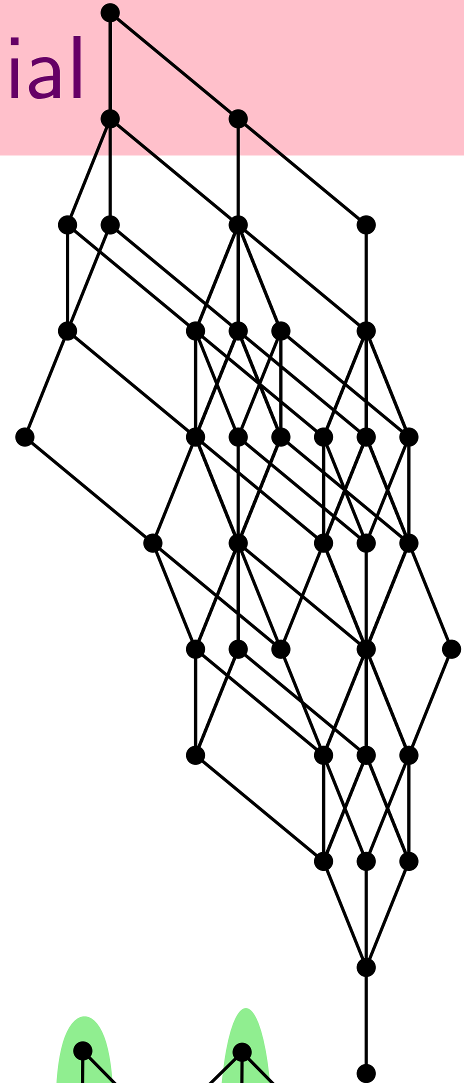
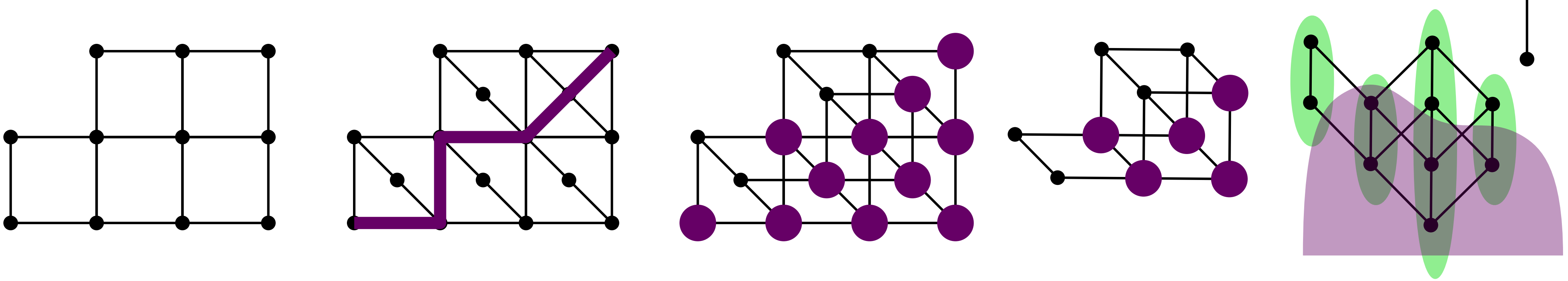
On lattice path matroid polytopes: integer points and Ehrhart polynomial



Jorge Luis Ramírez Alfonsín
Institut Montpelliérain Alexander Grothendieck,
Université de Montpellier

Leonardo Martínez-Sandoval
Faculté des Sciences
Sorbonne Université

Kolja Knauer
Universitat de Barcelona



seminario algebra conmutativa/geometria algebraica, febrer 2020

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

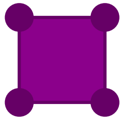
$$L_P(k) := \left|kP \cap \mathbb{Z}^{d'}\right|.$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$

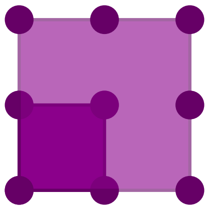
k	1
$L_{Q_2}(k)$	4

Q_2 

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$

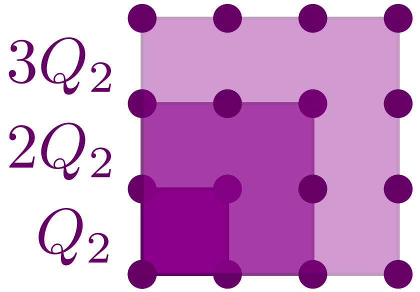


k	1	2
$L_{Q_2}(k)$	4	9

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$

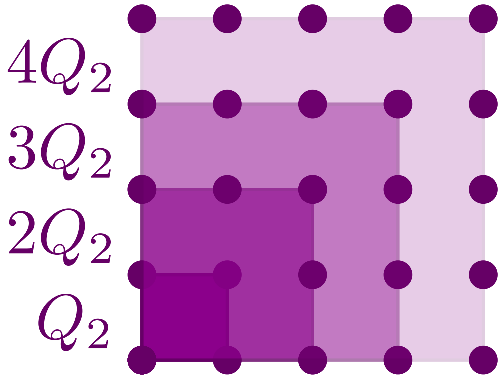


k	1	2	3
$L_{Q_2}(k)$	4	9	16

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$

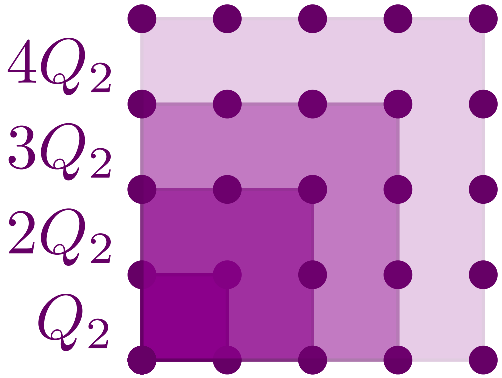


k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



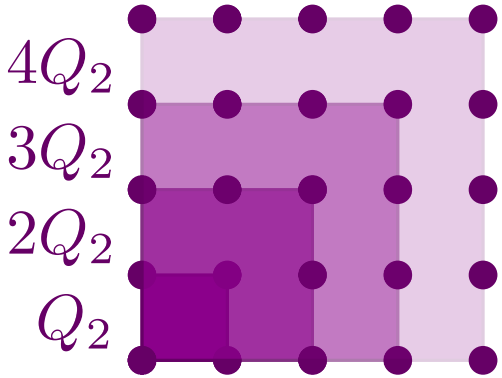
k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

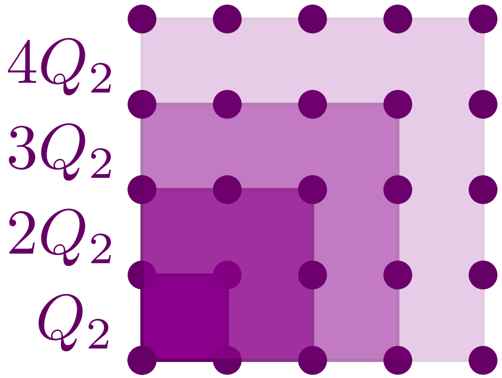
$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(k) = (k+1)^d = \sum_{i=0}^{d} \binom{d}{i} k^i$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

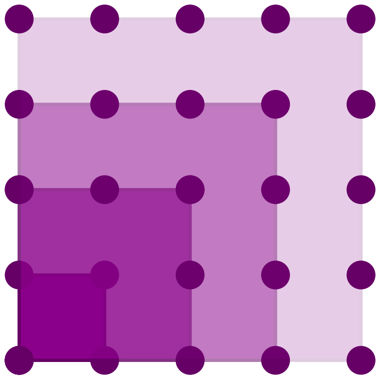
$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$$

- degree d polynomial
- rational coefficients

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$$

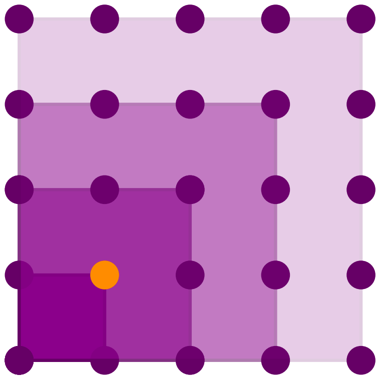
- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := \left|kP \cap \mathbb{Z}^{d'}\right|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$

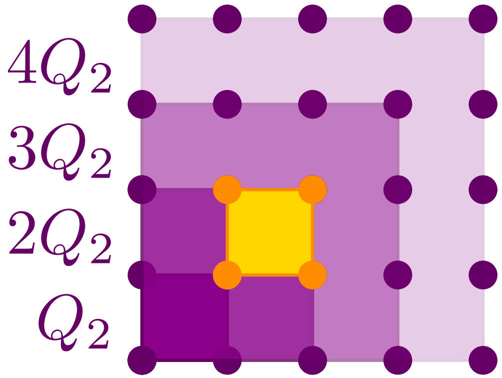
- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := \left|kP \cap \mathbb{Z}^{d'}\right|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$$

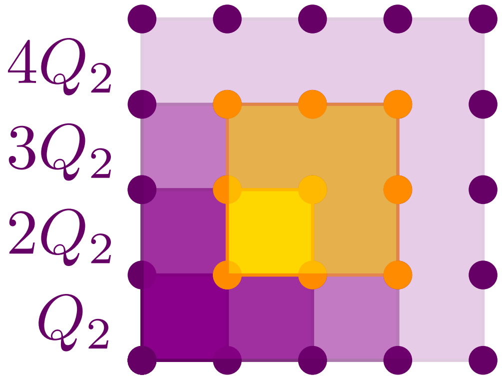
- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$

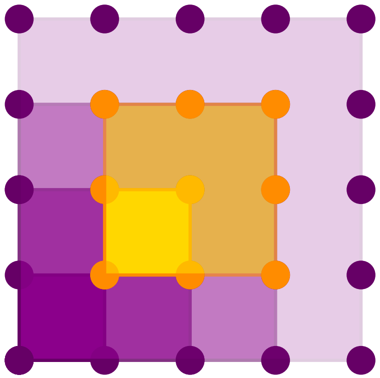
- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := \left|kP \cap \mathbb{Z}^{d'}\right|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$$

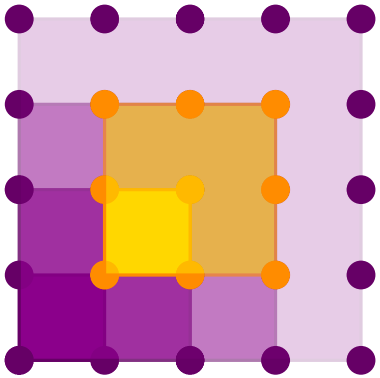
- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d$$
$$= (-1)^d L_{Q_d}(-k)$$

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$$

- degree d polynomial
- rational coefficients

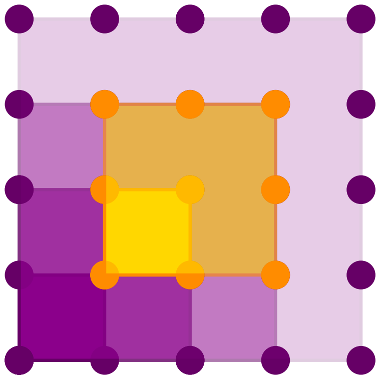
$$L_{Q_d^o}(k) = (k-1)^d = (-1)^d(1-k)^d = (-1)^d L_{Q_d}(-k)$$

Thm (Ehrhart '62):
$L_P(t)$ is a degree d polynomial with rational coefficients.
Moreover, $L_P(-k) = (-1)^d L_{P^o}(k)$ for all $k \in \mathbb{N}$.

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := \left|kP \cap \mathbb{Z}^{d'}\right|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^{d} \binom{d}{i} t^i$$

- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d = (-1)^d L_{Q_d}(-k)$$

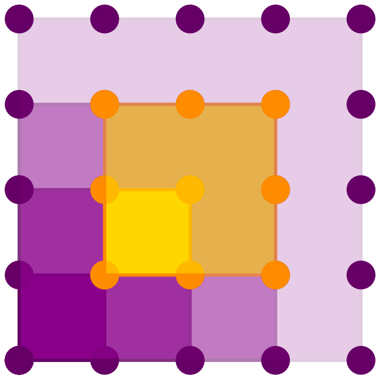
Thm (Ehrhart '62):
$L_P(t)$ is a degree d polynomial with rational coefficients.
Moreover, $L_P(-k) = (-1)^d L_{P^\circ}(k)$ for all $k \in \mathbb{N}$.

the *Ehrhart polynomial* of P

A little bit of Ehrhart-theory

Let $P \subset \mathbb{R}^{d'}$ be a d-dimensional integral convex polytope and $k \in \mathbb{N}$,

$$L_P(k) := |kP \cap \mathbb{Z}^{d'}|.$$



k	1	2	3	4
$L_{Q_2}(k)$	4	9	16	25

$$L_{Q_2}(k) = (k+1)^2$$

$$L_{Q_d}(t) = (t+1)^d = \sum_{i=0}^d \binom{d}{i} t^i$$

- degree d polynomial
- rational coefficients

$$L_{Q_d^\circ}(k) = (k-1)^d = (-1)^d(1-k)^d = (-1)^d L_{Q_d}(-k)$$

Thm (Ehrhart '62):
$L_P(t)$ is a degree d polynomial with rational coefficients.
Moreover, $L_P(-k) = (-1)^d L_{P^\circ}(k)$ for all $k \in \mathbb{N}$.

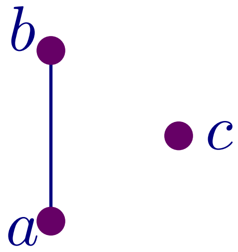
the *Ehrhart polynomial* of P

Thm (Stanley '91):
there are $h_0^*, \ldots, h_d^* \geq 0$ such that $L_P(t) = \sum_{i=0}^d h_i^* \binom{t+d-i}{d}$.

