Patterns in the large cardinal hierarchy

Philipp Moritz Lücke
Institut de Matemàtica, Universitat de Barcelona.

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Introduction

Despite their central role in modern set theory, large cardinals are still surrounded by many open conceptual questions:

- There is no widely accepted formal definition of the intuitive concept of large cardinals. Instead there are several common ways to formulate such principles (elementary embeddings, partition properties, etc.).
- Although the linearity of the ordering of large cardinal assumptions by their consistency strength seems to be a fundamental fact of mathematics, it has not been possible to prove the general validity of this principle and, without a formal definition for the concept of large cardinals, it is not even clear how such a proof should look like.
- Even though large cardinal assumptions answer many questions left open by ZFC in the desired way, the question whether they are true and should therefore be added to the standard axiomatization of set theory remains open.

Examples

- A cardinal κ is *supercompact* if for every cardinal $\lambda \geq \kappa$, there exists a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$.
- $0^\#$ exists if and only if for some uncountable ordinal δ, there exists an uncountable set of indiscernibles for L_δ.
- *Vopěnka's Principle* is the scheme of axioms stating that for every proper class of graphs, there are two members of the class with a homomorphism between them.

In order to address the issues discussed above, Bagaria introduced a framework of canonical strengthenings of the Downward Löwenheim-Skolem Theorem that aims to include various large cardinal axioms.

This framework is based on the following type of reflection principles:

Definition (Bagaria)

Given a class $\mathcal{C}$ of structures[1] of the same type and an infinite cardinal κ, we let $\mathrm{SR}_{\mathcal{C}}(\kappa)$ denote the statement that for every structure $\mathfrak{B}$ in $\mathcal{C}$, there exists a structure $\mathfrak{A}$ in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

[1] In the following, the term *structure* refers to structures for countable first-order languages.

Note that the Downward Löwenheim-Skolem Theorem implies that the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every elementary class $\mathcal{C}$ of structures and every uncountable cardinal κ.

We can extend this result by considering classes of structures defined by formulas of low set-theoretic complexity.

Definition

- A formula in the language $\mathcal{L}_\in$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic $\mathcal{L}_\in$-formulas and is closed under negation, disjunction and bounded quantification.

- An $\mathcal{L}_\in$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \ \neg\varphi(x)$ for some Σ_n-formula φ.

Proposition

$\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds for every uncountable cardinal* κ *and every class* $\mathcal{C}$ *of structures of the same type that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa)$*.*

Proof.

Fix a Σ_1-formula $\varphi(v_0, v_1)$ and $z \in \mathrm{H}(\kappa)$ with $\mathcal{C} = \{\mathfrak{A} \mid \varphi(\mathfrak{A}, z)\}$.

Pick $\mathfrak{B} \in \mathcal{C}$, a cardinal θ with $\mathfrak{B} \in \mathrm{H}(\theta)$ and an elementary submodel X of $\mathrm{H}(\theta)$ of cardinality less than κ with $\mathrm{tc}(\{z\}) \cup \{\mathfrak{B}\} \subseteq X$.

Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse and set $\mathfrak{A} = \pi(\mathfrak{B})$. Then $\pi(z) = z$ and the fact that $\varphi(\mathfrak{B}, z)$ holds in $\mathrm{H}(\theta)$ implies that $\varphi(\mathfrak{A}, z)$ holds in both M and V.

This shows that $\mathfrak{A}$ is a structure in $\mathcal{C}$ and, since $\mathrm{dom}(\mathfrak{A}) \subseteq M$, it follows that $\pi^{-1} \restriction \mathrm{dom}(\mathfrak{A})$ is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$. $\square$

We now present results that show that the large cardinal axioms listed above can be canonically characterized through the principle SR.

Theorem (Bagaria et al.)

