Partition properties and definability

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Jónsson and Rowbottom cardinals

Definition

- Given a non-empty set A, we let $[A]^{<\omega}$ denote the set of finite subsets of A.
- A pair $\langle A, f \rangle$ is an *algebra* if A is a non-empty set and $f : [A]^{<\omega} \longrightarrow A$ is a function.
- Given an algebra $\langle A, f \rangle$, a subset B of A is a *proper subalgebra* if B is a proper subset of A and B is closed under f, i.e., $f[[B]^{<\omega}] \subseteq B$.
- An algebra is a *Jónsson algebra* if it has no proper subalgebra of the same cardinality.
- An infinite cardinal κ is a *Jónsson cardinal* if there are no Jónsson algebras of cardinality κ.

Proposition

$\aleph_0$ is not a Jónsson cardinal.

Proof.

Define

$$f : [\mathbb{N}]^{<\omega} \longrightarrow \mathbb{N};\ a \longmapsto |a|.$$

If B is an infinite subset of $\mathbb{N}$, then $f[[A]^{<\omega}] = \mathbb{N}$. $\square$

Lemma

If an infinite cardinal κ is not Jónsson, then κ^+ is not Jónsson.

Question

Can $\aleph_\omega$ be Jónsson?

Theorem (Erdős-Hajnal, Keisler-Rowbottom)

The following statements are equivalent for every infinite cardinal κ:

- *κ is a Jónsson cardinal.*
- *Any structure for a countable first-order language with domain κ has a proper elementary substructure with domain of cardinality κ.*
- *$\kappa \longrightarrow [\kappa]^{<\omega}_{\kappa}$ holds, i.e., for every function $c : [\kappa]^{<\omega} \longrightarrow \kappa$, there exists $H \in [\kappa]^{\kappa}$ with $c[[H]^{<\omega}] \neq \kappa$.*

Definition

Given uncountable cardinals $\nu < \kappa$, the cardinal κ is *ν-Rowbottom* if $\kappa \longrightarrow [\kappa]^{<\omega}_{\lambda,<\nu}$ holds for every $\lambda < \kappa$, i.e., for every function $c : [\kappa]^{<\omega} \longrightarrow \lambda$, there exists $H \in [\kappa]^{\kappa}$ with $|c[[H]^{<\omega}]| < \nu$.

Rowbottom cardinals are $\aleph_1$-Rowbottom cardinals.

Lemma

- *A cardinal κ is ν-Rowbottom if and only if $\langle \kappa, \lambda \rangle \twoheadrightarrow \langle \kappa, {<}\nu \rangle$ holds for all $\lambda < \kappa$, i.e., given a countable first-order language $\mathcal{L}$ containing a unary predicate symbol $\dot{R}$, every $\mathcal{L}$-structure A with domain κ and $|\dot{R}^A| = \lambda$ has an elementary substructure B of cardinality κ with $|\dot{R}^B| < \nu$.*
- *If κ is ν-Rowbottom for some $\nu < \kappa$, then κ is Jónsson.*
- *If κ is the least Jónsson cardinal, then κ is ν-Rowbottom for some $\nu < \kappa$.*

Corollary

For all $n < \omega$, the cardinal $\aleph_n$ is not Rowbottom.

Question

Can $\aleph_\omega$ be Rowbottom?

The Σ_1-undefinability property

Definition

Given uncountable cardinals $\mu < \kappa$, we say that the cardinal κ has the $\Sigma_1(\mu)$-*undefinability property* if no ordinal α in the interval $[\mu, \kappa)$ has the property that the set $\{\alpha\}$ is definable by a Σ_1-formula with parameters in the set $\mathrm{H}(\mu) \cup \{\kappa\}$.

Proposition

A measurable cardinal κ *has the* $\Sigma_1(\mu)$*-undefinability property for every uncountable cardinal* $\mu < \kappa$*.*

Proof.

Assume that there is a Σ_1-formula $\varphi(v_0, v_1, v_1)$, an uncountable cardinal $\mu < \kappa$, $z \in \mathrm{H}(\mu)$ and an ordinal $\mu \leq \alpha < \kappa$ with the property that α is the unique ordinal ξ such that $\varphi(\xi, \kappa, z)$ holds.

Let X be an elementary substructure of $\mathrm{H}(\kappa^+)$ of cardinality less than μ with $\mathrm{tc}(\{z\}) \cup \{\kappa, \alpha\} \subseteq X$ and let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi(z) = z$ and $\pi(\alpha) < \alpha$.

Let U be a normal ultrafilter on κ and set $F = \pi[U \cap X]$. Then F is a weakly amenable M-ultrafilter and $\langle M, \in, F\rangle$ is ω_1-iterable.

Hence, we can find N transitive and an elementary embedding $j : M \longrightarrow N$ with $j(\pi(\kappa)) = \kappa$ and $j \restriction \pi(\kappa) = \mathrm{id}_{\pi(\kappa)}$.

