Huge reflection

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Logic Colloquium 2021, Poznań, 20. July 2021

Joint work with Joan Bagaria (Barcelona).

Introduction

Results of Bagaria and his collaborators establish deep connections between strong large cardinal axioms and set-theoretic extensions of the *Downward Löwenheim–Skolem Theorem*.

These results rely on the following *principle of structural reflection*:

Definition (Bagaria)

Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{SR}_{\mathcal{C}}(\kappa)$ denote the statement that for every structure $B \in \mathcal{C}$, there exists a structure $A \in \mathcal{C}$ of rank[1] less than κ and an elementary embedding of A into B.

[1] The *rank* of a structure is defined to be the rank of its domain.

For simply defined classes, the above principle is a consequence of the *Downward Löwenheim–Skolem Theorem*.

Proposition (Bagaria et al.)

$\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds for every uncountable cardinal* κ *and every class* $\mathcal{C}$ *of structures of the same type that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa)$.

In contrast, the work of Bagaria and his collaborators shows that the validity of the principle SR for classes of structures defined by more complex formulas closely corresponds to the existence of large cardinals.

Theorem (Bagaria et al.)

The following statements are equivalent for every infinite cardinal κ:

- κ *is the least supercompact cardinal.*
- κ *is the least cardinal such that* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *that is definable by a* Π_1*-formula without parameters.*
- κ *is the least cardinal such that* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds for every class* $\mathcal{C}$ *that is definable by a* Σ_2*-formula with parameters in* $\mathrm{H}(\kappa)$*.*

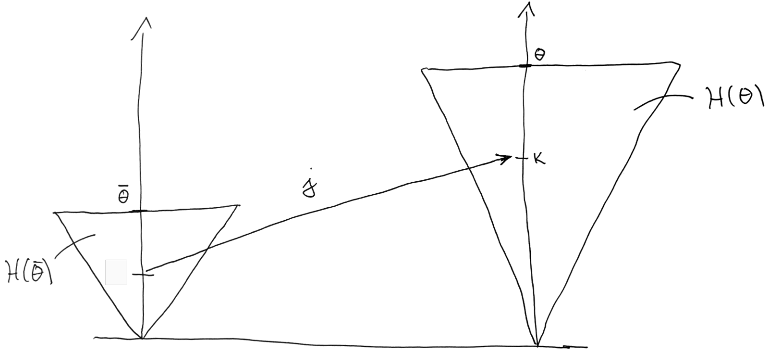
Bagaria and his collaborators extended the above result to Π_{n+1}- and Σ_{n+2}-definable classes of structures and so-called $C^{(n)}$*-extendible cardinals*.

A key component in the proof of the above equivalence is Magidor's classical characterization of supercompactness:

Theorem (Magidor)

The following statements are equivalent for every cardinal κ:

- *κ is supercompact.*
- *For every cardinal $\theta > \kappa$, there exists*
  - *a cardinals $\bar{\theta} < \kappa$, and*
  - *a non-trivial elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$ with $j(\mathrm{crit}(j)) = \kappa$.*



$\bar{\theta}$
$H(\bar{\theta})$
j
θ
$H(\theta)$
K

The results discussed above lead to a canonical characterization of *Vopěnka's Principle* through structural reflection.

Definition

Vopěnka's Principle is the scheme of sentences stating that every proper class of structures of the same type contains a structure that is elementary embeddable into another structure in the given class.

Theorem (Bagaria et al.)

The following schemes of sentences imply each other:

- *Vopěnka's Principle.*
- *For every $n > 0$, there exists a cardinal κ such that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every Π_n-definable class $\mathcal{C}$.*

In particular, no large cardinal property stronger than *Vopěnka's Principle* can be characterized through the principle SR.

Exact structural reflection

In order to characterize large cardinal notions stronger than *Vopěnka's Principle*, we study *principles of exact structural reflection*.

Definition

Given cardinals $\kappa < \lambda$ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ denote the assertion that for every $B \in \mathcal{C}$ of rank λ, there exists $A \in \mathcal{C}$ of rank κ and an elementary embedding from A into B.

