclude that this statement holds in V_λ. □

6. Strongly unfoldable product reflection

We shall give next a proof of Theorem 1.30. Namely, we will show that the following are equivalent for every cardinal κ:

(1) κ is either strongly unfoldable or a limit of strong cardinals.
(2) The principle $\text{WPSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.

The implication (1) ⇒ (2) follows from a combination of the following lemma and Theorem 1.22.

Lemma 6.1. *If κ is a strongly unfoldable cardinal, then* $\text{WPSR}_{\mathcal{C}}(\kappa)$ *holds for every non-empty class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Proof. First, note that, since all strongly unfoldable cardinals are elements of $C^{(2)}$ (see Theorem 1.10), Lemma 3.1 shows that $\mathcal{C} \cap V_\kappa \neq \varnothing$. Now, fix a substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality at most κ and a structure B in $\mathcal{C}$. Pick a cardinal $\delta \in C^{(2)}$ greater than κ, with $B \in V_\delta$, and an elementary submodel M of H_{κ^+} of cardinality κ with the property that $V_\kappa \cup \{X\} \cup {}^{<\kappa}M \subseteq M$. Since $\kappa \in C^{(2)}$, we know that $\mathcal{C} \cap V_\kappa \in M$. Using the strong unfoldability of κ, we can find a transitive set N with $V_\delta \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\text{crit}\,(j) = \kappa$ and $j(\kappa) > \delta$. Now notice that since δ is in $C^{(2)}$, and therefore in $C^{(1)}$ in the sense of N, and since $j(\kappa)$ is an inaccessible cardinal in N, we have that $V_\delta \prec_{\Sigma_1} V^N_{j(\kappa)}$, which implies $\mathcal{C} \cap V_\delta \subseteq j(\mathcal{C} \cap V_\kappa)$. Thus, $B \in \mathcal{C} \cap V_\delta \subseteq j(\mathcal{C} \cap V_\kappa)$, and the function

$$h : X \longrightarrow B;\ f \mapsto j(f)(B)$$

is a well-defined homomorphism. □

Proof of Theorem 1.30. Let κ be a cardinal with the property that $\text{WPSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ. Then Lemma 3.1 shows that κ is an element of $C^{(2)}$. Assume, towards a contradiction, that κ is neither strongly unfoldable nor a limit of strong cardinals. Pick an ordinal $\alpha < \kappa$ such that the interval $[\alpha, \kappa)$ contains no strong cardinals. Given a cardinal ρ that is not strong, we let η_ρ denote the least cardinal $\delta > \rho$ such that ρ is not δ-strong. Since the class of ordinals that are not strong cardinal is definable by a Σ_2-formula without parameters, the fact that $\kappa \in C^{(2)}$ implies that the interval (α, κ) is closed under the function $\rho \longmapsto \eta_\rho$, and therefore it contains unboundedly many cardinals ξ with the property that $\eta_\rho < \xi$ holds for all cardinals $\alpha \leq \rho < \xi$. Finally, since [23, Theorem 1.3] implies that κ is not *shrewd*, basic definability considerations allow us to find an $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, a limit ordinal $\theta > \kappa$ and $E \subseteq V_\kappa$ with the property that $\Phi(\kappa, E)$ holds in $V_{\theta+1}$ and for all $\beta < \gamma < \kappa$, the statement $\Phi(\beta, E \cap V_\beta)$ does not hold in $V_{\gamma+1}$.

Let $\mathcal{L}'$ be the first-order language that extends the language of set theory with a binary relation symbol $\dot{S}$, constant symbols $\dot{\kappa}$, $\dot{u}$, and $\dot{c}$, and a unary function symbol $\dot{e}$. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}'$ by an $(n+1)$-ary predicate symbol $\dot{T}_\varphi$ for every $\mathcal{L}'$-formula $\varphi(v_0,\ldots,v_n)$ with $(n+1)$-many free variables, and let $\mathcal{S}_\mathcal{L}$ be the class of structures as defined at the beginning of Section 5. Namely, $\mathcal{S}_\mathcal{L}$ is the class of all $\mathcal{L}$-structures A such that there exists a cardinal κ_A in $C^{(1)}$ and a limit ordinal $\theta_A > \kappa_A$ such that the following hold:

(1) The domain of A is V_{θ_A+1}.
(2) $\in^A = \in \restriction V_{\theta_A+1}$, $\dot{\kappa}^A = \kappa_A$ and $\dot{u}^A = \theta_A$.
(3) If $\varphi(v_0,\ldots,v_n)$ is an $\mathcal{L}'$-formula, then

$$\dot{T}^A_\varphi \ = \ \{\langle x_0,\ldots,x_n\rangle \in V^{n+1}_{\theta_A+1} \mid A \models \varphi(x_0,\ldots,x_n)\}.$$

Let $\mathcal{C}$ denote the class of all $A \in \mathcal{S}_\mathcal{L}$ such that the following hold:

- $\dot{c}^A = \alpha < \kappa_A$.
- The interval $[\alpha,\kappa_A)$ contains no strong cardinals.
- $\dot{S}^A = \{\langle\rho,\gamma\rangle \in \kappa_A \times \kappa_A \mid \rho \textit{ is a } \gamma\textit{-strong cardinal}\}$.
- If $\alpha < \delta < \kappa_A$ is a cardinal, then $\dot{e}^A(\delta)$ is a cardinal below κ_A and is the smallest cardinal ξ greater than δ that has the property that $\eta_\rho < \xi$ holds for all cardinals $\alpha \leq \rho < \xi$.

It is easily seen that the class $\mathcal{C}$ is definable by a Σ_2-formula with parameter α. In addition, the fact that κ is an element of $C^{(2)}$ implies that $\sup\{\kappa_A \mid A \in \mathcal{C} \cap V_\kappa\} = \kappa$ and there exists a structure B in $\mathcal{C}$ with $\kappa_B = \kappa$ and $\theta_B = \theta$. Let C be a cofinal subset of κ of order-type $\mathrm{cof}(\kappa)$. Given a set x, we define functions f_x and f^x with domain $\mathcal{C} \cap V_\kappa$ as in Section 5. Namely, we have $f^x(A) = x \cap V_{\kappa_A}$ for all $A \in \mathcal{C} \cap V_\kappa$, $f_x(A) = x$ for all $A \in \mathcal{C} \cap V_\kappa$ with $x \in V_{\kappa_A}$ and $f_x(A) = \dot{u}^A$ for all $A \in \mathcal{C} \cap V_\kappa$ with $x \notin V_{\kappa_A}$. Since κ is an element of $C^{(1)}$, we can find a substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality κ with the property that $f^E \in X$, $f^C \in X$, and $f_x, f^x \in X$ for all $x \in V_\kappa$. By our assumptions, there exists a homomorphism $h : X \longrightarrow B$ and we can define

$$\lambda \ = \ \min\{\mathrm{rnk}(x) \mid f_x \in X, \ h(f_x) = \dot{u}^B\}$$

and

$$\chi \ = \ \sup\{h(f_\beta) \mid \beta < \lambda\}.$$

Lemma 5.5 then shows that both λ and χ are less than or equal to κ. Moreover, Lemma 5.5.(1) shows that $h(f_\lambda) = \dot{u}^B \neq \lambda$, and this implies that $\alpha < \lambda$, because $f_\alpha(A) \neq \dot{u}^A$ holds for all $A \in \mathcal{C} \cap V_\kappa$ and, by Lemma 5.1.(1), this implies that $h(f_\alpha) \neq \dot{u}^B$. Now, let

$$\mu \ = \ \min\{\beta \leq \lambda \mid h(f_\beta) \neq \beta\}.$$

Since $\kappa_A < \kappa$ holds for all $A \in \mathcal{C} \cap V_\kappa$, we can apply Lemma 5.2.(5) to show that μ is an inaccessible cardinal. Moreover, Lemma 5.5.(2) implies that $h(f_x) = x$ holds for all $x \in V_\mu$.

Claim. *$\mu < \kappa$.*

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda = \kappa$ holds. Since Lemma 5.1.(1) implies $h(f^E) \subseteq \kappa$, we can apply Lemma 5.2.(1) to conclude that $h(f^E) = E$ and hence

$$B \models \Phi(\dot{\kappa}^B, h(f^E)).$$

Another application of Lemma 5.1.(1) then yields $A \in \mathcal{C} \cap V_\kappa$ with

$$A \models \Phi(\dot{\kappa}^A, f^E(A))$$

and this shows that $\Phi(\kappa_A, E \cap V_{\kappa_A})$ holds in V_{θ_A+1}. Since $\kappa_A < \theta_A < \kappa$, this contradicts the fact that E witnesses that κ is not a shrewd cardinal. □

Claim. *$\mu < \lambda$.*

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda < \kappa$. Then Lemma 5.2.(2) shows that $h(f^\mu) = \kappa$ and we can apply Lemma 5.4 to show that μ is a κ-strong cardinal. But this contradicts the fact that $\alpha < \mu < \eta_\mu < \kappa$. □

The above claim shows that $\mu < \lambda \leq \kappa$ and $h(f_\mu) \neq \dot{u}^B$. We let

$$j : V_\lambda \longrightarrow V_\chi;\ x \mapsto h(f_x)$$

denote the non-trivial Σ_1-elementary embedding with $\operatorname{crit}(j) = \mu$ that was introduced in Lemma 5.5.(3). As in the proof of Theorem 1.22, we can now use the Σ_1-elementarity of j and the *Kunen Inconsistency* to conclude that $\alpha < \mu$. Since $\alpha < \mu < \lambda \leq \kappa$, we can now pick a cardinal $\mu < \xi < \kappa$ that is the minimal cardinal above μ with the property that $\eta_\rho < \xi$ holds for all cardinals $\alpha \leq \rho < \xi$. Given $A \in \mathcal{C} \cap V_\kappa$ with $\mu < \kappa_A$, we then have $\xi = \dot{e}^A(\mu) < \kappa_A$. Using Lemma 5.1.(1), this shows that $h(f_\xi) \neq \dot{u}^B$ and $\xi < \lambda$.

Claim. *If $n < \omega$, then $j^n(\mu) < j^{n+1}(\mu) < \xi$ and $j^n(\mu)$ is a $j^{n+1}(\mu)$-strong cardinal.*

Proof of the Claim. Since $\mu < \kappa$ and Lemma 5.2.(3) shows that $j(\mu) = h(f^\mu)$, we can apply Lemma 5.4 to conclude that μ is $j(\mu)$-strong and this implies that $\mu < j(\mu) < \eta_\mu < \xi$. Now, assume that for some $n < \omega$, we have $j^n(\mu) < j^{n+1}(\mu) < \xi$ and $j^n(\mu)$ is a $j^{n+1}(\mu)$-strong cardinal. Then $\langle j^n(\mu), j^{n+1}(\mu)\rangle \in \dot{S}^A$ for all $A \in \mathcal{C} \cap V_\kappa$ with $\xi < \kappa_A$ and the fact that $h(f_\xi) \neq \dot{u}^B$ allows us to use Lemma 5.1.(1) to show that $\langle j^{n+1}(\mu), j^{n+2}(\mu)\rangle \in \dot{S}^B$. Hence, we know that $j^{n+1}(\mu)$ is a $j^{n+2}(\mu)$-strong cardinal and $j^{n+1}(\mu) < j^{n+2}(\mu) < \eta_{j^{n+1}(\mu)} < \xi$. □

We can now define

$$\tau \ = \ \sup_{n<\omega} j^n(\mu) \ \leq \ \xi \ < \ \lambda,$$

apply Lemma 5.5.(1) to show that λ is a limit ordinal, and use the Σ_1-elementarity of j to conclude that $j(V_{\tau+2}) = V_{\tau+2}$. Since this entails that $j \restriction V_{\tau+2} : V_{\tau+2} \longrightarrow V_{\tau+2}$ is a non-trivial elementary embedding, we again derived a contradiction to the *Kunen Inconsistency*.

The above computations yield a proof of the implication (2) ⇒ (1) of Theorem 1.30. The converse implication (1) ⇒ (2) follows directly from a combination of Theorem 1.22 and Lemma 6.1. □

7. A Lemma about Weak Product Structural Reflection

Recall (see Definition 1.29) that for a class $\mathcal{C}$ of structures of the same type and a cardinal κ, the principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ asserts that $\mathcal{C} \cap V_\kappa \neq \varnothing$ and for every substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality at most κ and every $B \in \mathcal{C}$, there exists a homomorphism from X to B. We now prove a lemma that will be used in the proof of Theorem 1.34, given in the next section.

Lemma 7.1. *Let δ be an uncountable cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ such that $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds. Then δ is inaccessible.*

Proof. We start by proving a series of claims.

