Weak compactness cardinals for abstract logics

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In this talk, I present work exploring connections between large cardinal axioms and compactness properties of strong logics.

I will focus on large cardinal assumptions whose validity is equivalent to the statement that all abstract logics possess certain compactness properties.

Such equivalences provide strong justifications for the acceptance of the given large cardinal axiom.

I will now discuss the main example of such a connection.

Vopěnka's Principle

Definition (Vopěnka)

Vopěnka's Principle is the scheme of axioms stating that for every proper class of structures of the same signature, there is an elementary embedding between two distinct members of the class.

This reflection principle was recently used to answer long-standing open questions in other areas of mathematics, like category theory, commutative algebra and homotopy theory.

Bagaria showed how the validity of Vopěnka's Principle can be characterized through large cardinal assumptions.

Definition (Bagaria)

Let n be a natural number.

- $C^{(n)}$ is the class of all ordinals α satisfying $\mathrm{V}_\alpha \prec_{\Sigma_n} \mathrm{V}$.
- A cardinal κ is $C^{(n)}$-*extendible* if for every ordinal $\lambda > \kappa$, there is an ordinal $\mu > \lambda$ and an elementary embedding $j : \mathrm{V}_\lambda \longrightarrow \mathrm{V}_\mu$ with $\mathrm{crit}(j) = \kappa$ and $\lambda < j(\kappa) \in C^{(n)}$.

Theorem (Bagaria)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *For every natural number* n*, there is a* $C^{(n)}$*-extendible cardinal.*

Vopěnka's Principle is closely related to compactness properties of strong logics.

An *abstract logic* is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ consisting of

- a class function $\mathcal{L}$ that maps signatures σ to sets $\mathcal{L}(\sigma)$ of $\mathcal{L}$-sentences, and
- a satisfaction relation $\models_{\mathcal{L}}$ that determines which $\mathcal{L}$-sentences $\phi \in \mathcal{L}(\sigma)$ hold in σ-structures

that satisfies certain canonical rules about invariance under isomorphic copies, extensions of signatures, and boundedness of the sizes of signatures generating sentences.

Given an abstract logic $\mathcal{L}$ and a cardinal κ, an $\mathcal{L}$-theory T is *$<\kappa$-satisfiable* if every subtheory of cardinality less than κ is satisfiable.

A cardinal κ is a *strong compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory is satisfiable.

Theorem (Makowsky)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *Every abstract logic has a strong compactness cardinal.*

Weak compactness cardinals

A cardinal κ is a *weak compactness cardinal* of an abstract logic $\mathcal{L}$ if every $<\kappa$-satisfiable $\mathcal{L}$-theory of cardinality κ is satisfiable.

Recent work of Boney, Dimopoulos, Gitman and Magidor connects this weaker property to the large cardinal notion of *subtleness*, introduced by Jensen and Kunen in their studies of strong diamond principles.

Definition (Jensen–Kunen)

A cardinal δ is *subtle* if for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Definition

We let "Ord *is subtle*" denote the scheme of axioms stating that for every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma \in \mathrm{Ord}\rangle$ and every closed unbounded class C of ordinals, there exist $\beta < \gamma$ in C with the property that $A_\beta = A_\gamma \cap \beta$.

Theorem (Boney–Dimopoulos–Gitman–Magidor)

The following schemes are equivalent over $\mathbf{ZFC}$ *together with the existence of a definable global well-ordering:*

- Ord *is subtle.*
- *Every abstract logic has a stationary class of weak compactness cardinals.*

This result raises two questions:

- Is it necessary to assume the existence of a global well-ordering?
- Can we characterize the existence of weak compactness cardinals for all abstract logics through large cardinal properties of Ord?

Proposition

The following statements are equivalent for every infinite cardinal δ:

- *The cardinal δ is subtle.*
- *For all closed unbounded subsets C of δ and all sequences $\langle \mathcal{E}_\gamma \mid \gamma < \delta \rangle$ with $\emptyset \neq \mathcal{E}_\gamma \subseteq \mathcal{P}(\gamma)$ for all $\gamma < \delta$, there are $\beta < \gamma$ in C and $E \in \mathcal{E}_\gamma$ with $E \cap \beta \in \mathcal{E}_\beta$.*

Definition (Bagaria–L.)

We let "Ord *is essentially subtle*" denote the scheme of axioms stating that for every closed unbounded class C of ordinals and every class sequence $\langle \mathcal{E}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ with $\emptyset \neq \mathcal{E}_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$, there exist $\alpha < \beta$ in C and $E \in \mathcal{E}_\beta$ with $E \cap \alpha \in \mathcal{E}_\alpha$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *Every abstract logic has a stationary class of weak compactness cardinals.*

Theorem (Bagaria–L.)

