SMALL EMBEDDING CHARACTERIZATIONS FOR LARGE CARDINALS

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ABSTRACT. We show that many large cardinal notions can be characterized in terms of the existence of certain elementary embeddings between transitive set-sized structures, that map their critical point to the large cardinal in question. As an application, we use such embeddings to provide new proofs of results of Christoph Weiß on the consistency strength of certain generalized tree properties. These new proofs eliminate problems contained in the original proofs provided by Weiß.

1. INTRODUCTION

Many large cardinal notions are characterized by the existence of non-trivial elementary embeddings with certain properties. There are two kinds of such characterizations, the first, more common one, where the large cardinal property of κ is characterized by the existence of elementary embeddings with critical point κ, and the second, less common one, where the large cardinal property of κ is characterized by the existence of elementary embeddings which map their critical point to κ. A classical result of Menachem Magidor ([8, Theorem 1]) provides the first example of a characterization of the second kind by showing that a cardinal κ is supercompact if and only if for every $\eta > \kappa$, there is a non-trivial elementary embedding $j : V_\alpha \longrightarrow V_\eta$ with $\alpha < \kappa$ and $j(\mathrm{crit}\,(j)) = \kappa$. Other prominent examples of large cardinal characterizations of the second kind are provided by the notion of *subcompactness* (introduced by Ronald Jensen) and its generalizations (see [2]), and also Ralf Schindler's *remarkable cardinals* (see [11]).

In this paper, we provide characterizations of various well-known large cardinals as the images of the critical points of certain elementary embeddings. All of these characterizations are based on the concept introduced in the next definition. In the following, we say that an elementary embedding $j : M \longrightarrow N$ between transitive classes is *non-trivial* if there is an ordinal $\alpha \in M$ with $j(\alpha) > \alpha$. In this case, we let $\mathrm{crit}\,(j)$ denote the least such ordinal.

Definition 1.1. Given cardinals $\kappa < \theta$, we say that a non-trivial elementary embedding $j : M \longrightarrow \mathrm{H}(\theta)$ is a *small embedding for* κ if $M \in \mathrm{H}(\theta)$ is transitive, and $j(\mathrm{crit}\,(j)) = \kappa$ holds.

The properties of cardinals κ studied in this paper usually state that for sufficiently large[1] cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with certain elements of $\mathrm{H}(\theta)$ in its range, and with the property that the domain model M satisfies certain correctness properties with respect to the universe of sets V,[2] sometimes in combination with some kind of smallness assumption about M.

A first example of such a characterization can easily be obtained by observing that the proof of Magidor's characterization of supercompactness in [8] also directly yields the equivalence stated below. Note that the requirement that $M = \mathrm{H}(\delta)$ in this characterization can easily be interpreted as a correctness property of M (since $\mathrm{V} = \mathrm{H}(\mathrm{On})$), and that $\delta < \kappa$ is a smallness assumption on M.

Theorem 1.2 (Magidor). *The following statements are equivalent for every cardinal κ:*

(i) *κ is supercompact.*
(ii) *For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with the property that $M = \mathrm{H}(\delta)$ for some cardinal $\delta < \kappa$.*

We will provide small embedding characterizations for various types of large cardinals, which will also have the property that small embeddings witnessing certain large cardinal properties relate in a way that parallels the implication structure of the corresponding large cardinals notions, that is, whenever there is a direct implication from some large cardinal property A to another large cardinal property B, then amongst the small embeddings witnessing Property A, we find small embeddings witnessing Property B. The verification of these statements is usually a routine adaption of the proof that the large cardinal property A implies the large cardinal property B, and we will therefore usually leave those verifications as easy exercises to the interested reader.

We will now summarize the contents of our paper. In Section 2, we will present small embedding characterizations for what we call *Mahlo-like cardinals*, that is notions of large cardinals that are characterized as being stationary limits of certain cardinals, in particular covering the cases of inaccessible and of Mahlo cardinals. Section 3 contains two technical lemmas that will be useful later on. In Section 4, we provide small embedding characterizations for Π_n^m-*indescribable cardinals* for all $0 < m, n < \omega$. The results of Section 5 provide such characterizations for *subtle*, for *ineffable*, and for *λ-ineffable cardinals*. The results of Section 6 provide small embedding characterizations for various filter based large cardinal notions, that is for *measurable*, for *λ-supercompact*, and for *n-huge cardinals*. In Section 7, we discuss some problems arising in the consistency proofs of certain generalized tree properties that are presented in [15] and [16]. We then use the theory of small embeddings developed in this paper to eliminate those problems and provide

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[1]Here, θ being a *sufficiently large cardinal* means that there is an $\alpha \geq \kappa$ such that the corresponding statement holds for all cardinals $\theta > \alpha$.

[2]We usually restrict ourselves to correctness properties mostly to avoid trivial small embedding characterizations. For example, without this requirement, one could propose the following equivalence: κ is measurable if and only if for all sufficiently large cardinals θ, there is a small embedding $j \colon M \to \mathrm{H}(\theta)$ for κ such that $\mathrm{crit}\,(j)$ is measurable in M. However $\mathrm{crit}\,(j)$ will in general not be measurable in V (consider for example the least measurable cardinal κ), hence this trivial characterization is ruled out by the above requirement. We will later present a non-trivial small embedding characterization of measurability (see Lemma 6.2).

new proofs of these consistency statements. We close the paper with some open questions in Section 8.

2. Mahlo-like cardinals

In this section, we provide small embedding characterizations for what we call *Mahlo-like* cardinals, that is notions of large cardinals that are characterized as being stationary limits of certain kinds of cardinals. The following lemma will directly yield these characterizations. Its proof and the corollary following it are very basic, and are certainly similar to and implicit in earlier results (see for example [12, Lemma 57] for the case of weakly inaccessible cardinals), and may perhaps be considered part of the set-theoretic folklore. For the benefit of the reader, and because they provide a kind of starting point for our later characterizations, we would nevertheless like to provide a complete proof.

Lemma 2.1. *Given an $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$, the following statements are equivalent for every cardinal κ and every set x:*

(i) *κ is a regular uncountable cardinal and the set of all ordinals $\lambda < \kappa$ such that $\varphi(\lambda, x)$ holds is stationary in κ.*

(ii) *For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with $\varphi(\mathrm{crit}\,(j), x)$ and $x \in \mathrm{ran}(j)$.*

Proof. First, assume that (i) holds, and pick a cardinal $\theta > \kappa$ with $x \in \mathrm{H}(\theta)$. Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ be a continuous and increasing sequence of elementary substructures of $\mathrm{H}(\theta)$ of cardinality less than κ with $x \in X_0$ and $\alpha \subseteq X_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. By (i), there is an $\alpha < \kappa$ such that $\alpha = X_\alpha \cap \kappa$ and $\varphi(\alpha, x)$ holds. Let $\pi : X_\alpha \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi^{-1} : M \longrightarrow \mathrm{H}(\theta)$ is a small embedding for κ with $\varphi(\mathrm{crit}\,(\pi^{-1}), x)$ and $x \in \mathrm{ran}(\pi^{-1})$.

Now, assume that (ii) holds. Then there is a cardinal $\theta > \kappa$ such that the formula φ is absolute between $\mathrm{H}(\theta)$ and V, and there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with the property that $\varphi(\mathrm{crit}\,(j), x)$ holds and there is a $y \in M$ with $x = j(y)$. Then κ is uncountable, because elementarity implies that $j \restriction (\omega + 1) = \mathrm{id}_{\omega+1}$. Next, assume that κ is singular. Then $\mathrm{crit}\,(j)$ is singular in M and there is a cofinal function $c : \mathrm{cof}(\mathrm{crit}\,(j))^M \longrightarrow \mathrm{crit}\,(j)$ in M. In this situation, elementarity implies that $j(c) = c$ is cofinal in κ, a contradiction. Finally, assume that there is a club C in κ such that $\neg\varphi(\lambda, x)$ holds for all $\lambda \in C$. Then elementarity and our choice of θ imply that, in M, there is a club D in $\mathrm{crit}\,(j)$ such that $\neg\varphi(\lambda, y)$ holds for all $\lambda \in D$. Again, by elementarity and our choice of θ, we know that $j(D)$ is a club in κ with the property that $\neg\varphi(\lambda, x)$ holds for all $\lambda \in j(D)$. But elementarity also implies that $\mathrm{crit}\,(j)$ is a limit point of $j(D)$ and therefore $\mathrm{crit}\,(j)$ is an element of $j(D)$ with $\varphi(\mathrm{crit}\,(j), x)$, a contradiction. $\square$

By varying the formula φ (and only using the empty set as a parameter), we can use the above lemma to obtain small embedding characterizations of some of the smallest notions of large cardinals. Moreover, one can also characterize regular uncountable cardinals in such a way. Using the above lemma, the statements listed in the next corollary can be easily derived from the fact that weakly inaccessible cardinals are exactly regular stationary limits of cardinals, and that inaccessible cardinals are exactly regular stationary limits of strong limit cardinals.

Corollary 2.2. *Let κ be a cardinal.*

(i) κ is uncountable and regular if and only if for all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ.

(ii) κ is weakly inaccessible if and only if for all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with the property that $\mathrm{crit}\,(j)$ is a cardinal.

(iii) κ is inaccessible if and only if for all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with the property that $\mathrm{crit}\,(j)$ is a strong limit cardinal.

(iv) κ is weakly Mahlo if and only if for all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with the property that $\mathrm{crit}\,(j)$ is a regular cardinal.

(v) κ is Mahlo if and only if for all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with the property that $\mathrm{crit}\,(j)$ is an inaccessible cardinal. □

Note that all of the small embedding characterizations provided by the above corollary rely on correctness properties. In general however, the characterizations provided by Lemma 2.1 are not necessarily based on correctness properties, as it may not be the case that $M \models \varphi(\mathrm{crit}\,(j), x)$. In particular, even though it is possible to characterize very large cardinals in the above way (e.g. stationary limits of supercompact cardinals), such characterizations do not satisfy our requirements (see Section 8 for further discussion).

Remark 2.3. In many important cases, and, in particular, in case of the characterizations provided by Corollary 2.2, the large cardinal properties in question can also be characterized by the existence of a single elementary embedding. For each of the above, it suffices to require the existence of a single appropriate small embedding $j : M \longrightarrow \mathrm{H}(\kappa^+)$, as can easily be seen from the proof of Lemma 2.1. For example, a cardinal κ is inaccessible if and only if there is a small embedding $j : M \longrightarrow \mathrm{H}(\kappa^+)$ for κ with the property that $\mathrm{crit}\,(j)$ is a strong limit cardinal. This will in fact be the case for several of the small embedding characterizations that will follow, however we will not make any further mention of this.

Note that Lemma 2.1 implies that small embedding characterizations as in its Statement (ii) (i.e. characterizations through the existence of small embeddings $j : M \longrightarrow \mathrm{H}(\theta)$ with $\varphi(\mathrm{crit}\,(j), x)$ and $x \in \mathrm{ran}(j)$ for a given formula $\varphi(v_0, v_1)$ and a set x) cannot provably characterize any notion of large cardinal that implies weak compactness, because stationary subsets of weakly compact cardinals reflect to smaller inaccessible cardinals and therefore for any weakly compact cardinal κ satisfying Lemma 2.1, (ii) – and hence by Lemma 2.1 also Lemma 2.1 (i) – there is, by stationary reflection, in fact a smaller cardinal that satisfies Lemma 2.1 (i) – and hence Lemma 2.1 (ii) – as well. In the remainder of this paper, we will however provide small embedding characterizations of a different form for many large cardinal notions that imply weak compactness, and, in particular, also for weak compactness itself.

3. Two lemmas

Before we continue with further small embedding characterizations, we need to interrupt for the sake of presenting two technical lemmas that will be of use in many places throughout our paper.