Claim. *δ is a limit cardinal.*

Proof of the Claim. Assume, towards a contradiction, that there exists a cardinal $\gamma < \delta$ satisfying $\gamma^+ = \delta$. Let $\mathcal{L}$ denote the trivial first-order language, let A be the $\mathcal{L}$-structure with domain V_γ and set $\mathcal{C} = \{A\}$. Then $\mathcal{C} \cap V_\kappa = \varnothing$ for all cardinals $\kappa < \delta$, contradicting our assumption. □

Claim. $\mathrm{cof}(\delta) > \omega$.

Proof of the Claim. Assume, towards a contradiction, that $\mathrm{cof}(\delta)$ is countable. Pick a strictly increasing sequence $\langle \delta_n \mid n < \omega \rangle$ of cardinals that is cofinal in δ and let $\mathcal{L}$ denote the first-order language that extends the language of group theory by a constant symbol $\dot{g}$. Given $1 < n < \omega$, fix an $\mathcal{L}$-structure G_n such that $\delta_n < \mathrm{rnk}(G_n) < \delta_{n+1}$ and the reduct of G_n to the language of group theory is the sum of δ_n-many copies of the cyclic group of order n and $\dot{g}^{G_n}$ is an element of order n in this group. Set $\mathcal{C} = \{G_n \mid 1 < n < \omega\} \subseteq V_\delta$. Then there exists a cardinal $\kappa < \delta$ with the property that $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds. Let X be a substructure of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality at most κ. Pick a prime number p with $\delta_p > \kappa$. Then our assumption yields a homomorphism $h : X \longrightarrow G_p$ and our setup ensures that $(\dot{g}^X)^{(p-1)!}$ is the neutral element of X. But this implies that $(\dot{g}^{G_p})^{(p-1)!}$ is the neutral element of G_p, contradicting the fact that $\dot{g}^{G_p}$ has order p in G_p. □

Claim. $|V_{\mathrm{cof}(\delta)}| \geq \delta$.

Proof of the Claim. Assume, towards a contradiction, that $|V_{\mathrm{cof}(\delta)}| < \delta$. Then $\mathrm{cof}(\delta) < \delta$ and we can pick a strictly increasing sequence $\langle \delta_\xi \mid \xi < \mathrm{cof}(\delta) \rangle$ of cardinals greater than $|V_{\mathrm{cof}(\delta)}|$ that is cofinal in δ. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}_\in$ by a constant symbol $\dot{c}$, a unary relation symbol $\dot{M}$ and an $(n+1)$-ary relation symbol $\dot{R}_\varphi$ for every $\mathcal{L}_\in$-formula $\varphi \equiv \varphi(v_0, \ldots, v_n)$ with $n+1$ free variables. Given $\xi < \mathrm{cof}(\delta)$, let A_ξ denote the unique $\mathcal{L}$-structure with $\mathcal{L}_\in$-reduct $\langle V_{\delta_\xi}, \in \rangle$ that satisfies $\dot{c}^{A_\xi} = \xi$, $\dot{M}^{A_\xi} = V_{\mathrm{cof}(\delta)}$ and

$$\dot{R}_\varphi^{A_\xi} \ = \ \{\langle x_0, \ldots, x_n \rangle \in V_{\mathrm{cof}(\delta)}^{n+1} \mid V_{\mathrm{cof}(\delta)} \models \varphi(x_0, \ldots, x_n)\}$$

for every $\mathcal{L}_\in$-formula $\varphi \equiv \varphi(v_0, \ldots, v_n)$. Set $\mathcal{C} = \{A_\xi \mid \xi < \mathrm{cof}(\delta)\} \subseteq V_\delta$. Then there exists a cardinal $\kappa < \delta$ with the property that $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds and the fact that $\mathcal{C} \cap V_\kappa \neq \varnothing$ directly implies that $\kappa > |V_{\mathrm{cof}(\delta)}|$. Fix $\zeta < \mathrm{cof}(\delta)$ with the property that $\delta_\zeta > \kappa$. Given $x \in V_{\mathrm{cof}(\delta)}$, let f_x denote the unique function with domain $\mathcal{C} \cap V_\kappa$ and $f_x(A) = x$ for all $A \in \mathcal{C} \cap V_\kappa$. Since $|V_{\mathrm{cof}(\delta)}| < \kappa$, we can find a substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ with $f_x \in X$ for all $x \in V_{\mathrm{cof}(\delta)}$. Our assumptions then yield a homomorphism $h : X \longrightarrow A_\zeta$ and, as in the proof of Lemma 5.1.(1), one can now prove the following statement.

Subclaim. *If $\varphi(v_0, \ldots, v_{n-1})$ is an $\mathcal{L}_\in$-formula and $g_0, \ldots, g_{n-1} \in X$ with the property that $g_0(A), \ldots, g_{n-1}(A) \in V_{\mathrm{cof}(\delta)}$ and $V_{\mathrm{cof}(\delta)} \models \varphi(g_0(A), \ldots, g_{n-1}(A))$ hold for all $A \in \mathcal{C} \cap V_\kappa$, then $h(g_0), \ldots, h(g_{n-1}) \in V_{\mathrm{cof}(\delta)}$ and $V_{\mathrm{cof}(\delta)} \models \varphi(h(g_0), \ldots, h(g_{n-1}))$.* □

If we now define

$$j : V_{\mathrm{cof}(\delta)} \longrightarrow V_{\mathrm{cof}(\delta)}; \ x \mapsto h(f_x),$$

then the above claim shows that j is an elementary embedding. Moreover, since

$$V_{\mathrm{cof}(\delta)} \models f_\zeta(A) \neq \dot{c}^A$$

holds for all $A \in \mathcal{C} \cap V_\kappa$, the subclaim shows that $j(\zeta) \neq \dot{c}^{A_\zeta} = \zeta$ and this allows us to conclude that j is a non-trivial embedding. But this contradicts the *Kunen Inconsistency*, because $\mathrm{cof}(\delta)$ is an uncountable regular cardinal. □

Claim. $|V_\xi| < \delta$ *for all* $\xi < \mathrm{cof}(\delta)$.

Proof of the Claim. Assume, towards a contradiction, that the statement of the claim fails and let $\xi < \mathrm{cof}(\delta)$ be minimal with $|V_\xi| \geq \delta$. The minimality of ξ then yields an ordinal η with $\xi = \eta + 1$ and $|V_\eta| < \delta$. Fix an injective enumeration $\langle x_\gamma \mid \gamma < \delta \rangle$ of subsets of V_η. Moreover, if there exists a limit cardinal λ of countable cofinality with $\eta \in \{\lambda, \lambda + 1\}$, then we also fix a cofinal function

$d : \omega \longrightarrow \lambda$. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}_\in$ by a constant symbol $\dot{c}$, a constant symbol $\dot{d}_n$ for every natural number n, a unary relation symbol $\dot{M}$ and an $(n+1)$-ary relation symbol $\dot{R}_\varphi$ for every $\mathcal{L}_\in$-formula $\varphi \equiv \varphi(v_0, \ldots, v_n)$ with $n+1$ free variables. Given $\gamma < \delta$, let A_γ denote an $\mathcal{L}$-structure such that the following statements hold:

- The $\mathcal{L}_\in$-reduct of A_γ is of the form $\langle V_\rho, \in \rangle$ for some cardinal $\max(\gamma, |V_\eta|) < \rho < \delta$.
- $\dot{c}^{A_\gamma} = x_\gamma$ and $\dot{M}^{A_\gamma} = V_\eta$.
- If $\eta \in \{\lambda, \lambda+1\}$ for a limit cardinal λ of countable cofinality, then $\dot{d}_n^{A_\gamma} = d(n)$.
- If $\varphi \equiv \varphi(v_0, \ldots, v_n)$ is an $\mathcal{L}_\in$-formula, then

$$\dot{R}_\varphi^{A_\xi} \ = \ \{\langle z_0, \ldots, z_n \rangle \in V_{\eta+1}^{n+1} \mid V_{\eta+1} \models \varphi(z_0, \ldots, z_n)\}.$$

Define $\mathcal{C} = \{A_\gamma \mid \gamma < \delta\} \subseteq V_\delta$ and pick a cardinal $\kappa < \delta$ with the property that $\mathrm{WPSR}_\mathcal{C}(\kappa)$ holds. Then $\kappa > |V_\eta|$, because $\mathcal{C} \cap V_\kappa$ is non-empty and the domain of every element of $\mathcal{C}$ is some V_ρ with ρ greater than $|V_\eta|$. Given $x \in V_{\eta+1}$, we define f_x to be the unique function with domain $\mathcal{C} \cap V_\kappa$ and $f_x(A) = x$ for all $A \in \mathcal{C} \cap V_\kappa$. Then there exists a substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality at most κ with $f_{V_\eta}, f_{x_\kappa} \in X$ and $f_x \in X$ for all $x \in V_\eta$. Moreover, our assumption yields a homomorphism $h : X \longrightarrow A_\kappa$. Then we again know that for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ and all $g_0, \ldots, g_{n-1} \in X$ such that $g_0(A), \ldots, g_{n-1}(A) \in V_{\eta+1}$ and $V_{\eta+1} \models \varphi(g_0(A), \ldots, g_{n-1}(A))$ hold for all $A \in \mathcal{C} \cap V_\kappa$, we have $h(g_0), \ldots, h(g_{n-1}) \in V_{\eta+1}$ and $V_{\eta+1} \models \varphi(h(g_0), \ldots, h(g_{n-1}))$. In particular, we know that $h(f_{V_\eta}) = V_\eta$ and the induced map

$$j : V_\eta \longrightarrow V_\eta; \ x \mapsto h(f_x)$$

is an elementary embedding.

Subclaim. *The embedding j is non-trivial.*

Proof of the Subclaim. Assume, towards a contradiction, that $j = \mathrm{id}_{V_\eta}$. We then have $\dot{c}^A \neq x_\kappa = f_{x_\kappa}(A)$ for all $A \in \mathcal{C} \cap V_\kappa$ and hence the above observations show that $h(f_{x_\kappa}) \neq \dot{c}^{A_\kappa} = x_\kappa$. Pick an element $x \in V_\eta$ that is contained in the symmetric difference of x_κ and $h(f_{x_\kappa})$. Since $h(f_x) = x$ holds, our earlier computations imply that x is an element of x_κ if and only if x is an element of $h(f_{x_\kappa})$, a contradiction. □

If we now define $\lambda = \sup_{n<\omega} j^n(\mathrm{crit}\,(j))$, then λ is a strong limit cardinal of countable cofinality and the *Kunen Inconsistency* implies that $\eta \in \{\lambda, \lambda+1\}$. But this shows that for some cofinal function $d : \omega \longrightarrow \lambda$, we have $\dot{d}_n^A = d(n)$ for all $n < \omega$ and $A \in \mathcal{C}$. In particular, this implies that $j(d(n)) = \dot{d}_n^{A_\kappa} = d(n)$ holds for all $n < \omega$, contradicting the fact that $\eta \leq \lambda + 1$. □

Claim. *The cardinal δ is regular.*

Proof of the Claim. Assume, towards a contradiction, that δ is singular. Since the above claims show that $|V_{\mathrm{cof}(\delta)}| \geq \delta > \mathrm{cof}(\delta)$, we can now find an ordinal $\eta < \mathrm{cof}(\delta)$ with $|V_\eta| \geq \mathrm{cof}(\delta)$. Fix an injective sequence $\langle x_\xi \mid \xi < \mathrm{cof}(\delta) \rangle$ of elements of V_η and a strictly increasing sequence $\langle \delta_\xi \mid \xi < \mathrm{cof}(\delta) \rangle$ of cardinals greater than $|V_{\eta+2}|$ that is cofinal in δ. We let $\mathcal{L}$ denote the first-order language extending $\mathcal{L}_\in$ by a constant symbol $\dot{c}$, a unary relation symbol $\dot{M}$ and an $(n+1)$-ary relation symbol $\dot{R}_\varphi$ for every $\mathcal{L}_\in$-formula $\varphi \equiv \varphi(v_0, \ldots, v_n)$ with $n+1$ free variables. For every $\xi < \mathrm{cof}(\delta)$, we let A_ξ denote the unique $\mathcal{L}$-structure with $\mathcal{L}_\in$-reduct $\langle V_{\delta_\xi}, \in \rangle$ such that $\dot{c}^{A_\xi} = x_\xi$, $\dot{M}^{A_\xi} = V_{\eta+2}$ and

$$\dot{R}_\varphi^{A_\xi} \ = \ \{\langle y_0, \ldots, y_n \rangle \in V_{\eta+2}^{n+1} \mid V_{\eta+2} \models \varphi(y_0, \ldots, y_n)\}$$

for every $\mathcal{L}_\in$-formula $\varphi \equiv \varphi(v_0, \ldots, v_n)$. Set $\mathcal{C} = \{A_\xi \mid \xi < \mathrm{cof}(\delta)\} \subseteq V_\delta$, pick a cardinal $\kappa < \delta$ with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds and fix $\zeta < \mathrm{cof}(\delta)$ with $\delta_\zeta > \kappa$. Given $x \in V_{\eta+2}$, we let f_x denote the unique function with domain $\mathcal{C} \cap V_\kappa$ and $f_x(A) = x$ for all $A \in \mathcal{C} \cap V_\kappa$. Since our setup ensures that $\kappa > |V_{\eta+2}|$, we can now find a homomorphism $h : X \longrightarrow A_\zeta$ for some substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ with $f_{x_\zeta} \in X$ and $f_x \in X$ for all $x \in V_{\eta+2}$. As above, we know that

$$j : V_{\eta+2} \longrightarrow V_{\eta+2};\ x \mapsto h(f_x)$$

is an elementary embedding and, by the *Kunen Inconsistency*, this map is trivial. But then

$$h(f_{x_\zeta}) \ = \ j(x_\zeta) \ = \ x_\zeta \ = \ \dot{c}^{A_\zeta}$$

and there exists $A \in \mathcal{C} \cap V_\kappa$ with $x_\zeta = f_{x_\zeta}(A) = \dot{c}^A$, a contradiction. □

The above arguments show that δ is a regular cardinal with $|V_\gamma| < \delta$ for all $\gamma < \delta$ and this directly implies that δ is inaccessible. □

We show how the above lemma can be combined with results in [9] to prove that the following statements are equivalent for every uncountable cardinal δ:

(1) δ is a Woodin cardinal.
(2) For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds.