Order polytopes

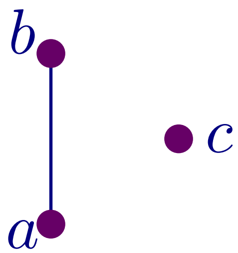
Order polytopes

poset X



Order polytopes

poset X

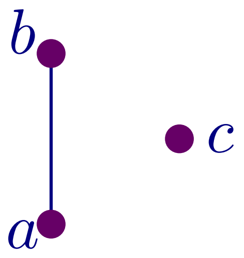


$I \subseteq X$ is an *ideal* if

$y \leq x \in I \implies y \in I$

Order polytopes

poset X



$I \subseteq X$ is an *ideal* if

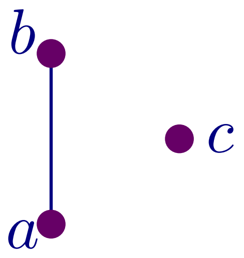
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X

ordered by inclusion

Order polytopes

poset X



$I \subseteq X$ is an *ideal* if

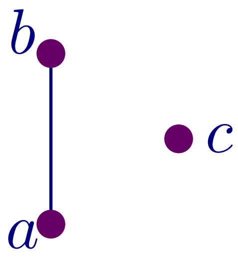
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$

Order polytopes

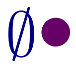
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

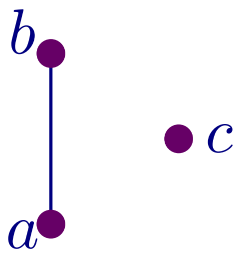
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$



Order polytopes

poset X

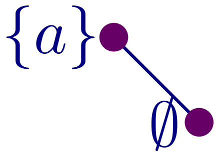


$I \subseteq X$ is an *ideal* if

$y \leq x \in I \implies y \in I$

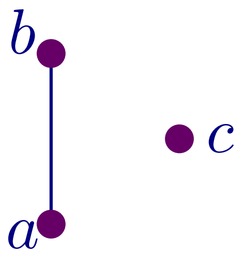
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

$\mathcal{I}(X)$



Order polytopes

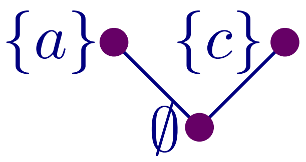
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

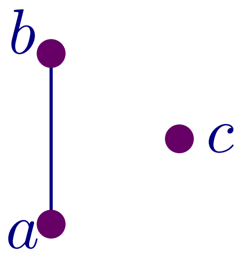
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$



Order polytopes

poset X

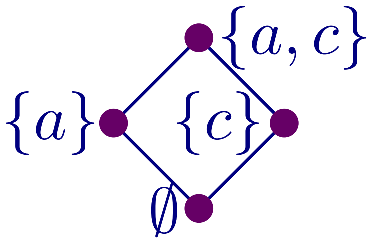


$I \subseteq X$ is an *ideal* if

$y \leq x \in I \implies y \in I$

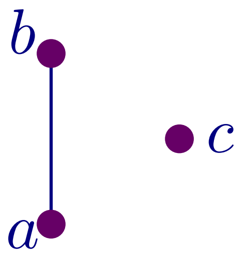
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

$\mathcal{I}(X)$



Order polytopes

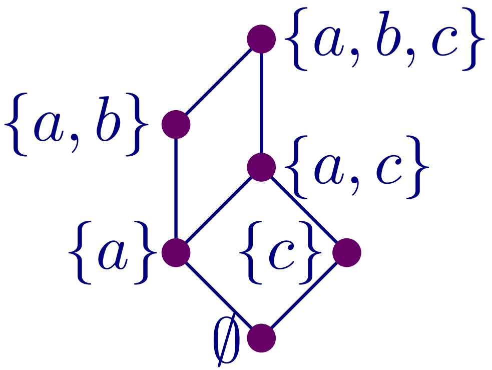
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

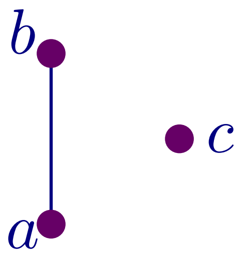
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$



Order polytopes

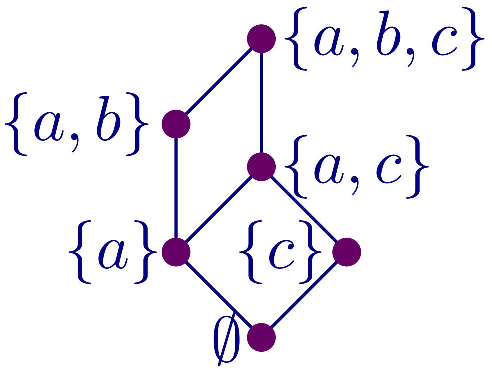
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

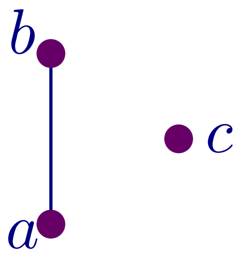
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



Order polytopes

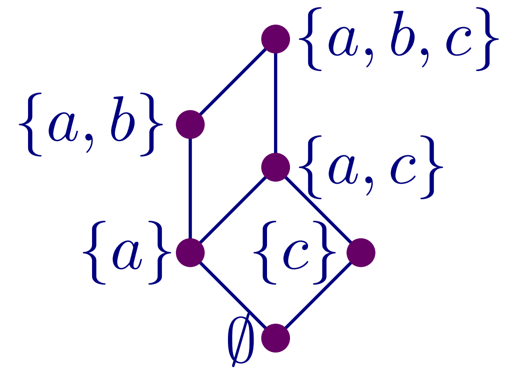
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

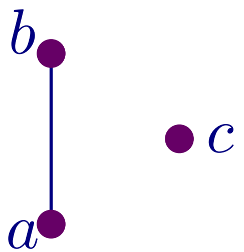
$\mathcal{I}(X)$ distributive lattice



order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$

Order polytopes

poset X

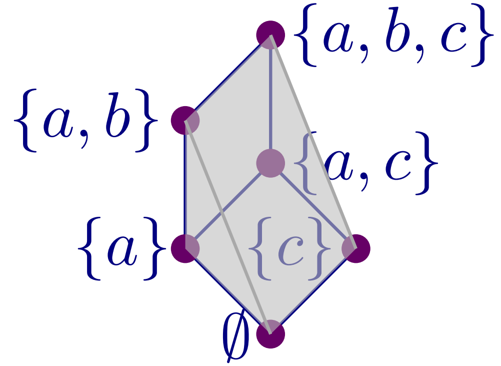


$I \subseteq X$ is an *ideal* if

$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

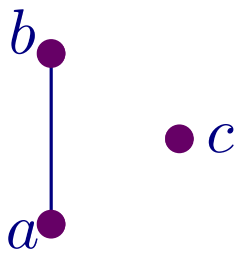
$\mathcal{I}(X)$ distributive lattice



order polytope $P_X =$ conv{characteristic vectors of $\mathcal{I}(X)$}

Order polytopes

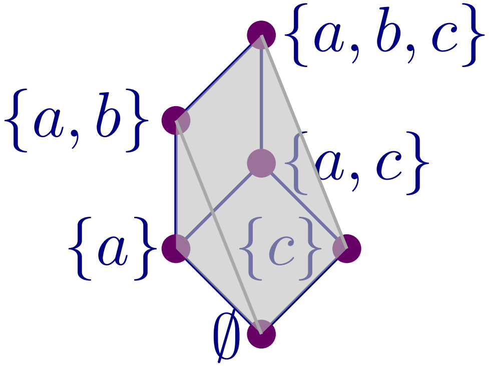
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

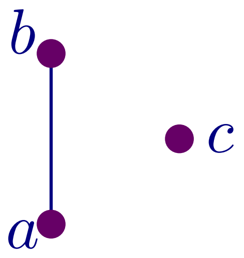
$\mathcal{I}(X)$ distributive lattice



order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

Order polytopes

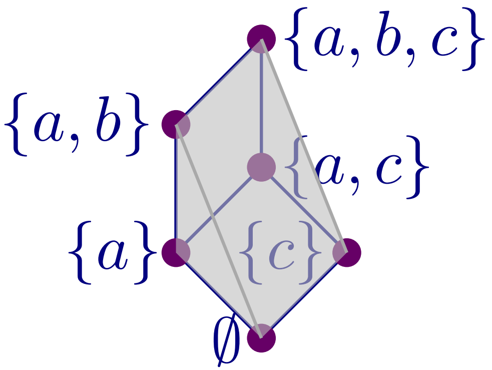
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

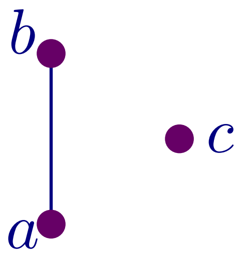


order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

Order polytopes

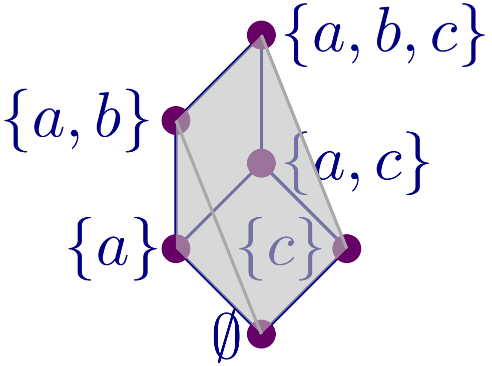
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



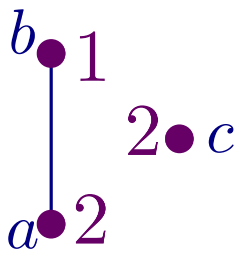
order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

$L_{P_X}(2) =$

Order polytopes

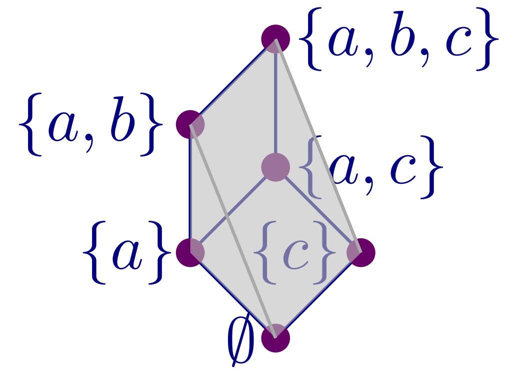
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



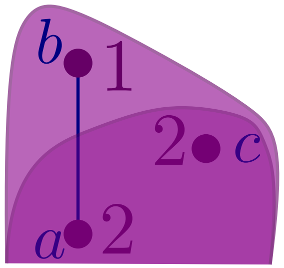
order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,2]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

$L_{P_X}(2) =$

Order polytopes

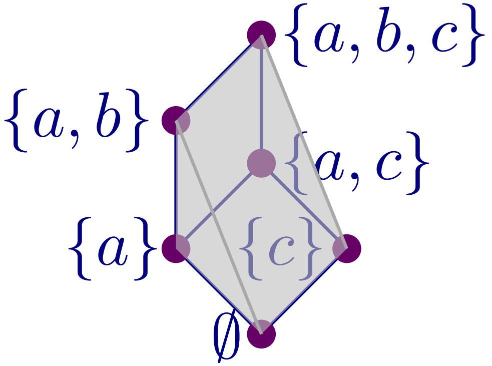
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



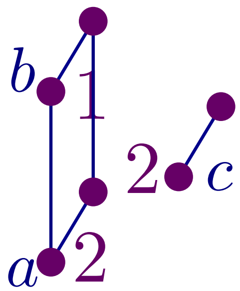
order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,2]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

$L_{P_X}(2) =$

Order polytopes

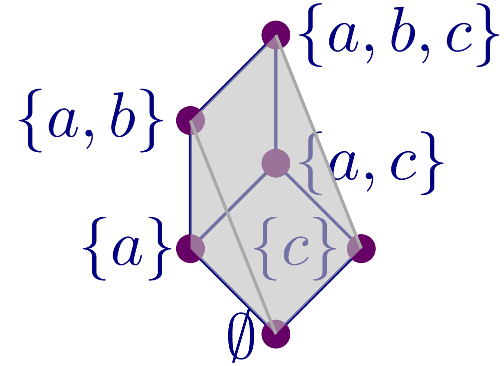
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



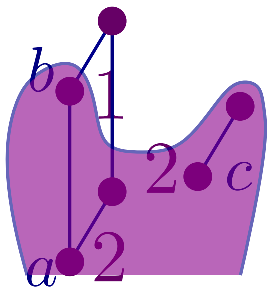
order polytope P_X = conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,2]^d \mid x_i \leq x_j$ if $i \geq j$ in $X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

$L_{P_X}(2) =$

Order polytopes

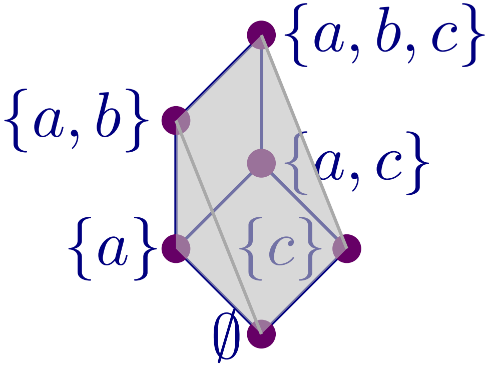
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



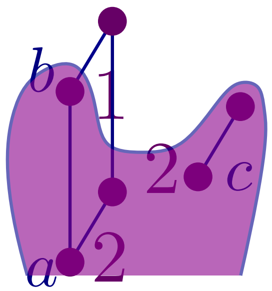
order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,2]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

