Weak compactness cardinals for abstract logics

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Introduction

I present work exploring connections between large cardinal axioms and compactness properties of strong logics.

Such connections provide strong justifications for the acceptance of large cardinal axioms.

In the following, I will focus on large cardinal assumptions that are equivalent to all abstract logics possessing certain compactness properties.

I now discuss the main example of such a connection.

Remember that *Vopěnka's Principle* is the scheme of axioms stating that for every proper class of graphs, there are two members of the class with a homomorphism between them.

Given an abstract logic $\mathcal{L}$ and a cardinal κ, an $\mathcal{L}$-theory T is *$<\kappa$-satisfiable* if every subtheory of cardinality less than κ is satisfiable.

A cardinal κ is a *strong compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory is satisfiable.

Theorem (Makowsky)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *Every abstract logic has a strong compactness cardinal.*

A cardinal κ is a *weak compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory of cardinality κ is satisfiable.

Recent work of Boney, Dimopoulos, Gitman and Magidor connects this weaker compactness property to the large cardinal notion of *subtleness*, introduced by Jensen and Kunen in their studies of strong diamond principles.

A cardinal δ is *subtle* if for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

We let "Ord *is subtle*" denote the scheme of axioms stating that for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma \in \mathrm{Ord}\rangle$ and every closed unbounded class C of ordinals, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Theorem (Boney–Dimopoulos–Gitman–Magidor)

The following schemes are equivalent over $\mathbf{ZFC}$ *together with the existence of a definable global well-ordering:*

- Ord *is subtle.*
- *Every abstract logic has a stationary class of weak compactness cardinals.*

This result raises two questions:

- Is it necessary to assume the existence of a global well-ordering?
- Can we characterize the existence of weak compactness cardinals for all abstract logics through large cardinal properties of Ord?

Proposition

The following statements are equivalent for every infinite cardinal δ:

- *The cardinal δ is subtle.*
- *For all closed unbounded subsets C of δ and all sequences $\langle E_\gamma \mid \gamma < \delta\rangle$ such that $\emptyset \neq E_\gamma \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma < \delta$, there are $\beta < \gamma$ in C and $A \in E_\gamma$ with $A \cap \beta \in E_\beta$.*

Definition

We let "Ord *is essentially subtle*" denote the scheme of axioms stating that for every closed unbounded class C of ordinals and every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ such that $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ holds for all $\alpha \in \mathrm{Ord}$, there exist $\alpha < \beta$ in C and $A \in E_\beta$ with $A \cap \alpha \in E_\alpha$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *Every abstract logic has a stationary class of weak compactness cardinals.*

Theorem (Bagaria–L.)

The following statements are equivalent for all cardinal δ with $\mathrm{V}_\delta \prec_{\Sigma_1} \mathrm{V}$:

- *For all $\xi < \delta$ and every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$, there are cardinals $\xi < \mu < \nu < \delta$ with $A_\mu = A_\nu \cap \mu$.*
- *The cardinal δ is either subtle or a limit of subtle cardinals.*
- *For all $\xi < \delta$ and every sequence $\langle E_\gamma \mid \gamma < \delta \rangle$ such that $\emptyset \neq E_\gamma \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma < \delta$, there exist cardinals $\xi < \mu < \nu < \delta$ and $A \in E_\nu$ with $A \cap \mu \in E_\mu$.*

Definition

We let "Ord *is essentially closure subtle*" denote the scheme of axioms stating that all $\xi \in \mathrm{Ord}$ and every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ such that $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ holds for all $\alpha \in \mathrm{Ord}$, there exist cardinals $\xi < \mu < \nu$ and $A \in E_\nu$ with $A \cap \mu \in E_\mu$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially closure subtle.*
- *Every abstract logic has a weak compactness cardinal.*

We now explore the differences between the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$$

and the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially closure subtle}\text{”}.$$

Proposition

If Φ is a sentence in the language of set theory with the property that $\mathbf{ZFC} + \Phi$ is consistent, then

$$\mathbf{ZFC} + \Phi \nvdash \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}.$$

Theorem

The following statements are equivalent:

- *There exists a sentence Φ in the language of set theory such that the theory $\mathbf{ZFC} + \Phi$ is consistent and*

$$\mathbf{ZFC} + \Phi \vdash \text{“}\mathrm{Ord} \textit{ is essentially closure subtle”}.$$

- $\mathbf{ZFC} + \text{“}\mathrm{Ord}$ *is essentially closure subtle”* $\nvdash$ “Ord *is essentially subtle”.*
- *The theory*

$$\mathbf{ZFC} + \textit{“There is a proper class of subtle cardinals”}$$

is consistent.