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is the least supercompact cardinal.*
- *The cardinal κ is the least cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Theorem (Bagaria)

The following statements are equivalent:

- $0^\#$ *exists.*
- *For every class* $\mathcal{C}$ *of constructible structures of the same type that is definable in* L*, there exists a cardinal* κ *with* $\mathrm{SR}_{\mathcal{C}}(\kappa)$*.*

Theorem (Bagaria et al.)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

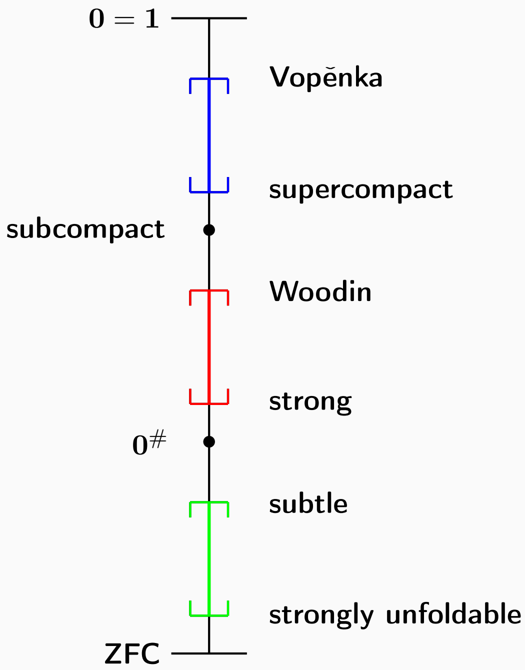
- *Vopěnka's Principle.*
- *For every class* $\mathcal{C}$ *of structures of the same type, there exists a cardinal* κ *with* $\mathrm{SR}_{\mathcal{C}}(\kappa)$*.*

Patterns in the large cardinal hierarchy

We now want to use principles of structural reflection to show that certain patterns recur throughout the large cardinal hierarchy.

These patterns can be viewed as evidence for the naturalness of large cardinal axioms.

Moreover, these characterizations can lead to both generalizations of results to other regions of the large cardinal hierarchy and to the weakening of large cardinal assumptions in existing results.


$0 = 1$
Vopěnka
supercompact
subcompact
Woodin
strong
$0^{\#}$
subtle
strongly unfoldable
ZFC

$$\frac{\textit{Vopěnka}}{\textit{supercompact}} = \frac{\textit{Woodin}}{\textit{strong}} = \frac{\textit{subtle}}{\textit{strongly unfoldable}}$$

Supercompact cardinals

The above characterization of the least supercompact cardinal can be strengthened in the following way:

Theorem (Bagaria)

The following statements are equivalent for every cardinal κ:

- *The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The cardinal κ is either supercompact or a limit of supercompact cardinals.*

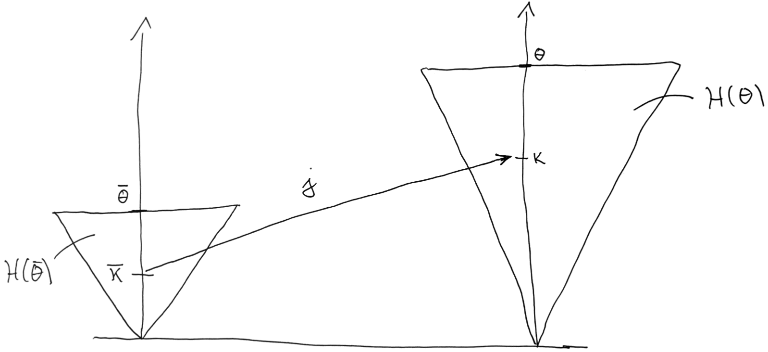
The proof of this equivalence is based on the following classical result of Magidor:

Lemma (Magidor)

The following statements are equivalent for every cardinal κ:

- κ *is a supercompact cardinal.*
- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$, and*
  - *an elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$*

  such that $\bar{\kappa} = \mathrm{crit}(j)$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.



θ
H(θ)
K
j
θ̄
H(θ̄)
K̄

Lemma

If κ is a supercompact cardinal, then $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.

Proof.

Pick a Σ_2-formula $\varphi(v_0, v_1)$ and $z \in \mathrm{V}_\kappa$ with $\mathcal{C} = \{\mathfrak{A} \mid \varphi(\mathfrak{A}, z)\}$.

