Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

Francesc Fité (MIT)   Xevi Guitart (UB)

SGA Barcelona 2020

A conjecture of Coleman

- Number Field: $k \subset \mathbb{C}$ with $[k\colon \mathbb{Q}] < \infty$.
  - e.g. $\mathbb{Q}$, $\mathbb{Q}(\sqrt{D})$
- A abelian variety over a number field k
  - $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) = \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- For $g, d \geq 1$ define

$$\mathcal{A}_{g,d} = \{\mathrm{End}^0_{\overline{\mathbb{Q}}}(A)\colon A/k \text{ of dimension } g \text{ and } [k : \mathbb{Q}] = d\}/\simeq$$

Conjecture (Coleman)

The set $\mathcal{A}_{g,d}$ is finite.

- In fact Coleman's conjecture is for endomorphism rings.
- Very little is known:
  - True for $\mathcal{A}_{1,d}$ (elliptic curves)
  - We are interested in $\mathcal{A}_{2,1}$ (abelian surfaces over $\mathbb{Q}$)

The case of elliptic curves (over $\mathbb{Q}$)

- $E/\mathbb{Q}$ elliptic curve $\rightsquigarrow \mathrm{End}^0_{\overline{\mathbb{Q}}}(E) \simeq \begin{cases} \mathbb{Q} \\ M = \mathbb{Q}(\sqrt{-D}), \ D \in \mathbb{Q}_{>0} \end{cases}$
- Constructing elliptic curves over $\mathbb{C}$ with CM
  - $\mathcal{O}$ = ring of integers of $\mathbb{Q}(\sqrt{-D})$ and $I \subset \mathcal{O}$ an ideal
  - $I \subset \mathbb{C}$ is a lattice and $\mathbb{C}/I$ is an elliptic curve with $\mathrm{End}(E) \simeq \mathcal{O}$

Theory of Complex Multiplication

- $\{E/\mathbb{C} \colon \mathrm{End}(E) \simeq \mathcal{O}\} \overset{1:1}{\longleftrightarrow} \{I \subset M \text{ fractional ideals}\}/I \sim \lambda I = \mathrm{Cl}(M)$
- If E has CM by $\mathcal{O}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)) : \mathbb{Q}] = \#\mathrm{Cl}(M)$

- If $E/\mathbb{Q}$ has CM by M then $j(E) \in \mathbb{Q} \Rightarrow \#\mathrm{Cl}(M) = 1$.
- $\#\mathrm{Cl}(\mathbb{Q}(\sqrt{-D})) = 1 \iff D = 1, 2, 3, 7, 11, 19, 43, 67, 163$
  - $\mathcal{A}_{1,1} = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\sqrt{-D}) \colon D = 1, 2, 3, 7, 11, 19, 43, 67, 163\}$
- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\#\mathrm{Cl}(M) = d$
  - $\mathcal{A}_{1,d}$ is finite for all d
  - For $d \leq 100$ the set $\mathcal{A}_{1,d}$ is known explicitly (Watkins)

The case of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1} = \{\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \colon A/\mathbb{Q},\ \dim(A) = 2\}/\simeq$
- $A/\mathbb{Q}$ an abelian surface
  - geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
  - geometrically split if $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$
- $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq \begin{cases} \mathbb{Q}, \mathbb{Q}(\sqrt{D}), \text{ CM field }, B/\mathbb{Q} \text{ def. quat. alg.} \\ \mathbb{Q}\times\mathbb{Q}, \mathbb{Q}\times\mathbb{Q}(\sqrt{-D}), \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}(\sqrt{-D})) \end{cases}$
- The case where A is geometrically simple is open:
  - There are 19 possibilities for the CM field (Murabayashi-Umegaki)
  - Nothing is known for the real quadratic field or quaternion algebra
- $\mathcal{A}^{\mathrm{split}}_{2,1} = \{\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \colon A/\mathbb{Q},\ \dim(A) = 2,\ A \text{ geom. split}\}$
- Theorem (Shafarevic): $\mathcal{A}^{\mathrm{split}}_{2,1}$ is finite
  - Our goal is to determine $\mathcal{A}^{\mathrm{split}}_{2,1}$ explicitly

Main Theorem

Theorem (Fité–G., 2018)

The set $\mathcal{A}_{2,1}^{\text{split}}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:

1. $\mathbb{Q} \times \mathbb{Q}$, $\mathrm{M}_2(\mathbb{Q})$;
2. $\mathbb{Q} \times M_1$, $M_1 \times M_2$, with M_i quadratic imag. fields of $\#\mathrm{Cl}(M_i) = 1$;
3. $\mathrm{M}_2(M)$ with M quadratic imaginary field, $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_2 \times \mathrm{C}_2$ and M distinct from

$$\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760})$$

In particular, the set $\mathcal{A}_{2,1}^{\text{split}}$ has cardinality 92.