Proof of Theorem 1.28. (1) ⇒ (2): Let δ be a Woodin cardinal and let $\mathcal{C} \subseteq V_\delta$ be a set of structures of the same type. Using [22, Theorem 26.14], we can find a cardinal $\kappa < \delta$ that is *γ-strong for $\mathcal{C}$* for all $\kappa < \gamma < \delta$, i.e. for all $\kappa < \gamma < \delta$, there exists a transitive class M with $V_\gamma \subseteq M$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \gamma$ and $j(\mathcal{C}) \cap V_\gamma = \mathcal{C} \cap V_\gamma$. Fix $B \in \mathcal{C}$ and pick an inaccessible cardinal $\kappa < \gamma < \delta$ with $B \in V_\gamma$ and an elementary embedding $j : V \longrightarrow M$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \gamma$ and $j(\mathcal{C}) \cap V_\gamma = \mathcal{C} \cap V_\gamma$. Then we have $B \in \mathcal{C} \cap V_\gamma = j(\mathcal{C}) \cap V_\gamma \subseteq j(\mathcal{C}) \cap V^M_{j(\kappa)} = j(\mathcal{C} \cap V_\kappa)$ and the map

$$h : \prod(\mathcal{C} \cap V_\kappa) \longrightarrow B;\ f \mapsto j(f)(B)$$

is a well-defined homomorphism.

(2) ⇒ (1): Assume that δ is an uncountable cardinal with the property that for every set $\mathcal{C} \subseteq V_\delta$ of structures of the same type, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds. Then Lemma 7.1 implies that δ is inaccessible. A direct adaptation of the proof of [9, Theorem 5.13] then shows that δ is a Woodin cardinal. □

8. Subtle product reflection

We will next give a proof of Theorem 1.34. Namely, we will show that the following statements are equivalent for every uncountable cardinal δ:

(1) δ is a subtle cardinal.
(2) For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds.

The implication (1) ⇒ (2) is given by the following lemma:

Lemma 8.1. *If δ is a subtle cardinal and $\mathcal{C} \subseteq V_\delta$ is a non-empty set of structures of the same type, then there exists an inaccessible cardinal $\kappa < \delta$ such that* $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ *holds.*

Proof. Assume, towards a contradiction, that the above conclusion fails. Given an inaccessible cardinal $\kappa < \delta$, fix a substructure X_κ of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality κ and an ordinal $\xi_\kappa < \delta$ such that there exists a structure $B_\kappa \in \mathcal{C} \cap V_{\xi_\kappa}$ with the property that there is no homomorphism from X_κ into B_κ. Then there exists a closed unbounded subset C of δ that consists of strong limit cardinals and has the property that whenever κ is an inaccessible cardinal in C, then $\xi_\kappa < \min(C \setminus (\kappa+1))$. In addition, fix a bijection $b_\kappa : \kappa \longrightarrow X_\kappa$ for every inaccessible cardinal $\kappa < \delta$.

Let $\mathcal{L}$ denote the signature of $\mathcal{C}$ and fix an enumeration $\langle \varphi_k \mid k < \omega \rangle$ of all $\mathcal{L}$-formulas. Pick a δ-list $\langle D_\alpha \mid \alpha < \delta \rangle$ such that the following statements hold for all $\alpha < \delta$:

- If α is inaccessible, then D_α is the set of all elements of α of the form[5] $\prec 0, k, \alpha_0, \ldots, \alpha_{n-1} \succ$, where φ_k is an n-ary $\mathcal{L}$-formula, $\alpha_0, \ldots, \alpha_{n-1} < \alpha$ and
$$X_\alpha \models \varphi_k(b_\alpha(\alpha_0), \ldots, b_\alpha(\alpha_{n-1})).$$
- If α is a singular limit cardinal of cofinality λ, then there exists a cofinal function $c_\alpha : \lambda \longrightarrow \alpha$ with $c_\alpha(0) = 0$ and $D_\alpha = \{\prec 1, \lambda, \xi, c(\xi) \succ \mid \xi < \lambda\}$.

Using the subtleness of δ, we can now find ordinals $\kappa < \theta$ in C with $D_\kappa = D_\theta \cap \kappa$.

Claim. *The ordinals κ and θ are both inaccessible cardinals.*

Proof of the Claim. First, note that the definition of C implies that both κ and θ are strong limit cardinals. Now, assume that κ is a singular strong limit cardinal. Then we have
$$\prec 1, \operatorname{cof}(\kappa), 0, 0 \succ \ \in \ D_\kappa \ \subseteq \ D_\theta$$
and this shows that θ is also singular with $\operatorname{cof}(\kappa) = \operatorname{cof}(\theta)$. Moreover, it follows that $c_\kappa = c_\theta$ and hence $\operatorname{ran}(c_\theta) \subseteq \kappa < \theta$, a contradiction. Thus we have shown that κ is inaccessible, and therefore there is some $k < \omega$ such that $\prec 0, k \succ \in D_\kappa \subseteq D_\theta$. By the definition of the list $\langle D_\alpha \mid \alpha < \delta \rangle$ this shows that θ is also an inaccessible cardinal. □

If we now define
$$b \ = \ b_\theta \circ b_\kappa^{-1} : X_\kappa \longrightarrow X_\theta,$$
then the above setup ensures that b is an elementary embedding. Note that the definition of C implies that $B_\kappa \in \mathcal{C} \cap V_\theta$ and therefore we know that the map
$$p : X_\kappa \longrightarrow B_\theta; \ f \mapsto b(f)(B_\kappa)$$
is a homomorphism of $\mathcal{L}$-structures. But this allows us to conclude that $p \circ b : X_\kappa \longrightarrow B_\kappa$ is a homomorphism, contradicting our initial assumptions. □

Proof of Theorem 1.34. Let δ be an uncountable cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds. Lemma 7.1 then shows that δ is inaccessible. Assume, towards a contradiction, that δ is not a subtle cardinal and fix a closed unbounded subset C of δ and a δ-list $\vec{E} = \langle E_\gamma \mid \gamma < \delta \rangle$ with the property that $E_\gamma \cap \beta \neq E_\beta$ holds for all $\beta < \gamma$ in C. Let α denote the least cardinal of uncountable cofinality in C.

Let $\mathcal{L}'$ denote the first-order language that extends the language of set theory by a unary predicate symbol $\dot{C}$, constant symbols $\dot{\kappa}$, $\dot{u}$, $\dot{c}$, and unary function symbols $\dot{e}$ and $\dot{s}$. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}'$ by an $(n+1)$-ary predicate symbol $\dot{T}_\varphi$ for every $\mathcal{L}'$-formula $\varphi(v_0, \ldots, v_n)$ with $(n+1)$-many free variables, and let $\mathcal{S}_{\mathcal{L}}$ be the class of $\mathcal{L}$-structures as defined at the beginning

[5] Here, we let $\prec \cdot, \ldots, \cdot \succ : \mathrm{Ord}^{n+1} \longrightarrow \mathrm{Ord}$ denote the iterated *Gödel Pairing Function*.

of Section 5. Namely, the class of $\mathcal{L}$-structures A such that there exists a cardinal κ_A in $C^{(1)}$ and a limit ordinal $\theta_A > \kappa_A$ such that:

(1) The domain of A is V_{θ_A+1}.
(2) $\in^A = \in \restriction V_{\theta_A} + 1$, $\dot{\kappa}^A = \kappa_A$ and $\dot{u}^A = \theta_A$.
(3) If $\varphi(v_0, \ldots, v_n)$ is an $\mathcal{L}'$-formula, then

$$\dot{T}^A_\varphi \ = \ \{\langle x_0, \ldots, x_n\rangle \in V^{n+1}_{\theta_A+1} \mid A \models \varphi(x_0, \ldots, x_n)\}.$$

Let $\mathcal{C}$ denote the set of all $A \in \mathcal{S}_\mathcal{L} \cap V_\delta$ such that the following hold:

- κ_A is a limit point of C above α.
- $\dot{c}^A = \alpha$ and $\dot{C}^A = C \cap \kappa_A$.
- If $\gamma < \kappa_A$, then $\dot{e}^A(\gamma) = E_\gamma$ and $\dot{s}^A(\gamma) = \min(C \setminus (\gamma + 1))$.

The fact that δ is inaccessible implies that $\mathcal{C}$ is non-empty and hence there exists a cardinal $\zeta < \delta$ with the property that $\mathrm{WPSR}_\mathcal{C}(\zeta)$ holds. Define

$$\kappa \ = \ \sup\{\kappa_A \mid A \in \mathcal{C} \cap V_\zeta\} \ \in \ (\alpha, \zeta]$$

and notice that $\kappa \in C^{(1)} \cap \mathrm{Lim}(C)$. Let D be some cofinal subset of κ of order-type $\mathrm{cof}(\kappa)$. Given a set x, we again define functions f_x and f^x with domain $\mathcal{C} \cap V_\zeta$ as in Section 5. Namely, we have $f^x(A) = x \cap V_{\kappa_A}$ for all $A \in \mathcal{C} \cap V_\zeta$, $f_x(A) = x$ for all $A \in \mathcal{C} \cap V_\zeta$ with $x \in V_{\kappa_A}$ and $f_x(A) = \dot{u}^A$ for all $A \in \mathcal{C} \cap V_\zeta$ with $x \notin V_{\kappa_A}$. Then there is a substructure X of $\prod(\mathcal{C} \cap V_\zeta)$ of cardinality κ with the property that $f^D, f^{E_\kappa} \in X$ and $f_x, f^x \in X$ for all $x \in V_\kappa$. Moreover, since δ is inaccessible, we can find a structure B in $\mathcal{C}$ with $\kappa_B > \zeta$. By $\mathrm{WPSR}_\mathcal{C}(\zeta)$, there is a homomorphism $h : X \longrightarrow B$. Using the results of Section 5, we can define

$$\lambda \ = \ \min\{\mathrm{rnk}(x) \mid f_x \in X,\, h(f_x) = \dot{u}^B\} \ \leq \ \kappa$$

as well as

$$\chi \ = \ \sup\{h(f_\alpha) \mid \alpha < \lambda\} \ \leq \ \kappa_B.$$

Since $h(f_\alpha) = \dot{c}^B = \alpha \neq \dot{u}^B$, we have that $\alpha < \lambda$.

Claim. $\lambda \in C$.