$L_{P_X}(2) =$

Order polytopes

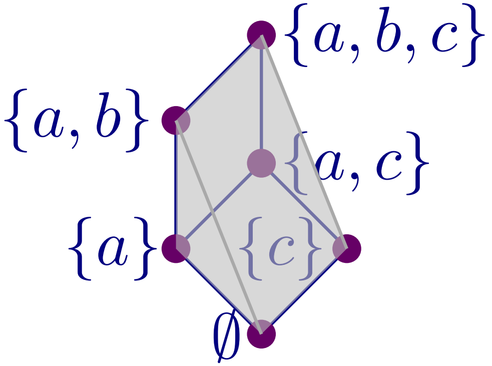
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



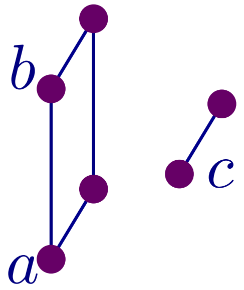
order polytope P_X = conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,2]^d \mid x_i \leq x_j$ if $i \geq j$ in $X\}$

$L_{P_X}(1) = |\mathcal{I}(X)|$

$L_{P_X}(2) = |\mathcal{I}(X \times P_2)|$

Order polytopes

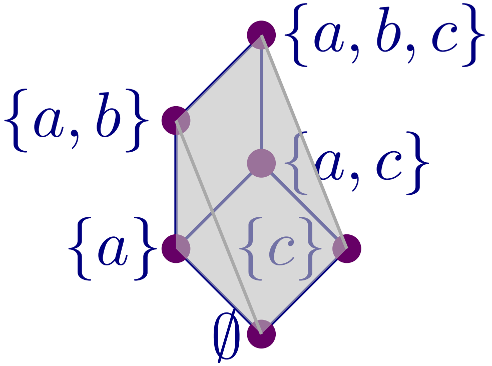
poset X



$I \subseteq X$ is an *ideal* if
$y \le x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

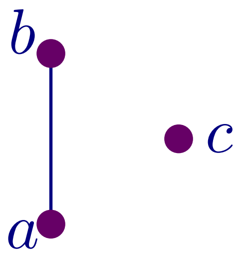


order polytope P_X = conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,1]^d \mid x_i \le x_j$ if $i \ge j$ in $X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

Order polytopes

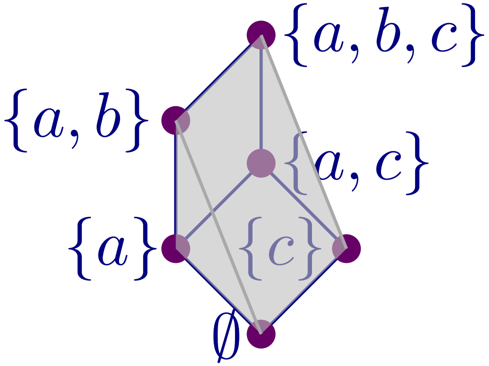
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



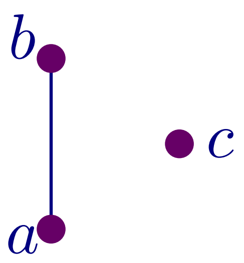
order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

$\mapsto$ *order polynomial* of X

Order polytopes

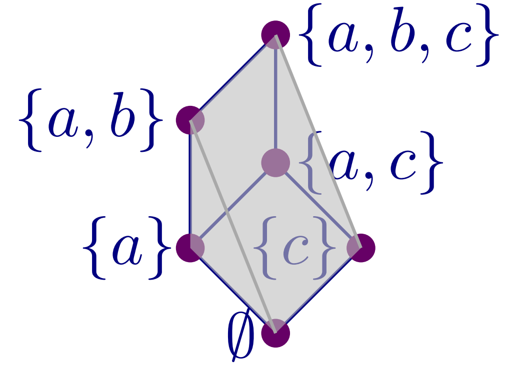
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

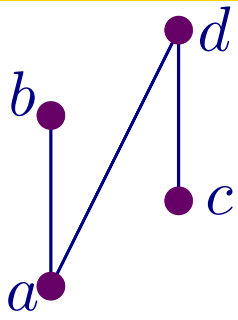
$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

Order polytopes

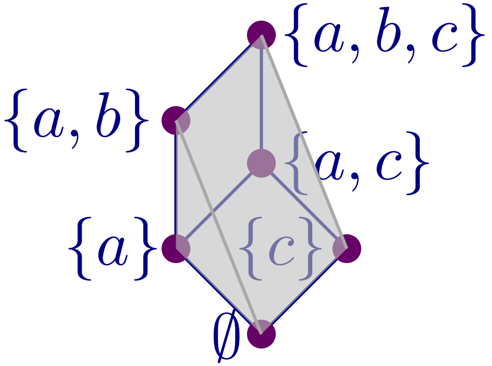
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

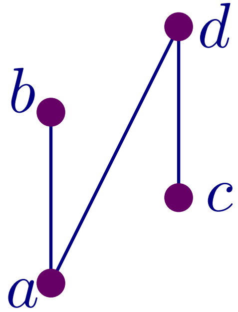
$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

Order polytopes

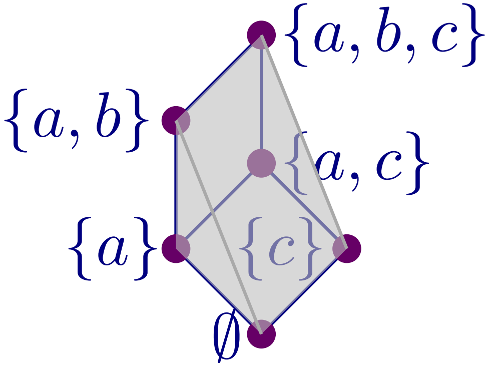
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



order polytope $P_X =$ conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,1]^d \mid x_i \leq x_j$ if $i \geq j$ in $X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

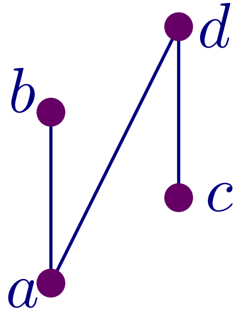
$\mapsto$*order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E'$ lin.ext of $X \mid i$ consecutive pairs in E' ordered different than $E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

$$E = abcd\bullet$$

Order polytopes

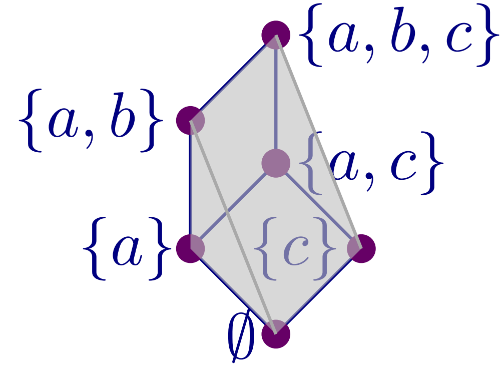
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

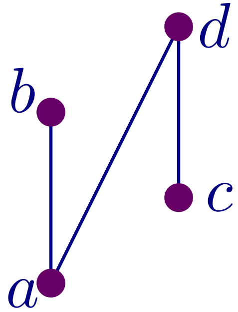
$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

Order polytopes

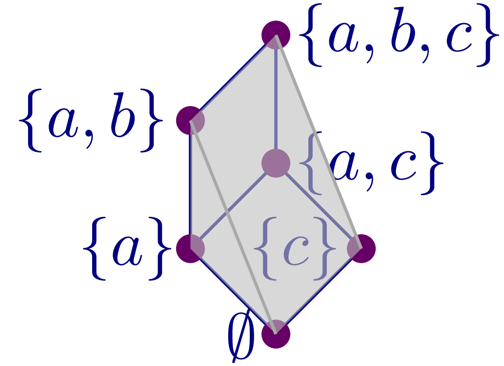
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

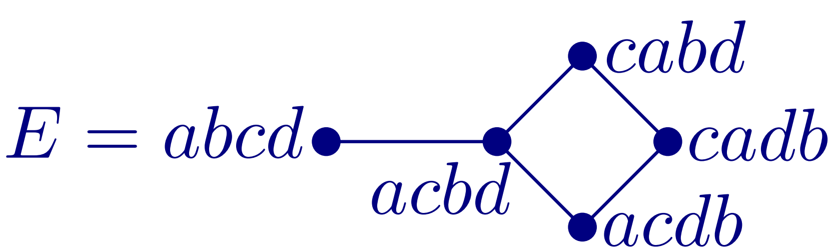


order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

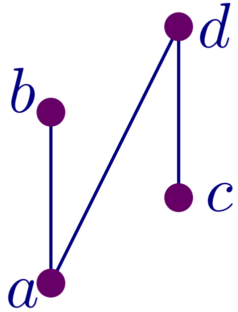
$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.



Order polytopes

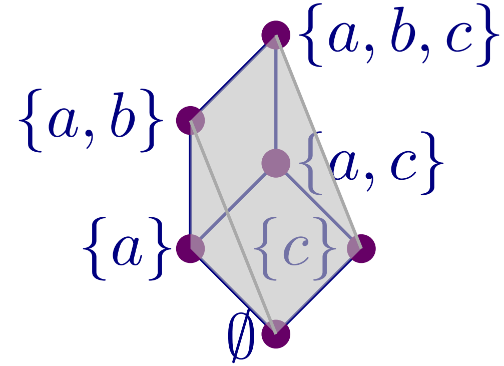
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

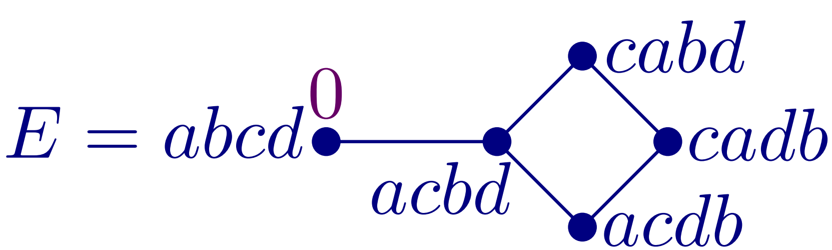


order polytope P_X = conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,1]^d \mid x_i \leq x_j$ if $i \geq j$ in $X\}$

$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

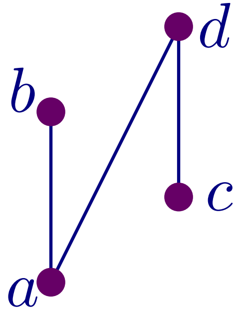
$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E'$ lin.ext of $X \mid i$ consecutive pairs in E' ordered different than $E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.



Order polytopes

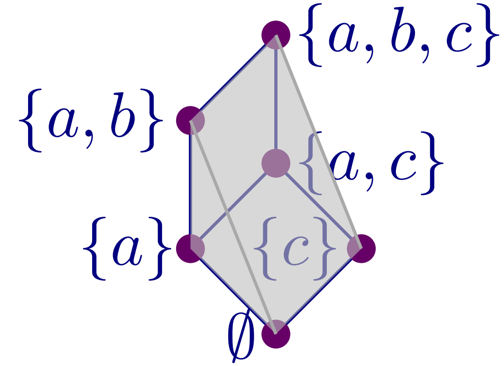
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

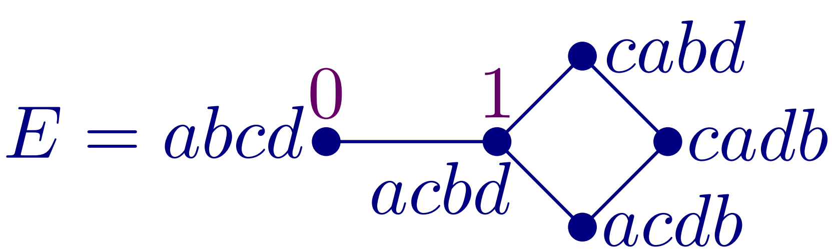


order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

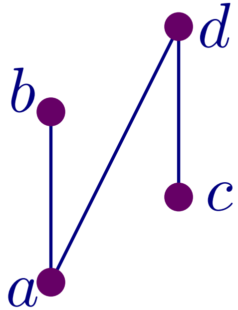
$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h^*_i$.



Order polytopes

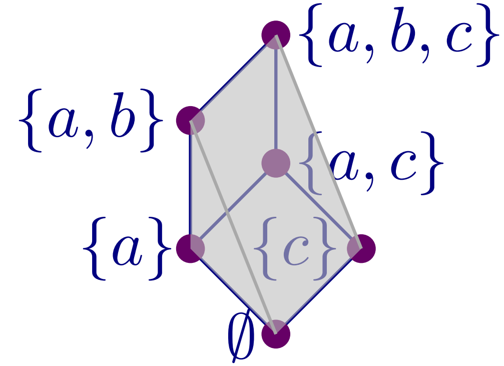
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

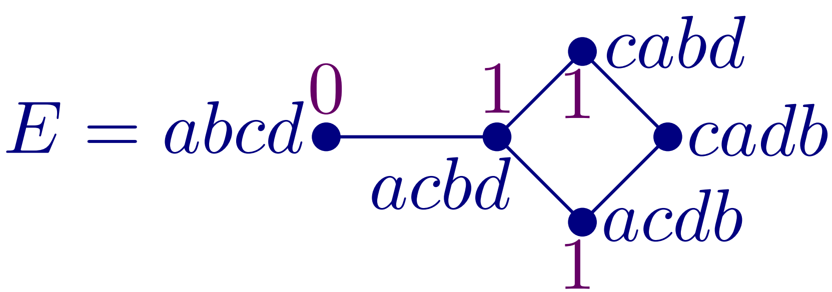


order polytope P_X = conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,1]^d \mid x_i \leq x_j$ if $i \geq j$ in $X\}$

$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

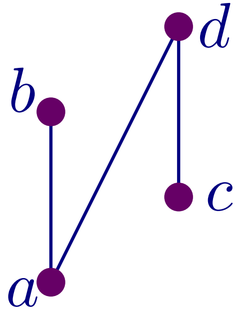
$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E'$ lin.ext of $X \mid i$ consecutive pairs in E' ordered different than $E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.