1. If $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$ or $A_{\overline{\mathbb{Q}}} \sim E_1^2$ with E_i non-CM
2. If $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$ and E_i can have CM
   - Here [FKRS] showed that each E_i can be defined over $\mathbb{Q}$
3. Here $A_{\overline{\mathbb{Q}}} \sim E^2$ with E with CM by M: here is where the work is

Squares of CM elliptic curves

Central question

If $A/\mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^2$ and E has CM by M, what are the possible M's?

Theorem (Fité–G., 2015)

Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_2$, or $\mathrm{C}_2 \times \mathrm{C}_2$

- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves
  - $K/\mathbb{Q}$ minimal such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathrm{End}(A_K) \rightsquigarrow E/K$ and $A_K \sim E^2$
  - $\sigma \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow ({}^\sigma E)^2 \sim {}^\sigma A_K = A_K \sim E^2$
    - ★ There is an isogeny $\mu_\sigma : {}^\sigma E \longrightarrow E$
  - Cohomology class $c_E \in H^2(\mathrm{Gal}(K/\mathbb{Q}), M^\times)$
    - ★ $c_E(\sigma, \tau) = \mu_\sigma \circ {}^\sigma \mu_\tau \circ \mu_{\sigma\tau}^{-1}$

Weil descent (up to isogeny)

For $L \subset K$, there exists C/L with $E \sim C_K \iff c_{E|\mathrm{Gal}(K/L)} = 1$.

- $A/\mathbb{Q}$ with $A_K \sim E^2$, and E has CM by M.

Theorem (Fité–G., 2015)

Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_2$, or $\mathrm{C}_2 \times \mathrm{C}_2$

- Known that $\mathrm{Gal}(K/M) \simeq \mathrm{C}_1, \mathrm{C}_r, \mathrm{D}_r$ with $r \in \{2, 3, 4, 6\}$.
- $A_K \sim E^2 \Rightarrow c_{E|\mathrm{Gal}(K/M)} \in H^2(\mathrm{Gal}(K/M), M^\times)[2] \simeq$

$$H^2(\mathrm{Gal}(K/M), \{\pm 1\}) \times \mathrm{Hom}(\mathrm{Gal}(K/M), P/P^2), \text{ where } P = M^\times/\{\pm 1\}$$

- $\exists\, N \subset K$ such that $\mathrm{Gal}(N/M) \simeq \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_2 \times \mathrm{C}_2$ and $c_{E\mathrm{Gal}(K/N)} = 1$
- Weil descent: E can be defined over N so $M(j(E)) \subset N$
- CM theory: $M(j(E)) =$ Hilbert class field of M
- $\mathrm{Cl}(M) \simeq \mathrm{Gal}(M(j(E))/M) = \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_2 \times \mathrm{C}_2$.

Now the question is

of these possible M's, which ones do really occur?

- Give a construction of A's for some M's and rule out the other M's

Constructing abelian surfaces: restriction of scalars

Weil's restriction of scalars

L/k a finite field extension and X/L a variety; $\mathcal{X} = \mathrm{Res}_{L/k} X$ is a variety over k representing the functor of k-schemes $S \mapsto X(S \times_k L)$.

- In particular, $\mathcal{X}(k) \simeq X(L)$
- If X/L is an abelian variety:
  - $\mathcal{X}$ is an abelian variety over k of dimension $[L : k] \dim X$
  - Y/k abelian variety $\rightsquigarrow \mathrm{Hom}(Y, \mathcal{X}) \simeq \mathrm{Hom}(Y_L, X)$
  - $\mathcal{X}_L = \prod_{\sigma \in \mathrm{Gal}(L/k)} {}^{\sigma}X$

Constructing abelian surfaces: the basic cases

Goal

Given M with $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_2$, construct $A/\mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^2$ and E with CM by M.