Lemma 3.1. *The following statements are equivalent for every small embedding* $j : M \longrightarrow \mathrm{H}(\theta)$ *for a cardinal* κ*:*

  (i) *κ is a strong limit cardinal.*
 (ii) *$\mathrm{crit}\,(j)$ is a strong limit cardinal.*
(iii) *$\mathrm{crit}\,(j)$ is a cardinal and $\mathrm{H}(\mathrm{crit}\,(j)) \subseteq M$.*

Proof. Assume that (i) holds and pick a cardinal $\nu < \mathrm{crit}\,(j)$. Since $\mathrm{crit}\,(j)$ is a strong limit cardinal in M, we have $(2^\nu)^M < \mathrm{crit}\,(j)$. But then

$$2^\nu \;=\; j((2^\nu)^M) \;=\; (2^\nu)^M \;<\; \mathrm{crit}\,(j)$$

and this shows that (ii) holds. In the other direction, assume (i) fails. By elementarity, there is a cardinal $\nu < \mathrm{crit}\,(j)$ and an injection of $\mathrm{crit}\,(j)$ into $\mathcal{P}(\nu)$ in M. Then this injection witnesses that (ii) fails.

Now, again assume that (i) holds. Then elementarity implies that, in M, there is a bijection $s : \mathrm{crit}\,(j) \longrightarrow \mathrm{H}(\mathrm{crit}\,(j))$ with the property that $\mathrm{H}(\delta) = s[\delta]$ holds for every strong limit cardinal $\delta < \mathrm{crit}\,(j)$. Since we already know that (i) implies (ii), we have $\mathrm{H}(\mathrm{crit}\,(j)) = j(s)[\mathrm{crit}\,(j)]$. Fix $x \in \mathrm{H}(\mathrm{crit}\,(j))$ and $\alpha < \mathrm{crit}\,(j)$ with $j(s)(\alpha) = x$. Since $\mathrm{crit}\,(j)$ is a strong limit cardinal in M, we have $j \restriction \mathrm{H}(\mathrm{crit}\,(j))^M = \mathrm{id}_{\mathrm{H}(\mathrm{crit}\,(j))^M}$ and this allows us to conclude that $x = j(s)(\alpha) = j(s(\alpha)) = s(\alpha) \in M$, and hence that (iii) holds.

Finally, assume for a contradiction that (iii) holds and (i) fails. Then, by elementarity, there is a minimal cardinal $\nu < \mathrm{crit}\,(j)$ such that either $(2^\nu)^M \geq \mathrm{crit}\,(j)$ or such that $\mathcal{P}(\nu)$ does not exist in M. By (iii), $\mathcal{P}(\nu) \subseteq M$. By elementarity, we may pick an injection $\iota : \mathrm{crit}\,(j) \longrightarrow \mathcal{P}(\nu)$ in M. Define $x = j(\iota)(\mathrm{crit}\,(j)) \in \mathcal{P}(\nu) \subseteq M$. Then $j(x) = x$, and elementarity yields an ordinal $\gamma < \mathrm{crit}\,(j)$ with $\iota(\gamma) = x$. But then $j(\iota)(\gamma) = x = j(\iota)(\mathrm{crit}\,(j))$, contradicting the injectivity of ι. $\qquad\square$

Next, we isolate a property of small embedding characterizations, that will be important throughout this paper.

Definition 3.2. Let $\Phi(v_0, v_1)$ be an $\mathcal{L}_\in$-formula and let x be a set. We call the pair (Φ, x) *restrictable* if for every cardinal κ, there is an ordinal α such that if

 - $j : M \longrightarrow \mathrm{H}(\theta)$ is a small embedding for κ with $\Phi(j, x)$ and $x \in \mathrm{ran}(j)$, and
 - ν is a cardinal in M with $\nu > \mathrm{crit}\,(j)$ and $j(\nu) > \alpha$,

then $\Phi(j \restriction \mathrm{H}(\nu)^M, x)$ holds.

Note that the small embedding characterizations (i) – (v) provided by Corollary 2.2 are given by pairs (Φ, x) such that $x = \emptyset$ and $\Phi(j, \emptyset)$ states that $\varphi(\mathrm{crit}\,(j))$ holds for some formula $\varphi(v)$. In particular, these pairs (Φ, x) are trivially restrictable. Next, note that the small embedding formulation of Magidor's characterization of supercompactness in Theorem 1.2 is given by the pair (Φ, x) with $x = \emptyset$ and

$$\Phi(v_0, v_1) \;\equiv\; \text{``}\exists \delta \; \mathrm{dom}(v_0) = \mathrm{H}(\delta)\text{''},$$

and this pair is obviously restrictable as well. Finally, we remark that the pairs (Φ, x) used in the small embedding characterizations provided in the remainder of this paper will all be restrictable. The verification of restrictability will be trivial in each case, and is thus left for the interested reader to check throughout. The following lemma will be the key consequence of restrictability.

Claim 3.3. *Let (Φ, x) be restrictable and assume that κ is a cardinal with the property that for sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow$*

$H(\theta)$ for κ with $\Phi(j,x)$ and $x \in \mathrm{ran}(j)$. Then for all sets z and sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with $\Phi(j,x)$ and $z \in \mathrm{ran}(j)$.

Proof. By our assumptions, there is an ordinal $\alpha > \kappa$ such that the following statements hold:

(i) For all cardinals $\theta > \alpha$, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with $\Phi(j,x)$ and $x \in \mathrm{ran}(j)$.

(ii) If $j : M \longrightarrow H(\theta)$ is a small embedding for κ such that $\Phi(j,x)$ holds, $\nu > \mathrm{crit}\,(j)$ is a cardinal in M, $x \in \mathrm{ran}(j)$ and $j(\nu) > \alpha$, then $\Phi(j \restriction H(\nu)^M, x)$ holds.

Assume for a contradiction that the conclusion of the lemma does not hold. Pick a strong limit cardinal $\theta > \alpha$ with the property that $H(\theta)$ is sufficiently absolute in V and fix a small embedding $j : M \longrightarrow H(\theta)$ for κ with the property that $\Phi(j,x)$ holds, and fix $y \in M$ with $j(y) = x$. In this situation, our assumptions, the absoluteness of $H(\theta)$ in V and the elementarity of j imply that there are $\beta, \vartheta, z \in M$ such that the following statements hold in M:

(a) If $k : N \longrightarrow H(\eta)$ is a small embedding for $\mathrm{crit}\,(j)$ such that $\Phi(k,y)$ holds, $\nu > \mathrm{crit}\,(k)$ is a cardinal in N, $y \in \mathrm{ran}(k)$ and $k(\nu) > \beta$, then $\Phi(k \restriction H(\nu)^N, y)$ holds.

(b) $\vartheta > \beta$ is a cardinal with $y, z \in H(\vartheta)$ and there is no small embedding $k : N \longrightarrow H(\vartheta)$ for $\mathrm{crit}\,(j)$ with $\Phi(k,y)$ and $z \in \mathrm{ran}(k)$.

By elementarity and our absoluteness assumptions on $H(\theta)$, the above implies that the following statements hold in V:

(a)$'$ If $k : N \longrightarrow H(\eta)$ is a small embedding for κ and $\mathrm{crit}\,(k) < \nu \in N$ is a cardinal in N such that $\Phi(k,x)$ holds, $x \in \mathrm{ran}(k)$ and $k(\nu) > j(\beta)$, then $\Phi(k \restriction H(\nu)^N, x)$ holds.

(b)$'$ $j(\vartheta) > j(\beta)$ is a cardinal with $x, j(z) \in H(j(\vartheta))$ and there is no small embedding $k : N \longrightarrow H(j(\vartheta))$ for κ with $\Phi(k,x)$ and $j(z) \in \mathrm{ran}(k)$.

Since $j(\vartheta) > j(\beta)$, we can apply the statement (a)$'$ to $j : M \longrightarrow H(\theta)$ and ϑ to conclude that $\Phi(j \restriction H(\vartheta)^M, x)$ holds in V. But we also have $j(z) \in \mathrm{ran}(j \restriction H(\vartheta)^M)$ and together these statements contradict (b)$'$. $\square$

We now show that the above claim implies a somewhat stronger statement, essentially allowing us to switch the quantifiers on z and on θ.

Lemma 3.4. *Let (Φ, x) be restrictable and assume that κ is a cardinal with the property that for sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with $\Phi(j,x)$ and $x \in \mathrm{ran}(j)$. Then for all sufficiently large cardinals θ and for all $z \in H(\theta)$, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with $\Phi(j,x)$ and $z \in \mathrm{ran}(j)$.*

Proof. Fix a sufficiently large cardinal θ and some $z \in H(\theta)$. By Claim 3.3, there is a cardinal θ' and a small embedding $j' : M' \longrightarrow H(\theta')$ for κ with $\Phi(j',x)$ and $z, \theta \in \mathrm{ran}(j')$. Let j be the restriction of j' to $M = H((j')^{-1}(\theta))^{M'}$. Then $j : M \longrightarrow H(\theta)$ is a small embedding for κ with $\Phi(j,x)$ and $z \in \mathrm{ran}(j)$. $\square$

4. INDESCRIBABLE CARDINALS

In this section, we provide small embedding characterizations for *indescribable* cardinals. Recall that, given $0 < m, n < \omega$, a cardinal κ is Π^m_n-*indescribable* if for

every Π_n^m-formula $\varphi(A_0, \ldots, A_{n-1})$ whose parameters $A_0, \ldots, A_{n-1}$ are subsets of V_κ, the assumption $V_\kappa \models \varphi(A_0, \ldots, A_{n-1})$ implies that there is a $\delta < \kappa$ such that $V_\delta \models \varphi(A_0 \cap V_\delta, \ldots, A_{n-1} \cap V_\delta)$ (see for example [4, p. 295] for a definition of the concepts used). Moreover, remember that, given an uncountable cardinal κ, a transitive set M of cardinality κ is a κ-model if $\kappa \in M$, $^{<\kappa}M \subseteq M$ and M is a model of ZFC$^-$. Our small embedding characterizations of indescribable cardinals build on the following embedding characterization of these cardinals by Kai Hauser (see [3, Theorem 1.3]).

Theorem 4.1 (Hauser). *The following statements are equivalent for every inaccessible cardinal κ and all $0 < m, n < \omega$:*

 (i) *κ is Π_n^m-indescribable.*
 (ii) *For every κ-model M, there is a transitive set N and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$ such that the following statements hold:*
   (a) *N has cardinality $\beth_{m-1}(\kappa)$, $^{<\kappa}N \subseteq N$ and $j, M \in N$.*
   (b) *If $m > 1$, then $^{\beth_{m-2}(\kappa)}N \subseteq N$.*
   (c) *We have*
$$V_\kappa \models \varphi \iff (V_\kappa \models \varphi)^N$$
   for all Π_{n-1}^m-formulas φ whose parameters are contained in $N \cap V_{\kappa+m}$.[3]

Note that, in case $m > 1$, the statement $j, M \in N$ in (a) is a direct consequence of (b). It is not explicitly mentioned, but easy to observe from the proof provided in [3] that this can also be equivalently required in case $m = 1$ (for weakly compact cardinals, this is in fact what became known as their *Hauser characterization*). The above theorem allows us to characterize indescribable cardinals through small embeddings, in two ways.

Lemma 4.2. *Given $0 < m, n < \omega$, the following statements are equivalent for every cardinal κ:*

 (i) *κ is Π_n^m-indescribable.*
 (ii) *For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with the property that*
$$(V_{\mathrm{crit}(j)} \models \varphi)^M \implies V_{\mathrm{crit}(j)} \models \varphi$$
   for every Π_n^m-formula φ whose parameters are contained in $M \cap V_{\mathrm{crit}(j)+1}$.
 (iii) *For all sufficiently large cardinals θ and all $x \in V_{\kappa+1}$, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with $x \in \mathrm{ran}(j)$ and with the property that*
$$(V_{\mathrm{crit}(j)} \models \varphi)^M \implies V_{\mathrm{crit}(j)} \models \varphi$$
   for every Π_n^m-formula φ using only $j^{-1}(x)$ as a parameter.

Proof. First, assume that (i) holds. Pick a cardinal $\theta > \beth_m(\kappa)$ and a regular cardinal $\vartheta > \theta$ with $H(\theta) \in H(\vartheta)$. Since κ is inaccessible, there is an elementary submodel X of $H(\vartheta)$ of cardinality κ with $\kappa + 1 \cup \{\theta\} \subseteq X$ and $^{<\kappa}X \subseteq X$. Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then M is a κ-model and Theorem 4.1 yields an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$ that satisfies the properties (a)–(c) listed in the Statement (ii) of Theorem 4.1. Note that the assumption $^{<\kappa}N \subseteq N$ implies that κ is inaccessible in N.