Proof of the Claim. Assume, towards a contradiction, that $\lambda \notin C$. We then have $\alpha \in C \cap \lambda \neq \varnothing$ and, if we define $\beta = \sup(C \cap \lambda)$, then $\beta < \lambda$ and $h(f_\beta) \neq \dot{u}^B$. Set $\gamma = \min(C \setminus (\beta + 1))$. If $A \in \mathcal{C} \cap V_\zeta$ is such that $f_\beta(A) \neq \dot{u}^A$, then $\beta < \kappa_A$, $\gamma = \dot{s}^A(\beta) < \kappa_A$, and hence $f_\gamma(A) = \gamma \neq \dot{u}^A$. Thus, for every $A \in \mathcal{C} \cap V_\zeta$, we have

$$A \models \text{“} f_\beta \neq \dot{u}^A \longrightarrow f_\gamma \neq \dot{u}^A\text{”}.$$

As $h(f_\beta) \neq \dot{u}^B$, Lemma 5.1.(1) shows that $h(f_\gamma) \neq \dot{u}^B$ and hence $\gamma < \lambda$, which yields a contradiction to the fact that γ is the least element of C greater than β, and therefore must be greater than λ. □

Now let

$$j : V_\lambda \longrightarrow V_\chi;\ x \mapsto h(f_x)$$

be the Σ_1-elementary embedding given by Lemma 5.5.(3). Lemma 5.5.(1) shows that $h(f_\lambda) = \dot{u}^B \neq \lambda$ and we can define $\mu \leq \lambda$ to be the minimal ordinal with $h(f_\mu) \neq \mu$. Then Lemma 5.5.(2) implies that $h(f_x) = x$ holds for all $x \in V_\mu$.

Claim. $\mu < \kappa$.

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda = \kappa$ holds. Since $f^{E_\kappa} \in X$, Lemma 5.2.(1) shows that $h(f^{E_\kappa}) \cap \kappa = E_\kappa = \dot{e}^B(\kappa)$. Since $\kappa \in C \cap \kappa_B = \dot{C}^B$, the cardinal κ therefore witnesses that

$$B \models \text{“}\exists\beta < \dot{\kappa}\ [\beta \in \dot{C} \wedge h(f^{E_\kappa}) \cap \beta = \dot{e}(\beta)]\text{”}.$$

Using Lemma 5.1.(1), we can now find $A \in \mathcal{C} \cap V_\zeta$ and $\beta \in C \cap \kappa_A$ with $E_\kappa \cap \beta = f^{E_\kappa}(A) \cap \beta = E_\beta$. But this yields a contradiction, because we have $\beta < \kappa_A \leq \kappa$ and hence κ_A and κ are distinct elements of C. □

Claim. $\mu < \lambda$.

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda$ holds. The above claims then show that $\mu \in C \cap \kappa$. Since $f^{E_\mu} \in X$, Lemma 5.2.(1) implies that $h(f^{E_\mu}) \cap \mu = E_\mu$ and Lemma 5.5.(1) shows that $h(f_\lambda) = \dot{u}^B$, the ordinal μ witnesses that

$$B \models \text{“}h(f_\mu) = \dot{u}\ \wedge\ \exists\beta < \dot{\kappa}\ [\beta \in \dot{C} \wedge h(f^{E_\mu}) \cap \beta = \dot{e}(\beta)]\text{”}.$$

Using Lemma 5.1.(1), this yields $A \in \mathcal{C} \cap V_\zeta$ with $\kappa_A \leq \mu$ and $\beta \in C \cap \kappa_A$ satisfying $E_\mu \cap \beta = f^{E_\mu}(A) \cap \beta = E_\beta$. Since $\beta < \mu$ are both elements of C, this yields a contradiction. □

By the above claims, we now know that $\mu < \lambda \leq \kappa$.

Claim. $\alpha < \mu$.

Proof of the Claim. Assume, towards a contradiction, that $\mu \leq \alpha$ holds. Since we have

$$j(\alpha) \ = \ h(f_\alpha) \ = \ \dot{c}^B \ = \ \alpha$$

we must have $\mu < \alpha$. But this implies that $j \restriction V_\alpha : V_\alpha \longrightarrow V_\alpha$ is a non-trivial elementary embedding and, since α has uncountable cofinality, this yields a contradiction via the *Kunen Inconsistency.* □

Claim. $\mu \in C$.

Proof of the Claim. Assume, towards a contradiction, that μ is not an element of C. Since the previous claim shows that $\alpha \in C \cap \mu$, we know that $\beta = \sup(C \cap \mu) < \mu$ and $\gamma = \min(C \setminus (\beta+1)) > \mu$. We then have $\gamma = \dot{s}^A(\beta) < \kappa_A$ for all $A \in \mathcal{C} \cap V_\zeta$ with $\mu < \kappa_A$. Thus for every $A \in \mathcal{C} \cap V_\zeta$, we have

$$A \models \text{“}f_\mu(A) \neq \dot{u}^A \longrightarrow \dot{s}^A(f_\beta(A)) < \dot{\kappa}^A\text{”}.$$

Thus, since an earlier claim shows $\mu < \lambda$ and therefore Lemma 5.1.(1) implies that $h(f_\mu) \neq \dot{u}^B$, we know that $\dot{s}^B(h(f_\beta)) < \kappa_B$. Hence, we can conclude that

$$j(\gamma) = h(f_\gamma)) = \dot{s}^B(h(f_\beta)) = \dot{s}^B(j(\beta)) = \dot{s}^B(\beta) = \gamma.$$

Since Lemma 5.5.(1) ensures that $\gamma + 2 < \lambda$, the Σ_1-elementarity of j implies that $j(V_{\gamma+2}) = V_{\gamma+2}$ and we can conclude that $j \restriction V_{\gamma+2} : V_{\gamma+2} \longrightarrow V_{\gamma+2}$ is a non-trivial elementary embedding, which yields a contradiction via *Kunen Inconsistency.* □

Now notice that since $\mu < \lambda$, Lemma 5.5.(2) yields that $h(f_\mu) \neq \dot{u}^B$, and since $\mu < j(\mu) = h(f_\mu)$, we have that

$$B \models \text{“}h(f_\mu) \neq \dot{u}\ \wedge\ \exists\beta < h(f_\mu)\ [\beta \in \dot{C} \wedge h(f^{E_\mu}) \cap \beta = \dot{e}(\beta)]\text{”}$$

as witnessed by μ. An application of Lemma 5.1.(1) then yields an $A \in \mathcal{C} \cap V_\zeta$ with $\kappa_A > \mu$ and $\beta \in C \cap \mu$ with the property that

$$E_\mu \cap \beta \ = \ f^{E_\mu}(A) \cap \beta \ = \ \dot{e}^A(\beta) \ = \ E_\beta.$$

Since β and μ are distinct elements of C, this yields a final contradiction.

The above computations yield the implication (2) $\Rightarrow$ (1) of the theorem. Since the implication (1) $\Rightarrow$ (2) follows from Lemma 8.1, this completes the proof of the theorem. □

9. Between strongly unfoldable and subtle cardinals

We shall give in this section a proof of Theorem 1.13. But first we shall prove some properties of $C^{(n)}$-strongly unfoldable cardinals, and give other equivalent reformulations of these cardinals in terms of elementary embeddings. Recall (see Definition 1.12) that an inaccessible cardinal κ is $C^{(n)}$*-strongly unfoldable* if for every ordinal $\lambda \in C^{(n)}$ greater than κ and every transitive ZF$^-$-model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$.

Proposition 9.1. *$C^{(n)}$-strongly unfoldable cardinals are elements of $C^{(n+1)}$.*

Proof. Let κ be a $C^{(n)}$-strongly unfoldable cardinal. Pick a Σ_{n+1}-formula $\varphi(v)$ and $z \in V_\kappa$ with the property that $\varphi(z)$ holds in V. Fix $\lambda \in C^{(n+1)}$ greater than κ, so that $\varphi(z)$ holds in V_λ, and fix an elementary submodel M of H_{κ^+} of cardinality κ with $(\kappa+1) \cup {}^{<\kappa}M \subseteq M$. By our assumption, there exists a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$. Thus, since $\varphi(z)$ holds in V_λ, it also holds in $V^N_{j(\kappa)}$, and hence elementarity implies that $\varphi(z)$ holds in V_κ. □

Clearly, a cardinal is strongly unfoldable if and only if it is $C^{(0)}$-strongly unfoldable. But more is true:

Proposition 9.2. *A cardinal is strongly unfoldable if and only if it is $C^{(1)}$-strongly unfoldable.*

Proof. Let κ be an inaccessible cardinal, let $\lambda \in C^{(1)}$ be greater than κ, let M be a transitive ZF$^-$-model of cardinality κ with $\{\kappa\} \cup {}^{<\kappa}M \subseteq M$, let N be a transitive set with $V_\lambda \subseteq N$, and let $j : M \longrightarrow N$ be an elementary embedding with $\operatorname{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. Then, in N, λ is also in $C^{(1)}$. So, since $j(\kappa)$ is an inaccessible cardinal in N, hence also in $C^{(1)}$ in the sense of N, we have that $V_\lambda \prec_{\Sigma_1} V^N_{j(\kappa)}$. □

Proposition 9.3. *Given a natural number n, every $C^{(n)}$-extendible cardinal is $C^{(n+1)}$-strongly unfoldable.*

Proof. Since $C^{(0)}$-extendibility coincides with $C^{(1)}$-extendibility and $C^{(0)}$-strong unfoldability coincides with $C^{(1)}$-strong unfoldability, we may assume that n is greater than 0. Let κ be a $C^{(n)}$-extendible cardinal, let $\lambda > \kappa$ be an element of $C^{(n+1)}$ and let M be a transitive ZF$^-$-model of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$. Then there is an ordinal μ and an elementary embedding $j : V_\lambda \longrightarrow V_\mu$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. Elementarity now implies that $N = j(M)$ is a transitive set with $V_{j(\kappa)} \subseteq N$, and our setup ensures that $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\kappa)} = V^N_{j(\kappa)}$. Moreover, the map $i = j \restriction M : M \longrightarrow N$ is an elementary embedding with $\operatorname{crit}(i) = \kappa$, $i(\kappa) > \lambda$ and $V_\lambda \prec_{\Sigma_{n+1}} V^N_{i(\kappa)}$. □

Theorem 9.4. *Given a natural number $n > 0$, the following statements are equivalent for every cardinal κ:*

1. *κ is $C^{(n)}$-strongly unfoldable.*
2. *For every $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$, every ordinal $\gamma \in C^{(n)}$ greater than κ, and every subset A of V_κ with the property that $\varphi(\kappa, A)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ with $\beta \in C^{(n)}$ and the property that $\varphi(\alpha, A \cap V_\alpha)$ holds in V_β.*

(3) *For every ordinal $\gamma \in C^{(n)}$ greater than κ and every $z \in V_\gamma$ there exist an ordinal $\bar{\gamma} \in C^{(n)} \cap \kappa$, a cardinal $\bar{\kappa} < \bar{\gamma}$, an elementary submodel X of $V_{\bar{\gamma}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and an elementary embedding $j : X \longrightarrow V_\gamma$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.*

(4) *For every ordinal $\lambda \in C^{(n)}$ greater than κ, every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, and every Π_{n-1}-formula $\psi(v_0, v_1)$ with the property that $\psi(M, \kappa)$ holds, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ such that $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$, and $\psi(N, j(\kappa))$ holds.*

Proof. First, note that, by a well-known result of Levy (see [22, Proposition 6.2]), the assumption (2) implies that κ is an inaccessible cardinal. And also each of (3) and (4) implies that κ is inaccessible, since so are the critical points of elementary embeddings of sufficiently reach transitive models.

(1) $\Rightarrow$ (2): Assume that (1) holds and fix an $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$, an ordinal $\gamma \in C^{(n)}$ greater than κ, and a subset A of V_κ such that $\varphi(\kappa, A)$ holds in V_γ. Pick an elementary submodel M of H_{κ^+} of cardinality κ with $\{\kappa, A\} \cup {}^{<\kappa}M \subseteq M$ and use our assumption to find a transitive set N with $V_\gamma \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \gamma$ and $V_\gamma \prec_{\Sigma_n} V^N_{j(\kappa)}$. The ordinals κ and γ then witness that, in N, there are ordinals $\alpha < \beta < j(\kappa)$ such that $V_\beta \prec_{\Sigma_n} V_{j(\kappa)}$ and $\varphi(\alpha, j(A) \cap V_\alpha)$ holds in V_β. Using the elementarity of j, we find ordinals $\alpha < \beta < \kappa$ with the property that $V_\beta \prec_{\Sigma_n} V_\kappa$ and $\varphi(\alpha, A \cap V_\alpha)$ holds in V_β. Since Proposition 9.1 shows that $\kappa \in C^{(n+1)}$, we also have that β is in $C^{(n)}$.