- If $\mathrm{Cl}(M) = 1$
  - take $E/\mathbb{Q}$ with CM by M and $A = E \times E$.
- If $\mathrm{Cl}(M) = \mathrm{C}_2$
  - If E has CM by $\mathcal{O}_M$ then $[\mathbb{Q}(j_E) : \mathbb{Q}] = 2$, so we can take $E/\mathbb{Q}(j(E))$
  - $A = \mathrm{Res}_{\mathbb{Q}(j_E)/\mathbb{Q}} E$ has dimension 2 and $A_{\overline{\mathbb{Q}}} \sim E \times {}^{\sigma}E$
  - If E has CM, then ${}^{\sigma}E_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}$ and therefore $A_{\overline{\mathbb{Q}}} \sim E^2$
- If $\mathrm{Cl}(M) = \mathrm{C}_2 \times \mathrm{C}_2$ then $[\mathbb{Q}(j_E) : \mathbb{Q}] = 4$ and $\mathrm{Res}_{\mathbb{Q}(j_E)/\mathbb{Q}} E$ has dim 4
  - Idea: choose E so that $\mathrm{Res}_{\mathbb{Q}(j_E)/\mathbb{Q}} \sim A^2$
  - We will take E to be a Gross $\mathbb{Q}$-curve

Case $\mathrm{C}_2 \times \mathrm{C}_2$: Gross's $\mathbb{Q}$-curves

- $M = \mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_2 \times \mathrm{C}_2$ for

$$D \in \{84, 120, 132, 168, 195, 228, 280, 312, 340, 372, 408, 435, 483, 520, 532, 555, 595, 627, 708, 715, 760, 795, 1012, 1435\}$$

- $H =$ Hilbert class field of M.
- A Gross $\mathbb{Q}$-curve is
  - E/H elliptic curve with CM by M s.t. ${}^{\sigma}E \sim E$ $\forall \sigma \in \mathrm{Gal}(H/\mathbb{Q})$

Theorem (Shimura–Nakamura)

If $\mathrm{Disc}(M) \neq -4 \times (\text{primes} \equiv 1 \pmod 4)$: $\exists$ Gross $\mathbb{Q}$-curve E/H

- The only exception is $D = 340$
- If $\mathcal{E} = \mathrm{Res}_{\mathbb{Q}(j_E)/\mathbb{Q}} E$ then $\mathrm{End}^0(\mathcal{E}) \simeq \mathbb{Q}^{c_E}[\mathrm{Gal}(\mathbb{Q}(j_E)/\mathbb{Q}]$
- For $D \neq 340$, Nakamura showed that:
  - For each D, Gross $\mathbb{Q}$-curves D give rise to 8 cohomology classes
  - Gave a method for computing all these cohomology classes c_E
- If $\mathrm{End}^0(\mathcal{E}) \simeq \mathrm{M}_2(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^2$ and we're done!

Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\mathrm{End}(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.
  1. For $D \in \{84, 120, 132, 168, 228, 280, 372, 408, 435, 483, 520, 532, 595, 627, 708, 795, 1012, 1435\}$ at least one of the $\mathbb{Q}$-curves has $\mathrm{End}^0(\mathcal{E}) \simeq \mathrm{M}_2(\mathbb{Q})$.
  2. For $D \in \{195, 312, 555, 715, 760\}$
     all $\mathbb{Q}$-curves have $\mathrm{End}^0(\mathcal{E}) \simeq \begin{cases} \text{number field} \\ \text{division quaternion algebra} \end{cases}$
     $\Rightarrow \mathcal{E}$ is simple over $\mathbb{Q}$ of dimension 4
- Need to show that for the fields M in 2, A does not exist

Ruling out abelian surfaces: the simplest case

- $M = \mathbb{Q}(\sqrt{-D})$ s.t. for any Gross $\mathbb{Q}$-curve E/H we know that $\mathrm{Res}_{H/\mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists\ A/\mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E_0^2$ and E_0 has CM by M.
- $K =$ minimal field where this decomposition takes place.
- If $\mathrm{Gal}(K/M) \simeq \mathrm{C}_2 \times \mathrm{C}_2$
  - $H \subset K$ and $\mathrm{Gal}(H/M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_2 \times \mathrm{C}_2 \Rightarrow K = H$
- Then E_0 is a Gross $\mathbb{Q}$-curve, but this is a contradiction!
  - $\mathrm{Hom}(A_H, E_0) \neq 0 \Rightarrow \mathrm{Hom}(A, \mathrm{Res}_{H/\mathbb{Q}} E_0) \neq 0$.
  - But the simple factors of $\mathrm{Res}_{H/\mathbb{Q}} E_0$ are of dimension 4.
- If $\mathrm{Gal}(H/K) \simeq \mathrm{C}_r, \mathrm{D}_r$ with $r \in \{3, 4, 6\}$
  - One needs to sharpen the argument...

Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

Francesc Fité (MIT)   Xevi Guitart (UB)

SGA Barcelona 2020