On the definability of almost disjoint families at uncountable cardinals

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Introduction

The work presented in this talk deals with the definability of the following type of combinatorial object:

Definition

Let κ be an infinite regular cardinal and let A be a set of unbounded subsets of κ.

- The set A is an *almost disjoint family in* $\mathcal{P}(\kappa)$ if $x \cap y$ is bounded in κ for all distinct $x, y \in A$.
- If A is an almost disjoint family of cardinality at least κ and A is not a proper subset of an almost disjoint family in $\mathcal{P}(\kappa)$, then A is a *maximal almost disjoint family*.

Maximal almost disjoint families are typical examples of *pathological sets*, i.e. sets whose construction requires non-trivial applications of the Axiom of Choice.

Our results are motivated by the following classical result of Mathias that shows that, in the case $\kappa = \omega$, such families cannot be defined by simple formulas.

Theorem (Mathias)

There are no analytic maximal almost disjoint families in $\mathcal{P}(\omega)$.

Our aim is to prove analogs of this result for uncountable regular cardinals.

We start by fixing a suitable definability class.

Definition

- A formula in the language $\mathcal{L}_\in$ of set theory is a Δ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under negation, disjunction and bounded quantification.
- An $\mathcal{L}_\in$-formula is a Σ_1-*formula* if it is of the form $\exists x\ \varphi(x)$ for some Δ_0-formula φ.

In the following, we consider the question whether maximal almost disjoint families at uncountable regular cardinals κ can be defined by Σ_1-formulas with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

The following observation shows that $\mathbf{ZFC}$ alone does not provide the desired answer to this question:

Proposition

Assume that $\mathrm{V} = \mathrm{L}$. *If* κ *is an infinite regular cardinal, then there exists a maximal almost disjoint family in* $\mathcal{P}(\kappa)$ *that is definable by a* Σ_1*-formula with parameter* κ.

Since the non-definability of pathological sets is a desirable feature of canonical extensions of $\mathbf{ZFC}$, we now aim to derive negative answers to the above question from such stronger theories.

The following result shows that a strong analog of Mathias' result holds at cardinals with sufficiently strong large cardinal properties.

Theorem

Let κ be a measurable limit of measurable cardinals and let A be a subset of $\mathcal{P}(\kappa)$ that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$.

If A has cardinality greater than κ, then there exist distinct $x, y \in A$ with the property that $x \cap y$ is unbounded in κ.

Sketch of the proof

Let κ be a measurable cardinal and let $\langle \kappa_\alpha \mid \alpha < \kappa \rangle$ be a strictly increasing sequence of measurable cardinals below κ.

Assume, towards a contradiction, that there exists an almost disjoint family A in $\mathcal{P}(\kappa)$ of cardinality greater than κ that is definable by a Σ_1-formula $\varphi(v_0, v_1)$ and the parameter κ.

Given $\alpha < \kappa$, fix a normal ultrafilter U_α on κ_α. Set $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$.

Then there exists $x_* \in A$ with the property that for all $\alpha < \kappa$, there exists an iteration

$$i : \langle \mathrm{V}, U_\alpha \rangle \longrightarrow \langle M, F \rangle$$

of length less than κ with $i(x_*) \neq x_*$.

We can now inductively construct

- a strictly increasing sequence $\langle \delta_n \mid n < \omega \rangle$ of cardinals less than κ, and
- a system $\langle I_s \mid s \in {}^{<\omega}2 \rangle$ of linear iterations of $\langle \mathrm{V}, \mathcal{U} \rangle$ of length less than κ with well-founded limit

with the property that the following statements hold:

- $i^{I_s}_{0,\infty}(\kappa) = \kappa$.
- If $s \subsetneq t$, then I_t properly extends I_s and

$$i^{I_s}_{0,\infty}(x_*) \restriction \delta_{\mathrm{lh}(s)} \; = \; i^{I_t}_{0,\infty}(x_*) \restriction \delta_{\mathrm{lh}(s)}.$$

- $i^{I_{s^\frown \langle 0 \rangle}}_{0,\infty}(x_*) \restriction \delta_{\mathrm{lh}(s)+1} \; \neq \; i^{I_{s^\frown \langle 1 \rangle}}_{0,\infty}(x_*) \restriction \delta_{\mathrm{lh}(s)+1}$.

Let G be $\mathrm{Add}(\omega,1)$-generic over V and set $x_G = \bigcup G : \omega \longrightarrow 2$.

In $\mathrm{V}[G]$, there exists a unique linear iteration I_G of $\langle \mathrm{V}, \mathcal{U} \rangle$ with

- I_G extends $I_{x_G \restriction n}$ for all $n < \omega$ and $\mathrm{lh}(I_G) = \sup_{n<\omega} \mathrm{lh}(I_{x_G \restriction n})$.
- I_G has a well-founded limit and $i^{I_G}_{0,\infty}(\kappa) = \kappa$.

