Definable clubs

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Introduction

The work presented in this talk contributes to a programme that aims to study strong combinatorial properties of uncountable cardinals through the interaction of these properties with set-theoretic definability.

More specifically, we study definable closed unbounded subsets of uncountable cardinals, focusing on large cardinals and singular cardinals, and the corresponding collections of *stationary* sets that intersect all of these definable closed unbounded sets.

Definition

- A formula in the language $\mathcal{L}_\in$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under negation, disjunction and bounded quantification.
- Given $n < \omega$, an $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \, \neg\varphi(x)$ for some Σ_n-formula φ.

Definition

A class X is *definable* by a formula $\varphi(v_0, \ldots, v_n)$ and parameters $z_0, \ldots, z_{n-1}$ if

$$X \;=\; \{y \mid \varphi(y, z_0, \ldots, z_{n-1})\}.$$

Definition

Let $n < \omega$, let κ be a uncountable cardinal and let S be a subset of κ.

- Given a class A, the set S is $\Sigma_n(A)$-*stationary in* κ if $C \cap S \neq \emptyset$ holds for every closed unbounded subset C of κ with the property that $\{C\}$ is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$.
- The subset S is Σ_n-*stationary in* κ if it is $\Sigma_n(\emptyset)$-stationary in κ.
- Given a class A, the subset S is $\boldsymbol{\Sigma}_n(A)$-*stationary in* κ if it is $\Sigma_n(A \cup \mathrm{H}(\kappa))$-stationary in κ.
- The subset S is $\boldsymbol{\Sigma}_n$-*stationary in* κ if it is $\boldsymbol{\Sigma}_n(\emptyset)$-stationary in κ.

We focus on the following two questions:

- How much can the collection of $\Sigma_n(A)$-stationary subsets of an uncountable cardinal κ differ from the collection of all stationary subsets of κ? What is the situation at cardinals of countable cofinality, where stationarity coincides with coboundedness?

- For which cardinals is it possible to develop a non-trivial structure theory for $\Sigma_n(A)$-stationary subsets?

Proposition

Assume that $\mathrm{V} = \mathrm{L}$ *holds and* κ *is an uncountable cardinal. Then a subset of* κ *is* $\Sigma_1(\kappa^+)$*-stationary in* κ *if and only if it is stationary in* κ*.*

Proposition

Assume that Martin's Maximum holds. Then a subset of ω_1 *is* $\boldsymbol{\Sigma}_1$*-stationary in* ω_1 *if and only if it is stationary in* ω_1*.*

Proof.

Woodin proved that *Martin's Maximum* implies *admissible club guessing*, i.e., every closed unbounded subset of ω_1 contains a closed unbounded subset of the form

$$\{\alpha < \omega_1 \mid \mathrm{L}_\alpha[x] \models \mathrm{KP}\}$$

for some $x \in \mathbb{R}$. $\square$

Proposition

If κ is a cardinal of uncountable cofinality, A is a set of cardinality at most $\mathrm{cof}(\kappa)$ and $0 < n < \omega$, then there is an unbounded, non-stationary subset of κ that is $\Sigma_n(A)$-stationary in κ.

Proposition

If κ is a singular cardinal of countable cofinality, A is a set of cardinality at most κ and $0 < n < \omega$, then there is an unbounded subset of κ that is $\Sigma_n(A)$-stationary in κ and whose complement in κ is unbounded in κ.

Proof.

By the *König's Lemma*, the set $[\kappa]^\omega$ contains an almost disjoint family of cardinality κ^+ that consists of unbounded subsets of κ and hence we can find an element $b \in [\kappa]^\omega$ that is unbounded in κ and has the property that no infinite subset of b is definable by a Σ_n-formula with parameters in $A \cup \{\kappa\}$. $\square$

The Σ_1-undefinability property

Definition

Given uncountable cardinals $\mu < \kappa$, we say that the cardinal κ has the $\Sigma_1(\mu)$-*undefinability property* if no ordinal α in the interval $[\mu, \kappa)$ has the property that the set $\{\alpha\}$ is definable by a Σ_1-formula with parameters in the set $\mathrm{H}(\mu) \cup \{\kappa\}$.

Lemma

Given uncountable cardinals $\mu < \kappa$ and $0 < n < \omega$, if the cardinal κ has the $\Sigma_n(\mu)$-undefinability property, then $\{\mu\}$ is $\Sigma_n(\mathrm{H}(\mu))$-stationary in κ.

Corollary

Let κ be a limit cardinal, let $0 < n < \omega$ and let E be the set of uncountable cardinals $\mu < \kappa$ with the property that κ has the $\Sigma_n(\mu)$-undefinability property for all $\mu \in E$. If E is unbounded in κ, then E is $\boldsymbol{\Sigma}_n$-stationary in κ.

