Cotas inferiores para sumas de potencias afines

Buscando paja en un pajar y pinchándome con las agujas

Ignacio García Marco

Universidad de La Laguna

Seminario GASIULL - 18/02/2019

Based on joint work with:

Pascal Koiran & **Timothée Pecatte**

ENS Lyon

The affine powers model: a way of expressing polynomials

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A complexity measure

For all $f \in \mathbb{F}[x]$, we set

$$c(f) := \min\{s \mid \quad f = \textstyle\sum_{i=1}^{s} \alpha_i (x + a_i)^{e_i} \text{ with } \alpha_i, a_i \in \mathbb{F} \text{ and } e_i \in \mathbb{N}\}$$

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Indeed, $c(f) = 2$.

An upper bound for $c(f)$

Lemma. Let $f \in \mathbb{F}[x]$ a (nonzero) polynomial of degree d. Then,

$$c(f) \leq d + 1$$

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Proof. By induction over d:

$$\left.\begin{array}{l} \text{If } d = 0,\ f = a_0 = a_0 x^0 \\ \text{if } d = 1,\ f = a_1 x + a_0 = a_1(x + (a_0/a_1))^1 \end{array}\right\} \Rightarrow c(f) = 1$$

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Tschirnhausen Transformation
$\deg(g) \leq d - 2$

Then, $c(f) \leq c(g) + 1 \leq \left\lceil \frac{d-1}{2} \right\rceil + 1 = \left\lceil \frac{d+1}{2} \right\rceil$ $\square$

The generic case

Theorem If $f \in \mathbb{F}[x]$ is a **generic** polynomial of degree d, then

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In other words

$$\left\{ (a_0, \ldots, a_d) \in \mathbb{F}^{d+1} \mid c\left(\sum_{i=0}^{d} a_i x^i \right) < \left\lceil \frac{d+1}{2} \right\rceil \right\} \subset X,$$

where $X \subsetneq \mathbb{F}^{d+1}$ is an algebraic set.

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Theorem If $f \in \mathbb{F}[x]$ is a **generic** polynomial of degree d, then

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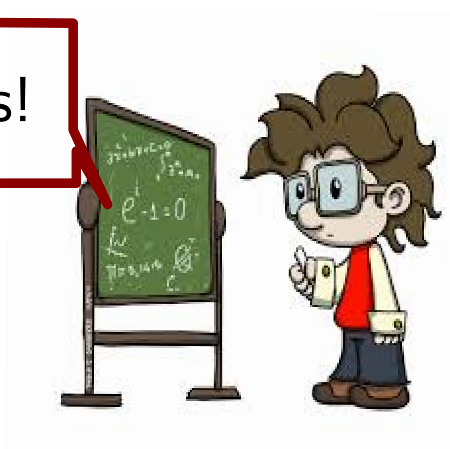
So, tell me one polynomial $f \in \mathbb{C}[x]$ of degree 100 such that $c(f) = 51$.

Almost anyone works!



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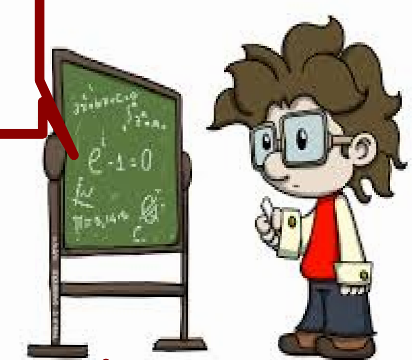

But, could you tell me one explicitly.

Mmmmmm, ...

The problem

For every $d \in \mathbb{N}$, find an **explicit** $f_d \in \mathbb{C}[x]$, such that

- f_d has degree d, and
- $c(f_d) = \lceil (d+1)/2 \rceil$.

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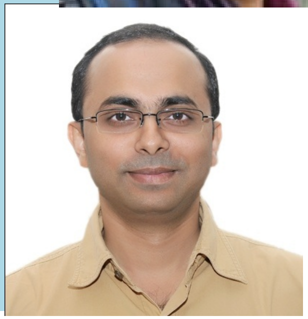
Theorem [Kayal, Koiran, Pecatte & Saha (2015)]

For all $k \in \mathbb{N}$ and all $a_1 \neq a_2 \in \mathbb{C}$, the polynomial

$$f = [(x+a_1)(x+a_2)]^k$$

of degree $d = 2k$ satisfies that $c(f) \geq (\sqrt{d})/2$.



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Today's goal: Improve this result for $\mathbb{F} = \mathbb{R}$.

This is, provide **explicit polynomials** $f \in \mathbb{R}[x]$ of degree d such that $c(f)$ is equal (or close) to $\lceil (d+1)/2 \rceil$.