Then $\varphi(\pi(\alpha), \kappa, z)$ holds in N and V, a contradiction. $\square$

Definition (Welch)

An uncountable regular cardinal κ is *stably measurable* if there exists a transitive set M with $\mathrm{H}(\kappa) \cup \{\kappa\} \subseteq M \prec_{\Sigma_1} \mathrm{H}(\kappa^+)$, a transitive set N with $M \cup {}^{<\kappa}N \subseteq N$ and a weakly amenable N-ultrafilter F on κ with the property that $\langle N, \in, F \rangle$ is ω_1-iterable.

Proposition

A stably measurable cardinal κ has the $\Sigma_1(\mu)$-undefinability property for every uncountable cardinal $\mu < \kappa$.

Theorem

If $\mathrm{V} = \mathrm{K}^{DJ}$, *then the following statements are equivalent for every cardinal* κ *bigger than* $\aleph_1$*:*

- *The cardinal* κ *is stably measurable.*
- *The set* $\{\mathrm{H}(\kappa)\}$ *is not definable by a* Σ_1*-formula with parameters in the set* $\mathrm{H}(\kappa) \cup \{\kappa\}$*.*
- *The cardinal* κ *has the* $\Sigma_1(\mu)$*-undefinability property for every uncountable cardinal* $\mu < \kappa$*.*

Theorem

Assume that there is no inner model with a measurable cardinal. If κ *is an inaccessible cardinal with the* $\Sigma_1(\mu)$*-undefinability property for all uncountable cardinals* $\mu < \kappa$*, then* κ *is stably measurable in* K^{DJ}*.*

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There is an uncountable regular cardinal* μ *with the property that* μ^+ *has the* $\Sigma_1(\mu)$*-undefinability property.*
- *There is a Mahlo cardinal.*

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a singular cardinal* μ *with the property that* μ^+ *has the* $\Sigma_1(\mu)$*-undefinability property.*
- *There are infinitely many measurable cardinals.*

Σ_1-undefinability at Rowbottom cardinals

Lemma

Let κ be a ν-Rowbottom cardinal with ν regular and let $\mathcal{L}$ be a first-order language of cardinality less than ν containing a unary predicate symbol $\dot{R}$ and let A be an $\mathcal{L}$-structure with domain κ and $|\dot{R}^A| = \nu$. Then A has an elementary substructure B of cardinality κ with $|\dot{R}^B| < \nu$.

Lemma

Let κ be a ν-Rowbottom cardinal with ν regular, let $y \in \mathrm{H}(\kappa^+)$ and let $z \in \mathrm{H}(\nu)$. Then there exists a transitive set M with $\kappa \in M$ and a non-trivial elementary embedding $j : M \longrightarrow \mathrm{H}(\kappa^+)$ satisfying $\mathrm{crit}(j) < \nu$, $y \in \mathrm{ran}(j)$, $j(\kappa) = \kappa$ and $j(z) = z$.

Corollary

- *If* $\aleph_\omega$ *is* $\aleph_n$*-Rowbottom for some* $0 < n < \omega$*, then* $\aleph_\omega$ *has the* $\Sigma_1(\aleph_n)$*-undefinability property.*
- *If* $\aleph_\omega$ *is Rowbottom, then* $\aleph_\omega$ *has the* $\Sigma_1(\aleph_n)$*-undefinability property for all* $0 < n < \omega$*.*
- *If* $\aleph_\omega$ *is Jónsson, then* $\aleph_\omega$ *has the* $\Sigma_1(\aleph_n)$*-undefinability property for all sufficiently large* $0 < n < \omega$*.*

Proposition

Assume that there is a natural number $m > 1$ such that there are no special $\aleph_m$-Aronszajn trees and for all $m < n < \omega$, there is a special $\aleph_n$-Aronszajn tree. Then the set $\{\aleph_m\}$ is definable by a Σ_1-formula with parameter $\aleph_\omega$ and hence $\aleph_\omega$ is not Rowbottom.

Corollary

In the standard models of strong forcing axioms, the cardinal $\aleph_\omega$ is not Rowbottom.

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *The cardinal* $\aleph_\omega$ *has the* $\Sigma_1(\aleph_n)$*-undefinability property for every natural number* $n > 0$*.*
- *There is a singular cardinal* λ *and an uncountable cardinal* $\mu < \lambda$ *such that no ordinal* α *in the interval* $[\mu, \lambda)$ *has the property the set* $\{\alpha\}$ *is definable by a* Σ_1*-formular with parameters in the set* $\mu \cup \{\lambda\}$*.*
- *There is a measurable cardinal.*

Lemma

If κ *is an infinite cardinal and* $c : \mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}} \longrightarrow \kappa$ *is the* $<_{\mathrm{K}^{DJ}}$*-least cofinal function, than the sets* $\{\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}\}$ *and* $\{c\}$ *are both definable by* Σ_1*-formulas with parameter* κ*.*

Thank you for listening!