The techniques developed in the proofs of the above results also allow us to show that the existence of weak compactness cardinals for all abstract logics does not imply the existence of large cardinals in V.

Theorem

The following schemes are equiconsistent over $\mathbf{ZFC}$:

- *There is a proper class of subtle cardinals.*
- Ord *is essentially closure subtle and there are no inaccessible cardinals.*

Weakly $C^{(n)}$-shrewd cardinals

For every natural number n, we let $C^{(n)}$ denote the Π_n-definable closed unbounded class of all of ordinals α with the property that $\mathrm{V}_\alpha \prec_{\Sigma_n} \mathrm{V}$.

Definition

Given a natural number $n > 0$, a cardinal κ is *weakly $C^{(n)}$-shrewd* if for every cardinal $\kappa < \theta \in C^{(n)}$ and every $z \in \mathrm{H}(\theta)$, there exists

- a cardinal $\bar{\theta} \in C^{(n)}$,
- a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$,
- an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
- an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

such that $\bar{\kappa} + 1 \subseteq X$, $j \upharpoonright \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially closure subtle.*
- *For every natural number* $n > 0$*, there is a proper class of weakly* $C^{(n)}$*-shrewd cardinals.*
- *Every logic has a weak compactness cardinal.*

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *For every natural number* $n > 0$*, there is a proper class of weakly* $C^{(n)}$*-shrewd cardinals that are elements of* $C^{(n+1)}$*.*
- *For every natural number* $n > 0$*, there is a weakly* $C^{(n)}$*-shrewd cardinal that is an element of* $C^{(n+1)}$*.*
- *Every logic has a stationary class of weak compactness cardinals.*

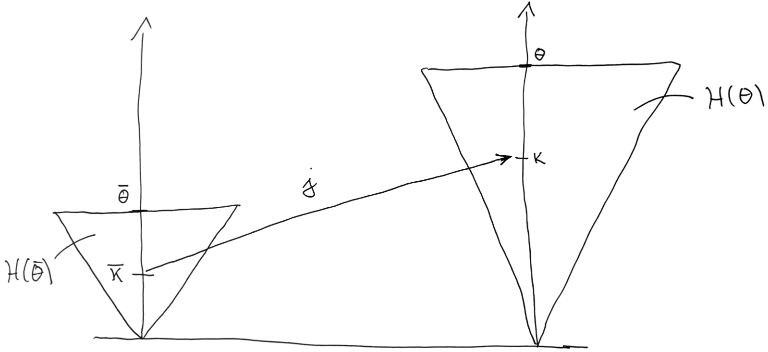
The above definition is motivated by the following classical result:

Theorem (Magidor)

The following statements are equivalent for every cardinal κ*:*

- κ *is supercompact.*
- *For every cardinal* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$, *there exist*
  - *cardinals* $\bar{\kappa} < \bar{\theta} < \kappa$, *and*
  - *an elementary embedding* $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$

  such that $\mathrm{crit}(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$ *and* $z \in \mathrm{ran}(j)$.


$\bar{\theta}$
$H(\bar{\theta})$
$\bar{\kappa}$
j
θ
$H(\theta)$
κ

Definition (Rathjen)

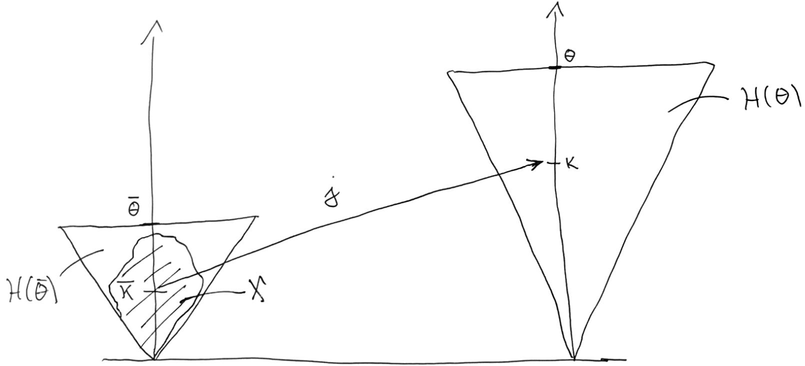
A cardinal κ is *shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every ordinal $\gamma > \kappa$ and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

Theorem

The following statements are equivalent for every cardinal κ:

- *κ is a shrewd cardinal.*
- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
    - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$,*
    - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
    - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

  such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.