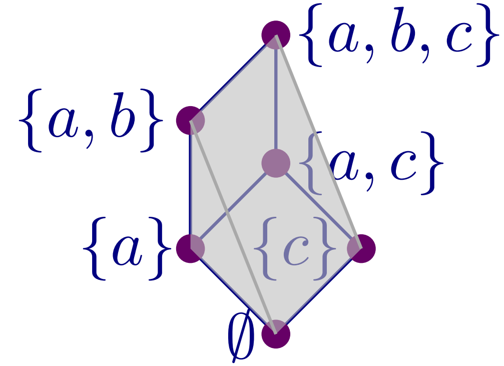
Order polytopes

poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice

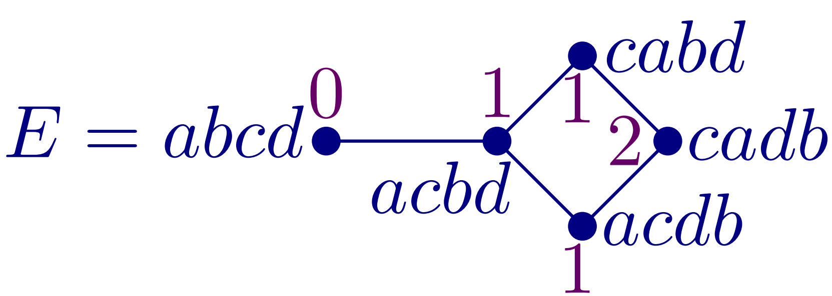


order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

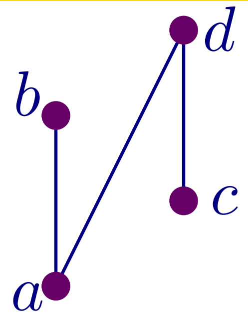
$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.



Order polytopes

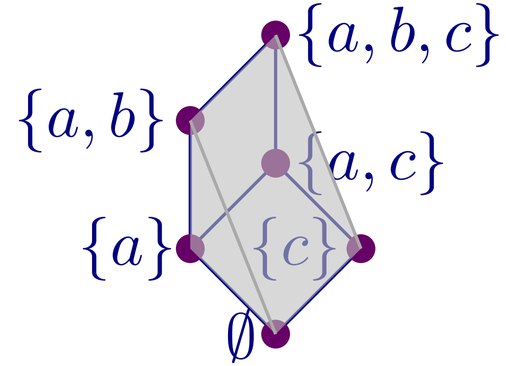
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



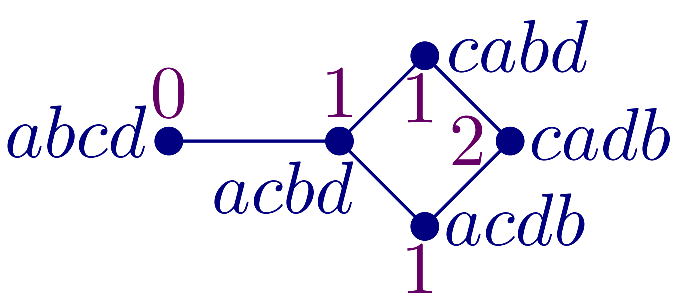
order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

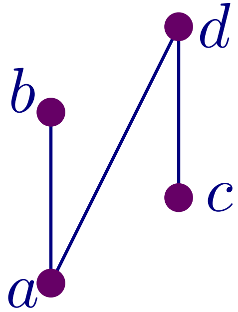
$E =$



i	0	1	2	3
ω_i	1	3	1	0

Order polytopes

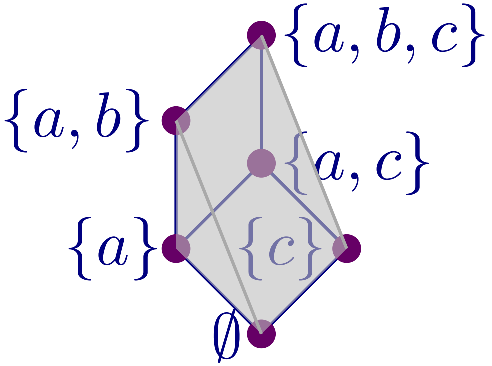
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



order polytope $P_X = \text{conv}\{\text{characteristic vectors of } \mathcal{I}(X)\}$
$= \{x \in [0,1]^d \mid x_i \leq x_j \text{ if } i \geq j \text{ in } X\}$

$$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$$

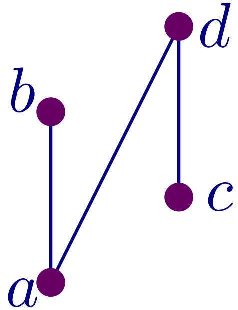
$\mapsto$ *order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E' \text{ lin.ext of } X \mid i \text{ consecutive pairs in } E' \text{ ordered different than } E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

Conjecture (Neggers '78): ω is unimodal.
(actually something stronger but false (Stembridge '06))

Order polytopes

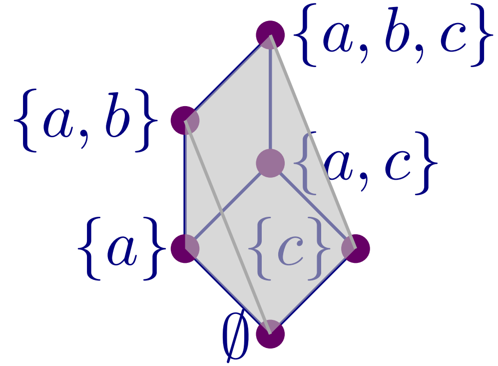
poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

$\mathcal{I}(X)$ distributive lattice



order polytope P_X = conv{characteristic vectors of $\mathcal{I}(X)$}
$= \{x \in [0,1]^d \mid x_i \leq x_j$ if $i \geq j$ in $X\}$

$L_{P_X}(k) = |\mathcal{I}(X \times P_k)| = L_{P_{X \times P_k}}(1)$

$\mapsto$*order polynomial* of X

Thm (Stanley '86): E linear extension of X,
$\omega_i = |\{E'$ lin.ext of $X \mid i$ consecutive pairs in E' ordered different than $E\}|$,
then $L_{P_X}(t) = \sum_{i=0}^{d-1} \omega_i \binom{d+t-i}{d}$, i.e., $\omega_i = h_i^*$.

Conjecture (Neggers '78): ω is unimodal.
(actually something stronger but false (Stembridge '06))

known for:
unions of chains (Simion '84) and graded posets (Reiner, Welker '05).

Matroid polytopes

A *matroid* is a pair $M = (E, \mathcal{B})$ with E finite and $\mathcal{B} \subseteq 2^E$ such that:

- $\mathcal{B}$ is non-empty.
- $\forall A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A : A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

Matroid polytopes

A *matroid* is a pair $M = (E, \mathcal{B})$ with E finite and $\mathcal{B} \subseteq 2^E$ such that:

- $\mathcal{B}$ is non-empty.
- $\forall A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A : A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

matroid (base) polytope $P_M = \text{conv}\{\text{characteristic vectors of } \mathcal{B}\}$

Matroid polytopes

A *matroid* is a pair $M = (E, \mathcal{B})$ with E finite and $\mathcal{B} \subseteq 2^E$ such that:

- $\mathcal{B}$ is non-empty.
- $\forall A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A : A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

matroid (base) polytope $P_M =$ conv{characteristic vectors of $\mathcal{B}$}

$P_M \subset \mathbb{R}^{|E|}$ is $(0,1)$-polytope of dimension $d = |E| -$ number components.

Matroid polytopes

A *matroid* is a pair $M = (E, \mathcal{B})$ with E finite and $\mathcal{B} \subseteq 2^E$ such that:

- $\mathcal{B}$ is non-empty.
- $\forall A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A : A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

matroid (base) polytope $P_M =$ conv{characteristic vectors of $\mathcal{B}$}

$P_M \subset \mathbb{R}^{|E|}$ is $(0,1)$-polytope of dimension $d = |E| -$ number components.

Conjecture (De Loera, Haws, Köppe '09): h^* of L_{P_M} is unimodal.

Matroid polytopes

A *matroid* is a pair $M = (E, \mathcal{B})$ with E finite and $\mathcal{B} \subseteq 2^E$ such that:

- $\mathcal{B}$ is non-empty.
- $\forall A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A : A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

matroid (base) polytope $P_M =$ conv{characteristic vectors of $\mathcal{B}$}

$P_M \subset \mathbb{R}^{|E|}$ is $(0,1)$-polytope of dimension $d = |E| -$ number components.

Conjecture (De Loera, Haws, Köppe '09): h^* of L_{P_M} is unimodal.

known for: $U_{2,n}$ (De Loera et. al. '09) and $U_{n,2n}$ (De Negri, Hibi '97).

Matroid polytopes

A *matroid* is a pair $M = (E, \mathcal{B})$ with E finite and $\mathcal{B} \subseteq 2^E$ such that:

- $\mathcal{B}$ is non-empty.
- $\forall A, B \in \mathcal{B}$ and $a \in A \setminus B$, $\exists b \in B \setminus A : A \setminus \{a\} \cup \{b\} \in \mathcal{B}$.

matroid (base) polytope $P_M =$ conv{characteristic vectors of $\mathcal{B}$}

$P_M \subset \mathbb{R}^{|E|}$ is $(0,1)$-polytope of dimension $d = |E| -$ number components.

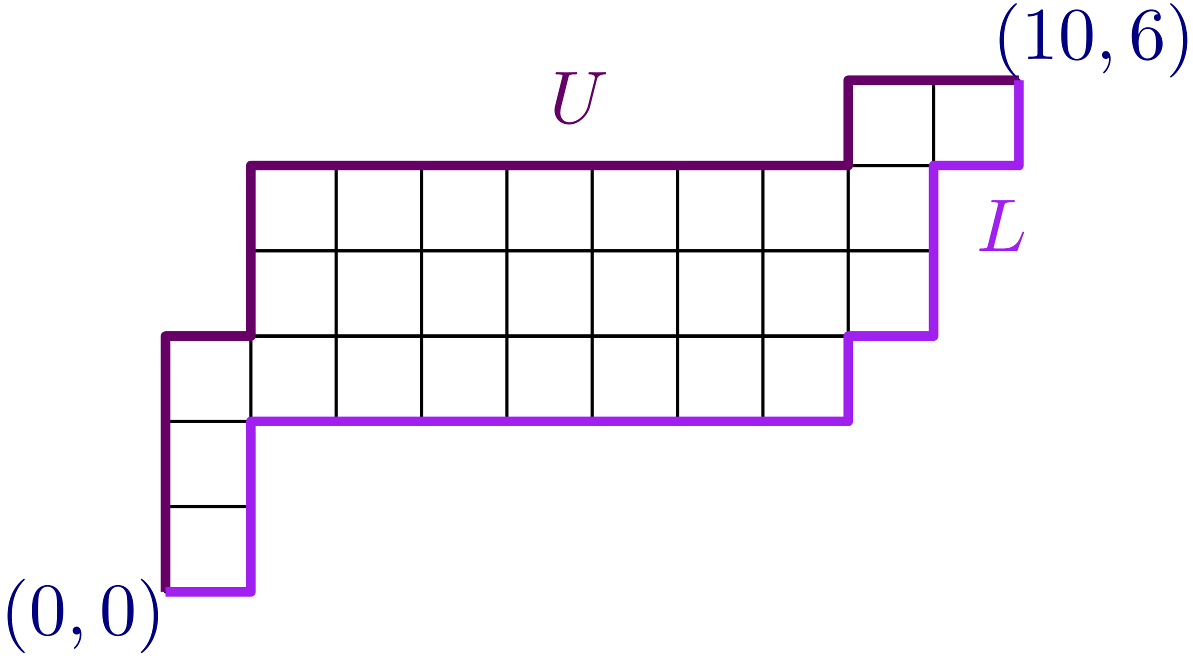
Conjecture (De Loera, Haws, Köppe '09): h^* of L_{P_M} is unimodal.

known for: $U_{2,n}$ (De Loera et. al. '09) and $U_{n,2n}$ (De Negri, Hibi '97).

We will now study P_M for *lattice path matroids* and in the end find some new families confirming the above conjecture.