(2) $\Rightarrow$ (3): Assume that (2) holds. Pick an ordinal $\gamma \in C^{(n)}$ greater than κ, and an element z of V_γ. Let $\varphi(v_0, v_1)$ be an $\mathcal{L}_\in$-formula with the property that for every ordinal λ and all $A, \delta \in V_\lambda$, the statement $\varphi(A, \delta)$ holds in V_λ if and only if λ is a limit ordinal and there exists an ordinal $\varepsilon > \delta$, a subset X of V_ε and a bijection $b : \delta \longrightarrow X$ such that the following statements hold:

- V_ε is Σ_n-correct.
- $V_\delta \cup \{\delta\} \subseteq X$.
- $b(0) = \delta$ and $b(\omega \cdot (1 + \alpha)) = \alpha$ for all $\alpha < \delta$.
- If $\alpha_0, \ldots, \alpha_{n-1} < \delta$ and $a \in \mathsf{Fml}$ represents a formula with n free variables, then
$$\begin{aligned}\langle a, \alpha_0, \ldots, \alpha_{n-1}\rangle \in A &\iff \mathsf{Sat}(X, \langle b(\alpha_0), \ldots, b(\alpha_{n-1})\rangle, a)\\ &\iff \mathsf{Sat}(V_\varepsilon, \langle b(\alpha_0), \ldots, b(\alpha_{n-1})\rangle, a),\end{aligned}$$
where Fml denotes the set of formalized $\mathcal{L}_\in$-formulas and Sat denotes the formalized satisfaction relation for $\mathcal{L}_\in$-formulas.[6]

Pick an ordinal $\lambda \in C^{(n)}$ greater than γ, an elementary submodel Y of V_γ of cardinality κ with $V_\kappa \cup \{\kappa, z\} \subseteq Y$ and a bijection $b : \kappa \longrightarrow Y$ with $b(0) = \kappa$, $b(1) = z$ and $b(\omega \cdot (1 + \alpha)) = \alpha$ for all $\alpha < \kappa$. Let A denote the set of all tuples $\langle a, \alpha_0, \ldots, \alpha_{n-1}\rangle$ with the property that $a \in \mathsf{Fml}$ represents an $\mathcal{L}_\in$-formula with n free variables, $\alpha_0, \ldots, \alpha_{n-1} < \kappa$ and $\mathsf{Sat}(Y, \langle b(\alpha_0), \ldots, b(\alpha_{n-1})\rangle, a)$. Then γ, Y and b witness that the statement $\varphi(A, \kappa)$ holds in V_λ, and we can use our assumption to find ordinals $\bar{\kappa} < \beta < \kappa$ such that $\beta \in C^{(n)}$ and $\varphi(A \cap V_{\bar{\kappa}}, \bar{\kappa})$ holds in V_β. Then there are ordinals $\bar{\kappa} < \bar{\gamma} < \beta$, a subset X of $V_{\bar{\gamma}}$ and a bijection $\bar{b} : \bar{\kappa} \longrightarrow X$ such that $V_{\bar{\gamma}} \prec_{\Sigma_n} V_\beta$, $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, $\bar{b}(0) = \bar{\kappa}$, $\bar{b}(\omega \cdot (1 + \alpha)) = \alpha$ for all $\alpha < \bar{\kappa}$ and

$$\begin{aligned}\langle a, \alpha_0, \ldots, \alpha_{n-1}\rangle \in A \cap V_{\bar{\kappa}} &\iff \mathsf{Sat}(X, \langle \bar{b}(\alpha_0), \ldots, \bar{b}(\alpha_{n-1})\rangle, a)\\ &\iff \mathsf{Sat}(V_{\bar{\gamma}}, \langle \bar{b}(\alpha_0), \ldots, \bar{b}(\alpha_{n-1})\rangle, a)\end{aligned}$$

[6] See [12, Section I.9]. Note that the classes Fml and Sat are defined by Σ_1-formulas without parameters.

for all $\alpha_0, \ldots, \alpha_{n-1} < \bar{\kappa}$ and every $a \in \mathsf{Fml}$ representing a formula with n free variables. Since $\beta \in C^{(n)}$ and $V_{\bar{\gamma}} \prec_{\Sigma_n} V_\beta$, we have that $\bar{\gamma} \in C^{(n)}$. Also, the displayed equivalences above show that X is an elementary submodel of $V_{\bar{\gamma}}$ and, for each $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ and $\alpha_0, \ldots, \alpha_{n-1} < \bar{\kappa}$, we have

$$X \models \varphi(\bar{b}(\alpha_0), \ldots, \bar{b}(\alpha_{n-1})) \iff \langle \ulcorner \varphi \urcorner, \alpha_0, \ldots, \alpha_{n-1} \rangle \in A \cap V_{\bar{\kappa}}$$
$$\iff \langle \ulcorner \varphi \urcorner, \alpha_0, \ldots, \alpha_{n-1} \rangle \in A \iff V_\gamma \models \varphi(b(\alpha_0), \ldots, b(\alpha_{n-1})),$$

where $\ulcorner \varphi \urcorner$ denotes the canonical element of Fml representing φ. In particular, we know that the map $j = b \circ \bar{b}^{-1} : X \longrightarrow V_\gamma$ is an elementary embedding that satisfies $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \operatorname{ran}(j)$. This shows that (3) holds.

(3) ⇒ (4): Assume that (3) holds. Fix ordinals $\lambda < \gamma$ in $C^{(n)}$ and greater than κ, a transitive ZF$^-$-model M of cardinality κ with $\{\kappa\} \cup {}^{<\kappa}M \subseteq M$, and a Π_{n-1}-formula $\psi(v_0, v_1)$ such that $\psi(M, \kappa)$ holds. Then there exists an ordinal $\bar{\gamma} \in C^{(n)} \cap \kappa$, a cardinal $\bar{\kappa} < \bar{\gamma}$, an elementary submodel X of $V_{\bar{\gamma}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and an elementary embedding $j : X \longrightarrow V_\gamma$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $M, \lambda \in \operatorname{ran}(j)$. Pick $\bar{M}, \bar{\lambda} \in X$ with $j(\bar{M}) = M$ and $j(\bar{\lambda}) = \lambda$. Then $\bar{M} \subseteq X$ and the fact that the class $C^{(n)}$ is Π_n-definable implies that $\bar{\lambda}$ is an also element of $C^{(n)}$. In particular, we know that $V_{\bar{\lambda}} \prec_{\Sigma_n} V_\kappa$. Thus, κ, M and $j \restriction \bar{M}$ witness that there exists a transitive set N and an elementary embedding $k : \bar{M} \longrightarrow N$ such that $\bar{\lambda} < k(\bar{\kappa})$, $V_{\bar{\lambda}} \subseteq N$, $V_{\bar{\lambda}} \prec_{\Sigma_n} V^N_{k(\bar{\kappa})}$, $\operatorname{crit}(k) = \bar{\kappa}$ and $\psi(N, k(\bar{\kappa}))$ holds. Since this statement can be formulated by a Σ_n-formula with parameters $\bar{\kappa}$, $\bar{M}$ and $V_{\bar{\lambda}}$, the fact that all these parameters are contained in X implies that the statement holds in X. The elementarity of j and the Σ_n-correctness of V_γ now yield a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $k : M \longrightarrow N$ such that $\operatorname{crit}(k) = \kappa$, $k(\kappa) > \lambda$, $V_\lambda \prec_{\Sigma_n} V^N_\rho$ and $\psi(N, k(\kappa))$ holds. This shows that (4) holds.

Since the implication (4) ⇒ (1) is trivial, this completes the proof that all four of the listed statements are equivalent. □

Remark 9.5. Note that, in the definition of κ being $C^{(n)}$-strongly unfoldable (Definition 1.12), we may only require, and obtain an equivalent definition, that for a *tail* of ordinals $\lambda \in C^{(n)}$ greater than κ and every transitive ZF$^-$-model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$. For given *any* $\lambda' \in C^{(n)}$ greater than κ, and given M as before, we can pick $\lambda \geq \lambda'$ in the tail so that the required N exists. Then N also works for λ, because $V_{\lambda'} \prec_{\Sigma_n} V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$, and therefore $V_{\lambda'} \prec_{\Sigma_n} V^N_{j(\kappa)}$.

The same applies to the equivalent reformulations of $C^{(n)}$-strong unfoldability given in Theorem 9.4, namely one gets equivalent statements by only requiring that they hold for a *tail* of λ in $C^{(n)}$. Let us see this for (2): so suppose (2) holds for a tail of $\lambda \in C^{(n)}$. Given $\varphi(v_0, v_1)$, $\gamma \in C^{(n)}$ greater than κ, and $A \subseteq V_\kappa$ such that $\varphi(\kappa, A)$ holds in V_γ, pick $\lambda \in C^{(n)}$ in the tail such that the sentence

$$\exists X, \delta \ [X = V_\delta \ \wedge \ \delta \in C^{(n)} \ \wedge \ V_\delta \models \varphi(\kappa, A)]$$

holds in V_λ (such a λ exists by the Reflection Theorem, since the sentence is true in V, and therefore true in V_β for a closed unbounded class class of β). So, there exist ordinals $\alpha < \beta < \kappa$ with $\beta \in C^{(n)}$ and the property that the sentence

$$\exists X, \delta \ [X = V_\delta \ \wedge \ \delta \in C^{(n)} \ \wedge \ V_\delta \models \varphi(\alpha, A \cap \alpha)]$$

holds in V_β. If δ witnesses this, then $\delta \in C^{(n)}$, because V_β is correct about this.

Corollary 9.6. *Given a nautural number $n > 1$, the following statements are equivalent for every cardinal κ:*

(1) *κ is $C^{(n)}$-strongly unfoldable.*
(2) *For every cardinal $\lambda \in C^{(n)}$ greater than κ, and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there exists an inaccessible cardinal $\rho \in C^{(n-1)}$ greater than λ,*[7] *a transitive set N with $\rho \in N$ and ${}^{<\rho}N \subseteq N$, and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$ and $j(\kappa) = \rho$.* □

Proof. Assuming (1), the statement (2) follows immediately from (4) of Theorem 9.4 by taking $\varphi(v_0, v_1)$ to be the formula that asserts that $v_1 \in C^{(n-1)}$ and v_0 is a transitive ZF^--model of cardinality v_1 with $v_1 \in v_0$ and ${}^{<v_1}v_0 \subseteq v_0$, and by taking $\rho = j(\kappa)$. That (2) implies (1) is immediate. □

In particular, this shows that all $C^{(2)}$-strongly unfoldable cardinals are *almost-hugely unfoldable*, as introduced in [19, Definition 4].

The characterizations and properties of $C^{(n)}$-strongly unfoldable cardinals given in Section 3 can now be put to use to extend the equivalences between strong unfoldability and weak forms of Structural Reflection given by Theorem 1.10 to any classes of structures of any degree of definitional complexity.

Lemma 9.7. *If κ is $C^{(n)}$-strongly unfoldable cardinal, then $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.*

Proof. Pick a Π_n-formula $\varphi(v_0, v_1, v_2)$ and an element z of V_κ with the property that

$$\mathcal{C} \ = \ \{A \mid \exists x \ \varphi(A, x, z)\}.$$

Fix a structure B in $\mathcal{C}$ of cardinality κ and an ordinal $\kappa < \gamma \in C^{(n)}$ with the property that $\varphi(B, y, z)$ holds for some $y \in V_\gamma$. By Theorem 9.4, we can now find an ordinal $\bar{\gamma} \in C^{(n)} \cap \kappa$, a cardinal $\bar{\kappa} < \bar{\gamma}$, an elementary submodel X of $V_{\bar{\gamma}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $j : X \longrightarrow V_\gamma$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $B, z \in \mathrm{ran}(j)$. Pick $A \in V_{\bar{\gamma}}$ with $j(A) = B$. Our setup then ensures that the domain of A is a subset of X, $j \restriction A : A \longrightarrow B$ is an elementary embedding and A is an element of $\mathcal{C} \cap H_\kappa$. □

We shall next prove Theorem 1.13, which is a generalization of Theorem 1.10 to arbitrary classes of structures. Namely, we will show that, for every $n > 1$, the following are equivalent for every cardinal κ:

(1) κ is either $C^{(n)}$-strongly unfoldable or a limit of $C^{(n-1)}$-extendible cardinals.
(2) The principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.
(3) $\kappa \in C^{(n+1)}$ and the principle $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.

The proof follows similar arguments as that of Theorem 1.10 (given in Section 3), but now using Proposition 9.1, Lemma 9.7, and the characterization of $C^{(n)}$-strongly unfoldable cardinals given in Theorem 9.4 3.

Proof of Theorem 1.13. Assume that κ is not $C^{(n)}$-strongly unfoldable and the principle $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with

[7] Note that these assumptions imply that $V_\lambda \prec_{\Sigma_n} V_\rho$.

parameters in V_κ. Lemma 3.1 then shows that $\kappa \in C^{(n+1)}$. In particular, we know that κ is a limit cardinal and $|V_\kappa| = \kappa$.