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[3]Note that we write $(V_\kappa \models \varphi)^N$ to denote satisfaction for the higher order formula φ in the model V_κ in N, i.e. k-th order variables are interpreted as elements of $V_{\kappa+k}^N$.

Claim. *We have*

$$(\mathrm{V}_\kappa \models \varphi)^M \implies (\mathrm{V}_\kappa \models \varphi)^N$$

for all Π_n^m-formulas φ whose parameters are contained in $M \cap \mathrm{V}_{\kappa+1}$.

Proof of the Claim. Assume that $(\mathrm{V}_\kappa \models \varphi)^M$ holds. This assumption implies that $\mathrm{V}_\kappa \models \varphi$ holds, because $\pi^{-1} \restriction \mathrm{V}_{\kappa+1} = \mathrm{id}_{\mathrm{V}_{\kappa+1}}$ and $\mathrm{V}_{\kappa+m} \in \mathrm{H}(\vartheta)$. By Statement (c) of Theorem 4.1, we can conclude that $(\mathrm{V}_\kappa \models \varphi)^N$ holds. $\square$

Set $\theta_* = \pi(\theta)$, $M_* = \mathrm{H}(\theta_*)^M$ and $j_* = j \restriction M_*$. Since $j, M \in N$, we also have $j_*, M_* \in N$. Moreover, in N, the map $j_* : M_* \longrightarrow \mathrm{H}(j(\theta_*))^N$ is a small embedding for $j(\kappa)$. If φ is a Π_n^m-formula with parameters in $M_* \cap \mathrm{V}_{\kappa+1}$ such that $(\mathrm{V}_\kappa \models \varphi)^{M_*}$ holds, then $\theta > \beth_m(\kappa)$ implies that $(\mathrm{V}_\kappa \models \varphi)^M$ holds, and we can use the above claim to conclude that $(\mathrm{V}_\kappa \models \varphi)^N$ holds. By elementarity, this shows that, in M, there is a small embedding $j' : M' \longrightarrow \mathrm{H}(\theta_*)$ for κ such that $\mathrm{crit}\,(j')$ is inaccessible and $\mathrm{V}_{\mathrm{crit}(j')} \models \varphi$ holds for every Π_n^m-formula φ with parameters in $M' \cap \mathrm{V}_{\mathrm{crit}(j)+1}$ with the property that $(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi)^{M'}$ holds. Since $\mathrm{V}_{\kappa+m} \in \mathrm{H}(\vartheta)$, we can conclude that $\pi^{-1}(j')$ is a small embedding for κ witnessing that (ii) holds for θ.

Next, Lemma 3.4 shows that (iii) is a consequence of (ii). Hence, assume, towards a contradiction, that (iii) holds and that there is a Π_n^m-formula $\varphi(x)$ with $x \in \mathrm{V}_{\kappa+1}$, $\mathrm{V}_\kappa \models \varphi(x)$ and $\mathrm{V}_\delta \models \neg\varphi(x \cap \mathrm{V}_\delta)$ for all $\delta < \kappa$. Pick a regular cardinal $\theta > \beth_m(\kappa)$ such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ that satisfies the statements listed in (iii) with respect to x. Since $\mathrm{V}_{\kappa+m} \in \mathrm{H}(\theta)$, elementarity yields that $(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(j^{-1}(x)))^M$. Thus our assumptions on j allow us to conclude that $\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(j^{-1}(x))$, contradicting the above assumption. $\square$

In the case $m = 1$, the equivalence between Statements (i) and (ii) in Lemma 4.2 can be rewritten in the following way, using the fact that we can canonically identify Σ_n-formulas using parameters in $\mathrm{H}(\mathrm{crit}\,(j)^+)$ with Σ_n^1-formulas using parameters in $\mathrm{V}_{\mathrm{crit}(j)+1}$, such that the given Σ_n-formula holds true in $\mathrm{H}(\mathrm{crit}\,(j)^+)$ if and only if the corresponding Σ_n^1-formula holds in $\mathrm{V}_{\mathrm{crit}(j)}$.

Corollary 4.3. *Given $0 < n < \omega$, the following statements are equivalent for every cardinal κ:*

 (i) κ is Π_n^1-indescribable.
(ii) For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ such that $\mathrm{H}(\mathrm{crit}\,(j)^+)^M \prec_{\Sigma_n} \mathrm{H}(\mathrm{crit}\,(j)^+)$. $\square$

Lemma 4.2 directly shows that small embeddings witnessing Π_n^m-indescribability also witness all smaller degrees of indescribability. The following is also easy to verify, which we leave for the interested reader to check.

Observation 4.4. *Let κ be weakly compact cardinal. Any collection of small embeddings witnessing the Π_1^1-indescribability of κ as in Corollary 4.3 (ii) also witnesses the Mahloness of κ as in Statement (v) of Corollary 2.2.* $\square$

5. Subtle, Ineffable and λ-Ineffable Cardinals

In this section, we provide small embedding characterizations for *ineffable*, for *subtle*, and for *λ-ineffable* cardinals. These large cardinal notions were introduced in [5] and in [9]. They all rely on the following definition.

Definition 5.1. Given a set A, a sequence $\langle d_a \mid a \in A \rangle$ is an A-*list* if $d_a \subseteq a$ holds for all $a \in A$.

An uncountable regular cardinal κ is *subtle* if for every κ-list $\langle d_\alpha \mid \alpha < \kappa \rangle$ and every club C in κ, there are $\alpha, \beta \in C$ with $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$.

Lemma 5.2. *The following statements are equivalent for every cardinal κ:*

(i) κ is subtle.

(ii) For all sufficiently large cardinals θ, for every κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ and for every club C in κ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ such that $\vec{d}, C \in \mathrm{ran}(j)$ and $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$ for some $\alpha \in C \cap \mathrm{crit}\,(j)$.

Proof. First, assume first that κ is subtle. Pick a cardinal $\theta > \kappa$, a club C in κ and a κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$. Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ be a continuous and increasing sequence of elementary substructures of $\mathrm{H}(\theta)$ of cardinality less than κ with $\vec{d}, C \in X_0$ and $\alpha \subseteq X_\alpha \cap \kappa \in \kappa$ for all $\alpha < \kappa$. Set $D = \{\alpha \in C \mid \alpha = M_\alpha \cap \kappa\}$. Then D is a club in κ and the subtlety of κ yields $\alpha, \beta \in D \subseteq C$ with $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$. Let $\pi : X_\beta \longrightarrow M$ denote the transitive collapse of X_β. Then $\pi^{-1} : M \longrightarrow \mathrm{H}(\theta)$ is a small embedding for κ with $\mathrm{crit}\,(\pi^{-1}) = \beta$, $\vec{d}, C \in \mathrm{ran}(\pi^{-1})$ and $d_\alpha = d_{\mathrm{crit}(\pi^{-1})} \cap \alpha$.

Now, assume that (ii) holds. Then Corollary 2.2 implies that κ is uncountable and regular. Fix a κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ and a club C in κ. Let θ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ such that $\vec{d}, C \in \mathrm{ran}(j)$ and $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$ for some $\alpha \in C \cap \mathrm{crit}\,(j)$. Since $C \in \mathrm{ran}(j)$, elementarity implies that $\mathrm{crit}\,(j)$ is a limit point of C and hence $\mathrm{crit}\,(j) \in C$. $\square$

Remark 5.3. Note that, unlike all other small embedding characterizations that we provide in this paper, the above characterization of subtle cardinals is not based on a correctness property between the domain model M and V. However, we think that the above characterization is still useful. This will be supported by the results of Section 7.

Adapting the proof of [1, Theorem 3.6.3], it is easy to verify the following. Since we will make use of this result in Section 7, we will provide a proof for the sake of completeness.

Lemma 5.4. *Let κ be a subtle cardinal. Then there is a κ-list $\vec{d}$ and a club subset C of κ with the property that whenever θ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ witnessing the subtlety of κ with respect to $\vec{d}$ and C, as in Statement (ii) of Lemma 5.2, then $\mathrm{crit}\,(j)$ is a totally indescribable cardinal.*

In particular, any family of small embeddings witnessing the subtlety of κ as in Statement (ii) of Lemma 5.2 witnesses that κ is a stationary limit of totally indescribable cardinals, as in Statement (ii) of Lemma 2.1.

Proof. Let C be the club $\{\alpha < \kappa \mid |\mathrm{V}_\alpha| = \alpha\}$ and let $h : \mathrm{V}_\kappa \longrightarrow \kappa$ be a bijection with $h[\mathrm{V}_\alpha] = \alpha$ for all $\alpha \in C$. Let $\prec\cdot, \cdot\succ$ denote the Gödel pairing function and let $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ be a κ-list with the following properties:

(i) If $\alpha \in C$ is not totally indescribable, then there is a Π_n^m-formula φ and a subset A of V_α such that these objects provide a counterexample to the Π_n^m-indescribability of α. Then $d_\alpha = \{\prec 0, \lceil \varphi \rceil \succ\} \cup \{\prec 1, h(a) \succ \mid a \in A\}$, where

$\ulcorner\varphi\urcorner \in \omega$ is the Gödel number of φ in some fixed Gödelization of mth order set theory.

(ii) Otherwise, d_α is the empty set.

Let θ be a sufficiently large cardinal and let $j : M \longrightarrow \mathrm{H}(\theta)$ be a small embedding for κ that witnesses the subtlety of κ with respect to $\vec{d}$ and C, as in Lemma 5.2. Then $\mathrm{crit}\,(j) \in C$. Assume for a contradiction that $\mathrm{crit}\,(j)$ is not totally indescribable. Then there is a Π_n^m-formula φ and a subset A of V_α such that $d_\alpha = \{\prec 0, \ulcorner\varphi\urcorner \succ\} \cup \{\prec 1, h(a) \succ \mid a \in A\}$, $\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(A)$ and $\mathrm{V}_\alpha \models \neg\varphi(A \cap \mathrm{V}_\alpha)$ for all $\alpha < \mathrm{crit}\,(j)$. By our assumptions, there is an $\alpha \in C \cap \mathrm{crit}\,(j)$ with $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$. In this situation, our definition of d_α ensures that the formula φ and the subset $A \cap \mathrm{V}_\alpha$ of V_α provide a counterexample to the Π_n^m-indescribability of α. In particular, we know that $\mathrm{V}_\alpha \models \varphi(A \cap \mathrm{V}_\alpha)$ holds, a contradiction.   □

Next, we consider small embedding characterizations of λ-ineffable cardinals. Remember that, given a regular uncountable cardinal κ and a cardinal $\lambda \geq \kappa$, the cardinal κ is λ-*ineffable* if for every $\mathcal{P}_\kappa(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$, there exists a subset D of λ such that the set $\{a \in \mathcal{P}_\kappa(\lambda) \mid d_a = D \cap a\}$ is stationary in $\mathcal{P}_\kappa(\lambda)$. Many of the ideas and techniques used in the remainder of this section are already well-known and appear in the work of Magidor, Viale, Weiß and others (see for example [9] and [14]). Nevertheless, these results fit nicely into the framework of this paper and their formulation in terms of small embeddings will lead to new applications presented in Section 7.