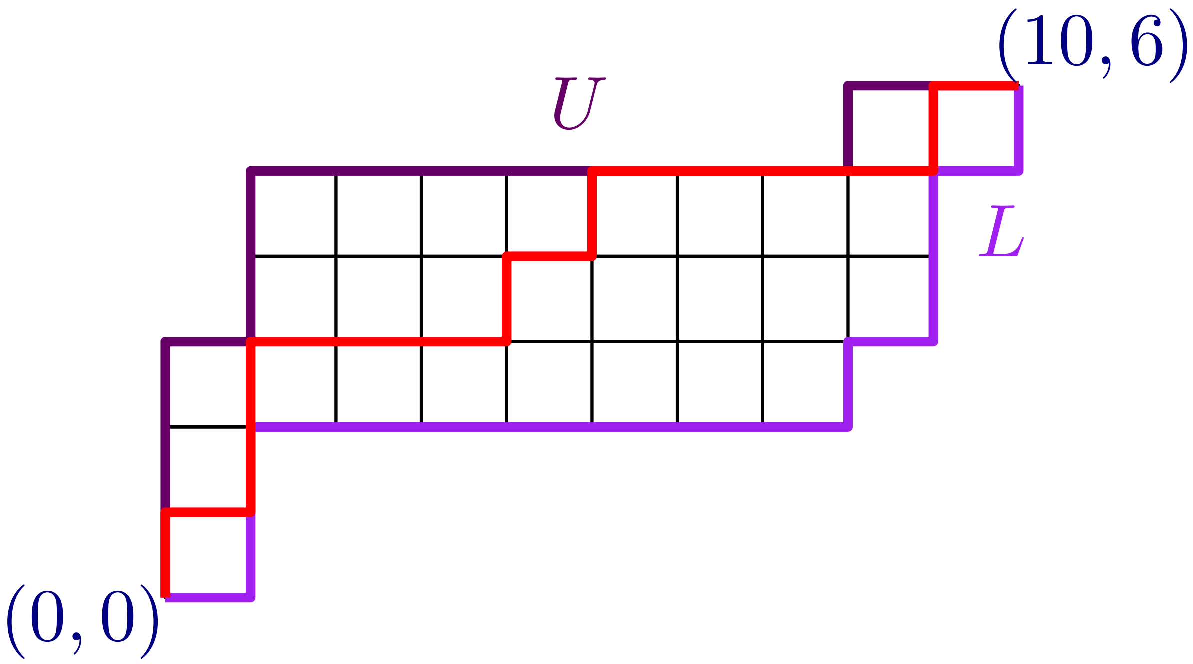
Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0, 0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0, 0)$ to (m, r).



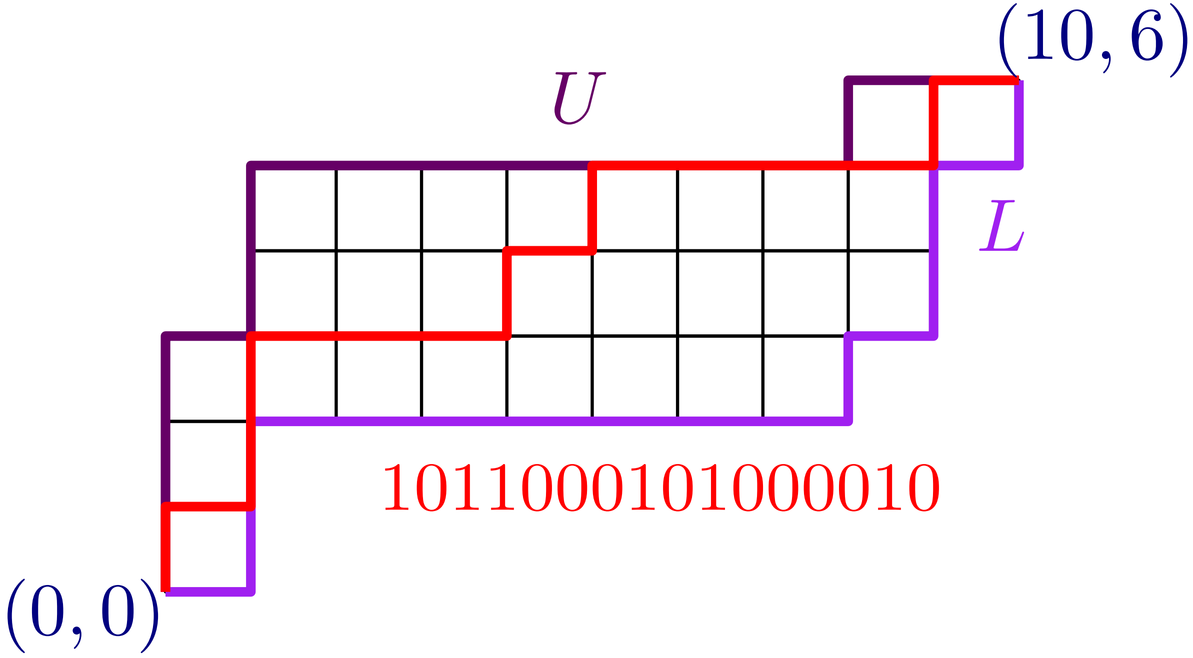
Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0, 0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0, 0)$ to (m, r).



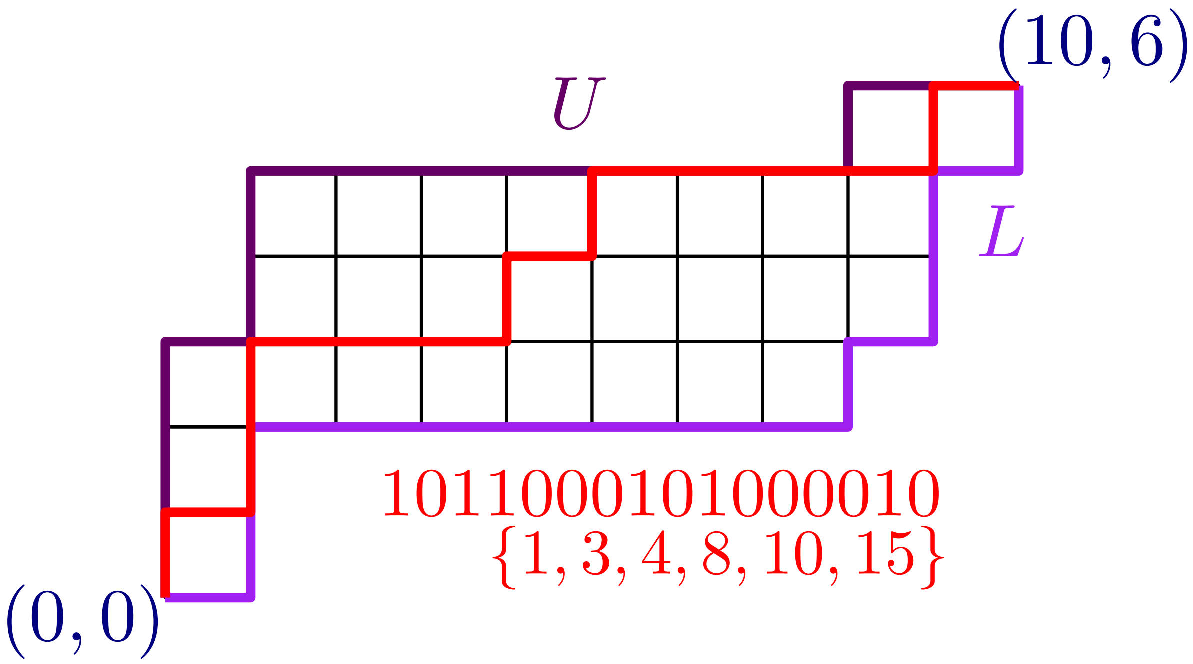
Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E = [r+m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



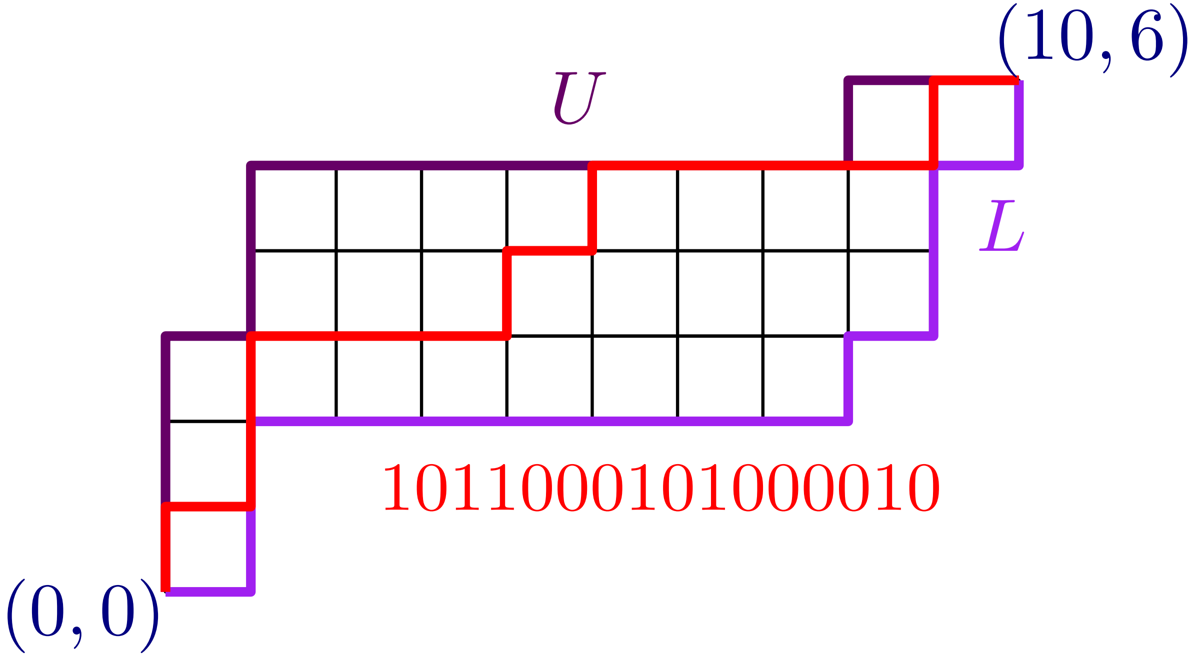
Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0, 0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0, 0)$ to (m, r).



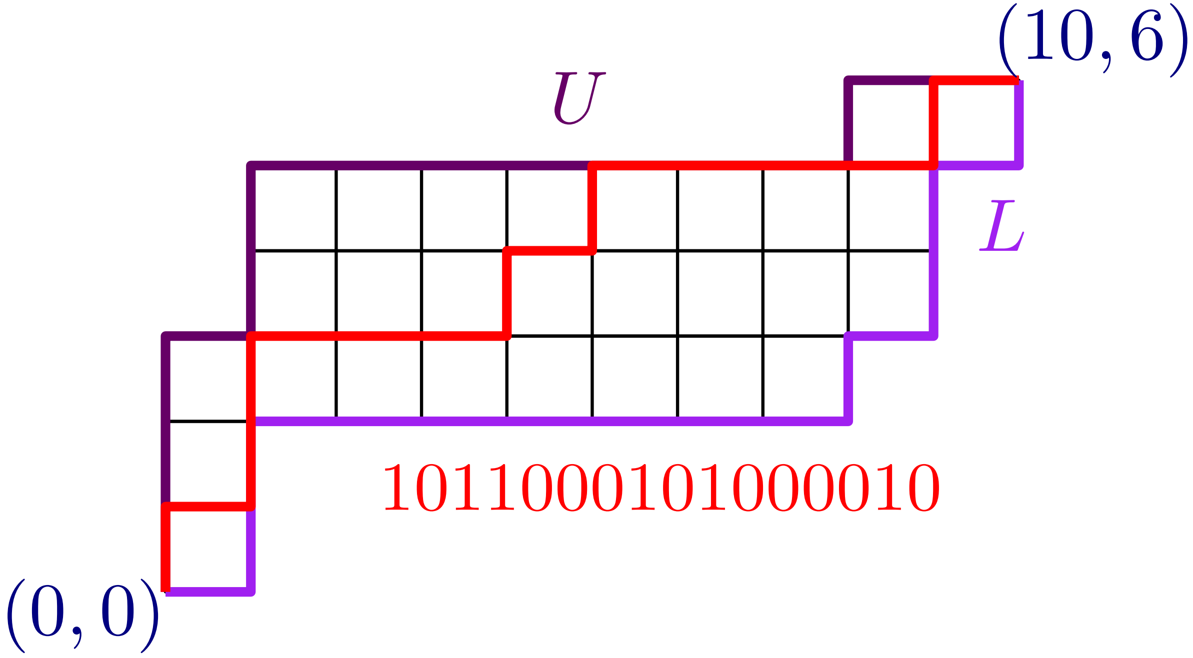
Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:

diagram from $(0,0)$ to (m,r) between L and U

$E = [r+m]$

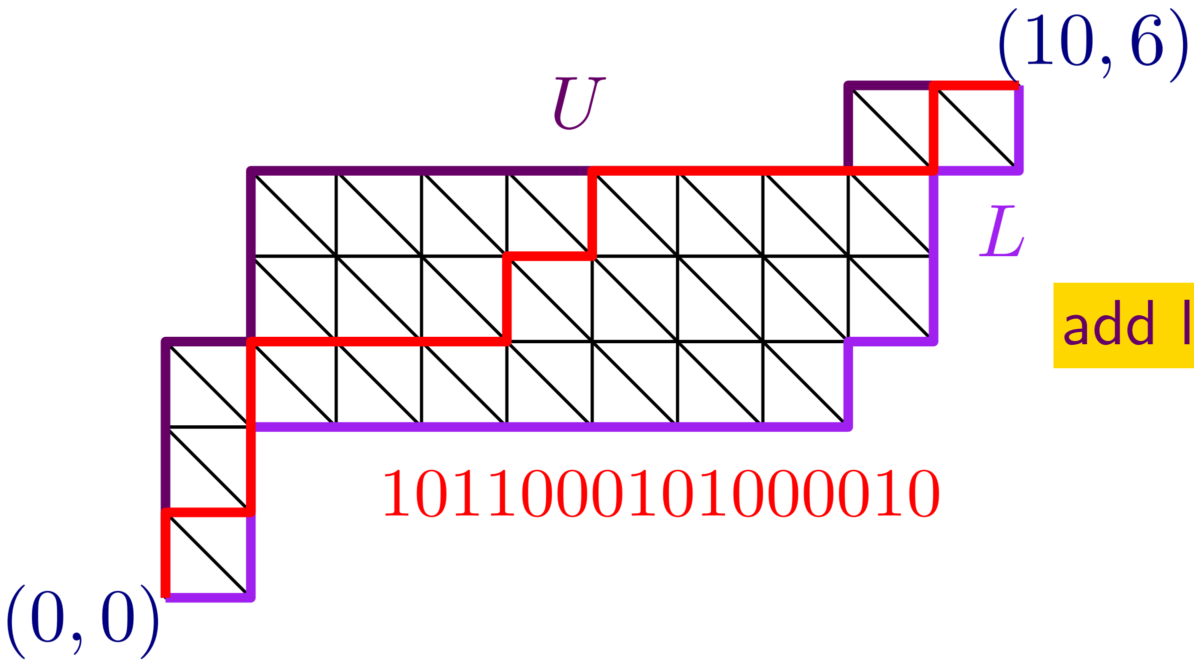
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