Claim. *If θ is a cardinal in $C^{(n)}$ greater than κ, $y \in V_\kappa$, and $z \in H_\theta$, then there are cardinals $\bar{\kappa} < \bar{\theta} < \kappa$ with $y \in V_{\bar{\kappa}}$ and $\bar{\theta} \in C^{(n)}$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and an elementary embedding $j : X \longrightarrow H_\theta$ such that $j(\bar{\kappa}) = \kappa$, $j(y) = y$ and $z \in \operatorname{ran}(j)$.*

Proof of the Claim. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}_\in$ by three constant symbols and let $\mathcal{C}$ denote the class of all $\mathcal{L}$-structures of the form $\langle M, \in, \mu, a, y\rangle$ such that μ is a cardinal in $C^{(n)}$, $y \in V_\mu$, and there exists a cardinal ν in $C^{(n)}$ greater than μ and an elementary submodel X of H_ν with $V_\mu \cup \{\mu\} \subseteq X$ and the property that M is the transitive collapse of X. Since the class $C^{(n)}$ is Π_n-definable, the class $\mathcal{C}$ is definable by a Σ_{n+1}-formula with parameter y. Now, let Y be an elementary submodel of H_θ of cardinality κ with $V_\kappa \cup \{\kappa, z\}$ and let $\tau : Y \longrightarrow N$ denote the corresponding transitive collapse. Thus θ and Y witness that $B = \langle N, \in, \kappa, \tau(z), y\rangle$ is an element of $\mathcal{C}$ of cardinality κ. Our assumption $\mathrm{WSR}_{\mathcal{C}}(\kappa)$ now yields cardinals $\bar{\kappa} < \bar{\theta}$ with $\bar{\kappa} \in C^{(n)} \cap \kappa$, an elementary submodel X of $H_{\bar{\theta}}$ of cardinality less than κ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and an elementary embedding $i : M \longrightarrow N$, with M being the transitive collapse of X, and with $i(\bar{\kappa}) = \kappa$, $i(y) = y$ and $\tau(z) \in \operatorname{ran}(i)$. Since $\kappa \in C^{(n+1)}$ and $M \in V_\kappa$, we may assume that $\bar{\theta} < \kappa$. Letting $\pi : X \longrightarrow M$ denote the transitive collapse, define

$$j \;=\; \tau^{-1} \circ i \circ \pi : X \longrightarrow H_\theta.$$

Then j is an elementary embedding with $j(\bar{\kappa}) = \kappa$, $j(y) = y$ and $z \in \operatorname{ran}(j)$. □

Claim. *If θ is a cardinal in $C^{(n)}$ greater than κ, $y \in V_\kappa$, and $z \in H_\theta$, then there are cardinals $\bar{\kappa} < \bar{\theta} < \kappa$ with $y \in V_{\bar{\kappa}}$ and $\bar{\theta} \in C^{(n)}$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and an elementary embedding $j : X \longrightarrow H_\theta$ such that $j(\bar{\kappa}) = \kappa$, $j(y) = y$, $z \in \operatorname{ran}(j)$, and $j \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$.*

Proof of the Claim. Since we assumed κ is not $C^{(n)}$-strongly unfoldable, Theorem 9.4.3 and Remark 9.5 show that there exists a cardinal ϑ in $C^{(n)}$ greater than θ, and $z' \in V_\vartheta$ such that for all cardinals $\bar{\kappa} < \bar{\vartheta}$, with $\bar{\vartheta} \in C^{(n)} \cap \kappa$, and all elementary submodels X of $H_{\bar{\vartheta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, there is no elementary embedding $j : X \longrightarrow H_\vartheta$ such that $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, and $z, z', \theta \in \operatorname{ran}(j)$. An application of our previous claim now yields cardinals $\bar{\kappa} < \bar{\vartheta} < \kappa$ with $\bar{\vartheta} \in C^{(n)}$ and $y \in V_{\bar{\kappa}}$, an elementary submodel Y of $H_{\bar{\vartheta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq Y$, and an elementary embedding $i : Y \longrightarrow H_\vartheta$ such that $i(\bar{\kappa}) = \kappa$, $i(y) = y$ and $z, z', \theta \in \operatorname{ran}(i)$. Therefore, we must have $i \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$. Pick $\bar{\theta} \in Y$ with $i(\bar{\theta}) = \theta$. Then elementarity implies that $\bar{\theta}$ is a cardinal. Set $X = Y \cap H_{\bar{\theta}}$ and $j = i \restriction X$. We can then conclude that $\bar{\kappa} < \bar{\theta} < \kappa$, X is an elementary submodel of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$ and $j : X \longrightarrow H_\theta$ is an elementary embedding with $j(\bar{\kappa}) = \kappa$, $j(y) = y$, $z \in \operatorname{ran}(j)$, and $j \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$. □

Claim. *There are unboundedly many cardinals below κ that are $C^{(n-1)}$-extendible.*

Proof of the Claim. Fix an uncountable regular cardinal $\rho < \kappa$ and a cardinal θ in $C^{(n)}$ above κ. By our previous claim, we can find cardinals $\rho < \bar{\kappa} < \bar{\theta} < \kappa$, with $\bar{\theta} \in C^{(n)}$, an elementary submodel X of $H_{\bar{\theta}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and an elementary embedding $i : X \longrightarrow H_\theta$ such that $i(\bar{\kappa}) = \kappa$, $i(\rho) = \rho$ and $i \restriction \bar{\kappa} \neq \mathrm{id}_{\bar{\kappa}}$. Set $j = i \restriction V_{\bar{\kappa}} : V_{\bar{\kappa}} \longrightarrow V_\kappa$. Since j is non-trivial, the *Kunen's Inconsistency* implies that $j \restriction V_\rho = \mathrm{id}_\rho$, and so $\operatorname{crit}(j) > \rho$.

Define $\lambda = \operatorname{crit}(j)$. We claim that $\lambda \in C^{(n)}$. For suppose $a \in V_\lambda$ and $\exists x\ \varphi(x, a)$ is a Σ_n-statement that uses the set a as a parameter and holds in V. First, notice that $\bar{\kappa} \in C^{(n)}$, because as $\theta \in C^{(n)}$, H_θ satisfies "$\kappa \in C^{(n)}$", and, by elementarity, the model X, and therefore also $H_{\bar{\theta}}$, satisfies "$\bar{\kappa} \in C^{(n)}$", and since $\bar{\theta} \in C^{(n)}$, this is true. Therefore, the statement $\exists x\ \varphi(x, a)$ holds in

$V_{\bar{\kappa}}$. Now, note that $\bar{\kappa} \leq \sup_{m<\omega} j^m(\lambda)$, because otherwise $\sup_{m<\omega} j^m(\lambda)$ would be a fixed point of j below $\bar{\kappa}$, which is impossible by the *Kunen Inconsistency*. Hence, we can pick $m > 0$ such that $V_{j^m(\lambda)}$ contains a witness for the existential statement $\exists x\ \varphi(x,a)$. Thus, we know that

$$V_{\bar{\kappa}} \models \exists x \in V_{j^m(\lambda)}\ \varphi(x,a).$$

Since $\kappa \in C^{(n)}$, and therefore $V_{\bar{\kappa}} \prec_{\Sigma_n} V_\kappa$, we also have

$$V_\kappa \models \exists x \in V_{j^m(\lambda)}\ \varphi(x,a),$$

which, by elementarity, directly implies that

$$V_{\bar{\kappa}} \models \exists x \in V_{j^{m-1}(\lambda)}\ \varphi(x,a).$$

By iterating this argument m-many times, we can conclude that

$$V_{\bar{\kappa}} \models \exists x \in V_\lambda\ \varphi(x,a)$$

and, since $\bar{\kappa} \in C^{(n)}$, this shows that the statement $\exists x \in V_\lambda \varphi(x,a)$ holds in V, as wanted.

For each ordinal α in the interval $(\lambda, \bar{\kappa})$, the restricted map $j \restriction V_\alpha : V_\alpha \longrightarrow V_{j(\alpha)}$ is an elementary embedding with critical point λ. Note that $j(\lambda) \in C^{(n)}$, because $\lambda \in C^{(n)}$ implies that $\lambda \in (C^{(n)})^{V_{\bar{\kappa}}}$, elementarity ensures that $j(\lambda) \in (C^{(n)})^{V_\kappa}$ and the fact that $\kappa \in C^{(n)}$ allows us to conclude that $j(\lambda) \in C^{(n)}$. Now, for each $\alpha \in (\lambda, \bar{\kappa})$, the statement "*There exists an ordinal β and an elementary embedding $i : V_\alpha \longrightarrow V_\beta$ with* $\operatorname{crit}(i) = \lambda$ *and* $i(\lambda) \in C^{(n-1)}$" can be formulated by a Σ_n-formula with parameters α and λ, and it holds in V. Hence, it also holds in $V_{\bar{\kappa}}$. Thus, in $V_{\bar{\kappa}}$, for every ordinal α greater than λ, there is an ordinal β and an elementary embedding $i : V_\alpha \longrightarrow V_\beta$ with $\operatorname{crit}(i) = \lambda$ and $i(\lambda) \in C^{(n-1)}$. Elementarity then implies that, in V_κ, for every ordinal α greater than $j(\lambda)$, there is an ordinal β and an elementary embedding $i : V_\alpha \longrightarrow V_\beta$ with $\operatorname{crit}(i) = j(\lambda)$ and $i(j(\lambda)) \in C^{(n-1)}$. Since κ is an element of $C^{(n+1)}$, this statement also holds in V and we can conclude that $j(\lambda)$ is a cardinal in the interval (ρ, κ) that is $C^{(n-1)}$-extendible. □

The above computations directly yield the implication (2) ⇒ (1) of the theorem. Moreover, the implication (1) ⇒ (3) follows directly from a combination of Theorem 1.4, Proposition 9.1 and Lemma 9.7. This completes the proof of the theorem, because the implication (3) ⇒ (2) is immediate. □

We end this section by determining the class-principle corresponding to weak SR, i.e., we show that the following schemes are equivalent over ZFC:

(1) Ord is essentially subtle.
(2) For every natural number n, there exists a $C^{(n)}$-strongly unfoldable cardinal.
(3) For every natural number n, there exists a proper class of $C^{(n)}$-strongly unfoldable cardinals.
(4) For every natural number n and every class $\mathcal{C}$ of structures of the same type, there exists a cardinal $\kappa \in C^{(n)}$ with the property that that $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds.

Proof of Theorem 1.17. (1) ⇒ (2): Assume that for some natural number $m > 1$, there are no $C^{(m)}$-strongly unfoldable cardinals. Using a canonical coding of both Gödel numbers of $\mathcal{L}_\in$-formulas and subsets of V_κ into subsets of κ for $\kappa \in C^{(1)}$, our assumption allows us to apply Theorem 9.4 to find an $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$ with the property that for every cardinal κ in $C^{(1)}$, there exists a subset E of κ and an ordinal $\kappa < \gamma \in C^{(m)}$ with the property that $\varphi(\kappa, E)$ holds in V_γ and $\neg\varphi(\alpha, E \cap \alpha)$ holds in V_β for all $\alpha < \beta < \kappa$ with $\alpha \in C^{(1)}$ and $\beta \in C^{(m)}$. Now, let $\mathcal{E}$ be the unique class function with domain Ord such that the following statements hold:

- If $\gamma \in \mathrm{Ord} \setminus C^{(1)}$, then $\mathcal{E}(\gamma) = \mathcal{P}(\gamma)$.

- If $\kappa \in C^{(1)}$, then $\mathcal{E}(\kappa)$ consists of all subsets E of κ with the property that there exists an ordinal $\kappa < \gamma \in C^{(m)}$ with the property that $\varphi(\kappa, E)$ holds in V_γ and $\neg\varphi(\alpha, E \cap \alpha)$ holds in V_β for all $\alpha < \beta < \kappa$ with $\alpha \in C^{(1)}$ and $\beta \in C^{(m)}$.

Assume, towards a contradiction, that there are $\rho < \kappa$ in $C^{(m+1)}$ and $E \in \mathcal{E}(\kappa)$ with $E \cap \rho \in \mathcal{E}(\rho)$. Then there exists an ordinal $\rho < \gamma \in C^{(m)}$ with the property that $\varphi(\rho, E \cap \rho)$ holds in V_γ. Since κ is an element of $C^{(m+1)}$, we can find such a γ that is smaller than κ. But this contradicts the fact that E is an element of $\mathcal{E}(\kappa)$. These arguments show that Ord is not essentially subtle in this case.