Fix $\mathfrak{B} \in \mathcal{C}$. Then $\varphi(\mathfrak{B}, z)$ holds and there exists a cardinal $\theta > \kappa$ such that $\mathfrak{B} \in \mathrm{H}(\theta)$ and $\varphi(\mathfrak{B}, z)$ holds in $\mathrm{H}(\theta)$.

Then there exists a cardinal $\bar{\theta} < \kappa$ and a non-trivial elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$ with $j(\mathrm{crit}(j)) = \kappa$ and $\mathfrak{B}, z \in \mathrm{ran}(j)$.

Pick $\mathfrak{A} \in \mathrm{H}(\bar{\theta})$ with $j(\mathfrak{A}) = \mathfrak{B}$. Then $z \in \mathrm{H}(\bar{\theta})$ with $j(z) = z$ and $\varphi(\mathfrak{A}, z)$ holds in $\mathrm{H}(\bar{\theta})$ and V.

This allows us to conclude that $\mathfrak{A} \in \mathcal{C}$ and $j \restriction \mathrm{dom}(\mathfrak{A})$ is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$. $\square$

Strongly unfoldable cardinals

We know aim to show that the above connection between large cardinal properties and structural reflection can be generalized to other regions of the large cardinal hierarchy.

The following weakening of strongness was introduced by Villaveces in his model-theoretic investigations of models of set theory.

Definition (Villaveces)

An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^{-}-model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.

The next result shows that strong unfoldability can also be seen as a miniature version of supercompactness.

Lemma (Džamonja–Hamkins)

An inaccessible cardinal κ is strongly unfoldable if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with ${}^{\lambda}N \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.

Strongly unfoldable cardinals turn out to have a very rich structure theory.

In particular, many important results about supercompact and strong cardinals have analogs for strongly unfoldable cardinals.

Theorem (Hamkins–Johnstone)

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There exists a strongly unfoldable cardinal.*
- *The restriction of the Proper Forcing Axiom to the class of proper partial orders that preserve either* $\aleph_2$ *or* $\aleph_3$*.*

Definition (Rathjen)

A cardinal κ is *shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every ordinal $\gamma > \kappa$ and every subset A of V_κ such that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

Theorem (L.)

A cardinal is shrewd if and only if it is strongly unfoldable.

Theorem (Džamonja–Hamkins)

If the existence of a strongly unfoldable cardinal is consistent with the axioms of $\mathbf{ZFC}$*, then a failure of the principle* $\diamondsuit_{Reg}(\kappa)$ *at a strongly unfoldable cardinal* κ *is consistent with these axioms.*

Theorem (L.)

If κ *is a strongly unfoldable cardinal with* $\mathcal{P}(\kappa) \subseteq \mathrm{HOD}_z$ *for some* $z \subseteq \kappa$*, then* $\diamondsuit_{Reg}(\kappa)$ *holds.*

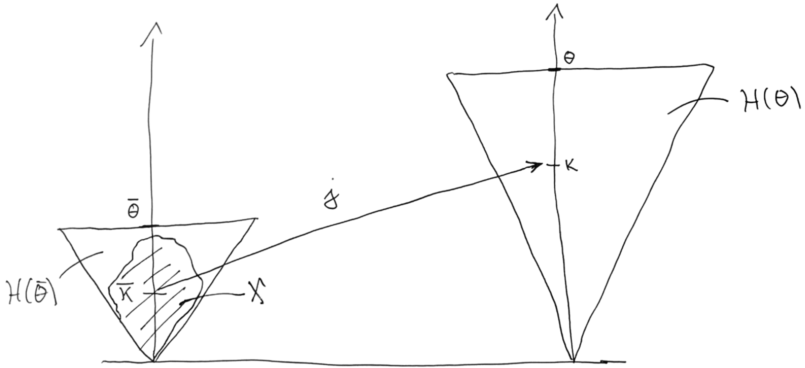
The following result shows how Magidor's characterization of supercompactness can be adapted to strongly unfoldable cardinals.