generalized lattice paths $\mathcal{C}_M$

Lattice path matroid polytopes

Lattice path matroid $M[U,L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E = [r+m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



generalized lattice paths $\mathcal{C}_M$

add lines $\ell_i := \{(x,y) \mid x+y=i\}$

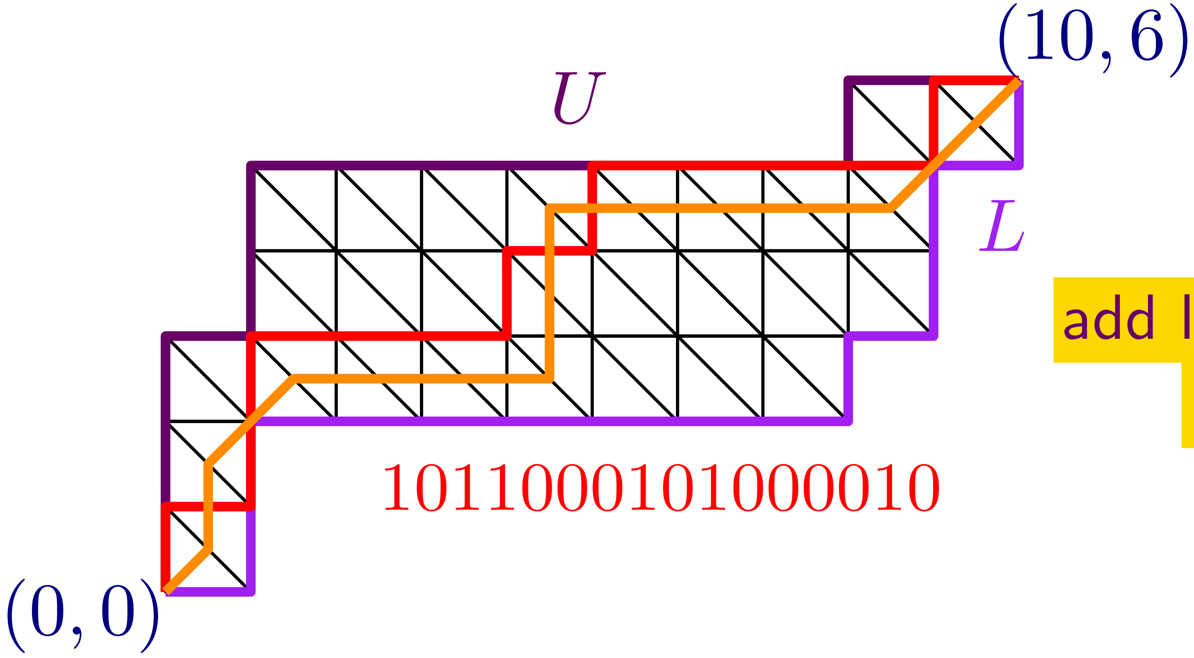
Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:

diagram from $(0,0)$ to (m,r) between L and U

$E = [r+m]$

$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



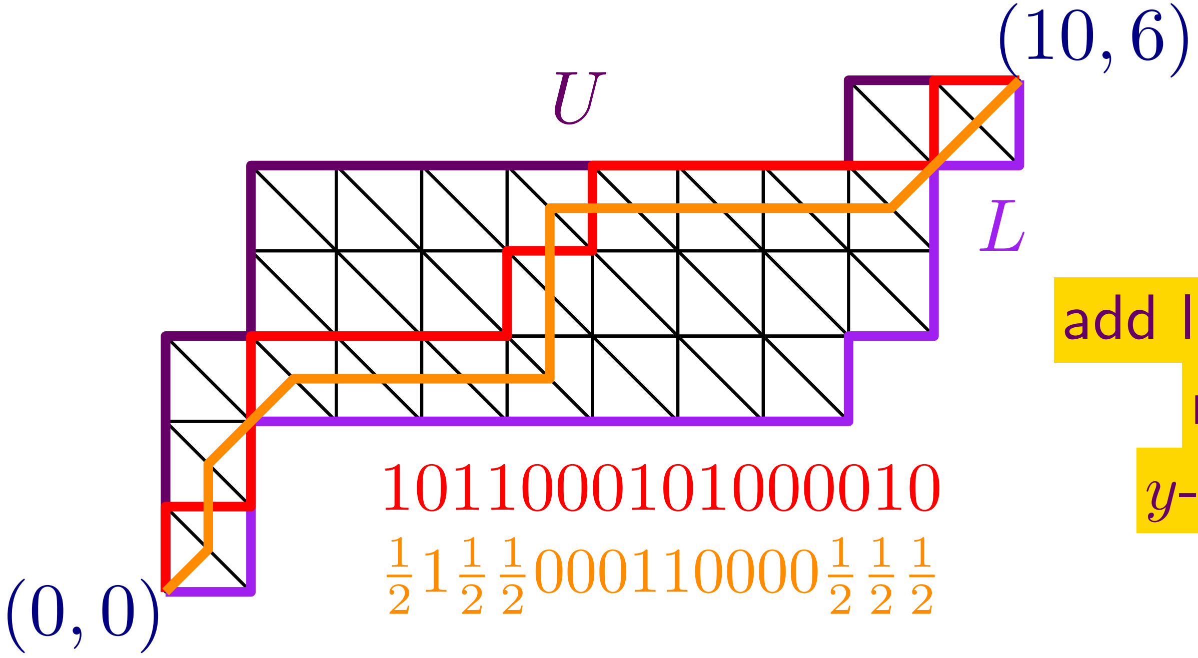
generalized lattice paths $\mathcal{C}_M$

add lines $\ell_i := \{(x,y) \mid x+y=i\}$

monotone, bending only on ℓ_i

Lattice path matroid polytopes

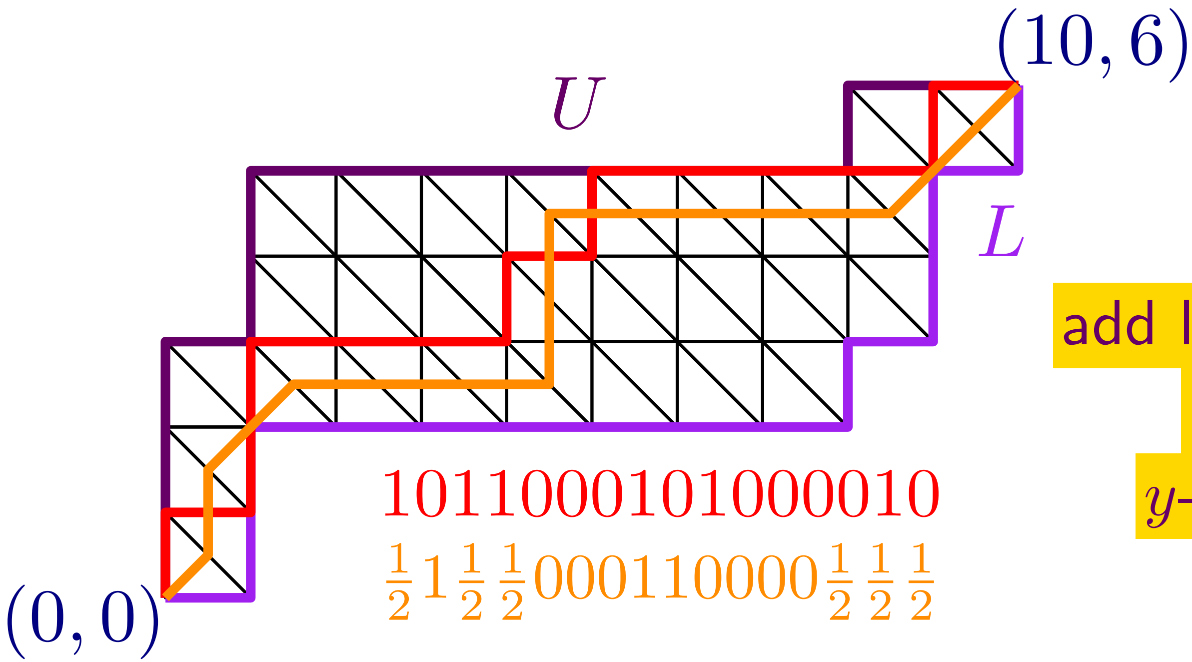
Lattice path matroid $M[U, L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E = [r+m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x,y) \mid x+y=i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E = [r+m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).

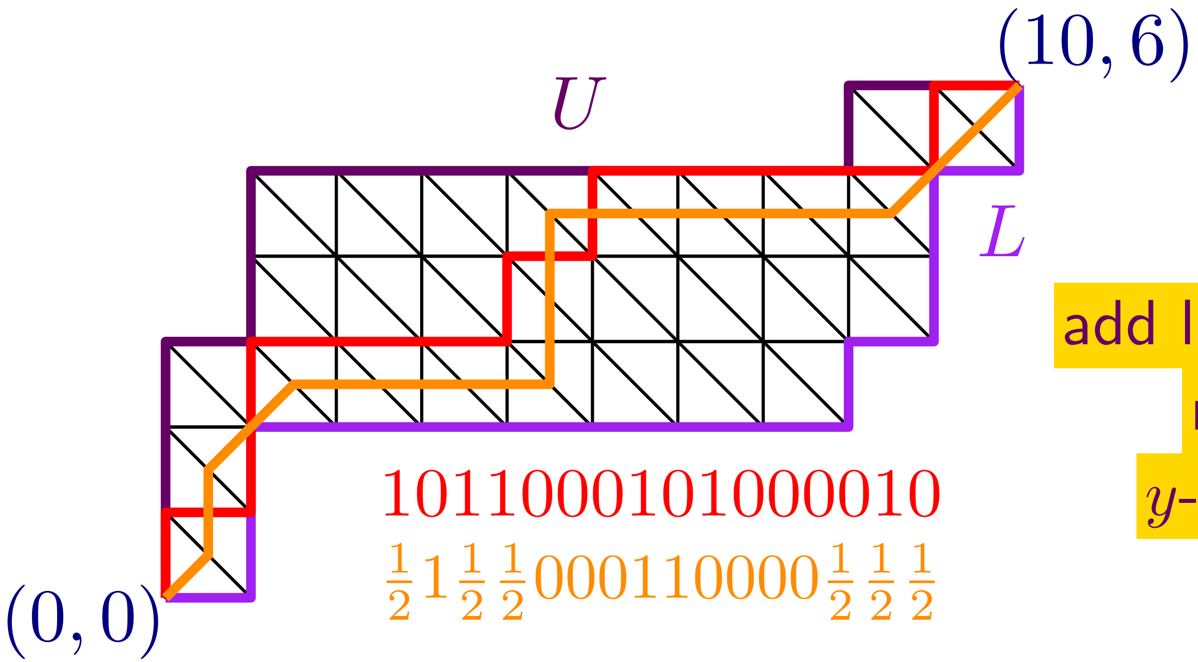


generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x,y) \mid x+y=i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.

Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0,0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m, r).



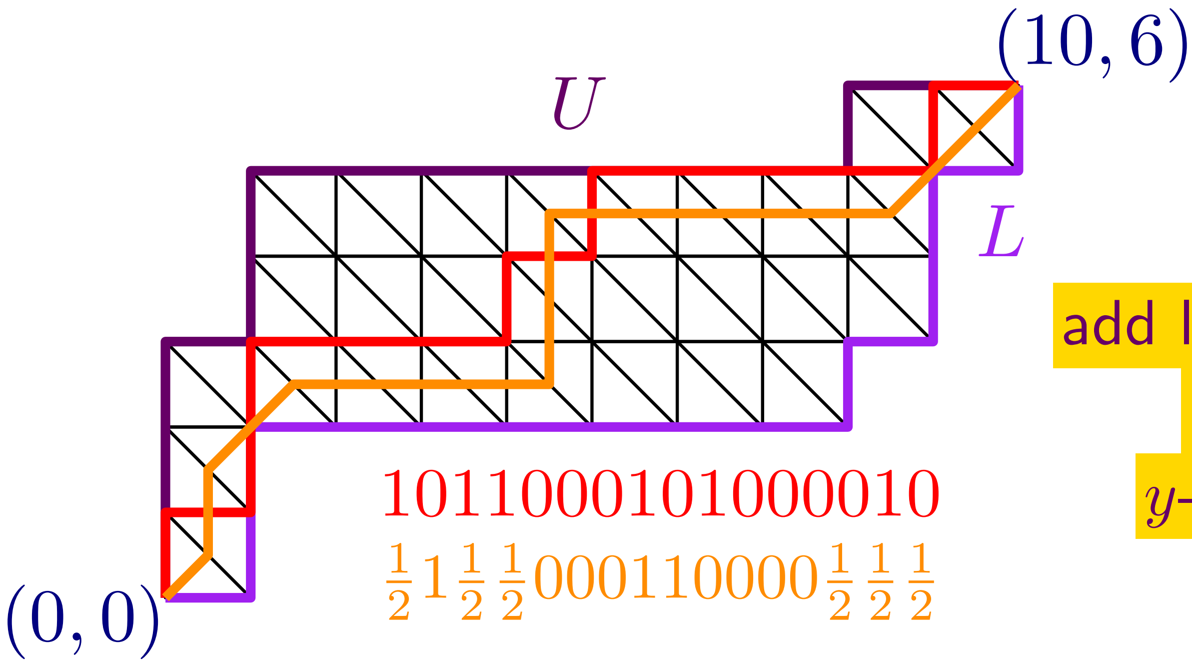
generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x, y) \mid x + y = i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r + m]\}$.