Lemma 5.5. *The following statements are equivalent for all cardinals $\kappa \leq \lambda$:*

(i) *κ is λ-ineffable.*

(ii) *For all sufficiently large cardinals θ and every $\mathcal{P}_\kappa(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ and $\delta \in M \cap \kappa$ such that $j(\delta) = \lambda$, $\vec{d} \in \mathrm{ran}(j)$ and $j^{-1}[d_{j[\delta]}] \in M$.*[4]

Proof. Assume first that κ is λ-ineffable. Fix a $\mathcal{P}_\kappa(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ and a cardinal θ with $\mathcal{P}_\kappa(\lambda) \in \mathrm{H}(\theta)$. Then the λ-ineffability of κ yields a subset D of λ such that the set $S = \{a \in \mathcal{P}_\kappa(\lambda) \mid d_a = D \cap a\}$ is stationary in $\mathcal{P}_\kappa(\lambda)$. In this situation, we can find $X \prec \mathrm{H}(\theta)$ of cardinality less than κ such that $\vec{d}, D \in X$, $X \cap \kappa \in \kappa$ and $X \cap \lambda \in S$. Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi(\lambda) < \kappa$ and $\pi^{-1} : M \longrightarrow \mathrm{H}(\theta)$ is a small embedding for κ with $\vec{d} \in \mathrm{ran}(\pi^{-1})$. Moreover, we have

$$\pi[d_{\pi^{-1}[\pi(\lambda)]}] \ = \ \pi[d_{X \cap \lambda}] \ = \ \pi[D \cap X] \ = \ \pi(D) \ \in M.$$

Now, assume that (ii) holds, and let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ be a $\mathcal{P}_\kappa(\lambda)$-list. Pick a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ and $\delta \in M \cap \kappa$ with $j(\delta) = \lambda$, $\vec{d} \in \mathrm{ran}(j)$ and $d = j^{-1}[d_{j[\delta]}] \in \mathcal{P}(\delta)^M$. We define $S = \{a \in \mathcal{P}_\kappa(\lambda) \mid d_a = j(d) \cap a\} \in \mathrm{ran}(j)$. Assume for a contradiction that the set S is not stationary in $\mathcal{P}_\kappa(\lambda)$. Then there is a function $F : \mathcal{P}_\omega(\lambda) \longrightarrow \mathcal{P}_\kappa(\lambda)$ with $\mathrm{Cl}_F \cap S = \emptyset$, where Cl_F denotes the set of all $a \in \mathcal{P}_\kappa(\lambda)$ with $F(b) \subseteq a$ for all $b \in \mathcal{P}_\omega(a)$. Since $S \in \mathrm{ran}(j)$, elementarity yields a function $F_0 : \mathcal{P}_\omega(\delta) \longrightarrow \mathcal{P}_{\mathrm{crit}(j)}(\delta)$ in M with $\mathrm{Cl}_{j(F_0)} \cap S = \emptyset$. Pick $b \in \mathcal{P}_\omega(j[\delta])$.

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[4]Note that requiring $\delta < \kappa$ below should be seen as a smallness requirement on the domain model M of the embedding. It can be read off from the proof below that we could equivalently require that $|M|$ is less than κ.

Then $b \in \mathrm{ran}(j)$, and hence $j^{-1}(b) = j^{-1}[b] \in M$, and there is $a \in \mathrm{Cl}^{\mathrm{M}}_{\mathrm{F}_0}$ with $j^{-1}[b] \subseteq a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)^M$. In this situation, we have

$$j(F_0)(b) = j(F_0(j^{-1}[b])) \subseteq j(a) = j[a] \subseteq j[\delta].$$

These computations show that $j[\delta] \in \mathrm{Cl}_{j(F_0)}$. But we also have

$$j(d) \cap j[\delta] = j[d] = d_{j[\delta]},$$

and this shows that $j[\delta] \in \mathrm{Cl}_{j(F_0)} \cap S$, a contradiction. $\square$

A regular uncountable cardinal κ is *ineffable* if for every κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$, there exists a subset D of κ such that the set $\{\alpha < \kappa \mid d_\alpha = D \cap \alpha\}$ is stationary in κ. Since κ is a club in $\mathcal{P}_\kappa(\kappa)$ for every uncountable regular cardinal κ, it is easy to see that a cardinal κ is ineffable if and only if it is κ-ineffable. The above thus in particular yields the following small embedding characterization of ineffability.

Corollary 5.6. *The following statements are equivalent for every cardinal κ:*

(i) *κ is ineffable.*

(ii) *For all sufficiently large cardinals θ and for every κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with $\vec{d} \in \mathrm{ran}(j)$ and $d_{\mathrm{crit}(j)} \in M$.* $\square$

The following two observations are again easy to check. Note that the least ineffable cardinal is not Π^1_3-indescribable.

Observation 5.7. *Assume that κ is ineffable, and that J is a family of small embeddings witnessing the ineffability of κ, as in Statement (ii) of Corollary 5.6. Then J witnesses that κ is subtle, as in Statement (ii) of Lemma 5.2, and that κ is Π^1_2-indescribable, as in Statement (iii) of Lemma 4.2.* $\square$

Observation 5.8. *Assume that κ is λ-ineffable, and that $\kappa \leq \bar{\lambda} < \lambda$. Then any family of small embeddings witnessing that κ is λ-ineffable also witnesses that κ is $\bar{\lambda}$-ineffable, both in the sense of Statement (ii) of Lemma 5.5.* $\square$

The next result reformulates the proof of [14, Proposition 3.2] to derive a strengthening of Lemma 3.1 for many small embeddings witnessing λ-ineffability. We will make use of this in Section 7 below.

Lemma 5.9. *Let κ be a λ-ineffable cardinal. If $\lambda = \lambda^{<\kappa}$, then there is a $\mathcal{P}_\kappa(\lambda)$-list $\vec{d}$ and a set x with the property that whenever θ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ and $\delta \in M \cap \kappa$ witnessing the λ-ineffability of κ with respect to $\vec{d}$, as in Statement (ii) of Lemma 5.5, then $x \in \mathrm{ran}(j)$ implies that $\mathrm{crit}(j)$ is an inaccessible cardinal and $\mathcal{P}_{\mathrm{crit}(j)}(\delta) \subseteq M$.*

Proof. Fix a bijection $f : \mathcal{P}_\kappa(\lambda) \longrightarrow \lambda$. Then Lemma 5.4 yields a club C in κ and a κ-list $\vec{e} = \langle e_\alpha \mid \alpha < \kappa \rangle$ with the property that whenever θ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ witnessing the subtlety of κ with respect to $\vec{e}$ and C as in Statement (ii) of Lemma 5.2, then $\mathrm{crit}(j)$ is an inaccessible cardinal.

Let A denote the set of all $a \in \mathcal{P}_\kappa(\lambda)$ with the property that there is a cardinal $\vartheta_a > \lambda$ and an elementary submodel X_a of $\mathrm{H}(\vartheta_a)$ such that $f \in X_a$, $\alpha_a = X_a \cap \kappa \in C$ is inaccessible and $\mathcal{P}_{\alpha_a}(X_a \cap \lambda) \nsubseteq X_a$. Given $a \in A$, pick $x_a \in \mathcal{P}_{\alpha_a}(X_a \cap \lambda) \setminus X_a$. Next, let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda) \rangle$ denote the unique $\mathcal{P}_\kappa(\lambda)$-list such that $d_a = x_a$ for all $a \in A$, $d_a = e_{a \cap \kappa}$ for all $a \in \mathcal{P}_\kappa(\lambda) \setminus A$ with $a \cap \kappa \in C$, and $d_a = \emptyset$ otherwise.

Now, let θ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ and $\delta \in M \cap \kappa$ witnessing the λ-ineffability of κ with respect to $\vec{d}$, as in Statement (ii) of Lemma 5.5, such that f, $\vec{e}$ and C are contained in $\mathrm{ran}(j)$. Assume for a contradiction that either $\mathrm{crit}\,(j)$ is not inaccessible or $\mathcal{P}_{\mathrm{crit}(j)}(\delta) \nsubseteq M$.

Next, assume also that $j[\delta] \notin A$. Since $j[\delta] \cap \kappa = \mathrm{crit}\,(j) \in C$, $j^{-1}[d_{j[\delta]}] \in M$ implies that $e_{\mathrm{crit}(j)} \in M$. In this situation, the combination of Lemma 5.4 and Observation 5.7 yields that $\mathrm{crit}\,(j) = j[M] \cap \kappa$ is inaccessible. Since our assumptions imply that $\mathcal{P}_{\mathrm{crit}(j)}(j[M] \cap \lambda) \nsubseteq j[M]$, we can conclude that $j[M]$ witnesses that $j[\delta] \in A$, a contradiction.

Hence $j[\delta] \in A$. Since we know that $\mathrm{crit}\,(j) = \alpha_{j[\delta]}$ and $j^{-1}[d_{j[\delta]}] \in M$, we know that $x_{j[\delta]} \in j[M]$. But this allows us to conclude that $f(x_{j[\delta]}) \in \lambda \cap j[M] \subseteq X_{j[\delta]}$ and hence $x_{j[\delta]} \in X_{j[\delta]}$, a contradiction. $\qquad\square$

6. Filter-based large cardinals

Next, we show that several large cardinal notions defined through the existence of certain normal filters can also be characterized through the existence of small embeddings. As for Section 2, most of the ideas used in the proofs of the results presented in this section are quite elementary and are already present in earlier work (see for example [13, Section 3]). However, in our present setting, these arguments show that many more large cardinal notions fit into our uniform framework of small embedding characterizations. We start by considering λ-supercompact cardinals.

Lemma 6.1. *The following statements are equivalent for all cardinals $\kappa \leq \lambda$:*

(i) *κ is λ-supercompact.*
(ii) *For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ and $\delta \in M \cap \kappa$ such that $j(\delta) = \lambda$ and*
$$\{A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M \mid j[\delta] \in j(A)\} \ \in \ M.$$

Proof. Assume that there is a normal ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ witnessing the λ-supercompactness of κ. Let $j_U : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ denote the corresponding ultrapower embedding. Then $\lambda < j_U(\kappa)$. Fix a cardinal θ with $U \in \mathrm{H}(\theta)$ and an elementary submodel X of $\mathrm{H}(\theta)$ of cardinality λ with $\{U\} \cup (\lambda + 1) \subseteq X$. Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Then the closure of $\mathrm{Ult}(\mathrm{V}, U)$ under λ-sequences in V implies that the map $k = j_U \circ \pi^{-1} : N \longrightarrow \mathrm{H}(j_U(\theta))^{\mathrm{Ult}(\mathrm{V}, U)}$ is an element of $\mathrm{Ult}(\mathrm{V}, \mathcal{U})$, and this map is a small embedding for $j_U(\kappa)$ with $\mathrm{crit}\,(k) = \kappa$ and $k(\lambda) = j_U(\lambda)$ in $\mathrm{Ult}(\mathrm{V}, U)$. Then $k[\lambda] = j_U[\lambda]$ and therefore we have

$$k[\lambda] \in k(A) \iff j_U[\lambda] \in j_U(\pi^{-1}(A)) \iff \pi^{-1}(A) \in U \iff A \in \pi(U)$$

for all $A \in \mathcal{P}(\mathcal{P}_\kappa(\lambda))^N$. These computations show that

$$\pi(U) \ = \ \{A \in \mathcal{P}(\mathcal{P}_\kappa(\lambda))^N \mid k[\lambda] \in k(A)\} \ \in \ N.$$

In this situation, we can use elementarity between V and $\mathrm{Ult}(\mathrm{V}, U)$ to find a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ and $\delta \in M$ such that $\delta < \kappa$, $j(\delta) = \lambda$ and $\{A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M \mid j[\delta] \in j(A)\} \in M$.

Now, assume that (ii) holds. Fix a cardinal θ such that $\mathcal{P}(\mathcal{P}_\kappa(\lambda)) \in \mathrm{H}(\theta)$ and such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ and $\delta \in M \cap \kappa$ as in (ii). Then the set U of all $A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M$ with $j[\delta] \in j(A)$ is an element of M and the assumption $\delta < \kappa$ implies that this set is a normal ultrafilter on $\mathcal{P}_{\mathrm{crit}(j)}(\delta)$ in

M. Since $\mathcal{P}(\mathcal{P}_\kappa(\lambda)) \in \mathrm{H}(\theta)$, we can conclude that $j(U)$ is a normal filter on $\mathcal{P}_\kappa(\lambda)$ that witnesses the λ-supercompactness of κ. $\qquad\square$

In particular, the above easily yields the following small embedding characterization of measurable cardinals.

Corollary 6.2. *The following statements are equivalent for every cardinal κ:*

(i) *κ is measurable.*

(ii) *For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ with*

$$\{A \in \mathcal{P}(\mathrm{crit}\,(j))^M \mid \mathrm{crit}\,(j) \in j(A)\} \ \in \ M. \qquad\square$$

The following two observations are again based on well-known implications between the relevant large cardinals, and easy to verify.