Lemma

Given uncountable cardinals $\mu < \kappa$ and $0 < n < \omega$, if the cardinal κ has the $\Sigma_n(\mu)$-undefinability property, then $\{\mu\}$ is $\Sigma_n(\mathrm{H}(\mu))$-stationary in κ.

Proof.

Let C be a closed unbounded subset of κ with the property that the set $\{C\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\mu) \cup \{\kappa\}$.

Assume, towards a contradiction, that μ is not an element of C. Set $\nu = \min(C \setminus \mu) > \mu$.

Then $C \cap \mu \neq \emptyset$, because otherwise $\nu = \min(C)$ and this would imply that $\{\nu\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\mu) \cup \{\kappa\}$.

Define $\rho = \max(C \cap \mu) < \mu$. Then $\nu = \min(C \setminus (\rho + 1))$ and hence $\{\nu\}$ is definable by a Σ_n-formula with parameters in $\mathrm{H}(\mu) \cup \{\kappa\}$, a contradiction. □

Large cardinals

Proposition

A measurable cardinal κ has the $\Sigma_1(\mu)$-undefinability property for every uncountable cardinal $\mu < \kappa$.

Proof.

Assume that there is a Σ_1-formula $\varphi(v_0, v_1, v_1)$, an uncountable cardinal $\mu < \kappa$, $z \in \mathrm{H}(\mu)$ and an ordinal $\mu \leq \alpha < \kappa$ with the property that α is the unique ordinal ξ such that $\varphi(\xi, \kappa, z)$ holds.

Let X be an elementary substructure of $\mathrm{H}(\kappa^+)$ of cardinality less than μ with $\mathrm{tc}(\{z\}) \cup \{\kappa, \alpha\} \subseteq X$ and let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi(z) = z$ and $\pi(\alpha) < \alpha$.

Let U be a normal ultrafilter on κ and set $F = \pi[U \cap X]$. Then F is a weakly amenable M-ultrafilter and $\langle M, \in, F \rangle$ is ω_1-iterable.

Hence, we can find N transitive and an elementary embedding $j : M \longrightarrow N$ with $j(\pi(\kappa)) = \kappa$ and $j \restriction \pi(\kappa) = \mathrm{id}_{\pi(\kappa)}$.

Then $\varphi(\pi(\alpha), \kappa, z)$ holds in N and V, a contradiction. $\square$

Definition (Welch)

An uncountable regular cardinal κ is *stably measurable* if there exists a transitive set M with $\mathrm{H}(\kappa) \cup \{\kappa\} \subseteq M \prec_{\Sigma_1} \mathrm{H}(\kappa^+)$, a transitive set N with $M \cup {}^{<\kappa}N \subseteq N$ and a weakly amenable N-ultrafilter F on κ with the property that $\langle N, \in, F \rangle$ is ω_1-iterable.

Lemma

A stably measurable cardinal κ has the $\Sigma_1(\mu)$-undefinability property for every uncountable cardinal $\mu < \kappa$.

Theorem

Let κ be a stably measurable cardinal.

- *If E is an unbounded subset of κ that consists of cardinals, then E is $\mathbf{\Sigma}_1$-stationary in κ.*
- *If S is a $\mathbf{\Sigma}_1$-stationary subset of κ and $r : \kappa \longrightarrow \kappa$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then r is constant on a $\mathbf{\Sigma}_1$-stationary subset of S.*

We let K^{DJ} denote the *Dodd-Jensen core model*.

Theorem

If $\mathrm{V} = \mathrm{K}^{DJ}$, then the following statements are equivalent for every limit cardinal κ:

- *The cardinal κ is stably measurable.*
- *Every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary in κ.*

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$:

- *There exists a stably measurable cardinal.*
- *There exists a limit cardinal κ with the property that every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary in κ.*

By replacing iterations of small models with iterations of V, we can extend the above conclusions to, possibly singular, limits of measurable cardinals:

Theorem

Let κ be a cardinal that is a limit of measurable cardinals.

- *Every unbounded subset S of κ consisting of cardinals is $\mathbf{\Sigma}_1(Ord)$-stationary.*
- *If S is a $\mathbf{\Sigma}_1(Ord)$-stationary subset of κ and $r : \kappa \longrightarrow \kappa$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup Ord$, then r is constant on a $\mathbf{\Sigma}_1(Ord)$-stationary subset of S.*

Undefinability and the Dodd-Jensen Core model

Lemma

If κ is an infinite cardinal and $c : \mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}} \longrightarrow \kappa$ is the $<_{\mathrm{K}^{DJ}}$-least cofinal function, than the sets $\{\mathrm{cof}(\kappa)^{\mathrm{K}^{DJ}}\}$ and $\{c\}$ are both definable by Σ_1-formulas with parameter κ.