(2) $\Rightarrow$ (3): Assume that for some natural number m, there are only boundedly many $C^{(m)}$-strongly unfoldable cardinals, and let λ denote the least upper bound of the set of all $C^{(m)}$-strongly unfoldable cardinals. Since the set $\{\lambda\}$ is definable by an $\mathcal{L}_\in$-formula without parameter, Proposition 9.1 implies that for all sufficiently large natural numbers n, there is no $C^{(n)}$-strongly unfoldable cardinal.

(3) $\Rightarrow$ (4): This implication follows directly from Proposition 9.1 and Lemma 9.7.

(4) $\Rightarrow$ (1): Assume that for every natural number n and every class $\mathcal{C}$ of structures of the same type, there exists a cardinal $\kappa \in C^{(n)}$ with the property that $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds. Let C be a closed unbounded class of ordinals and let $\mathcal{E}$ be a class function on the ordinals with the property that $\varnothing \neq \mathcal{E}(\gamma) \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma \in \mathrm{Ord}$. Then there is a natural number $n > 0$ such that every element κ of $C^{(n)}$ with the property that V_κ contains the parameters used in the definition of C is a limit point of C. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}_\in$ by a constant symbol $\dot{d}_n$ for every natural number n, unary function symbols $\dot{e}$ and $\dot{s}$, and unary relation symbols $\dot{C}$ and $\dot{E}$. Define $\mathcal{C}$ to be the class of $\mathcal{L}$-structures of the form $\langle V_\rho, \in, \langle d_n \mid n < \omega\rangle, e, s, C \cap \rho, E\rangle$ such that the following statements hold:

- $\rho \in C^{(n)}$ and V_ρ contains the parameters used in the definition of C.
- $E \in \mathcal{E}(\rho)$.
- If $\mathrm{cof}(\rho) = \omega$, then $\langle d_n \mid n < \omega\rangle$ is a strictly increasing cofinal sequence in ρ.
- $e(\gamma) \in \mathcal{E}(\gamma)$ for all $\gamma < \rho$.
- $s(\gamma) = \min(C \setminus (\gamma + 1))$ for all $\gamma < \rho$.

By our assumption, there exists $\kappa \in C^{(n)}$ with the property that $\mathrm{HSR}^-_{\mathcal{C}}(\kappa)$ holds. Then there exists B in $\mathcal{C}$ with domain V_κ and, since $|V_\kappa| = \kappa$, we find $A \in \mathcal{C} \cap H_\kappa$ such that there exists an elementary embedding j of A into B. Pick $\rho \in C^{(n)}$ with the property that V_ρ is the domain of A. If j is the trivial embedding, then $\rho < \kappa$ are elements of C and $\dot{E}^B$ is an element of $\mathcal{E}(\kappa)$ with $\dot{E}^B \cap \rho = \dot{E}^A \in \mathcal{E}(\rho)$. In the following, assume that the embedding j is non-trivial and set $\lambda = \mathrm{crit}\,(j) < \rho$.

Claim. $\lambda \in C$.

Proof of the Claim. Assume, towards a contradiction, that $\lambda \notin C$. Since elementarity implies that $j(\min(C)) = \min(C)$, the *Kunen Inconsistency* ensures that $\lambda > \min(C)$. Set $\beta = \sup(C \cap \lambda) < \lambda$ and $\gamma = \min(C \setminus (\beta+1)) \in (\lambda, \rho)$. Then $\dot{s}^A(\beta) = \gamma$ and hence we have $j(\gamma) = \gamma$ and $j(V_{\gamma+2}) = V_{\gamma+2}$, contradicting the *Kunen Inconsistency*. □

By elementarity, we now know that $\lambda < j(\lambda)$ are elements of C and $j(\dot{e}^A(\lambda)) = \dot{e}^B(\lambda)$ is an element of $\mathcal{E}(j(\lambda))$ with $j(\dot{e}^A(\lambda)) \cap \lambda = \dot{e}^A(\lambda) \in \mathcal{E}(\lambda)$. These computations show that (1) holds in this case. □

10. On WPSR for arbitrary classes of structures

We shall deal next with the extension of Theorem 1.30 to arbitrary classes of structures. This is given by Theorem 1.32. Namely, we will show that for every natural number $n > 1$, the following statements are equivalent for every cardinal κ:

(1) κ is either $C^{(n)}$-strongly unfoldable or a limit of Σ_{n+1}-strong cardinals.
(2) The principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.

The proof will use similar arguments to those in the proof of Theorem 1.30, given in section 6. The implication (1) $\Rightarrow$ (2) follows from the following $C^{(n)}$-version of Lemma 6.1, which can be proved in the same way, using Proposition 9.1.

Lemma 10.1. *If $n > 0$ is a natural number and κ is a $C^{(n)}$-strongly unfoldable cardinal, then $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds for every non-empty class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ.*

Proof. Since Proposition 9.1 ensures that $\kappa \in C^{(n+1)}$, we have $\varnothing \neq \mathcal{C} \cap V_\kappa \in H_{\kappa^+}$. Pick a substructure X of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality at most κ, a structure B in $\mathcal{C}$, a cardinal $\kappa < \delta \in C^{(n+1)}$ with $B \in V_\delta$, and an elementary submodel M of H_{κ^+} of cardinality κ with $V_\kappa \cup \{\mathcal{C} \cap V_\kappa, X\} \cup {}^{<\kappa}M \subseteq M$. Fix a transitive set N with $V_\delta \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \delta$ and $V_\delta \prec_{\Sigma_n} V^N_{j(\kappa)}$. This setup ensures that $B \in \mathcal{C} \cap V_\delta \subseteq j(\mathcal{C} \cap V_\kappa)$ and hence the function

$$h : X \longrightarrow B;\ f \mapsto j(f)(B)$$

is a well-defined homomorphism. $\square$

Proof of Theorem 1.32. Let $n > 0$ be a natural number and let κ be a cardinal with the property that $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_{n+1}-formula with parameters in V_κ. Then Lemma 3.1 shows that $\kappa \in C^{(n+1)}$. Assume, towards a contradiction, that κ is neither $C^{(n)}$-strongly unfoldable nor a limit of Σ_{n+1}-strong cardinals. Pick an ordinal $\alpha < \kappa$ such that the interval $[\alpha, \kappa)$ contains no Σ_{n+1}-strong cardinals. Given a cardinal ρ that is not Σ_{n+1}-strong, we let η_ρ denote the least cardinal $\delta > \rho$ such that ρ is not δ-Σ_{n+1}-strong. Since the property of not being a Σ_{n+1}-strong cardinal can be defined by a Σ_{n+1}-formula, and since $\kappa \in C^{(n+1)}$, the interval (α, κ) is closed under the function $\rho \longmapsto \eta_\rho$, and therefore it contains unboundedly many cardinals ξ with the property that $\eta_\rho < \xi$ holds for all cardinals $\alpha \leq \rho < \xi$. Finally, using Theorem 9.4.(2) and some standard fact about definability in models of the form V_η, we can find an $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, an ordinal $\theta > \kappa$ in $C^{(n)}$, and $E \subseteq V_\kappa$ with the property that $\Phi(\kappa, E)$ holds in $V_{\theta+1}$ and for all $\beta < \gamma < \kappa$ with $\gamma \in C^{(n)}$ the statement $\Phi(\beta, E \cap V_\beta)$ does not hold in $V_{\gamma+1}$.

Let $\mathcal{L}'$ be the first-order language that extends the language of set theory with a binary relation symbol $\dot{S}$, constant symbols $\dot{\kappa}$, $\dot{u}$, and $\dot{c}$, and a unary function symbol $\dot{e}$. Let $\mathcal{L}$ denote the first-order language that extends $\mathcal{L}'$ by an $(n+1)$-ary predicate symbol $\dot{T}_\varphi$ for every $\mathcal{L}'$-formula $\varphi(v_0, \ldots, v_n)$ with $(n+1)$-many free variables, and let $\mathcal{S}_{\mathcal{L}}$ be the class of structures A such that there exists a cardinal κ_A in $C^{(n)}$ and a cardinal $\theta_A > \kappa_A$ also in $C^{(n)}$ such that the following hold:

(1) The domain of A is V_{θ_A+1}.
(2) $\in^A = \in \restriction V_{\theta_A+1}$, $\dot{\kappa}^A = \kappa_A$ and $\dot{u}^A = \theta_A$.
(3) If $\varphi(v_0, \ldots, v_n)$ is an $\mathcal{L}'$-formula, then

$$\dot{T}^A_\varphi \ = \ \{\langle x_0, \ldots, x_n\rangle \in V^{n+1}_{\theta_A+1} \mid A \models \varphi(x_0, \ldots, x_n)\}.$$

Let $\mathcal{C}$ denote the class of all $A \in \mathcal{S}_{\mathcal{L}}$ such that the following statements hold:

- $\dot{c}^A = \alpha < \kappa_A$.
- The interval $[\alpha, \kappa_A)$ contains no Σ_{n+1}-strong cardinals.
- $\dot{S}^A = \{\langle \rho, \gamma\rangle \in \kappa_A \times \kappa_A \mid \rho \textit{ is a } \gamma\text{-}\Sigma_{n+1}\textit{-strong cardinal}\}$.
- If $\alpha < \delta < \kappa_A$ is a cardinal, then $\dot{e}^A(\delta)$ is a cardinal below κ_A and is the smallest cardinal ξ greater than δ that has the property that $\eta_\rho < \xi$ holds for all cardinals $\alpha \leq \rho < \xi$.

It is easily seen that the class $\mathcal{C}$ is Σ_{n+1}-definable with parameter α. In addition, the fact that κ is an element of $C^{(n+1)}$ implies that $\sup\{\kappa_A \mid A \in \mathcal{C} \cap V_\kappa\} = \kappa$ and there exists a structure B in $\mathcal{C}$ with $\kappa_B = \kappa$ and $\theta_B = \theta$. Let C be a cofinal subset of κ of order-type $\operatorname{cof}(\kappa)$. Let X be a substructure of $\prod(\mathcal{C} \cap V_\kappa)$ of cardinality κ with the property that $f^C \in X$, $f^E \in X$, and $f_x, f^x \in X$ for all $x \in V_\kappa$. By our assumption, there is a homomorphism $h : X \longrightarrow B$ and we define

$$\lambda \ = \ \min\{\operatorname{rnk}(x) \mid f_x\{\kappa\}, \ h(f_x) = \dot{u}^B\}$$

and

$$\chi \ = \ \sup\{h(f_\beta) \mid \beta < \lambda\}.$$

Note that the results of Section 5 imply that $\lambda, \chi \leq \kappa$. Moreover, Lemma 5.5.(2) shows that $h(f_\lambda) = \dot{u}^B \neq \lambda$ and this implies that $\alpha < \lambda$ (see the proof of Theorem 1.30). Let

$$\mu \ = \ \min\{\beta \leq \lambda \mid h(f_\beta) \neq \beta\}.$$

and apply Lemma 5.2.(5) to show that μ is an inaccessible cardinal. Moreover, Lemma 5.5.(2) implies that $h(f_x) = x$ holds for all $x \in V_\mu$.

Claim. *$\mu < \kappa$.*

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda = \kappa$ holds. Since Lemma 5.1.(1) implies $h(f^E) \subseteq \kappa$, we can apply Lemma 5.2.(1) to conclude that $h(f^E) = E$ and hence

$$B \models \Phi(\dot{\kappa}, h(f^E)).$$

Another application of Lemma 5.1.(1) then yields $A \in \mathcal{C} \cap V_\kappa$ with

$$A \models \Phi(\dot{\kappa}, f^E(A))$$

and this shows that $\Phi(\kappa_A, E \cap V_{\kappa_A})$ holds in V_{θ_A+1}. Since $\kappa_A < \theta_A < \kappa$ and $\theta_A \in C^{(n)}$, this contradicts the choice of E. □

Claim. *$\mu < \lambda$.*

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda < \kappa$. Then Lemma 5.2.(2) shows that $h(f^\mu) = \kappa$ and we can apply Lemmas 5.3 and 5.4 to show that μ is a κ-strong cardinal. The claim will be proved once we show that μ is a κ-Σ_{n+1}-strong cardinal, for this contradicts the fact that $\alpha < \mu < \eta_\mu < \kappa$.