Observation 6.3. *Assume that κ is measurable.*

(i) *Every embedding j witnessing the measurability of κ as in Statement (ii) of Corollary 6.2 also witnesses that κ is a stationary limit of Ramsey cardinals, as in Statement (ii) of Lemma 2.1.*

(ii) *Assume that for every $z \in \mathrm{V}_{\kappa+1}$, J_z is a family of small embeddings j witnessing the measurability of κ as in Statement (ii) of Corollary 6.2, with $z \in \mathrm{ran}(j)$.[5] Then the family $J = \bigcup\{J_z \mid z \in \mathrm{V}_{\kappa+1}\}$ witnesses that κ is Π_1^2-indescribable, as in Statement (iii) of Lemma 4.2, and that κ is ineffable, as in Statement (ii) of Corollary 5.6.* $\qquad\square$

Observation 6.4. *Let κ be a λ-supercompact cardinal, and assume that $\kappa \leq \bar{\lambda} < \lambda$.*

(i) *If j is a small embedding witnessing the λ-supercompactness of κ, as in Statement (ii) of Lemma 6.1, and with $\bar{\lambda} \in \mathrm{ran}(j)$, then j witnesses the $\bar{\lambda}$-supercompactness of κ, again as in Statement (ii) of Lemma 6.1.*

(ii) *Assume that for every $z \in \mathrm{V}_{\kappa+1}$, J_z is a family of small embeddings j witnessing the λ-supercompactness of κ, as in Statement (ii) of Lemma 6.1, and with $z \in \mathrm{ran}(j)$. Then $J = \bigcup_{z \in \mathrm{V}_{\kappa+1}} J_z$ witnesses the λ-ineffability of κ, as in Statement (ii) of Lemma 5.5.* $\qquad\square$

The next proposition shows that the domain models of small embeddings witnessing λ-supercompactness possess certain closure properties. These closure properties connect the characterization of supercompactness provided by Lemma 6.1 with Magidor's characterization of supercompactness mentioned in the introduction to this paper.

Proposition 6.5. *Let κ be a λ-supercompact cardinal and let $j : M \longrightarrow \mathrm{H}(\theta)$ be a small embedding for κ witnessing the λ-supercompactness of κ, as in Statement (ii) of Lemma 6.1. If $\delta \in M \cap \kappa$ with $j(\delta) = \lambda$ and $x \in \mathcal{P}(\mathrm{crit}\,(j))^M$, then $j(x) \cap \delta \in M$. Moreover, if λ is a strong limit cardinal, then δ is a strong limit cardinal and $\mathrm{H}(\delta) \in M$.*

Proof. Fix some $x \in \mathcal{P}(\mathrm{crit}\,(j))^M$. Given $\gamma < \delta$, set

$$A_\gamma = \{a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)^M \mid \gamma \in a, \ \mathrm{otp}\,(a \cap \gamma) \in x\}.$$

Then

$$j[\delta] \in j(A_\gamma) \iff \mathrm{otp}\,(j[\delta] \cap j(\gamma)) \in j(x) \iff \gamma \in j(x)$$

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[5]This latter additional assumption is harmless by Lemma 3.4.

for all $\gamma < \delta$. By our assumptions, these equivalences imply that the subset $j(x) \cap \delta$ is definable in M.

Now, assume that λ is a strong limit cardinal. Fix a sequence $s = \langle s_\alpha \mid \alpha < \mathrm{crit}\,(j) \rangle$ in M such that $s_\alpha : (2^{|\alpha|})^M \longrightarrow \mathcal{P}(\alpha)^M$ is a bijection for every $\alpha < \mathrm{crit}\,(j)$. Define

$$x \ = \ \{\prec\alpha, \prec\beta, \gamma\succ\succ \ \mid \ \alpha < \mathrm{crit}\,(j), \ \beta < 2^{|\alpha|}, \gamma \in s_\alpha(\beta)\} \ \in \ \mathcal{P}(\mathrm{crit}\,(j))^M.$$

Elementarity implies that δ is a strong limit cardinal in M, and the above computations show that $j(x) \cap \delta$ is an element of M. Assume for a contradiction that δ is not a strong limit cardinal. Pick a cardinal $\nu < \delta$ with $2^\nu \geq \delta$. Then the injection $j(s)_\nu \restriction \delta : \delta \longrightarrow \mathcal{P}(\nu)$ can be defined from $j(x) \cap \delta$, and therefore this function is contained in M, a contradiction. Since the above computations show that the sequence $\langle j(s)_\alpha \mid \alpha < \delta \rangle$ can be defined from the subset $j(x) \cap \delta$ of δ, and this subset is contained in M, it follows that $\mathrm{H}(\delta)$ is an element of M. $\qquad\square$

Next, we turn our attention to *huge* cardinals and their generalizations. Remember that, given $0 < n < \omega$, an uncountable cardinal κ is *n-huge* if there is a sequence $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ of cardinals and a κ-complete normal ultrafilter U on $\mathcal{P}(\lambda_n)$ with $\{a \in \mathcal{P}(\lambda_n) \mid \mathrm{otp}\,(a \cap \lambda_{i+1}) = \lambda_i\} \in U$ for all $i < n$. A cardinal is *huge* if it is 1-huge. Note that, if $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ and U witness the n-hugeness of κ and $j_U : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ is the induced ultrapower embedding, then $\mathrm{crit}\,(j_U) = \kappa$, $j_U(\lambda_i) = \lambda_{i+1}$ for all $i < n$, $U = \{A \in \mathcal{P}(\mathcal{P}(\lambda_n)) \mid j_U[\lambda_n] \in j_U(A)\}$ and $\mathrm{Ult}(\mathrm{V}, U)$ is closed under λ_n-sequences. In particular, each λ_i is measurable. Moreover, since U concentrates on the subset $[\lambda_n]^{\lambda_{n-1}}$ of all subsets of λ_n of order-type λ_{n-1}, we may as well identify U with an ultrafilter on this set of size λ_n.

The proof of the next lemma is essentially as the proof for the analogous statement about λ-supercompactness above, and will thus be omitted.

Lemma 6.6. *Given $0 < n < \omega$, the following statements are equivalent for all cardinals κ:*

 (i) κ *is n-huge.*
 (ii) For all sufficiently large cardinals θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)$ for κ such that $j^i(\mathrm{crit}\,(j)) \in M$ for all $i \leq n$ and

$$\{A \in \mathcal{P}(\mathcal{P}(j^n(\mathrm{crit}\,(j))))^M \mid j[j^n(\mathrm{crit}\,(j))] \in j(A)\} \ \in \ M. \qquad\square$$

The next lemma shows that the domain models of small embeddings witnessing n-hugeness also possess certain closure properties. These closure properties will directly imply that these embeddings also witness weaker large cardinal properties in the observation below.

Lemma 6.7. *Let $0 < n < \omega$, let κ be an n-huge cardinal and let $j : M \longrightarrow \mathrm{H}(\theta)$ be a small embedding for κ witnessing the n-hugeness of κ, as in Statement (ii) of Lemma 6.6. Then $\mathcal{P}(j^n(\mathrm{crit}\,(j))) \cap \mathrm{ran}(j)$ is contained in M. In particular, $\mathrm{H}(j^n(\mathrm{crit}\,(j)))$ is an element of M.*

Proof. Fix $A \in \mathcal{P}(j^{n-1}(\mathrm{crit}\,(j)))^M$. Given $\gamma < j^n(\mathrm{crit}\,(j))$, define

$$A_\gamma \ = \ \{a \in \mathcal{P}(j^n(\mathrm{crit}\,(j)))^M \mid \gamma \in a, \ \mathrm{otp}\,(a \cap \gamma) \in A\}.$$

For each $\gamma < j^n(\mathrm{crit}\,(j))$, we then have

$$A_\gamma \in U \ \Longleftrightarrow \ j[j^n(\mathrm{crit}\,(j))] \in j(A_\gamma) \ \Longleftrightarrow \ \mathrm{otp}\,(j(\gamma) \cap j[j^n(\mathrm{crit}\,(j))]) \in j(A)$$
$$\Longleftrightarrow \ \mathrm{otp}\,(j[\gamma]) \in j(A) \ \Longleftrightarrow \ \gamma \in j(A).$$

This shows that $j(A)$ is equal to the set $\{\gamma < j^n(\mathrm{crit}\,(j)) \mid A_\gamma \in U\}$. Since the sequence $\langle A_\gamma \mid \gamma < j^n(\mathrm{crit}\,(j))\rangle$ is an element of M, this shows that $j(A) \in M$.

The final statement of the lemma follows from the fact that elementarity implies that there is a subset of $j^n(\mathrm{crit}\,(j))$ in $\mathrm{ran}(j)$ that codes all elements of $\mathrm{H}(j^n(\mathrm{crit}\,(j)))$. $\square$

Observation 6.8. *Let $0 < n < \omega$, let κ be an n-huge cardinal, and let j be a small embedding witnessing the n-hugeness of κ, as in Statement (ii) of Lemma 6.6. If $0 < m < n$, then j also witnesses the m-hugeness of κ, as in Statement (ii) of Lemma 6.6. If $\kappa \leq \lambda < j(\kappa)$ and $\lambda \in \mathrm{ran}(j)$, then j also witnesses the λ-supercompactness of κ, as in Statement (ii) of Lemma 6.1.* $\square$

7. On a theorem by Christoph Weiß

In the remainder of this paper, we use the theory developed above to study the consistency of certain generalized tree properties at small cardinals. These properties are obtained by restricting the large cardinal properties defining λ-ineffable and subtle cardinals through the notion of *slenderness* to obtain strong combinatorial principles that can consistently be valid at smaller cardinals. The concept of *slenderness* originates from work of Saharon Shelah, and was isolated and studied by Christoph Weiß in [15] and [16]. The following definition contains the formulations of the relevant concepts used in this section.

Definition 7.1. Let κ be an uncountable regular cardinal and let $\lambda \geq \kappa$ be a cardinal.

(i) A κ-list $\langle d_\alpha \mid \alpha < \kappa\rangle$ is *slender* if there is a club C in κ with the property that for every $\gamma \in C$ and every $\alpha < \gamma$, there is a $\beta < \gamma$ with $d_\gamma \cap \alpha = d_\beta \cap \alpha$.
(ii) $\mathrm{SSP}(\kappa)$ is the statement that for every slender κ-list $\langle d_\alpha \mid \alpha < \kappa\rangle$ and every club C in κ, there are $\alpha, \beta \in C$ such that $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$.
(iii) A $\mathcal{P}_\kappa(\lambda)$-list $\langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$ is *slender* if for every sufficiently large cardinal θ, there is a club C in $\mathcal{P}_\kappa(\mathrm{H}(\theta))$ with $b \cap d_{X\cap\lambda} \in X$ for all $X \in C$ and all $b \in X \cap \mathcal{P}_{\omega_1}(\lambda)$.
(iv) $\mathrm{ISP}(\kappa, \lambda)$ is the statement that for every slender $\mathcal{P}_\kappa(\lambda)$-list $\langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)\rangle$, there exists $D \subseteq \lambda$ such that the set $\{a \in \mathcal{P}_\kappa(\lambda) \mid d_a = D \cap a\}$ is stationary in $\mathcal{P}_\kappa(\lambda)$.

The following theorem summarizes the upper bounds for the consistency strength of the principles $\mathrm{SSP}(\kappa)$ and $\mathrm{ISP}(\kappa, \lambda)$ presented in [15] and [16]. Remember that, given transitive classes $M \subseteq N$, the pair (M, N) satisfies the ω_1-*approximation property* if $A \in M$ holds for all $A \in N$ with $A \subseteq M$ and $A \cap x \in M$ for every $x \in M$ which is countable in M. Moreover, such a pair (M, N) satisfies the ω_1-*covering property* if whenever $A \in N$ is countable in N and $A \subseteq M$, then there is a $B \in M$ which is countable in M and satisfies $A \subseteq B$.