The above principle turns out to be very strong, even for classes of low complexities.

Proposition

There is a class $\mathcal{C}$ of structures of the same type that is definable by a Σ_0-formula without parameters with the property that, if $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for uncountable cardinals $\kappa < \lambda$, then $x^{\#}$ exists for all $x \in \mathbb{R}$.

Proposition

If δ is a Ramsey cardinal, then there are uncountable cardinals $\kappa < \lambda < \delta$ with the property that, in V_{δ}, the principle $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ that is definable by a Σ_1-formula with parameters in V_{κ}.

For complexity classes beyond Σ_1, principles of exact structural reflection turn out to imply the existence of very large cardinals.

Theorem

If $\kappa < \lambda$ are uncountable cardinals such that $\mathrm{ESR}\mathcal{C}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures that is definable by a Π_1-formula without parameters, then there exists an almost huge cardinal below κ.

Theorem

If κ is a huge cardinal, then there are inaccessible cardinals $\kappa < \lambda < \theta$ such that, in V_θ, the principle $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures that is definable by a Π_1-formula with parameters in V_κ.

The next result provides an upper bound for the consistency strength of principles of exact reflection.

Theorem

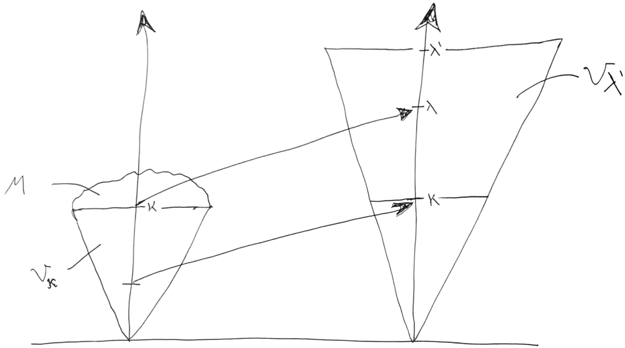
If κ is an almost 2-huge cardinal, then there are inaccessible cardinals $\kappa < \lambda < \theta$ such that, in V_θ, the principle $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures that is definable by a formula with parameters in V_κ.

The proofs of the above results make use of the following large cardinal property:

Definition

A cardinal κ is *weakly parametrically exact for a cardinal* $\lambda > \kappa$ if for every $A \in V_{\lambda+1}$, there exists

- a transitive set M with $\mathrm{V}_\kappa \cup \{\kappa\} \subseteq M \prec_{\Pi_1(\mathrm{V}_{\kappa+1})} \mathrm{V}$,
- a cardinal $\lambda' > \beth_\lambda$, and
- an elementary embedding $j : M \longrightarrow \mathrm{V}_{\lambda'}$ with $j(\mathrm{crit}(j)) = \kappa$, $j(\kappa) = \lambda$ and $A \in \mathrm{ran}(j)$.



M
κ
V_κ
λ'
λ
κ
$V_{\lambda'}$

Theorem

The following statements are equivalent for every infinite cardinal κ:

- *κ is the least cardinal that is weakly parametrically exact for some $\lambda > \kappa$.*
- *κ is the least cardinal such that there exists a cardinal $\lambda > \kappa$ with the property that $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures that is definable by a Π_1-formula without parameters.*
- *κ is the least cardinal such that there exists a cardinal $\lambda > \kappa$ with the property that $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures that is definable by a Π_1-formula with parameters in V_κ.*

The above notions and results have direct generalizations to higher complexities that allow analogous characterizations of the validity of the principle ESR for Π_n- and Σ_{n+1}-definable classes of structures.

Further work

Motivated by *Chang's Conjecture*, we also formulated *sequential* versions of exact structural reflection.

Different versions of these modified principles imply the existence of n-huge cardinals and even $I3$-embeddings.

We do not know if the strongest versions of these assumptions are inconsistent with $\mathbf{ZFC}$, because they seem to avoid the *Kunen Inconsistency*.

But we can show that the existence of a *Berkeley cardinal* implies the validity of all of these principles in $\mathbf{ZF}$.

Thank you for listening!