Now, [9, Proposition 5.5] shows that μ is a κ-Σ_{n+1}-strong cardinal if and only if there exists a transitive M and an elementary embedding $j : V \longrightarrow M$ with $\operatorname{crit}(j) = \mu$, $j(\mu) > \kappa$, $V_\kappa \subseteq M$, and $M \models$ "$\kappa \in C^{(n)}$". Thus, letting $\mathcal{E}$ be the (μ, κ)-extender given by Lemma 5.3, and letting $j_{\mathcal{E}} : V \longrightarrow M_{\mathcal{E}}$ be as in Lemma 5.4, it only remains to show that $\bar{M}_{\mathcal{E}} \models$ "$\kappa \in C^{(n)}$", where $\bar{M}_{\mathcal{E}}$ is the transitive collapse of $M_{\mathcal{E}}$. Since κ is a limit point of $C^{(n)}$, it suffices to show that if $\gamma < \kappa$ belongs to $C^{(n)}$, then $\bar{M}_{\mathcal{E}} \models$ "$\gamma \in C^{(n)}$". So, fix such an ordinal γ.

Let $f : [\mu]^1 \longrightarrow \mu$ be such that $f(\{x\}) = x$. It is well known that $k_{\{\gamma\}}([f]_{E_{\{\gamma\}}}) = \gamma$, where $k_{\{\gamma\}} : M_{\{\gamma\}} \longrightarrow \bar{M}_{\mathcal{E}}$ is the standard map given by $k_{\{\gamma\}}([g]_{E_{\{\gamma\}}}) = \pi([\{\gamma\}, [g]_{E_{\{\gamma\}}}])$, for any $g : [\mu]^1 \longrightarrow V$, and where $\pi : M_{\mathcal{E}} \longrightarrow \bar{M}_{\mathcal{E}}$ is the transitive collapse (see [22, Lemma 26.2(a)]).

Since being a singleton whose unique element belongs to $C^{(n)}$ is a predicate in the language of every structure $A \in \mathcal{C} \cap V_\kappa$, letting $Z = \{\{x\} \in [\mu]^1 \mid x \in C^{(n)}\}$, by Lemma 5.1.(1) and, since $h(f^\mu) = \kappa$ and $\kappa \in C^{(n)}$, we have that h maps f^Z to the set $\{\{x\} \in [\kappa]^1 \mid x \in C^{(n)} \cap V_\kappa\}$. Thus, $\{\gamma\} \in h(Z)$, and therefore $Z \in E_{\{\gamma\}}$. Hence, $M_{\{\gamma\}} \models$ "$[f]_{E_{\{\gamma\}}} \in C^{(n)}$", and therefore we can conclude that $M_{\mathcal{E}} \models$ "$[\{\gamma\}, [f]_{\mathcal{E}}] \in C^{(n)}$", which yields $\bar{M}_{\mathcal{E}} \models$ "$\gamma \in C^{(n)}$", as desired. □

The claim above shows that $\mu < \lambda \leq \kappa$ and $h(f_\mu) \neq \dot{u}^B$. So let

$$j : V_\lambda \longrightarrow V_\chi;\ x \mapsto h(f_x)$$

be the non-trivial Σ_1-elementary embedding with $\operatorname{crit}(j) = \mu$ given by Lemma 5.5.(3). Like in the proof of Theorem 1.22 we may now use the Σ_1-elementarity of j and the *Kunen Inconsistency* to conclude that $\alpha < \mu$. Since $\alpha < \mu < \lambda \leq \kappa$, we can pick a cardinal $\mu < \xi < \kappa$ that is minimal above μ with the property that $\eta_\rho < \xi$ holds for all cardinals $\alpha \leq \rho < \xi$. Given $A \in \mathcal{C} \cap V_\kappa$ with $\mu < \kappa_A$, we then have $\xi = \dot{e}^A(\mu) < \kappa_A$. Using Lemma 5.1.(1), this shows that $h(f_\xi) \neq \dot{u}^B$ and $\xi < \lambda$.

Claim. *If $i < \omega$, then $j^i(\mu) < j^{i+1}(\mu) < \xi$ and $j^i(\mu)$ is a $j^{i+1}(\mu)$-Σ_{n+1}-strong cardinal.*

Proof of the Claim. Since $\mu < \kappa$ and Lemma 5.2.(3) shows that $j(\mu) = h(f^\mu)$, we can apply Lemma 5.4 and argue as in the last claim above to conclude that μ is $j(\mu)$-Σ_{n+1}-strong, and this implies that $\mu < j(\mu) < \eta_\mu < \xi$. Now, assume that for some $i < \omega$, we have $j^i(\mu) < j^{i+1}(\mu) < \xi$ and $j^i(\mu)$ is a $j^{i+1}(\mu)$-Σ_{n+1}-strong cardinal. Then $\langle j^i(\mu), j^{i+1}(\mu)\rangle \in \dot{S}^A$ for all $A \in \mathcal{C} \cap V_\kappa$ with $\xi < \kappa_A$ and the fact that $h(f_\xi) \neq \dot{u}^B$ allows us to use Lemma 5.1.(1) to show that $\langle j^{i+1}(\mu), j^{i+2}(\mu)\rangle \in \dot{S}^B$. Hence, we know that $j^{i+1}(\mu)$ is a $j^{i+2}(\mu)$-Σ_{n+1}-strong cardinal and $j^{i+1}(\mu) < j^{i+2}(\mu) < \eta_{j^{i+1}(\mu)} < \xi$. □

Now, define

$$\tau \ = \ \sup_{i<\omega} j^i(\mu) \ \leq \ \xi \ < \ \lambda,$$

and apply Lemma 5.5.(2) to show that λ is a limit ordinal and use the Σ_1-elementarity of j to conclude that $j(V_{\tau+2}) = V_{\tau+2}$. Since this entails that $j \restriction V_{\tau+2} : V_{\tau+2} \longrightarrow V_{\tau+2}$ is a non-trivial elementary embedding, we obtain a contradiction to the *Kunen Inconsistency*.

The above computations prove the implication (2) $\Rightarrow$ (1) of the theorem. The implication (1) $\Rightarrow$ (2) follows directly from Lemma 10.1. □

We conclude this section with the proof of Theorem 1.33, i.e., we prove that the following schemas are equivalent over ZFC:

(1) Ord is essentially subtle.
(2) For every non-empty class $\mathcal{C}$ of structures of the same type, there exists a cardinal κ such that $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds.

Proof of Theorem 1.33. The implication (1) $\Rightarrow$ (2) follows from a combination of Theorem 1.17 and Lemma 10.1.

To show that (2) $\Rightarrow$ (1), assume, aiming for a contradiction, that there is a closed unbounded class C of ordinals and a class function $\mathcal{E}$ with domain Ord and the property that $\varnothing \neq \mathcal{E}(\gamma) \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma \in \mathrm{Ord}$ and $E \cap \beta \notin \mathcal{E}(\beta)$ holds for all $\beta < \gamma$ in C and $E \in \mathcal{E}(\gamma)$. As in the proof of Theorem 1.34, let α denote the least cardinal of uncountable cofinality in C and let $\mathcal{L}'$ denote the first-order language that extends $\mathcal{L}_\in$ by a unary predicate symbol $\dot{C}$, constant symbols $\dot{\kappa}$, $\dot{c}$ and $\dot{u}$, and unary function symbols $\dot{e}$ and $\dot{s}$. Then let $\mathcal{L}$ denote the first-order language extending $\mathcal{L}'$ by an $(n+1)$-ary predicate symbol $\dot{T}_\varphi$ for every $\mathcal{L}'$-formula $\varphi(v_0, \ldots, v_n)$ with $(n+1)$-many free variables, and let $\mathcal{S}_\mathcal{L}$ be the class of $\mathcal{L}$-structures as defined at the beginning of Section 5. Now, let $\mathcal{C}$ denote the class of all $A \in \mathcal{S}_\mathcal{L}$ such that the following statements hold:

- κ_A is a limit point of C above α.
- $\dot{c}^A = \alpha$ and $\dot{C}^A = C \cap \kappa_A$.
- If $\gamma < \kappa_A$, then $\dot{e}^A(\gamma) \in \mathcal{E}(\gamma)$ and $\dot{s}^A(\gamma) = \min(C \setminus (\gamma+1))$.

Then $\mathcal{C} \neq \varnothing$. Assume, towards a contradiction, that there is a cardinal ζ with the property that $\mathrm{WPSR}_\mathcal{C}(\zeta)$ holds. Set

$$\kappa \ = \ \sup\{\kappa_A \mid A \in \mathcal{C} \cap V_\zeta\} \ \in \ \mathrm{Lim}(C) \cap C^{(1)} \cap (\alpha, \zeta].$$

Let D be a cofinal subset of κ of order-type $\mathrm{cof}(\kappa)$, and let E be an element of $\mathcal{E}(\kappa)$.

For each set x, define functions f_x and f^x as in Section 5. Pick $B \in \mathcal{C}$ with $\kappa_B > \zeta$ and $\dot{e}^B(\kappa) = E$. Fix a substructure X of $\prod(\mathcal{C} \cap V_\zeta)$ of cardinality κ such that $f^D, f^E \in X$ and $f_x, f^x \in X$ for all $x \in V_\kappa$. Then there is a homomorphism $h : X \longrightarrow B$ and we can define

$$\lambda \ = \ \min\{\mathrm{rnk}(x) \mid f_x \in X, \, h(f_x) = \dot{u}^B\} \ \leq \ \kappa$$

as well as

$$\chi \ = \ \sup\{h(f_\alpha) \mid \alpha < \lambda\} \ \leq \ \kappa_B.$$

Our setup then ensures that $\alpha < \lambda$ and an argument presented in the proof of Theorem 1.34 shows that $\lambda \in C$. Using Lemma 5.5.(3), we know that

$$j : V_\lambda \longrightarrow V_\chi; \ x \mapsto h(f_x)$$

is a Σ_1-elementary embedding. Next, Lemma 5.5.(1) allows us to define $\mu \leq \lambda$ to be the minimal ordinal with $h(f_\mu) \neq \mu$. By Lemma 5.5.(2), we then have $h(f_x) = x$ for all $x \in V_\mu$.

Claim. *$\mu < \kappa$.*

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda = \kappa$ holds. Lemma 5.2.(1) now shows that $h(f^E) \cap \kappa = E = \dot{e}^B(\kappa)$ and hence the cardinal $\kappa \in C \cap \kappa_B = \dot{C}^B$ witnesses that

$$B \models \text{“}\exists \beta < \dot{\kappa} \ [\beta \in \dot{C} \wedge h(f^E) \cap \beta = \dot{e}(\beta)]\text{”}.$$

Then Lemma 5.1.(1) yields $A \in \mathcal{C} \cap V_\zeta$ and $\beta \in C \cap \kappa_A$ with $E \cap \beta = f^E(A) \cap \beta = \dot{e}^A(\beta) \in \mathcal{E}(\beta)$. Since $\beta < \kappa_A \leq \kappa$, this yields a contradiction. □

Define $E_0 = \dot{e}^B(\mu) \in \mathcal{E}(\mu)$.

Claim. *$\mu < \lambda$.*

Proof of the Claim. Assume, towards a contradiction, that $\mu = \lambda$ holds. Then $\mu \in C \cap \kappa$ and $f^{E_0} \in X$. Lemma 5.2.(1) now implies that $h(f^{E_0}) \cap \mu = E_0$ and, since Lemma 5.5.(1) shows that $h(f_\lambda) = \dot{u}^B$, we have

$$B \models \text{“}h(f_\mu) = \dot{u} \ \wedge \ \exists \beta < \dot{\kappa} \ [\beta \in \dot{C} \wedge h(f^{E_0}) \cap \beta = \dot{e}(\beta)]\text{”}.$$

By Lemma 5.1.(1), there is $A \in \mathcal{C} \cap V_\zeta$ with $\kappa_A \leq \mu$ and $\beta \in C \cap \kappa_A$ with $E_0 \cap \beta = f^{E_0}(A) \cap \beta = \dot{e}^A(\beta) \in \mathcal{E}(\beta)$, a contradiction. □

This shows that $\mu < \lambda \leq \kappa$. In addition, arguments already contained in the proof of Theorem 1.34 prove that $\alpha < \mu \in C$. Then $\mu < j(\mu) = h(f_\mu) \neq \dot{u}^B$ and μ witnesses that

$$B \models \text{“}h(f_\mu) \neq \dot{u} \ \wedge \ \exists \beta < h(f_\mu) \ [\beta \in \dot{C} \wedge h(f^{E_0}) \cap \beta = \dot{e}(\beta)]\text{”}.$$

By Lemma 5.1.(1), there is $A \in \mathcal{C} \cap V_\zeta$ with $\kappa_A > \mu$ and $\beta \in C \cap \mu$ with $E_0 \cap \beta = f^{E_0}(A) \cap \beta = \dot{e}^A(\beta) \in \mathcal{E}(\beta)$, a contradiction. □

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