Theorem 7.2 (Weiß, [15, Theorem 2.3.1] & [16, Theorem 5.4])**.** *Let $\tau < \kappa \leq \lambda$ be cardinals with τ uncountable and regular, and let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \kappa\rangle\rangle$ be a forcing iteration such that the following statements hold for all inaccessible cardinals $\eta \leq \kappa$:*

(i) $\vec{\mathbb{P}}_{<\eta} \subseteq \mathrm{H}(\eta)^6$ *is the direct limit of* $\langle\langle\mathbb{P}_{<\alpha} \mid \alpha < \eta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \eta\rangle\rangle$ *and satisfies the η-chain condition.*

(ii) *If G is $\vec{\mathbb{P}}_{<\kappa}$-generic over* V *and G_η is the filter on $\vec{\mathbb{P}}_{<\eta}$ induced by G, then the pair* $(\mathrm{V}[G_\eta], \mathrm{V}[G])$ *satisfies the ω_1-approximation property.*

(iii) *If $\alpha < \eta$, then $\mathbb{P}_{<\alpha}$ is definable in* $\mathrm{H}(\eta)$ *from the parameters τ and α.*

Then the following statements hold:

(1) *If κ is a subtle cardinal, then* $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash \mathrm{SSP}(\check{\kappa})$.

(2) *If κ is an ineffable cardinal, then* $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash \mathrm{ISP}(\check{\kappa}, \check{\kappa})$.

(3) *Assume that $\vec{\mathbb{P}}$ also satisfies the following statement for all inaccessible cardinals $\eta \leq \kappa$:*

(iv) *If G_η is $\vec{\mathbb{P}}_{<\eta}$-generic over* V, *then the pair* $(\mathrm{V}, \mathrm{V}[G_\eta])$ *satisfies the ω_1-covering property.*

Then, if κ is $\lambda^{<\kappa}$-ineffable for some cardinal $\lambda \geq \kappa$, then $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash \mathrm{ISP}(\check{\kappa}, \check{\lambda})$.

As pointed out in [16, Section 5], William Mitchell's classical proof of the consistency of the tree property at successors of regular cardinals in [10] shows that for every uncountable regular cardinal τ and every inaccessible cardinal $\kappa > \tau$, there is a forcing iteration $\vec{\mathbb{P}}$ satisfying the Statements (i)-(iv) listed in Theorem 7.2 such that $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash$ "$\check{\kappa} = \check{\tau}^+$" and forcing with $\vec{\mathbb{P}}_{<\kappa}$ preserves all cardinals less than or equal to τ.

In the following, we discuss what appears to be a serious problem in the arguments used to derive the above statements in [15] and [16]. Afterwards, we present new proofs for (slight strengthenings of) the statements listed in Theorem 7.2. These arguments will make heavy use of the small embedding characterizations of subtlety and of λ-ineffability from Section 5.

We would first like to point out where the problematic step in Weiß's proof of Statements (2) and (3) seems to be, and argue that it is indeed a problem, for Weiß's proof would in fact yield a stronger result, one that is provably wrong. Let κ be a λ-ineffable cardinal with $\lambda = \lambda^{<\kappa}$, let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \kappa\rangle\rangle$ be a forcing iteration satisfying Statements (i)-(iv) listed in Theorem 7.2, let G be $\vec{\mathbb{P}}_{<\kappa}$-generic over V, and let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda)^{\mathrm{V}[G]}\rangle$ be a slender $\mathcal{P}_\kappa(\lambda)$-list in V[$G$]. The proofs of [15, Theorem 2.3.1] and [16, Theorem 5.4] then claim that there is a stationary subset T of $\mathcal{P}_\kappa(\lambda)$ in V and $d \in \mathcal{P}(\lambda)^{\mathrm{V}[G]}$ such that $d_a = d \cap a$ holds for all $a \in T$. Since $\vec{\mathbb{P}}_{<\kappa}$ satisfies the κ-chain condition in V and therefore preserves the stationarity of T, this argument would actually yield a strengthening of $\mathrm{ISP}(\kappa, \lambda)$ stating that every instance of the principle is witnessed by a stationary subset of $\mathcal{P}_\kappa(\lambda)$ contained in the ground model V. In particular, this conclusion would imply that if G is $\vec{\mathbb{P}}_{<\kappa}$-generic over V and $\langle d_\alpha \mid \alpha < \kappa\rangle$ is a κ-list in V[G], then there is a stationary subset S of κ in V such that $d_\alpha = d_\beta \cap \alpha$ holds for all $\alpha, \beta \in S$ with $\alpha < \beta$. The following proposition shows that this statement provably fails if forcing with $\vec{\mathbb{P}}_{<\kappa}$ destroys the ineffability of κ.

Proposition 7.3. *Let $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \kappa\rangle\rangle$ be a forcing iteration with the property that κ is an uncountable regular cardinal, $\vec{\mathbb{P}}_{<\kappa}$ is a direct limit and $\vec{\mathbb{P}}_{<\kappa}$ satisfies the κ-chain condition. Let G be $\vec{\mathbb{P}}_{<\kappa}$-generic over* V *and, given $\alpha < \kappa$, let*

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[6]Following [16], we make use of the convention that conditions in forcing iterations are only defined on their support.

G_α denote the filter on $\vec{\mathbb{P}}_{<\alpha}$ induced by G. Then one of the following statements holds:

(i) There is an $\alpha < \kappa$ such that for all $\alpha \leq \beta < \kappa$, the partial order $\dot{\mathbb{P}}_\beta^{G_\beta}$ is trivial.

(ii) There is a slender κ-list $\langle d_\alpha \mid \alpha < \kappa \rangle$ in $\mathrm{V}[G]$ with the property that for every stationary subset S of κ in V, there are $\alpha, \beta \in S$ with $\alpha < \beta$ and $d_\alpha \neq d_\beta \cap \alpha$.

Proof. Pick a sequence $\langle \langle \dot{q}_\alpha^0, \dot{q}_\alpha^1 \rangle \mid \alpha < \kappa \rangle$ in V such that the following statements hold for all $\alpha < \kappa$:

(i) $\dot{q}_{\alpha,0}$ and $\dot{q}_{\alpha,1}$ are both $\vec{\mathbb{P}}_{<\alpha}$-names for a condition in $\dot{\mathbb{P}}_\alpha$.

(ii) If H is $\vec{\mathbb{P}}_{<\alpha}$-generic over V, then the conditions $\dot{q}_{\alpha,0}^H$ and $\dot{q}_{\alpha,1}^H$ are compatible in $\dot{\mathbb{P}}_\alpha^H$ if and only if the partial order $\dot{\mathbb{P}}_\alpha^H$ is trivial.

Now, assume that (i) fails, and work in $\mathrm{V}[G]$. Let $g : \kappa \longrightarrow \kappa$ denote the unique function with the property that for all $\beta < \kappa$, $g(\beta)$ is the minimal ordinal greater than or equal to $\sup_{\alpha < \beta} g(\alpha)$ such that $\dot{\mathbb{P}}_{g(\beta)}^{G_{g(\beta)}}$ is a non-trivial partial order. Since $\vec{\mathbb{P}}_{<\kappa}$ satisfies the κ-chain condition, there is a club subset C of κ in V with $g(\alpha) < \beta$ for all $\alpha < \beta$ whenever $\beta \in C$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ denote the unique κ-list with the property that

$$d_\alpha = 0 \iff d_\alpha \neq 1 \iff \dot{q}_{g(\alpha),0}^{G_{g(\alpha)}} \in G^{g(\alpha)}$$

holds for every $\alpha < \kappa$, where G^β denotes the filter on $\dot{\mathbb{P}}_\beta^{G_\beta}$ induced by G for all $\beta < \kappa$. Then $\vec{d}$ is a slender κ-list.

Assume for a contradiction that there is a stationary subset S of κ in V such that $d_\alpha = d_\beta \cap \alpha$ holds for all $\alpha, \beta \in S$ with $\alpha < \beta$. Then there is an $i < 2$ with $d_\alpha = i$ for all $\alpha \in S$. Let $\dot{g}$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a function from κ to κ with $g = \dot{g}^G$ and let $\dot{d}$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a κ-list with $\vec{d} = \dot{d}^G$. Let p be a condition in G forcing all of the above statements. Pick a condition q in $\vec{\mathbb{P}}_{<\kappa}$ below p. Then there is $\alpha \in C \cap S$ with $q \in \vec{\mathbb{P}}_{<\alpha}$. By density, we can find a condition $s \in G$ below q, and $\alpha \leq \beta < \kappa$ with $g(\alpha) = \beta$, $s \in \vec{\mathbb{P}}_{<\beta+1}$ and $s(\beta) = \dot{q}_{\beta,1-i}$. But then $\dot{\mathbb{P}}_\beta^{G_\beta}$ is non-trivial, $\dot{q}_{\beta,1-i}^{G_\beta} \in G^\beta$ and $d_\alpha = 1 - i$, a contradiction. $\square$

In the argument that is supposed to prove Theorem 7.2 (2) and (3), Weiß constructs a club C in $\mathcal{P}_\kappa(\lambda)$ in V such that $d_a \in \mathrm{V}[G_{a \cap \kappa}]$ holds for every $a \in C$ with the property that $a \cap \kappa$ is an inaccessible cardinal in V. The problematic step then seems to be his conclusion that there exists a sequence $\langle \dot{d}_a \mid a \in C \rangle$ in V with the property that for all $a \in C$ with $a \cap \kappa$ inaccessible in V, $\dot{d}_a$ is a $\vec{\mathbb{P}}_{<(a \cap \kappa)}$-name with $d_a = \dot{d}_a^G$. Assuming the existence of such a sequence of names in V, it is easy to code the name $\dot{d}_a$ as a subset of a and then use the λ-ineffability of κ in V to obtain a stationary subset of $\mathcal{P}_\kappa(\lambda)$ in V that witnesses the strengthening of $\mathrm{ISP}(\kappa, \lambda)$ formulated above. Therefore, the above proposition shows that such a sequence cannot exist in the ground model V. The same kind of problematic step is used in the proof of Statement (1) of Theorem 7.2 presented in [15].

In the following, we will use the theory of small embeddings developed in this paper to present a different proof of Theorem 7.2. In fact, our results will yield a slight strengthening of the statements listed in Theorem 7.2, because, in contrast to these statements, our proofs do not rely on any kind of definability assumption and, in the case of λ-ineffable cardinals, we will not need to assume any kind of covering property of our iteration.

The following result shows how the consistency of $\mathrm{SSP}(\kappa)$ at a successor cardinal κ can be established from a subtle cardinal – by the results of [10], there are forcing iterations with the properties listed below, and which turn an inaccessible cardinal into the successor of an uncountable regular cardinal.

Theorem 7.4. *Let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \kappa\rangle\rangle$ be a forcing iteration with κ an uncountable and regular cardinal, such that the following statements hold for all inaccessible $\nu \leq \kappa$:*

(i) *$\vec{\mathbb{P}}_{<\nu} \subseteq \mathrm{H}(\nu)$ is the direct limit of $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha < \nu\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \nu\rangle\rangle$ and satisfies the ν-chain condition.*

(ii) *If G is $\vec{\mathbb{P}}_{<\kappa}$-generic over V and G_ν is the filter on $\vec{\mathbb{P}}_{<\nu}$ induced by G, then the pair $(\mathrm{V}[G_\nu], \mathrm{V}[G])$ satisfies the ω_1-approximation property.*

Then, if κ is a subtle cardinal, $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash \mathrm{SSP}(\check{\kappa})$.

Proof. Let $\dot{d}$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a slender κ-list, let $\dot{C}_0$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a club in κ that witnesses the slenderness of $\dot{d}$, and let $\dot{C}_1$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a club in κ. Since $\vec{\mathbb{P}}_{<\kappa}$ satisfies the κ-chain condition, there is a club $C \subseteq \mathrm{Lim}$ in κ such that $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash$ "$\check{C} \subseteq \dot{C}_0 \cap \dot{C}_1$", and such that all elements of C are closed under the Gödel pairing function $\prec\cdot,\cdot\succ$. Given $\alpha < \kappa$, let $\dot{d}_\alpha$ be a $\vec{\mathbb{P}}_{<\kappa}$-nice name for the α-th component of $\dot{d}$. Pick a regular cardinal $\theta > 2^\kappa$ with $\dot{d}, C, \dot{C}_0, \dot{C}_1, \vec{\mathbb{P}} \in \mathrm{H}(\theta)$, which is sufficiently large with respect to Statement (ii) in Lemma 5.2. Define A to be the set of all inaccessible cardinals $\nu < \kappa$ with the property that there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta)^\mathrm{V}$ for κ with $\mathrm{crit}(j) = \nu$ and $\dot{d}, C, \dot{C}_0, \dot{C}_1, \vec{\mathbb{P}} \in \mathrm{ran}(j)$. Note that elementarity implies that A is a subset of C.