y-increase between ℓ_i and ℓ_{i+1} in $[0, 1]$

Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E = [r+m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x,y) \mid x+y=i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

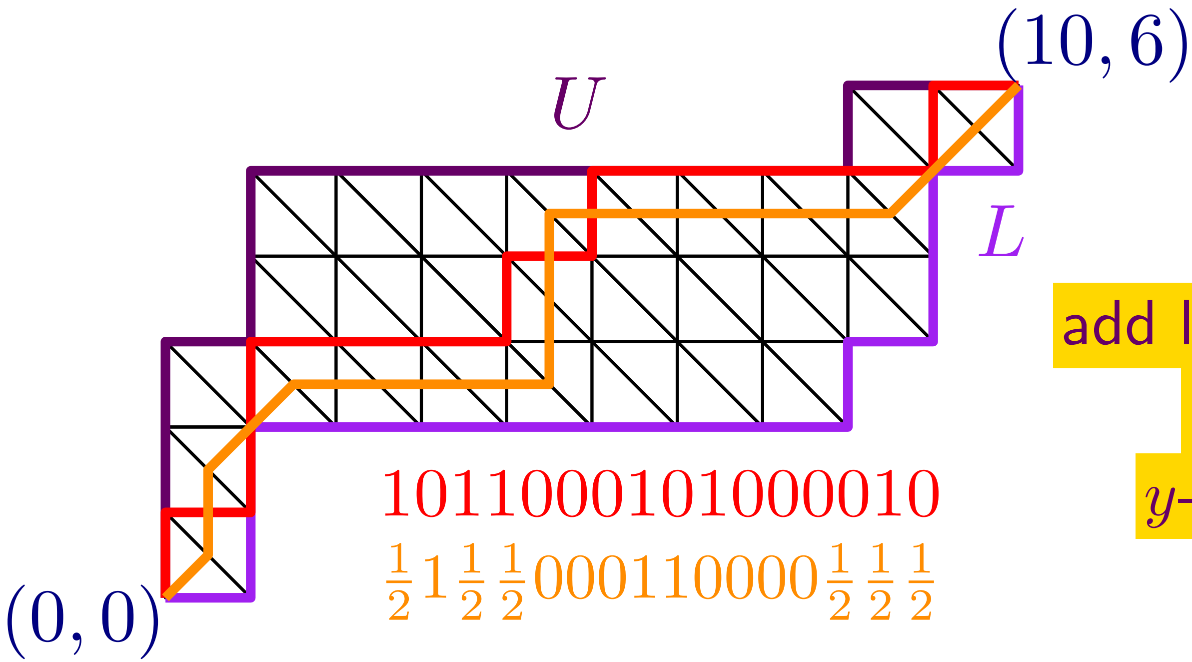
Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.

y-increase between ℓ_i and ℓ_{i+1} in $[0,1]$

stay between L and U

Lattice path matroid polytopes

Lattice path matroid $M[U,L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E=[r+m]$
$\mathcal{B}=$monotone lattice paths from $(0,0)$ to (m,r).

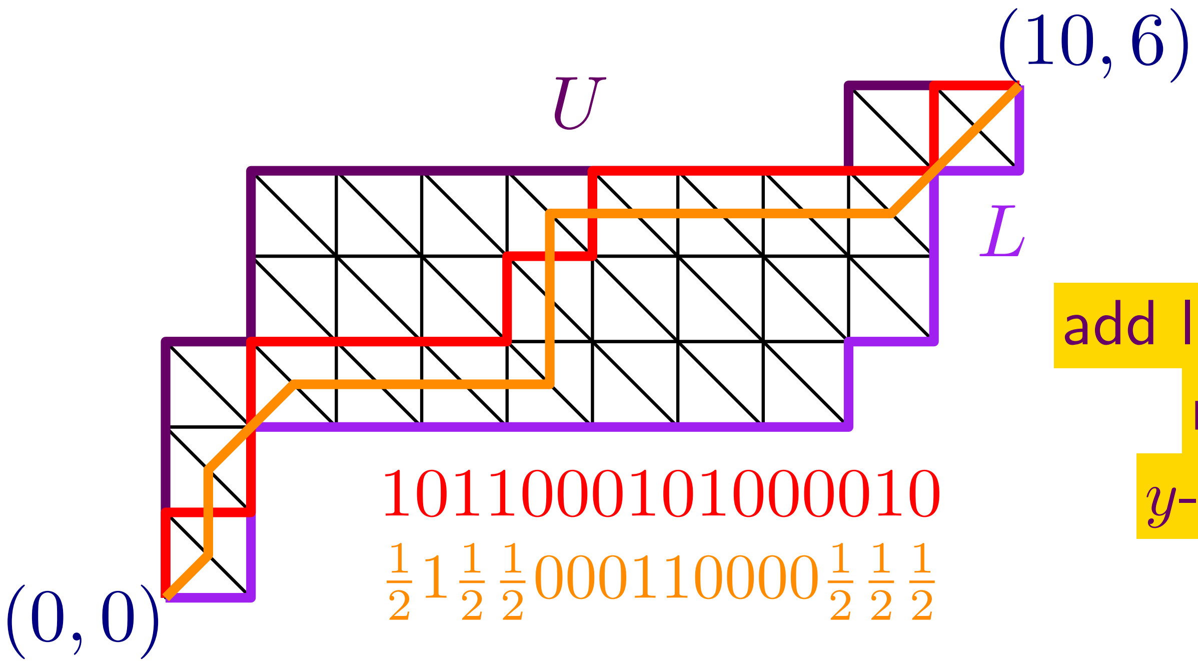


generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x,y) \mid x+y=i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M=M[U,L]$ a lattice path matroid, then
$P_M=\mathcal{C}_M \overset{\checkmark}{=} \{p\in[0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i\in[r+m]\}.$

Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0, 0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0, 0)$ to (m, r).



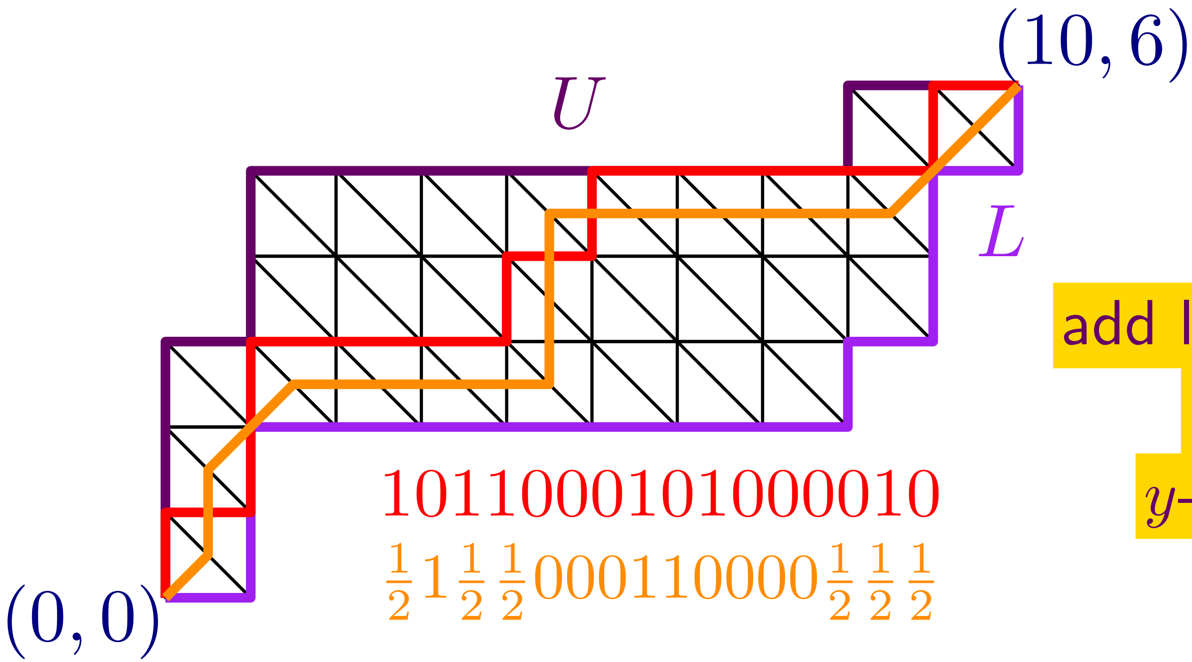
generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x, y) \mid x + y = i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M \stackrel{\checkmark}{=} \{p \in [0, 1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r + m]\}$.

$P_M = \text{conv}\{\text{characteristic vectors of } \mathcal{B}\} \subseteq \text{conv}(\mathcal{C}_M) = \mathcal{C}_M$.

Lattice path matroid polytopes

Lattice path matroid $M[U,L]$:
diagram from $(0,0)$ to (m,r) between L and U
$E=[r+m]$
$\mathcal{B}$ =monotone lattice paths from $(0,0)$ to (m,r).



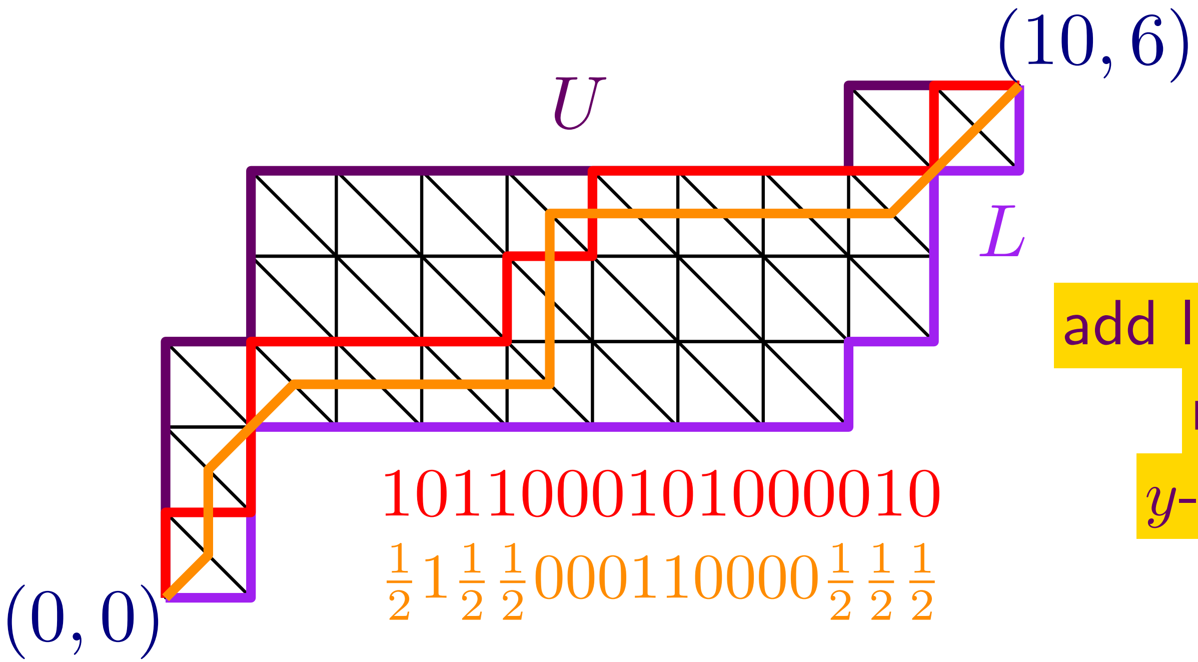
generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x,y) \mid x+y=i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M=M[U,L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M \overset{\checkmark}{=} \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.

$P_M \supseteq \mathcal{C}_M$: easy induction.

Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0, 0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0, 0)$ to (m, r).

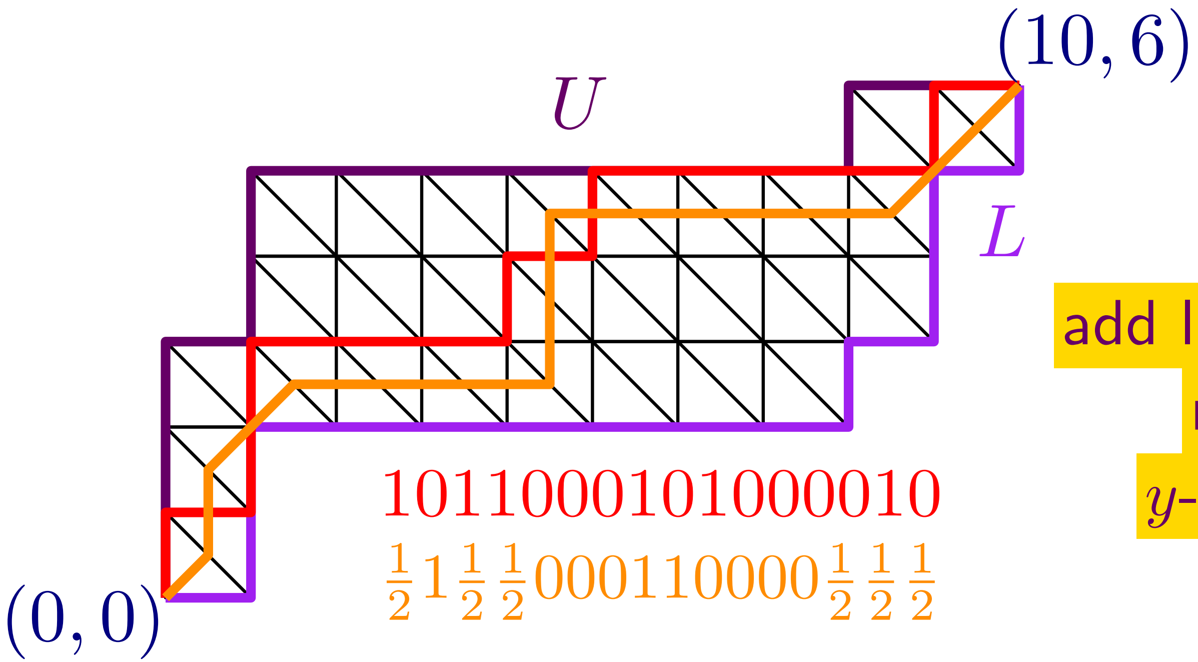


generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x, y) \mid x + y = i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M \overset{✓}{=} \mathcal{C}_M \overset{✓}{=} \{p \in [0, 1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r + m]\}$.