An easy observation

Let $f \in \mathbb{F}[x]$, if:

$$f = \sum_{i=1}^{s} \alpha_i (x + a_i)^{e_i}$$

and

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Then,

$$\left\{(x + a_i)^{e_i} \,|\, 1 \le i \le s\right\} \cup \left\{(x + b_j)^{d_j} \,|\, 1 \le j \le \ell\right\},$$

is a **linearly dependent** set.

Our approach

If $I := \{(x + a_i)^{e_i} \mid 1 \leq i \leq s\}$ satisfies the following **nice property**:

For **all** $L = \{(x + b_i)^{d_i} \mid 1 \leq i \leq \ell\}$ $\mathbb{F}$-linearly independent with $\ell < s \Rightarrow\ L \cup I$ is also $\mathbb{F}$-linearly independent.

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Taking $\alpha_1, \ldots, \alpha_s \in \mathbb{F} \setminus \{0\}$ and setting

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Goal: find a set I with the **nice property** and s large.

To carry out this, we need a **good criterion for deciding $\mathbb{F}$-linear independence** of polynomials of the form $(x+a_i)^{e_i}$.

Our approach: study linear independence of affine powers

Let $\ell_1, \ldots, \ell_s \in \mathbb{F}[x]$ be affine powers:

$$\ell_i = (x + a_i)^{e_i} = \sum_{j=0}^{e_i} \binom{e_i}{j} a_i^{e_i - j} x^j.$$

Are $\ell_1, \ldots, \ell_s$ linearly independent?

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Take $d := \max(e_i)$, $B := \{1, \frac{x}{1!}, \ldots, \frac{x^j}{j!}, \ldots, \frac{x^d}{d!}\}$ and set A_i the vector of coordinates of ℓ_i with respect to B, that is

$$A_i = \left(a_i^{e_i}, \ldots, \frac{e_i!}{(e_i - j)!} a_i^{e_i - j}, \ldots, e_i!, 0, \ldots, 0 \right) \in \mathbb{F}^{d+1}$$

Let A be the matrix with rows $A_1, \ldots, A_s$. Then,

$$\dim_{\mathbb{F}} \langle \ell_1, \ldots, \ell_s \rangle = \mathrm{rk}(A).$$

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It is not easy to determine when A has maximal row rank!!!

Birkhoff interpolation

Birkhoff interpolation studies the set of polynomials $g \in \mathbb{F}[x]$ of degree $\leq d$ satisfying a system of equations of the form

$$\left.\begin{array}{rcl} g^{(k_1)}(a_1) & = & 0 \\ \cdots & & \\ g^{(k_s)}(a_s) & = & 0 \end{array}\right\}$$

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Definition.

The **Birkhoff interpolation problem** is **regular** if its set of solutions $g \in \mathbb{F}[x]_{\leq d}$ is a vector space of dimension $d + 1 - s$.

From linear independence to Birkhoff interpolation

Example: Consider the affine powers $(x+1)^4, (x-1)^4, x^3, x$.

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We associate the following interpolation problem:

We look for polynomials $g \in \mathbb{F}[x]$ of degree $d \leq 4$ such that

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The set of solutions is $\langle x^4 - 1, x^2 - 1 \rangle$.

The **Birkhoff interpolation problem** is **not regular**.

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Given $\ell_1, \dots, \ell_s \in \mathbb{F}[x]$ with $\ell_i = (x + a_i)^{e_i}$ and $d := \max(e_i)$.
We associate the following **Birkhoff interpolation problem**:

Find $g \in \mathbb{F}_{\le d}[x]$ such that

$$g^{(d-e_i)}(a_i) = 0 \text{ for } 1 \le i \le s.$$

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Proposition

Set $r := \dim_{\mathbb{F}} \langle \ell_1, \ldots, \ell_s \rangle$ with $\ell_i = (x + a_i)^{e_i}$. Then,

$$\dim_{\mathbb{F}} \{g \in \mathbb{F}_{\leq d}[x];\ g^{(d-e_i)}(a_i) = 0 \text{ for all } i\} = d + 1 - r.$$

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Proof. We know that $r = \mathrm{rk}(A)$.

If $g = \sum_{j=0}^{d} g_j \frac{x^j}{j!}$ with $g_j \in \mathbb{F}$

$\Searrow$

$$g^{(d-e_i)} = \sum_{j=0}^{e_i} g_{d-e_i+j} \frac{x^j}{j!}.$$

From linear independence to Birkhoff interpolation

Then,

$$g^{(d-e_i)}(a_i) = 0 \text{ for all } 1 \leq i \leq s \Longleftrightarrow B \cdot \begin{pmatrix} g_d \\ g_{d-1} \\ \vdots \\ g_0 \end{pmatrix} = 0,$$

where B is the $s \times (d+1)$ matrix whose i-th row is

$$B_i := \left(\frac{a_i^{e_i}}{e_i!}, \ldots, \frac{a_i^{e_i - j}}{(e_i - j)!}, \ldots, a_i, 1, 0, \ldots, 0 \right) \in \mathbb{F}^{d+1}.$$