θ
H(θ)
κ
j
θ̄
H(θ̄)
κ̄
X

Definition

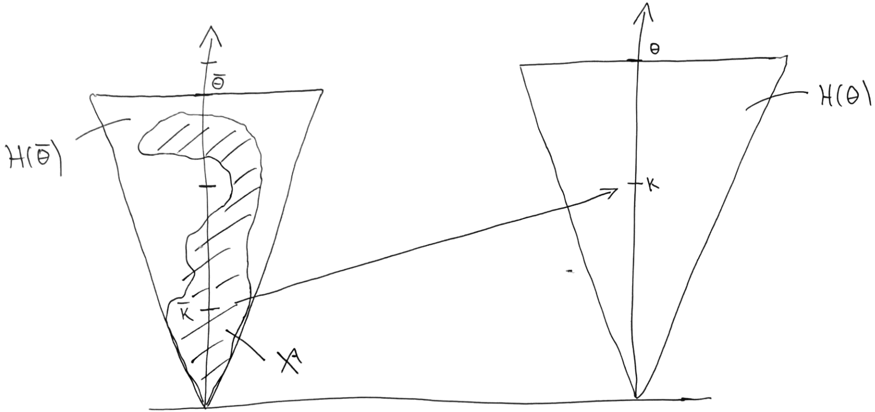
An infinite cardinal κ is *weakly shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every cardinal $\theta > \kappa$ and every subset A of κ with the property that $\Phi(A, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ with the property that $\bar{\kappa} < \kappa$ and $\Phi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

Lemma

The following statements are equivalent for every infinite cardinal κ:

- *κ is a weakly shrewd cardinal.*
- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta}$,*
  - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
  - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

  with $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $z \in \mathrm{ran}(j)$.



$\bar{\theta}$
$H(\bar{\theta})$
$\bar{K}$
X
θ
$H(\theta)$
K

Theorem

The following statements are equivalent for all weakly shrewd cardinals κ*:*

- κ *is a shrewd cardinal.*
- κ *is an element of* $C^{(2)}$*.*

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a weakly shrewd cardinal that is not shrewd.*
- *There is a subtle cardinal.*

The following weakening of strongness was introduced by Villaveces in his investigation of chains of end elementary extensions of models of set theory.

Definition (Villaveces)

An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.

Theorem

A cardinal is strongly unfoldable if and only if it is shrewd.

Definition (Bagaria–L.)

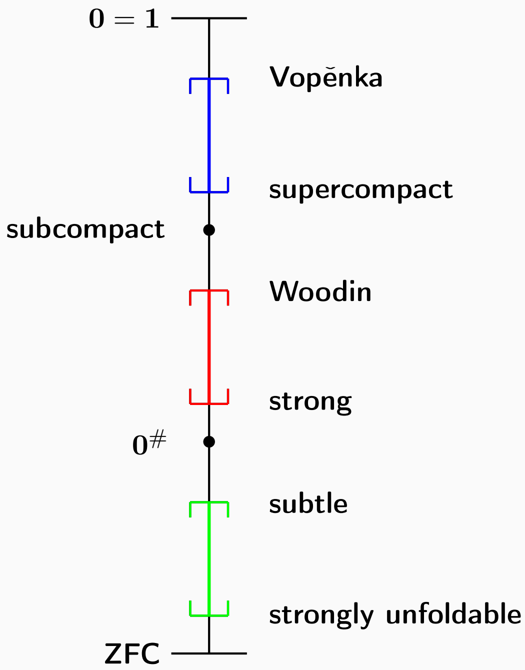
Given a natural number n, an inaccessible cardinal κ is $C^{(n)}$*-strongly unfoldable* if for every ordinal $\lambda \in C^{(n)}$ greater than κ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \prec_{\Sigma_n} V^N_{j(\kappa)}$.