The following statements are equivalent for every cardinal $\delta \in C^{(1)}$:

- *For every sequence $\langle A_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and all $\xi < \delta$, there are cardinals $\xi < \mu < \nu < \delta$ with $A_\mu = A_\nu \cap \mu$.*
- *The cardinal δ is either subtle or a limit of subtle cardinals.*
- *For every sequence $\langle \mathcal{E}_\gamma \mid \gamma < \delta \rangle$ such that $\emptyset \neq \mathcal{E}_\gamma \subseteq \mathcal{P}(\gamma)$ holds for all $\gamma < \delta$ and all $\xi < \delta$, there exist cardinals $\xi < \mu < \nu < \delta$ and $E \in \mathcal{E}_\nu$ with $E \cap \mu \in \mathcal{E}_\mu$.*

Definition

We let "Ord *is essentially closure subtle*" denote the scheme of axioms stating that every class sequence $\langle \mathcal{E}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ such that $\emptyset \neq \mathcal{E}_\alpha \subseteq \mathcal{P}(\alpha)$ holds for all $\alpha \in \mathrm{Ord}$ and all $\xi \in \mathrm{Ord}$, there exist cardinals $\xi < \mu < \nu$ and $E \in \mathcal{E}_\nu$ with $E \cap \mu \in \mathcal{E}_\mu$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially closure subtle.*
- *Every abstract logic has a weak compactness cardinal.*

We now explore the differences between the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$$

and the assumption

$$\text{“}\mathrm{Ord}\ \textit{is essentially closure subtle}\text{”}.$$

Proposition

If Φ is a sentence in the language of set theory with the property that $\mathbf{ZFC} + \Phi$ is consistent, then

$$\mathbf{ZFC} + \Phi \nvdash \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}.$$

Theorem

The following statements are equivalent:

- *There exists a sentence Φ in the language of set theory such that the theory $\mathbf{ZFC} + \Phi$ is consistent and*

$$\mathbf{ZFC} + \Phi \vdash \text{“}\mathrm{Ord} \textit{ is essentially closure subtle”}.$$

- $\mathbf{ZFC} + \text{“}\mathrm{Ord}$ *is essentially closure subtle”* $\nvdash$ “Ord *is essentially subtle”.*
- *The theory*

$$\mathbf{ZFC} + \textit{“There is a proper class of subtle cardinals”}$$

is consistent.

The techniques developed in the proofs of the above results also allow us to show that the existence of weak compactness cardinals for all abstract logics does not imply the existence of strongly inaccessible cardinals in V.

Theorem

The following schemes are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a proper class of subtle cardinals.*
- Ord *is essentially closure subtle and there are no inaccessible cardinals.*

Weakly $C^{(n)}$-shrewd cardinals

We now relate the existence of weak compactness cardinals to large cardinal properties.

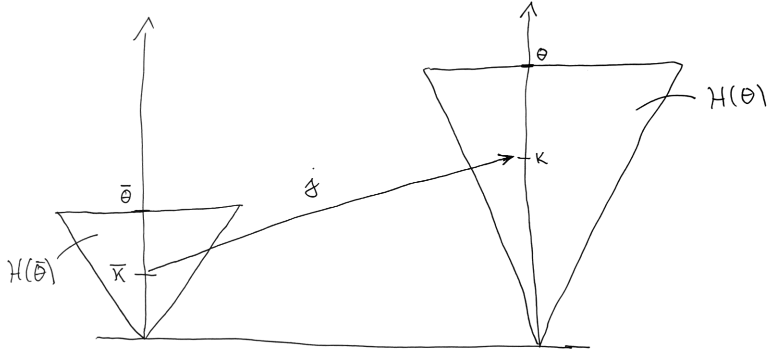
The starting point of these results is the following classical result:

Theorem (Magidor)

The following statements are equivalent for every cardinal κ:

- κ *is supercompact.*
- *For every cardinal $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$, and*
  - *an elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$*

  such that $\mathrm{crit}(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.



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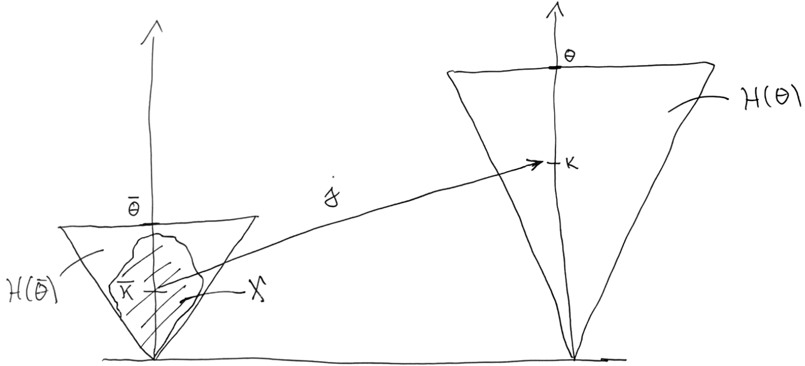

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X

Theorem

The following statements are equivalent for every cardinal κ:

- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
    - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$,*
    - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
    - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