Claim. *If $\nu \in A$ and G is $\vec{\mathbb{P}}_{<\kappa}$-generic over V, then $\dot{d}_\nu^G \in \mathrm{V}[G_\nu]$.*

Proof of the Claim. Fix a countable set x in $\mathrm{V}[G_\nu]$. By our assumptions on $\vec{\mathbb{P}}$, the cardinal ν is uncountable and regular in $\mathrm{V}[G_\nu]$. Hence, there is an $\alpha < \nu$ with $x \cap \nu \subseteq \alpha$. Since $\nu \in C \subseteq \dot{C}_0^G$, the slenderness of $\dot{d}^G$ in $\mathrm{V}[G]$ yields a $\beta < \nu$ with $\dot{d}_\nu^G = \dot{d}_\beta^G \cap \alpha$. Let $j : M \longrightarrow \mathrm{H}(\theta)^\mathrm{V}$ be a small embedding for κ in V that witnesses that ν is an element of A. Since $\dot{d} \in \mathrm{ran}(j)$ and $\beta < \nu = \mathrm{crit}(j)$, we know that $\dot{d}_\beta \in M$ is a $\vec{\mathbb{P}}_{<\nu}$-name with $j(\dot{d}_\beta) = \dot{d}_\beta$. Next, note that our assumptions on $\vec{\mathbb{P}}$ imply that $\vec{\mathbb{P}}_{<\nu} \in M$, $j(\vec{\mathbb{P}}_{<\nu}) = \vec{\mathbb{P}}_{<\kappa}$ and $j \upharpoonright \vec{\mathbb{P}}_{<\nu} = \mathrm{id}_{\vec{\mathbb{P}}_{<\nu}}$. In particular, if we define $j_G(\dot{x}^{G_\nu}) = j(\dot{x})^G$ for all $\vec{\mathbb{P}}_{<\nu}$-names $\dot{x}$ in M, then the resulting map $j_G : M[G_\nu] \longrightarrow \mathrm{H}(\theta)^{\mathrm{V}[G]}$ is a small embedding for κ in $\mathrm{V}[G]$ that extends j and satisfies $\dot{d}^G, \dot{C}^G \in \mathrm{ran}(j_G)$. But this allows us to conclude that

$$\dot{d}_\beta^G \;=\; j(\dot{d}_\beta)^G \;=\; j_G(\dot{d}_\beta^{G_\nu}) \;=\; \dot{d}_\beta^{G_\nu} \;\in\; M[G_\nu] \;\subseteq\; \mathrm{V}[G_\nu]$$

and hence $\dot{d}_\nu^G \cap x = \dot{d}_\beta^G \cap x = \dot{d}_\beta^{G_\nu} \cap x \in \mathrm{V}[G_\nu]$. Since the pair $(\mathrm{V}[G_\nu], \mathrm{V}[G])$ satisfies the σ-approximation property, the above computations show that $\dot{d}_\nu^G \in \mathrm{V}[G_\nu]$. $\square$

Now, work in V, fix a condition p in $\vec{\mathbb{P}}_{<\kappa}$, and let A_* denote the set of all $\nu \in A$ with $p \in \vec{\mathbb{P}}_{<\nu}$. With the help of the above claim and the fact that $\vec{\mathbb{P}}_{<\kappa}$ satisfies the κ-chain condition, we find a function $g : A_* \longrightarrow \kappa$ and sequences $\langle q_\nu \mid \nu \in A_*\rangle$, $\langle r_\nu \mid \nu \in A_*\rangle$ and $\langle \dot{e}_\nu \mid \nu \in A_*\rangle$, such that the following statements hold for all $\nu \in A_*$:

(1) $g(\nu) > \nu$ and $\dot{d}_\nu$ is a $\vec{\mathbb{P}}_{<g(\nu)}$-name.

(2) q_ν is a condition in $\vec{\mathbb{P}}_{<\nu}$ below p.

(3) $\dot{r}_\nu$ is a $\vec{\mathbb{P}}_{<\nu}$-name for a condition in the corresponding tail forcing $\dot{\mathbb{P}}_{[\nu,g(\nu))}$.[7]

(4) $\dot{e}_\nu$ is a $\vec{\mathbb{P}}_{<\nu}$-name for a subset of ν with the property that

$$\langle q_\nu, \dot{r}_\nu \rangle \Vdash_{\vec{\mathbb{P}}_{<\nu} * \dot{\mathbb{P}}_{[\nu,g(\nu))}} \text{``} \dot{d}_\nu = \dot{e}_\nu \text{''}.$$

Given $\nu \in A_*$, let E_ν denote the set of all triples $\langle s, \beta, i \rangle \in \vec{\mathbb{P}}_{<\nu} \times \nu \times 2 \subseteq \mathrm{H}(\nu)$ satisfying

$$s \Vdash_{\vec{\mathbb{P}}_{<\nu}} \text{``} \check{\beta} \in \dot{e}_\nu \longleftrightarrow i = 1 \text{''}.$$

Using Lemma 5.4, we find a κ-list $\vec{c} = \langle c_\alpha \mid \alpha < \kappa \rangle$ and a club C_* in κ with the property that $\mathrm{crit}\,(j)$ is a totally indescribable cardinal whenever $j : M \longrightarrow \mathrm{H}(\theta)$ is a small embedding for κ witnessing the subtlety of κ with respect to $\vec{c}$ and C_*, as in Statement (ii) of Lemma 5.2. Fix a bijection $f : \kappa \longrightarrow \mathrm{H}(\kappa)$ with $f[\nu] = \mathrm{H}(\nu)$ for every inaccessible cardinal $\nu < \kappa$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ be the unique κ-list such that the following statements hold for all $\alpha < \kappa$:

(a) If $\alpha \in A_*$, then

$$d_\alpha = \{\prec 0, 0 \succ\} \cup \{\prec f^{-1}(q_\alpha), 1 \succ\} \cup \{\prec f^{-1}(e), 2 \succ \mid e \in E_\alpha\} \subseteq \alpha.$$

(b) If $\omega \subseteq \alpha \notin A_*$ and α is closed under $\prec \cdot, \cdot \succ$, then

$$d_\alpha = \{\prec 1, 0 \succ\} \cup \{\prec \beta, 1 \succ \mid \beta \in c_\alpha\} \subseteq \alpha.$$

(c) Otherwise, d_α is the empty set.

Let $j : M \longrightarrow \mathrm{H}(\theta)$ be a small embedding for κ witnessing the subtlety of κ with respect to $\vec{d}$ and $C \cap C_*$, as in Statement (ii) of Lemma 5.2, such that $\vec{c}, \vec{d}, f, g, p, C, C_*, \dot{C}_0, \dot{C}_1, \vec{\mathbb{P}} \in \mathrm{ran}(j)$. Set $\nu = \mathrm{crit}\,(j)$ and pick $\alpha \in C \cap C_* \cap \nu$ with $d_\alpha = d_\nu \cap \alpha$. Then $\omega \leq \alpha < \nu$ and both α and ν are closed under $\prec \cdot, \cdot \succ$.

Claim. $\nu \in A_*$.

Proof of the Claim. Assume for a contradiction that $\nu \notin A_*$. This implies that $\prec 1, 0 \succ \in d_\alpha$ and therefore $\alpha \notin A_*$. But then $c_\alpha = c_\nu \cap \alpha$, and j witnesses the subtlety of κ with respect to $\vec{c}$ and C_*, as in Statement (ii) of Lemma 5.2. By the choices of $\vec{c}$ and C_*, this implies that ν is inaccessible, and hence j witnesses that ν is an element of A_*, a contradiction. $\square$

The above claim shows that $\nu \in A_*$, $\prec 0, 0 \succ \in d_\nu \cap \alpha = d_\alpha$, $\alpha \in A_*$, $g(\alpha) < \nu$, $q_\alpha = q_\nu \in \vec{\mathbb{P}}_{<\alpha}$ and $E_\alpha \subseteq E_\nu$. Pick a condition u in $\vec{\mathbb{P}}_{<\kappa}$ such that the canonical condition in $\vec{\mathbb{P}}_{<\alpha} * \dot{\mathbb{P}}_{[\alpha,\nu)}$ corresponding to $u \upharpoonright \nu$ is stronger than $\langle q_\alpha, \dot{r}_\alpha \rangle$ and the canonical condition in $\vec{\mathbb{P}}_{<\nu} * \dot{\mathbb{P}}_{[\nu,\kappa)}$ corresponding to u is stronger than $\langle u \upharpoonright \nu, \dot{r}_\nu \rangle$. Let G be $\vec{\mathbb{P}}_{<\kappa}$-generic over V with $u \in G$. Then we have $\dot{d}_\alpha^G = \dot{e}_\alpha^{G_\alpha} \in \mathrm{V}[G_\alpha]$, and $\dot{d}_\nu^G = \dot{e}_\nu^{G_\alpha} \in \mathrm{V}[G_\nu]$. If $\beta \in \dot{d}_\alpha^G$, then there is $s \in G_\alpha \subseteq G_\nu$ with $\langle s, \beta, 1 \rangle \in E_\alpha \subseteq E_\nu$, and this implies that $\beta \in \dot{d}_\nu^G$. In the other direction, if $\beta \in \alpha \setminus \dot{d}_\alpha^G$, then there is $s \in G_\alpha$ with $\langle s, \beta, 0 \rangle \in E_\alpha$, and hence $\beta \notin \dot{d}_\nu^G$. Therefore we have $\alpha < \nu$, $\alpha, \nu \in A_* \subseteq C \subseteq \dot{C}_1^G$ and $\dot{d}_\alpha^G = \dot{d}_\nu^G \cap \alpha$. Through a standard density argument, these computations now imply the conclusion of the theorem. $\square$

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[7]Let us point out that the problematic argument in Weiß's original proof can be seen as him assuming that the name $\dot{r}_\nu$ is just the name for the trivial condition in the corresponding tail forcing.

A variation of the above proof, using Lemma 5.9, allows us to establish the consistency of the principle $\mathrm{ISP}(\kappa, \lambda)$ for accessible cardinals κ with the help of small embeddings. Note that since $\lambda^{<\kappa} = (\lambda^{<\kappa})^{<\kappa}$ and $\mathrm{ISP}(\kappa, \lambda^{<\kappa})$ implies $\mathrm{ISP}(\kappa, \lambda)$ (see [16, Proposition 3.4]), the following result implies Statements (2) and (3) of Theorem 7.2. Moreover, note that results of Chris Johnson in [6] show that if κ is λ-ineffable and $\mathrm{cof}(\lambda) \geq \kappa$, then $\lambda = \lambda^{<\kappa}$ (see also [15, Proposition 1.5.4]).