Theorem

The following statements are equivalent for every natural number $n > 0$ and every weakly $C^{(n)}$-shrewd cardinal κ:

- *κ is $C^{(n)}$-strongly unfoldable.*
- *κ is an element of $C^{(n+1)}$.*

In combination with results of Bagaria and Wilson, this shows that certain patterns repeat in all parts of the large cardinal hierarchy.


$\mathbf{0 = 1}$
Vopěnka
supercompact
subcompact
Woodin
strong
$\mathbf{0^\#}$
subtle
strongly unfoldable
ZFC

Weak compactness cardinals for abstract logics

Definition

Given a natural number $n > 0$, a cardinal κ is *weakly $C^{(n)}$-shrewd* if for every cardinal $\kappa < \theta \in C^{(n)}$ and every $z \in \mathrm{H}(\theta)$, there exist a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

In the remainder of this talk, I want to outline the proof of the following implication:

Lemma

Assume that for every natural number $n > 0$, there exist unboundedly many weakly $C^{(n)}$-shrewd cardinals. Then every abstract logic has unboundedly many weak compactness cardinals.

- A *language* is a tuple $\tau = \langle \mathfrak{C}_\tau, \mathfrak{F}_\tau, \mathfrak{R}_\tau, \mathfrak{a}_\tau \rangle$, where $\mathfrak{C}_\tau$, $\mathfrak{F}_\tau$ and $\mathfrak{R}_\tau$ are pairwise disjoint sets and $\mathfrak{a}_\tau : \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \omega \setminus \{0\}$ is a function.

  We then call $\mathfrak{C}_\tau$ the *set of constant symbols* of τ, $\mathfrak{F}_\tau$ the *set of function symbols* of τ, $\mathfrak{R}_\tau$ the *set of relation symbols* of τ and $\mathfrak{a}_\tau$ the *arity function* of τ.

- Given a language τ, a τ*-structure* is a tuple

  $$M \ = \ \langle |M|, (c^M)_{c \in \mathfrak{C}_\tau}, (f^M)_{f \in \mathfrak{F}_\tau}, (R^M)_{R \in \mathfrak{R}_\tau} \rangle,$$

  where $|M|$ is a non-empty set, $c^M \in |M|$ for $c \in \mathfrak{C}_\tau$,
  $f^M : |M|^{\mathfrak{a}_\tau(f)} \longrightarrow |M|$ for $f \in \mathfrak{F}_\tau$ and $R^M \subseteq |M|^{\mathfrak{a}_\tau(R)}$ for $R \in \mathfrak{R}_\tau$.

  We let $Str(\tau)$ denote the class of all τ-structures.

- A *morphism* between languages τ and υ is an injection

$$h : \mathfrak{C}_\tau \cup \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \mathfrak{C}_\upsilon \cup \mathfrak{F}_\upsilon \cup \mathfrak{R}_\upsilon$$

  with $h[\mathfrak{C}_\tau] \subseteq \mathfrak{C}_\upsilon$, $h[\mathfrak{F}_\tau] \subseteq \mathfrak{F}_\upsilon$, $h[\mathfrak{R}_\tau] \subseteq \mathfrak{R}_\upsilon$, $\mathfrak{a}_\upsilon(h(f)) = \mathfrak{a}_\tau(f)$ for all $f \in \mathfrak{F}_\tau$ and $\mathfrak{a}_\upsilon(h(R)) = \mathfrak{a}_\tau(R)$ for all $R \in \mathfrak{R}_\tau$.

  Such a morphism is a *renaming* if it is bijective. Given a renaming h from τ to υ, we let

$$h_* : Str(\tau) \longrightarrow Str(\upsilon)$$

  denote the unique bijection with the property that $|h_*(M)| = |M|$ and $h(x)^{h_*(M)} = x^M$ for all $M \in Str(\tau)$ and $x \in \mathfrak{C}_\tau \cup \mathfrak{F}_\tau \cup \mathfrak{R}_\tau$.