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Then,

$$\dim\{g \in \mathbb{F}_{\le d}[x]\, ;\, g^{(d-e_i)}(a_i) = 0 \text{ for } 1 \le i \le s\} = d+1-\mathrm{rk}(B).$$

We observe that $A_i = e_i!\, B_i \Longrightarrow r = \mathrm{rk}(A) = \mathrm{rk}(B)$. $\square$

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Corollary
The set $\{\ell_1, \ldots, \ell_s\}$ is **linearly independent** if and only if the **Birkhoff interpolation problem**

$$g \in \mathbb{F}_{\leq d}[x];\ g^{(d-e_i)}(a_i) = 0\ for\ 1 \leq i \leq s$$

is **regular**.

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Bringing results from Birkhoff interpolation

Given $\ell_1, \ldots, \ell_s$ with $\ell_i = (x + a_i)^{e_i}$.

Set $N_j := \sharp\{i \mid e_i < j\}$.

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Remark:

If $\{\ell_1, \ldots, \ell_s\}$ are lin. indep. $\Longrightarrow$ $N_j \leq j$ for all $j \geq 1$

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And as a consequence of **Atkinson-Sharma theorem (1969)** for **real Birkhoff interpolation**:

Theorem. For $\mathbb{F} = \mathbb{R}$.

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Bringing results from Birkhoff interpolation

Given $\ell_1, \dots, \ell_s$ with $\ell_i = (x + a_i)^{e_i}$.

Set $N_j := \sharp\{i \mid e_i < j\}$.

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Not true for $\mathbb{F} = \mathbb{C}$

Main results

Theorem. Let $e, s \in \mathbb{N}$ with $s \leq (e+2)/4$, $a_1, \ldots, a_s \in \mathbb{R}$ distinct. Then,

$$I := \{(x + a_i)^e \mid 1 \leq i \leq s\}$$

has the *nice* property.

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Moreover,

$$(x+1)^{d+1} - (x-1)^{d+1} = \sum_{\substack{i\ odd \\ 1 \leq i \leq d+1}} 2\binom{d}{i} x^{d+1-i}.$$

$\lceil \frac{d+1}{2} \rceil$ affine powers

The complex case

Over $\mathbb{C}$ one can generalize the identity:

$$(x+1)^d - (x-1)^d = \sum_{\substack{i\ odd \\ 1\leq i\leq d}} 2\binom{d}{i} x^{d-i}.$$

Proposition

Take $k \in \mathbb{Z}^+$ and let ξ be a **k-th primitive root of unity**. Then, for all $d \in \mathbb{Z}^+$ the following equality holds:

$$\sum_{j=1}^{k} \xi^j (x+\xi^j)^d = \sum_{\substack{i\equiv -1 \pmod{k} \\ 0\leq i\leq d}} k\binom{d}{i} x^{d-i} \in \mathbb{R}[x]$$

Open questions

- Find explicit $f \in \mathbb{C}[x]$ such that $c(f)$ is linear in d.
- Find a *good* sufficient condition for $\ell_1, \ldots, \ell_s \in \mathbb{C}[x]$ to be $\mathbb{C}$-linearly independent with $\ell_i = (x + a_i)^{e_i}$, $a_i \in \mathbb{C}$.
- Find a *good* sufficient condition for a Birkhoff interpolation problem to be regular over $\mathbb{C}$.
- Does Corollary 1 hold over $\mathbb{C}$? and Corollary 2?
- Devise algorithms computing $c(f)$ for a given polynomial f.
- What about multivariate polynomials?

Open questions

Proposition 1

For any family $F = \{(x - a_i)^{e_i} \mid 1 \leq i \leq s\}$ with $a_i \in \mathbb{R}$. If $e_i \geq 2s - 4$, then F is linearly independent.

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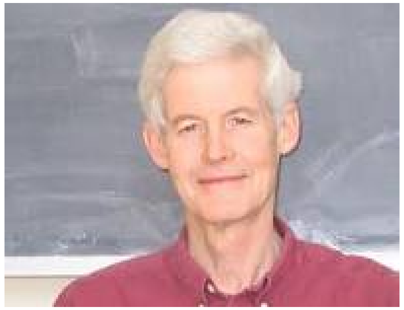
Conjecture. There are constants a and b such that Proposition 2 can be proved for $e_i \geq as + b$

Motivation

In 1971, Stephen Cook introduces: 'Is $P = NP$?'