Lemma

The following statements are equivalent for every cardinal κ:

- κ *is a strongly unfoldable cardinal.*
- *For all cardinals* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$*, there exist*
  - *cardinals* $\bar{\kappa} < \bar{\theta} < \kappa$*,*
  - *an elementary submodel* X *of* $\mathrm{H}(\bar{\theta})$*, and*
  - *an elementary embedding* $j : X \longrightarrow \mathrm{H}(\theta)$

  such that $\bar{\kappa} + 1 \subseteq X$*,* $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$*,* $j(\bar{\kappa}) = \kappa$ *and* $z \in \mathrm{ran}(j)$*.*


θ
H(θ)
κ
j
θ̄
H(θ̄)
κ̄
X

We now introduce the weakened reflection principles that characterize strong unfoldability.

Definition (Bagaria–Väänänen)

Let $\mathcal{C}$ be a non-empty class of structures of the same type and let κ be an infinite cardinal.

- $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ denotes the statement that for every structure $\mathfrak{B}$ in $\mathcal{C}$ of cardinality κ, there exists a structure $\mathfrak{A}$ in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$.
- $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ denotes the statement that $\mathcal{C}$ contains a structure of cardinality less than κ.

The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ obviously implies both $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$.

No other implication between these principles holds in general.

Theorem

The following statements are equivalent for every cardinal κ:

- *The principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The cardinal κ is either strongly unfoldable or a limit of supercompact cardinals.*

Corollary

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is the least strongly unfoldable cardinal.*
- *The cardinal κ is the least cardinal with the property that the principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Corollary

The following statements are equivalent for every singular cardinal κ:

- *The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The cardinal κ is a limit of supercompact cardinals.*

Vopěnka cardinals

Above, we showed how the validity of the first-order Vopěnka's Principle can be characterized through principles of structural reflection.

We now want to characterize the validity of the second-order Vopěnka's Principle in initial segments of the set-theoretic universe.

Definition

An inaccessible cardinal δ is a *Vopěnka cardinal* if for every set $\mathcal{C} \in \mathrm{V}_{\delta+1} \setminus \mathrm{V}_{\delta}$ of graphs, there are two members of the class with a homomorphism between them.

Theorem

The following statements are equivalent for every uncountable cardinal δ:

- *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds.*
- *The cardinal δ is a Vopěnka cardinal.*

The main challenge in the proof of the above equivalence is to show that the given reflection property implies inaccessibility. The following observation presents the main idea of the proof of this implication.

Observation

If δ is a singular cardinal of countable cofinality, then there is a set $\mathcal{C} \subseteq \mathrm{V}_\delta$ of groups with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ fails for every cardinal $\kappa < \delta$.

Proof.

Let $\langle \kappa_n \mid n < \omega \rangle$ denote a strictly increasing sequence of infinite cardinals that is cofinal δ.

Given $1 < n < \omega$, let G_n denote the sum of κ_n-many copies of $\mathbb{Z}/n\mathbb{Z}$.

Define $\mathcal{C} = \{G_n \mid 1 < n < \omega\} \subseteq \mathrm{V}_\delta$.

Given a cardinal $\kappa < \delta$ and a prime p with $\kappa_p > \kappa$, there is no elementary embedding of a group of cardinality less than κ in $\mathcal{C}$ into G_p. □

Subtle cardinals

Earlier, we showed that, by replacing all occurrences of the principle SR by the conjunction of the principles SR^- and SR^{--}, we obtain a characterization of strong unfoldability from the given characterization of supercompactness.

We now show that the same modification turns the above characterization of Vopěnka cardinals into a characterization of another well-studied notion from the lower reaches of the large cardinal hierarchy.

Definition (Jensen–Kunen)

An infinite cardinal δ is *subtle* if for every sequence $\langle E_\gamma \subseteq \gamma \mid \gamma < \delta \rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with the property that $E_\beta = E_\gamma \cap \beta$.

Subtle cardinals were introduced by Jensen and Kunen in their analysis of combinatorial principles in Gödel's constructible universe L.

Lemma

If δ is a subtle cardinal, then there are stationary-many $\kappa < \delta$ with the property that κ is strongly unfoldable in V_δ.