Corollary

Assume that there is no inner model with a measurable cardinal. If κ is a singular cardinal, then the following statements hold:

- *There is an unbounded subset of κ that consists of cardinals and is not Σ_1-stationary.*
- *There exists a regressive function $r : \kappa \longrightarrow \kappa$ that is definable by a Σ_1-formula with parameter κ and is not constant on any unbounded subset of κ.*
- *If κ has countable cofinality, then the complement of $\boldsymbol{\Sigma}_1$-stationary subset of κ is not $\boldsymbol{\Sigma}_1$-stationary.*

Theorem

If U is a normal ultrafilter on κ and G is generic over V for Prikry forcing with G, then the following statements hold in $V[G]$:

- *Every unbounded subset of κ that consists of cardinals is $\mathbf{\Sigma}_1$-stationary in κ.*
- *If S is a $\mathbf{\Sigma}_1$-stationary subset of κ and $r : \kappa \longrightarrow \kappa$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$, then r is constant on a $\mathbf{\Sigma}_1$-stationary subset of S.*

Corollary

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a measurable cardinal.*
- *There is a singular cardinal* κ *with the property that every unbounded subset of* κ *that consists of cardinals is* $\mathbf{\Sigma}_1$*-stationary in* κ*.*
- *There is a singular cardinal* κ *with the property that every regressive function* $r : \kappa \longrightarrow \kappa$ *that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa) \cup \{\kappa\}$ *is constant on an unbounded subset of* κ*.*
- *There is a singular cardinal* κ *of countable cofinality with the property that there exists a subset* E *of* κ *such that both* E *and* $\kappa \setminus E$ *are* $\mathbf{\Sigma}_1$*-stationary in* κ*.*

Partition properties

Remember that, given uncountable cardinals $\mu < \kappa$, the cardinal κ is μ-*Rowbottom* if the square brackets partition relation

$$\kappa \longrightarrow [\kappa]^{<\omega}_{\lambda,<\mu}$$

holds true for all $\lambda < \kappa$, i.e., for every $\lambda < \kappa$ and every function

$$c : [\kappa]^{<\omega} \longrightarrow \lambda,$$

there exists $H \in [\kappa]^\kappa$ with $|c[[H]^{<\omega}]| < \mu$.

Lemma

If ω_ω is an ω_n-Rowbottom cardinal, then ω_ω has the $\Sigma_1(\omega_n)$-undefinability property.

Recall that a cardinal κ is *Jónsson* if for every function $f : [\kappa]^{<\omega} \longrightarrow \kappa$ there is a proper subset H of κ of cardinality κ with $f[[H]^{<\omega}] \subseteq H$.

Theorem

If ω_ω is a Jónsson cardinal, then the following statements hold:

- *Every infinite subset of $\{\omega_n \mid n < \omega\}$ is $\mathbf{\Sigma}_1$-stationary in ω_ω.*
- *If $r : \omega_\omega \longrightarrow \omega_\omega$ is a regressive function that is definable by a Σ_1-formula with parameters in $\mathrm{H}(\aleph_\omega) \cup \{\omega_\omega\}$, then r is constant on an infinite subset of $\{\omega_n \mid n < \omega\}$.*

We can use the above results to reduce the class of models of set theory in which ω_ω possesses strong partition properties.

In particular, we will show that ω_ω is not ω_2-Rowbottom in the standard models of strong forcing axioms, where the given axiom was forced over a model of the GCH by turning some large cardinal into ω_2.

Lemma

Assume that there is a natural number $m > 1$ such that there are no special ω_m-Aronszajn trees and for all $m < n < \omega$, there is a special ω_n-Aronszajn tree. Then the set $\{\omega_m\}$ is definable by a Σ_1-formula with parameter ω_ω and the cardinal ω_ω is not ω_m-Rowbottom.

Theorem

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There is a measurable cardinal.*
- ω_ω *has the* $\Sigma_1(\omega_n)$*-undefinability property for every* $0 < n < \omega$*.*
- *Every infinite subset of* $\{\omega_n \mid n < \omega\}$ *is* $\mathbf{\Sigma}_1$*-stationary in* ω_ω*.*
- *Every regressive function* $r : \omega_\omega \longrightarrow \omega_\omega$ *that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\aleph_\omega) \cup \{\omega_\omega\}$ *is constant on an infinite subset of* $\{\omega_n \mid n < \omega\}$*.*
- *There exists a subset* E *of* ω_ω *such that both* E *and* $\omega_\omega \setminus E$ *are* $\mathbf{\Sigma}_1$*-stationary in* ω_ω*.*

Thank you for listening!