P y NP are complexity clases of **decision problems**



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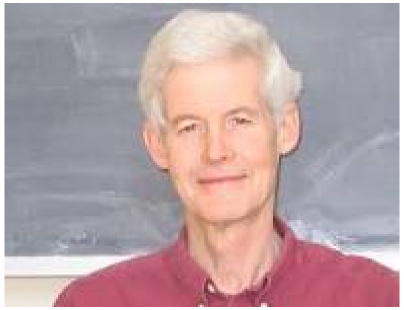

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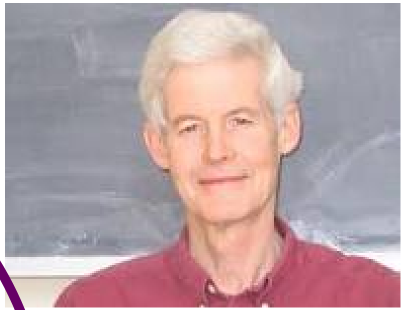

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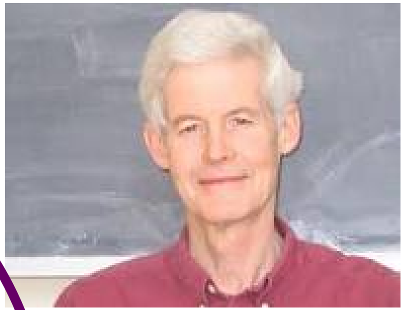

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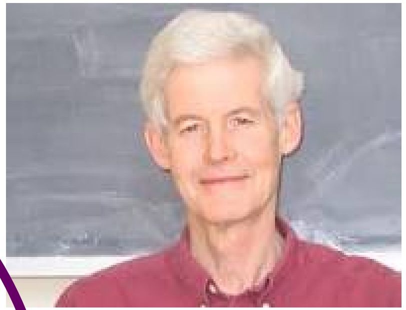

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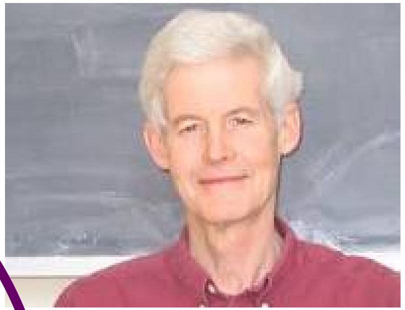

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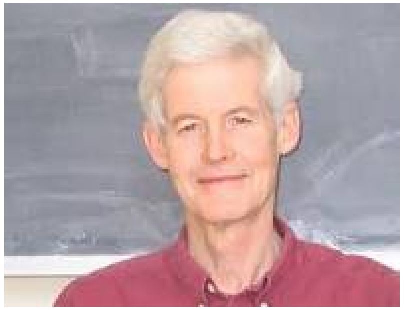

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- General thought: $\mathsf{P} \subsetneq \mathsf{NP}$
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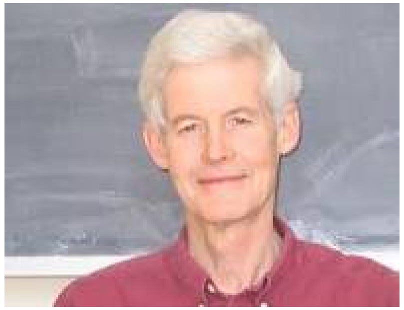

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- 45 before 2050
- 17 before 2100
- 12 after 2100
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Survey by W.I. Gasarch

An algebraic variant of P vs. NP

In 1979, Leslie Valiant presents the problem: Is $\mathsf{VP} = \mathsf{VNP}$?

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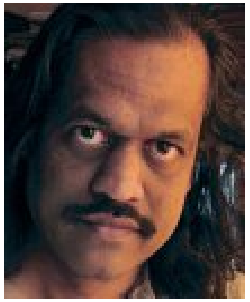
Bad news:

Landsberg et al. (2016) show that the *shifted partial derivatives method* is not enough to separate VP from VNP.

Good news:

Results by [Agrawal & Vinay (2008), Tavenas (2014)] show that a **generic vs. explicit** result for any of the following model of computation of polynomials imply that VP $\subsetneq$ VNP.



1. $f = \sum_{i=1}^{k} f_i^{e_i} \in \mathbb{C}[x]$
   where f_i have at most t monomials and $e_i \in \mathbb{N}$
2. $f = \sum_{i=1}^{k} a_i f_i^{e_i} \in \mathbb{F}[x_1, \ldots, x_n]$
   where $a_i \in k$ and f_i have at most t monomials
3. $f = \sum_{i=1}^{s} \prod_{j=1}^{m} L_{i,j} \in \mathbb{F}[x_1, \ldots, x_n]$
   where $L_{i,j}$ are linear forms.

Si $P = NP$ el mundo sería muy diferente de como creemos que es. La creatividad no tendría un valor especial: no habría diferencia entre alguien que sepa apreciar a Mozart y el propio genio, todo el que pudiera entender una demostración sería Gauss...

Scott Aaronson

¡Muchas gracias!