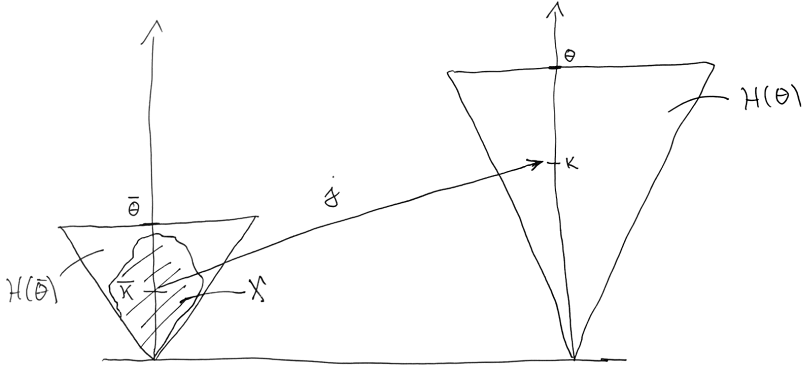
  such that $\bar{\kappa} + 1 \subseteq X$, $j \upharpoonright \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.
- *κ is a shrewd cardinal.*
- *κ is a strongly unfoldable cardinal.*

Definition (Rathjen)

A cardinal κ is *shrewd* if for every $\mathcal{L}_{\in}$-formula $\Phi(v_0, v_1)$, every ordinal $\gamma > \kappa$ and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

Definition (Villaveces)

An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.



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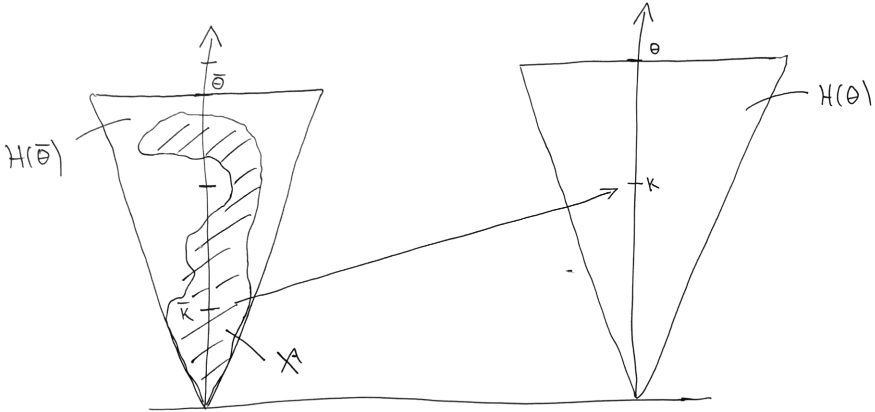

$\bar{\theta}$
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Definition

An infinite cardinal κ is *weakly shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every cardinal $\theta > \kappa$ and every subset A of κ with the property that $\Phi(A, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ with the property that $\bar{\kappa} < \kappa$ and $\Phi(A \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

Lemma

The following statements are equivalent for every infinite cardinal κ:

- *κ is a weakly shrewd cardinal.*
- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta}$,*
  - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
  - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

  with $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa > \bar{\kappa}$ and $z \in \mathrm{ran}(j)$.

Definition

Given a natural number $n > 0$, a cardinal κ is *weakly $C^{(n)}$-shrewd* if for every cardinal $\kappa < \theta \in C^{(n)}$ and every $z \in \mathrm{H}(\theta)$, there exists

- a cardinal $\bar{\theta} \in C^{(n)}$,
- a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$,
- an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and
- an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$

such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially closure subtle.*
- *For every natural number* $n > 0$*, there is a proper class of weakly* $C^{(n)}$*-shrewd cardinals.*
- *Every logic has a weak compactness cardinal.*

Theorem

The following schemes of sentences are equivalent over $\mathbf{ZFC}$*:*

- Ord *is essentially subtle.*
- *For every natural number* $n > 0$*, there is a proper class of weakly* $C^{(n)}$*-shrewd cardinals that are elements of* $C^{(n+1)}$*.*
- *For every natural number* $n > 0$*, there is a weakly* $C^{(n)}$*-shrewd cardinal that is an element of* $C^{(n+1)}$*.*
- *Every logic has a stationary class of weak compactness cardinals.*

Proposition

Given a natural number $n > 0$ and a cardinal κ, the cardinal κ is not an element of $C^{(n+1)}$ if and only if there is a cardinal $\delta > \kappa$ such that the set $\{\delta\}$ is definable by a Σ_{n+1}-formula with parameters in $\mathrm{H}(\kappa)$.

Theorem

Let $n > 0$ be natural numbers, let κ be a weakly $C^{(n)}$-shrewd cardinal that is not an element of $C^{(n+1)}$ and let $\delta > \kappa$ be a cardinal such that $\{\delta\}$ is definable by a Σ_{n+1}-formula with parameters in $\mathrm{H}(\kappa)$.

- *If $m > 0$ is a natural number and $\alpha < \kappa$, then the interval (α, δ) contains a weakly $C^{(m)}$-shrewd cardinal.*
- *There is an ordinal γ in the interval $(\kappa, \delta]$ that is a subtle cardinal in L.*

Thank you for listening!