Theorem (Jensen–Kunen)

If δ is a subtle cardinal, then $\diamondsuit_{Reg}(\delta)$ holds.

Theorem

The following statements are equivalent for every uncountable cardinal δ:

- *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principles $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold.*
- *The cardinal δ is either subtle or a limit of subtle cardinals.*

Corollary

The following statements are equivalent for every uncountable cardinal δ:

- *The cardinal δ is the least subtle cardinal.*
- *The cardinal δ is the least cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold.*

The middle reaches

We now present variations of the principle SR that can be used to obtain canonical characterizations of strong and Woodin cardinals through reflection principles.

The formulation of these principles is motivated by Trevor Wilson's work on the consistency strength of the *weak Vopěnka principle* studied in category theory.

In order to formulate these variations, we need to recall some basic model-theoretic constructions.

Definition

Given a first-order language $\mathcal{L}$, a *homomorphism* from an $\mathcal{L}$-structures $\mathfrak{A}$ to an $\mathcal{L}$-structure $\mathfrak{B}$ is a map $h : \operatorname{dom}(\mathfrak{A}) \longrightarrow \operatorname{dom}(\mathfrak{B})$ that satisfies the following statements:

- If $\dot{c}$ is a constant symbol in $\mathcal{L}$, then $h(\dot{c}^{\mathfrak{A}}) = \dot{c}^{\mathfrak{B}}$.
- If $\dot{f}$ is an $(n+1)$-ary function symbol in $\mathcal{L}$ and $a_0, \ldots, a_n \in \operatorname{dom}(\mathfrak{A})$, then $h(\dot{f}^{\mathfrak{A}}(a_0, \ldots, a_n)) = \dot{f}^{\mathfrak{B}}(h(a_0), \ldots, h(a_n))$.
- If $\dot{R}$ is an $(n+1)$-ary relation symbol in $\mathcal{L}$ and $a_0, \ldots, a_n \in \operatorname{dom}(\mathfrak{A})$ with $\langle a_0, \ldots, a_n \rangle \in \dot{R}^{\mathfrak{A}}$, then $\langle h(a_0), \ldots, h(a_n) \rangle \in \dot{R}^{\mathfrak{B}}$.

Definition

Given a first-order language $\mathcal{L}$ and a non-empty set $\mathcal{S}$ of $\mathcal{L}$-structures, we let $\prod \mathcal{S}$ denote the unique $\mathcal{L}$-structure satisfying the following statements:

- The domain of $\prod \mathcal{S}$ is the set $\prod_{\mathfrak{A} \in \mathcal{S}} \operatorname{dom}(\mathfrak{A})$.
- If $\dot{c}$ is a constant symbol in $\mathcal{L}$, then $\dot{c}^{\prod \mathcal{S}}(\mathfrak{A}) = \dot{c}^{\mathfrak{A}}$ for all $\mathfrak{A} \in \mathcal{S}$.
- If $\dot{f}$ is an $(n+1)$-ary function symbol in $\mathcal{L}$ and $g_0, \ldots, g_n \in \operatorname{dom}(\prod \mathcal{S})$, then
$$\dot{f}^{\prod \mathcal{S}}(g_0, \ldots, g_n)(\mathfrak{A}) \;=\; \dot{f}^{\mathfrak{A}}(g_0(\mathfrak{A}), \ldots, g_n(\mathfrak{A}))$$
for all $\mathfrak{A} \in \mathcal{S}$.
- If $\dot{R}$ is an $(n+1)$-ary relation symbol in $\mathcal{L}$ and $g_0, \ldots, g_n \in \operatorname{dom}(\prod \mathcal{S})$, then
$$\langle g_0, \ldots, g_n \rangle \in \dot{R}^{\prod \mathcal{S}} \iff \forall \mathfrak{A} \in \mathcal{S} \; \langle g_0(\mathfrak{A}), \ldots, g_n(\mathfrak{A}) \rangle \in \dot{R}^{\mathfrak{A}}.$$

The following reflection principle originates from Trevor Wilson's work on the consistency strength of the *weak Vopěnka principle*.