An *abstract logic* is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ consisting of a class function $\mathcal{L}$ whose domain is the class of all languages and a binary relation $\models_{\mathcal{L}}$ satisfying:

- If $M \models_{\mathcal{L}} \phi$, then there is a language τ with $M \in Str(\tau)$ and $\phi \in \mathcal{L}(\tau)$.
- Given a language υ that extends a language τ, we have $\mathcal{L}(\tau) \subseteq \mathcal{L}(\upsilon)$ and, for all $\phi \in \mathcal{L}(\tau)$ and $M \in Str(\upsilon)$, we have $M \models_{\mathcal{L}} \phi$ if and only if $M \upharpoonright \tau \models_{\mathcal{L}} \phi$.
- Given a language τ, isomorphic $M, N \in Str(\tau)$ and $\phi \in \mathcal{L}(\tau)$, we have $M \models_{\mathcal{L}} \phi$ if and only if $N \models_{\mathcal{L}} \phi$.
- If h is a renaming of a language τ into a language υ, then there is a unique bijection $h_* : \mathcal{L}_\tau \longrightarrow \mathcal{L}_\rho$ with the property that

  $$M \models_{\mathcal{L}} \phi \iff h^*(M) \models_{\mathcal{L}} h_*(\phi)$$

  holds for every τ-structure M and all $\phi \in \mathcal{L}(\tau)$.
- There exists a minimal cardinal $o(\mathcal{L})$ (the *occurrence number* of $\mathcal{L}$) such that for every language υ and all $\phi \in \mathcal{L}(\upsilon)$, there is a language τ with the property that υ extends τ, τ has less than $o(\mathcal{L})$-many symbols and ϕ is an element of $\mathcal{L}(\tau)$.

Let $(\mathcal{L}, \models_{\mathcal{L}})$ be an abstract logic. Pick $\mu \in C^{(1)}$ satisfying:

- $\mathrm{H}(\mu)$ contains all parameters used in the definition of $(\mathcal{L}, \models_{\mathcal{L}})$.
- If $\tau \in \mathrm{H}(o(\mathcal{L}))$ is a language, then $\mathcal{L}(\tau) \in \mathrm{H}(\mu)$.
- If $\tau \in \mathrm{H}(o(\mathcal{L}))$ is a language and π is a non-trivial permutation of $\mathcal{L}(\tau)$, then there exists a τ-structure $M_{\tau,\pi} \in \mathrm{H}(\mu)$ and $\phi_{\tau,\pi} \in \mathcal{L}(\tau)$ with the property that

$$M_{\tau,\pi} \models_{\mathcal{L}} \phi_{\tau,\pi} \iff M_{\tau,\pi} \not\models_{\mathcal{L}} \pi(\phi_{\tau,\pi}).$$

In the following, fix a sufficiently large natural number n and a weakly $C^{(n)}$-shrewd cardinal κ greater than $|\mathrm{H}(\mu)|$.

Pick a $<\kappa$-satisfiable theory T of cardinality at most κ. Without loss of generality, we may assume that there exists a language $\tau \subseteq \mathrm{H}(\kappa)$ of cardinality at most κ with $T \subseteq \mathcal{L}(\tau)$. Pick $\kappa < \theta \in C^{(n)}$ with $\mathcal{L}(\tau) \in \mathrm{H}(\theta)$.

By our assumption, we can now find $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and the property that $\mathrm{ran}(j)$ contains the language τ and the theory T.

This setup ensures that $\bar{\kappa}$ is a regular cardinal greater than μ and $\mathrm{H}(\mu)$ is a subset of X. By elementarity, the fact that $\tau \subseteq \mathrm{H}(\kappa)$ implies that $\tau \cap \mathrm{H}(\bar{\kappa}) \in X$ and $j(\tau \cap \mathrm{H}(\bar{\kappa})) = \tau$.

Elementarity yields a function $b \in X$ with domain $\bar{\kappa}$ and the property that $j(b)$ is a surjection from κ onto T.

Claim

$b = j(b) \restriction \bar{\kappa}$.

Since $b[\bar{\kappa}] \in \mathcal{P}_\kappa(T)$, our assumptions imply that the set $b[\bar{\kappa}] \in X \subseteq \mathrm{H}(\bar{\theta})$ is a satisfiable $\mathcal{L}$-theory. By our choice of n, this implies that $b[\bar{\kappa}]$ is satisfiable in both $\mathrm{H}(\bar{\theta})$ and X. Finally, elementarity and our choice of n then imply that $T = j(b[\bar{\kappa}])$ is satisfiable in both $\mathrm{H}(\theta)$ and V.

Thank you for listening!