Let $\dot{x}$ be an $\mathrm{Add}(\omega,1)$-name for an unbounded subset of κ with

$$\dot{x}^G \ = \ i^{I_G}_{0,\infty}(x_*)$$

holds whenever G is $\mathrm{Add}(\omega,1)$-generic over V. Then

$$\mathbb{1}_{\mathrm{Add}(\omega,1)} \ \Vdash^{\mathrm{V}} \ \varphi(\dot{x}, \check{\kappa})$$

and mutual genericity implies that

$$\mathbb{1}_{\mathrm{Add}(\omega,1)\times\mathrm{Add}(\omega,1)} \ \Vdash^{\mathrm{V}} \ \dot{x}_l \neq \dot{x}_r.$$

Let $j : \mathrm{V} \longrightarrow M$ be an embedding given by a normal ultrafilter on κ.

Let $\dot{\gamma}$ be an $\mathrm{Add}(\omega, 1)$-name for an ordinal less than $j(\kappa)$ such that

$$\dot{\gamma}^G \;=\; \min(j_G(\dot{x}^G) \setminus \kappa)$$

holds, whenever G is $\mathrm{Add}(\omega, 1)$-generic over V and $j_G : \mathrm{V}[G] \longrightarrow M[G]$ is the corresponding lifting of j.

Claim

$\mathbb{1}_{\mathrm{Add}(\omega,1)\times\mathrm{Add}(\omega,1)} \;\Vdash^{\mathrm{V}}\; \dot{\gamma}_l \neq \dot{\gamma}_r$.

This claim yields a contradiction, because, if $\vec{G}$ is $\mathrm{Add}(\omega, (2^\kappa)^+)$-generic over V, then

$$\iota : (2^\kappa)^+ \longrightarrow j(\kappa);\; \varepsilon \longmapsto \dot{\gamma}^{\vec{G}(\varepsilon)}$$

is an injection in $\mathrm{V}[\vec{G}]$.

Proof of the Claim.

Assume that $\langle p, q\rangle \Vdash^{\mathrm{V}}_{\mathrm{Add}(\omega,1)\times\mathrm{Add}(\omega,1)} \dot{\gamma}_l = \dot{\gamma}_r$.

Then $\langle p, q\rangle$ forces the intersection of $\dot{x}_l$ and $\dot{x}_r$ to be unbounded in κ.

Let X be a countable elementary submodel of some $\mathrm{H}(\theta)$ containing all relevant objects and let $\pi : X \longrightarrow N$ be the corresponding collapse.

By using the pointwise image of a normal ultrafilter on κ, we can find a transitive set N' and an elementary embedding $e : N \longrightarrow N'$ with $\mathrm{crit}(e) = \pi(\kappa)$ and $(e \circ \pi)(\kappa) = \kappa$. Set $\dot{y} = (e \circ \pi)(\dot{x})$.

Let $H_0 \times H_1$ be $(\mathrm{Add}(\omega, 1) \times \mathrm{Add}(\omega, 1))$-generic over N' containing $\langle p, q\rangle$.

Then $\dot{y}^{H_0} \neq \dot{y}^{H_1}$, $\dot{y}^{H_0} \cap \dot{y}^{H_1}$ is unbounded in κ and Σ_1-upwards absoluteness implies that both $\varphi(\dot{y}^{H_0}, \kappa)$ and $\varphi(\dot{y}^{H_1}, \kappa)$ hold.

This shows that $\dot{y}^{H_0}$ and $\dot{y}^{H_1}$ are distinct elements of A with unbounded intersection, a contradiction. $\square$

Smaller cardinals

Using results of Woodin on the Π_2-maximality of $\mathbb{P}_{max}$-extensions of $\mathrm{L}(\mathbb{R})$ and unpublished work of Chan–Jackson–Trang on the non-existence of maximal almost disjoint families in $\mathrm{L}(\mathbb{R})$, it is possible to prove analogs of the above results for ω_1.

Theorem

Let A be an almost disjoint family in $\mathcal{P}(\omega_1)$ of cardinality greater than $\aleph_1$.

- *If there exists a measurable cardinal above infinitely many Woodin cardinals, then A is not definable by a Σ_1-formula with parameters in $\mathrm{H}(\aleph_1) \cup \{\omega_1\}$.*
- *If Woodin's Axiom $(*)$ holds, then A is not an element of $\mathrm{HOD}(\mathbb{R})^{\mathrm{L}(\mathcal{P}(\omega_1))}$.*

The following observation shows that the above implications cannot be generalized from ω_1 to ω_2.

Proposition

- *If the* $\mathbf{BPFA}$ *holds, then there exists an almost disjoint family of cardinality* $2^{\aleph_2}$ *in* $\mathcal{P}(\omega_2)$ *that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\aleph_2) \cup \{\omega_2\}$*.*
- *If there is a supercompact cardinal, then, in a generic extension of the ground model, there exists exists an almost disjoint family of cardinality* $2^{\aleph_2}$ *in* $\mathcal{P}(\omega_2)$ *that is definable by a* Σ_1*-formula and the parameter* ω_2*.*

Thank you for listening!