Definition

Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ denote the statement that $\mathcal{C} \cap \mathrm{V}_\kappa \neq \emptyset$ and for every $\mathfrak{B} \in \mathcal{C}$, there exists a homomorphism $h : \prod(\mathcal{C} \cap V_\kappa) \longrightarrow \mathfrak{B}$.

Proposition

If κ is a strong cardinal, then $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds for every non-empty class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.

Proof.

Fix a Σ_2-formula $\varphi(v_0, v_1)$ and $z \in V_\kappa$ with $\mathcal{C} = \{\mathfrak{A} \mid \varphi(\mathfrak{A}, z)\}$, and pick a structure $\mathfrak{B}$ in $\mathcal{C}$. Since $\mathrm{V}_\kappa \prec_{\Sigma_2} \mathrm{V}$, we know that $\mathcal{C} \cap \mathrm{V}_\kappa \neq \emptyset$.

Pick $\lambda > \kappa$ with $\mathfrak{B} \in \mathrm{V}_\lambda \prec_{\Sigma_2} \mathrm{V}$. Then there exists an inner model M with $\mathrm{V}_\lambda \subseteq M$ and an elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$.

Then $\mathfrak{B} \in \mathcal{C} \cap \mathrm{V}_\lambda \subseteq j(\mathcal{C} \cap \mathrm{V}_\kappa)$ and the map

$$h : \prod(\mathcal{C} \cap \mathrm{V}_\kappa) \longrightarrow \mathfrak{B};\ f \longmapsto j(f)(\mathfrak{B})$$

is a well-defined homomorphism. □

The following result extend earlier work by Bagaria and Wilson.

Theorem

The following statements are equivalent for every cardinal κ:

- *The principle $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The cardinal κ is either strong or a limit of strong cardinals.*

Corollary (Bagaria–Wilson)

The following statements are equivalent for every cardinal κ:

- *κ is the least strong cardinal.*
- *κ is the least cardinal with the property that $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

The next result again extend previous work by Bagaria and Wilson.

Theorem

The following statements are equivalent for every uncountable cardinal δ:

- *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq \mathrm{V}_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{PSR}_{\mathcal{C}}(\kappa)$ holds.*
- *The cardinal δ is a Woodin cardinal.*

Back to the lower reaches

Definition

Given an infinite cardinal κ and a class $\mathcal{C}$ of structures of the same type, we let $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ denote the statement that $\mathcal{C} \cap \mathrm{V}_\kappa \neq \emptyset$ and for every substructure $\mathfrak{X}$ of $\prod(\mathcal{C} \cap \mathrm{V}_\kappa)$ of cardinality at most κ and every $\mathfrak{B} \in \mathcal{C}$, there exists a homomorphism $h : \mathfrak{X} \longrightarrow \mathfrak{B}$.

Theorem

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is either strongly unfoldable or a limit of strong cardinals.*
- *The principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

As before, the fact that strong cardinals are strongly unfoldable directly leads to the following canonical characterization of the least strongly unfoldable cardinal.

Corollary

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is the least strongly unfoldable cardinal.*
- *The cardinal κ is the least cardinal with the property that the principles $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.* □

Theorem

The following statements are equivalent for every uncountable cardinal δ:

- *The cardinal δ is a subtle cardinal.*
- *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq \mathrm{V}_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principle $\mathrm{WPSR}_{\mathcal{C}}(\kappa)$ holds.*

Theorem

The following statements are equivalent for every cardinal δ:

- *The cardinal δ is subtle.*
- *For every function $F : \delta \longrightarrow \mathrm{H}(\delta)$, there exists a cardinal $\kappa < \delta$ with the following properties:*
  - $F[\kappa] \subseteq \mathrm{H}(\kappa)$.
  - *For every $\gamma < \delta$ and every transitive set M of cardinality κ with $\kappa \cup \{\kappa, F \restriction \kappa\} \subseteq M$, there exists*
    - *a transitive set N with $\gamma \in N$, and*
    - *a non-trivial elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$ and $j(F \restriction \kappa) \restriction \gamma = F \restriction \gamma$.*

Thank you for listening!