Theorem 7.5. *Let κ be a cardinal, and let $\vec{\mathbb{P}} = \langle\langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \kappa\rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \kappa\rangle\rangle$ be a forcing iteration satisfying the statements listed in Theorem 7.4. If κ is a λ-ineffable cardinal with $\lambda = \lambda^{<\kappa}$, then $\mathbb{1}_{\vec{\mathbb{P}}_{<\kappa}} \Vdash \mathrm{ISP}(\check\kappa, \check\lambda)$.*

Proof. Let $\dot{d}$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a slender $\mathcal{P}_\kappa(\lambda)$-list, let $\theta > 2^\lambda$ be a cardinal and let $\dot{F}$ be a $\vec{\mathbb{P}}_{<\kappa}$-name for a function from $\mathcal{P}_\omega(\mathrm{H}(\theta))$ to $\mathcal{P}_\kappa(\mathrm{H}(\theta))$ such that the club $\mathrm{Cl}_{\dot{F}}$ in $\mathcal{P}_\kappa(\mathrm{H}(\theta))$ witnesses the slenderness of $\dot{d}$. Given $a \in \mathcal{P}_\kappa(\lambda)$, let $\dot{d}_a$ be a $\vec{\mathbb{P}}_{<\kappa}$-nice name for the a-th component of $\dot{d}$. Fix a bijection $f : \kappa \longrightarrow \mathrm{H}(\kappa)$ with $f[\nu] = \mathrm{H}(\nu)$ for every inaccessible cardinal $\nu < \kappa$. Pick a regular cardinal $\vartheta > \theta$ with $\dot{d}, \dot{F}, \vec{\mathbb{P}} \in \mathrm{H}(\vartheta)$, which is sufficiently large with respect to Statement (ii) in Lemma 5.5. Define A to be the set of all $a \in \mathcal{P}_\kappa(\lambda)$ with the property that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for κ and an ordinal $\delta \in M \cap \kappa$ such that $j(\delta) = \lambda$, $a = j[\delta]$, $\nu(a) = a \cap \kappa$ is an inaccessible cardinal, $\mathrm{crit}(j) = \nu(a)$, $\mathcal{P}_{\nu(a)}(\delta) \subseteq M$ and $\dot{d}, f, \dot{F}, \vec{\mathbb{P}} \in \mathrm{ran}(j)$.

Claim. *If $a \in A$ and G is $\vec{\mathbb{P}}_{<\kappa}$-generic over V, then $\dot{d}_a^G \in \mathrm{V}[G_{\nu(a)}]$.*

Proof of the Claim. Fix a countable set x in $\mathrm{V}[G_{\nu(a)}]$. Pick a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ for κ in V and an ordinal $\delta \in M \cap \kappa$ that witness that a is an element of A. Then elementarity yields an ordinal $\eta \in M$ with $j(\eta) = \theta$. Moreover, we have $\vec{\mathbb{P}}_{<\nu(a)} \in M$, $j(\vec{\mathbb{P}}_{<\nu(a)}) = \vec{\mathbb{P}}_{<\kappa}$, $j \upharpoonright \vec{\mathbb{P}}_{<\nu(a)} = \mathrm{id}_{\vec{\mathbb{P}}_{<\nu(a)}}$ and hence there is a canonical small embedding $j_G : M[G_{\nu(a)}] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ for κ in $\mathrm{V}[G]$ that extends j. Set $y = j^{-1}[a \cap x] \in \mathcal{P}_{\omega_1}(\delta)^{\mathrm{V}[G_{\nu(a)}]}$. Since $\mathcal{P}_{\nu(a)}(\delta)^{\mathrm{V}} \subseteq M$ and $\vec{\mathbb{P}}_{<\nu(a)}$ satisfies the $\nu(a)$-chain condition in V, we also know that $\mathcal{P}_{\nu(a)}(\delta)^{\mathrm{V}[G_{\nu(a)}]} \subseteq M[G_{\nu(a)}]$ and therefore $y \in \mathcal{P}_{\nu(a)}(\delta)^{M[G_{\nu(a)}]}$. Moreover, we have $\dot{F}^G \in \mathrm{ran}(j_G)$ and, as in the proof of Lemma 5.5, this implies that $j_G[\mathrm{H}(\eta)^{M[G_{\nu(a)}]}] \in \mathrm{Cl}_{\dot{F}^G}$. Since

$$a \cap x \;=\; j[y] \;=\; j_G(y) \;\in\; j_G[\mathrm{H}(\eta)^{M[G_{\nu(a)}]}] \cap \mathcal{P}_{\omega_1}(\lambda)^{\mathrm{V}[G]},$$

the slenderness of $\dot{d}^G$ in $\mathrm{V}[G]$ yields a $d \in \mathcal{P}_{\omega_1}(\delta)^{M[G_{\nu(a)}]}$ with $j_G(d) = \dot{d}_a^G \cap x$. But this allows us to conclude that $\dot{d}_a^G \cap x = j_G(d) = j[d] \in \mathrm{V}[G_{\nu(a)}]$. Since the pair $(\mathrm{V}[G_{\nu(a)}], \mathrm{V}[G])$ satisfies the σ-approximation property, these computations show that $\dot{d}_a^G$ is an element of $\mathrm{V}[G_{\nu(a)}]$. $\square$

In the following, we work in V. Fix a condition p in $\vec{\mathbb{P}}_{<\kappa}$ and let A_* denote the set of all $a \in A$ with $q \in \vec{\mathbb{P}}_{<\nu(a)}$. Then all elements of A_* are closed under $\prec\cdot, \cdot\succ$ and, with the help of the above claim and the fact that $\vec{\mathbb{P}}_{<\kappa}$ satisfies the κ-chain condition, we find sequences $\langle q_a \mid a \in A_*\rangle$, $\langle \dot{r}_a \mid a \in A_*\rangle$ and $\langle \dot{e}_a \mid a \in A_*\rangle$ such that the following statements hold for all $a \in A_*$:

(1) $q(a)$ is a condition in $\vec{\mathbb{P}}_{<\nu(a)}$ below p.

(2) $\dot{r}_a$ is a $\vec{\mathbb{P}}_{<\nu(a)}$-name for a condition in the corresponding tail forcing $\dot{\mathbb{P}}_{[\nu(a),\kappa)}$.

(3) $\dot{e}_a$ is a $\vec{\mathbb{P}}_{<\nu(a)}$-nice name for a subset of a with $\langle q_a, \dot{r}_a\rangle \Vdash_{\vec{\mathbb{P}}_{<\nu(a)} * \dot{\mathbb{P}}_{[\nu(a),\kappa)}} $ "$\dot{d}_a = \dot{e}_a$".

Let $\vec{c} = \langle c_a \mid a \in \mathcal{P}_\kappa(\lambda) \rangle$ be the $\mathcal{P}_\kappa(\lambda)$-list given by an application of Lemma 5.9 and define $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\kappa(\lambda) \rangle$ to be the unique $\mathcal{P}_\kappa(\lambda)$-list with

$$d_a \;=\; \{\prec\! f^{-1}(s), \beta\!\succ \;\mid\; \langle \check{\beta}, s \rangle \in \dot{e}_a\} \;\subseteq\; a$$

for all $a \in A_*$ and $d_a = c_a$ for all $a \in \mathcal{P}_\kappa(\lambda) \setminus A_*$. Pick a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for κ and $\delta \in M \cap \kappa$ that witness the λ-ineffability of κ with respect to $\vec{d}$, as in Statement (ii) of Lemma 5.5, such that $\vec{c}, \dot{d}, f, \dot{F}, p, \vec{\mathbb{P}} \in \mathrm{ran}(j)$.

Claim. $j[\delta] \in A_*$.

Proof of the Claim. Assume for a contradiction that $j[\delta] \notin A_*$. Then $d_{j[\delta]} = c_{j[\delta]}$ and therefore $j^{-1}[c_{j[\delta]}] \in M$. This shows that j and δ witness the λ-ineffability of κ with respect to $\vec{c}$. By the definition of $\vec{c}$, this implies that $\mathrm{crit}\,(j)$ is an inaccessible cardinal and $\mathcal{P}_{\mathrm{crit}(j)}(\delta) \subseteq M$. But this shows that the embedding j and the ordinal δ witness that $j[\delta]$ is an element of A_*, a contradiction. $\qquad\square$

Set $\nu = \mathrm{crit}\,(j) = j[\delta] \cap \kappa = \nu(j[\delta])$, and pick a condition u in $\vec{\mathbb{P}}_{<\kappa}$ such that the canonical condition in $\vec{\mathbb{P}}_{<\nu} * \dot{\mathbb{P}}_{[\nu,\kappa)}$ corresponding to u is stronger than $\langle q_{j[\delta]}, \dot{r}_{j[\delta]} \rangle$. Let G be $\vec{\mathbb{P}}_{<\kappa}$-generic over V with $u \in G$ and set $S = j^{-1}[\dot{d}_{j[\delta]}^G] \subseteq \delta$.

Claim. $S \in M[G_\nu]$.

Proof of the Claim. Since $j[\delta] \in A_*$, an earlier claim yields $\dot{d}_{j[\delta]}^G = \dot{e}_{j[\delta]}^{G_j} \in \mathrm{V}[G_\nu]$. Given $\gamma < \delta$, we know that $j(\gamma) \in \dot{d}_{j[\delta]}^G$ if and only if there is an $s \in G_\nu$ with $\prec\! f^{-1}(s), j(\gamma)\!\succ \;\in d_{j[\delta]}$. Since $f \upharpoonright \nu \in M$ with $j(f \upharpoonright \nu) = f$ and $j \upharpoonright G_\nu = \mathrm{id}_{G_\nu}$, this shows that S is equal to the set of all $\gamma < \delta$ with the property that there is an $s \in G_\nu$ with $\prec\!(f \upharpoonright \nu)^{-1}(s), \gamma\!\succ \;\in j^{-1}[d_{j[\delta]}]$. Since $j^{-1}[d_{j[\delta]}] \in M$, we can conclude that S is an element of $M[G_\nu]$. $\qquad\square$

Now, work in V$[G]$ and let $j_G : M[G_\nu] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ denote the canonical small embedding for κ extending j. Assume, towards a contradiction, that there is a function $F : \mathcal{P}_\omega(\lambda) \longrightarrow \mathcal{P}_\kappa(\lambda)$ with the property that for every element a of Cl_F, the corresponding entry of $\dot{d}^G$ is different from $a \cap j_G(S)$. By elementarity, there is a function $F_0 : \mathcal{P}_\omega(\delta) \longrightarrow \mathcal{P}_\nu(\delta)$ in $M[G_\nu]$ with the property that $j(F_0)$ is a function with the properties listed above. But then elementarity implies $j[\delta] \in \mathrm{Cl}_{j(F_0)}$ with

$$j[\delta] \cap j_G(S) \;=\; j[\delta] \cap j_G(j^{-1}[\dot{d}_{j[\delta]}^G]) \;=\; \dot{d}_{j[\delta]}^G,$$

a contradiction. A density argument now yields the conclusion of the theorem. $\quad\square$

8. Open Questions and concluding remarks

Clearly, our paper suggests the task to characterize further important types of large cardinals through the existence of certain small embeddings. For example, one may consider large cardinals defined through stronger partition properties.

Question 8.1. Is there a small embedding characterization for Ramsey cardinals?

Moreover, it would be interesting to obtain small embedding characterizations for large cardinal notions whose first-order definitions rely on the existence of certain extenders.

Question 8.2. Is there a small embedding characterization for strong cardinals?

Another important type of large cardinals usually defined through the existence of certain extenders are Woodin cardinals. By combining Lemma 2.1 with a theorem of Woodin (see [7, Theorem 26.14]) stating that a cardinal κ is Woodin if and only if for any $A \subseteq V_\kappa$, the set $\{\alpha < \kappa \mid \alpha$ is γ-strong for A for every $\gamma < \kappa\}$ is stationary in κ, we directly obtain the following characterization of Woodinness.

Corollary 8.3. *The following statements are equivalent for every cardinal κ:*

(i) *κ is a Woodin cardinal.*
(ii) *For all sufficiently large cardinals θ and any $A \subseteq V_\kappa$, there is a small embedding $j : M \longrightarrow H(\theta)$ for κ with the property that $A \in \mathrm{ran}(j)$, and that for every $\gamma < \kappa$, $\mathrm{crit}\,(j)$ is γ-strong for A.*[8]   □

Note that the above characterization is not based on a correctness property, because the first Woodin cardinal is not even weakly compact. Therefore, we naturally arrive at the following question.

Question 8.4. Is there a small embedding characterization of Woodinness that relies on a correctness property?

Among the large cardinal properties characterized through small embeddings in this paper, subtlety is the only property whose characterization does not rely on a correctness property. This motivates the following question.

Question 8.5. Is there a small embedding characterization of subtlety that relies on a correctness property?

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[8]A cardinal κ is *γ-strong for A* if there is an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{crit}\,(j) = \kappa$, $\gamma < j(\kappa)$, $V_{\kappa+\gamma} \subseteq M$, and $A \cap V_{\kappa+\gamma} = j(A) \cap V_{\kappa+\gamma}$.

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