Lattice path matroid polytopes

Lattice path matroid $M[U, L]$:
diagram from $(0, 0)$ to (m, r) between L and U
$E = [r + m]$
$\mathcal{B}$ =monotone lattice paths from $(0, 0)$ to (m, r).



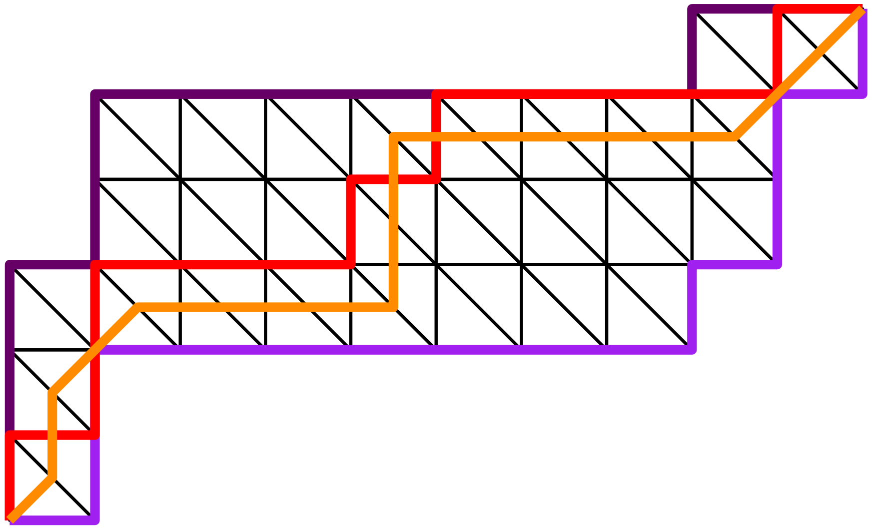
generalized lattice paths $\mathcal{C}_M$
add lines $\ell_i := \{(x, y) \mid x + y = i\}$
monotone, bending only on ℓ_i
y-increase between ℓ_i and ℓ_{i+1}

Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M \overset{\checkmark}{=} \mathcal{C}_M \overset{\checkmark}{=} \{p \in [0, 1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r + m]\}$.

Cor: $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\} =: \mathcal{C}_M^k$.

Distributive polytopes

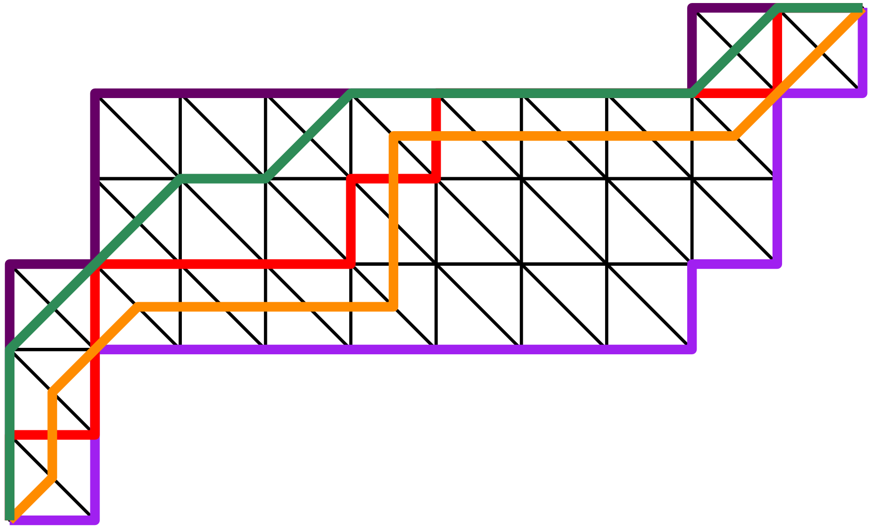
Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.
and $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\} =: \mathcal{C}_M^k$.



poset on $\mathcal{C}_M$

Distributive polytopes

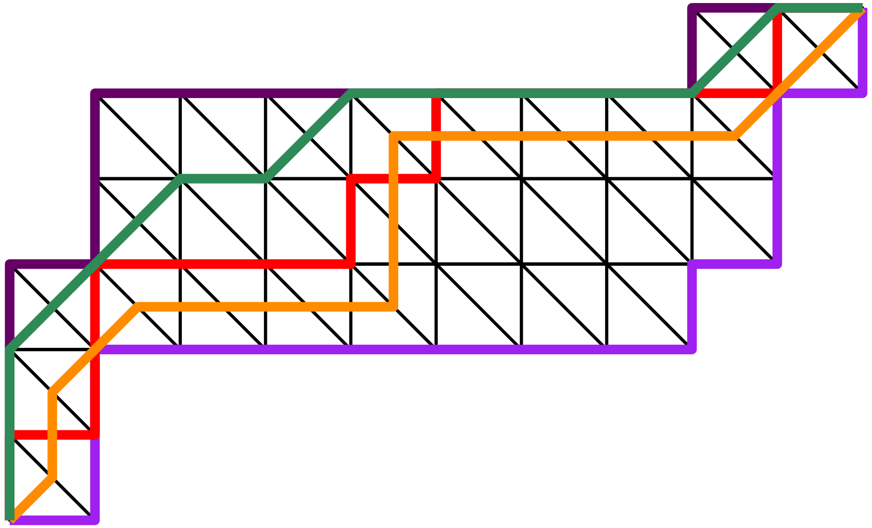
Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.
and $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\}$=:$\mathcal{C}_M^k$.



poset on $\mathcal{C}_M$
$P \leq P' \Leftrightarrow P$ below P'.

Distributive polytopes

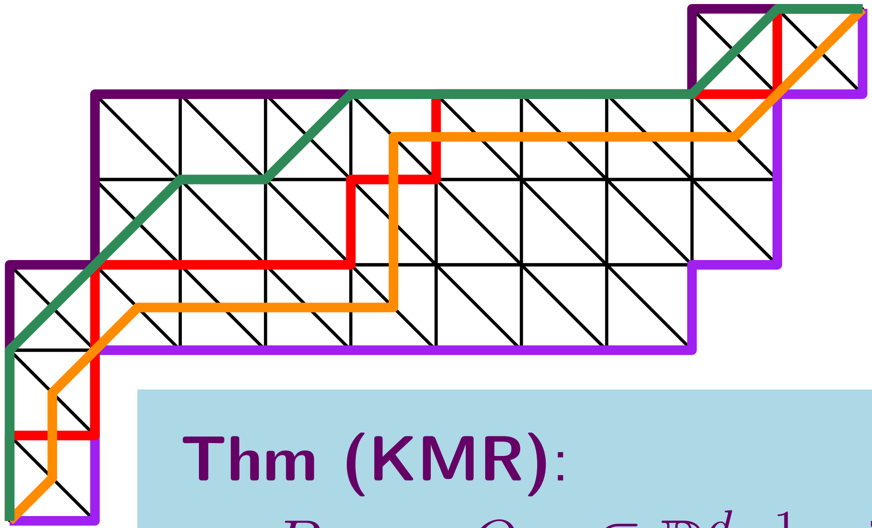
Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.
and $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\}{=:}\mathcal{C}_M^k$.



poset on $\mathcal{C}_M$
$P \leq P' \Leftrightarrow P$ below P'.
$\Leftrightarrow \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p'_j;\ \forall i \in [r+m]\}$.

Distributive polytopes

Thm (KMR): Let $M = M[U, L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.
and $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\} =: \mathcal{C}_M^k$.

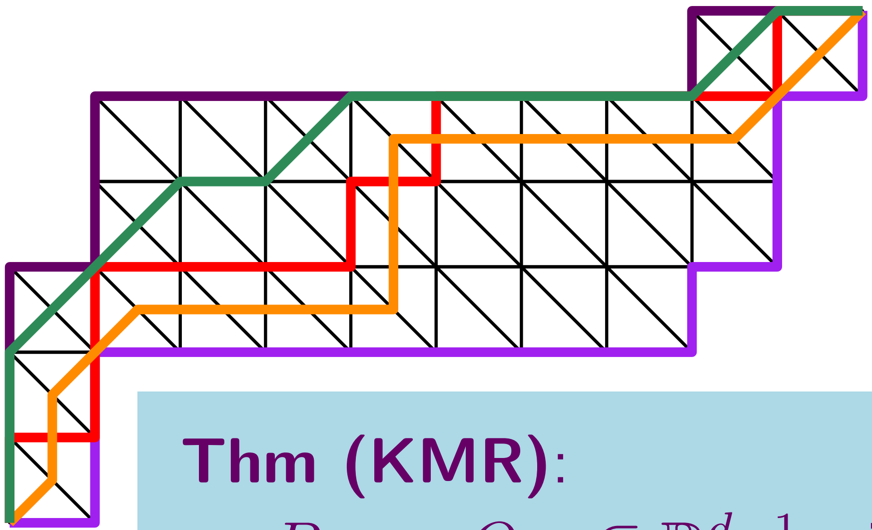


poset on $\mathcal{C}_M$
$P \leq P' \Leftrightarrow P$ below P'.
$\Leftrightarrow \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p'_j;\ \forall i \in [r+m]\}$.

Thm (KMR):
$\pi : P_M \to Q_M \subseteq \mathbb{R}^{d-1}$ with $\pi(p)_i := \sum_{j=1}^{i} p_j;\ \forall i \in [r+m-1]$
gives a polytope Q_M, with $|kP_M \cap \mathbb{Z}^d| = |kQ_M \cap \mathbb{Z}^{d-1}|$.
Moreover, Q_M ordered componentwise is distributive lattice, corresponding to the order on $\mathcal{C}_M$.

Distributive polytopes

Thm (KMR): Let $M = M[U,L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.
and $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\} =: \mathcal{C}_M^k$.



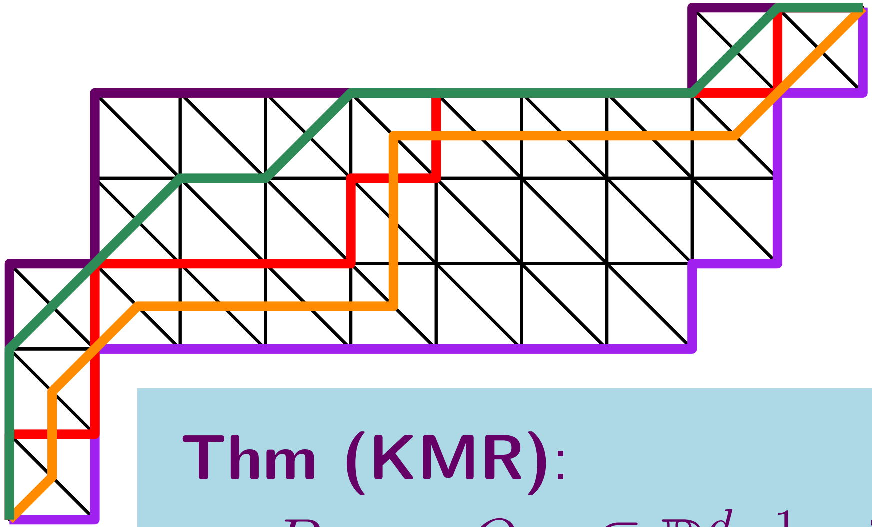
poset on $\mathcal{C}_M$
$P \leq P' \Leftrightarrow P$ below P'.
$\Leftrightarrow \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p'_j;\ \forall i \in [r+m]\}$.

Thm (KMR):
$\pi : P_M \to Q_M \subseteq \mathbb{R}^{d-1}$ with $\pi(p)_i := \sum_{j=1}^{i} p_j;\ \forall i \in [r+m-1]$
gives a polytope Q_M, with $|kP_M \cap \mathbb{Z}^d| = |kQ_M \cap \mathbb{Z}^{d-1}|$.
Moreover, Q_M ordered componentwise is distributive lattice, corresponding to the order on $\mathcal{C}_M$.

show that closed under componentwise minimum and maximum

Distributive polytopes

Thm (KMR): Let $M = M[U,L]$ a lattice path matroid, then
$P_M = \mathcal{C}_M = \{p \in [0,1]^{r+m} \mid \sum_{j=1}^{i} L_j \leq \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} U_j;\ \forall i \in [r+m]\}$.
and $kP_M \cap \mathbb{Z}^d = \{$gen. lattice paths y-increase in $\frac{i}{k}$, $0 \leq i \leq k\} =: \mathcal{C}_M^k$.



poset on $\mathcal{C}_M$
$P \leq P' \Leftrightarrow P$ below P'.
$\Leftrightarrow \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p'_j;\ \forall i \in [r+m]\}$.

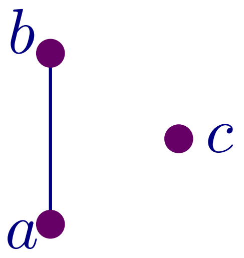
Thm (KMR):
$\pi : P_M \to Q_M \subseteq \mathbb{R}^{d-1}$ with $\pi(p)_i := \sum_{j=1}^{i} p_j;\ \forall i \in [r+m-1]$
gives a polytope Q_M, with $|kP_M \cap \mathbb{Z}^d| = |kQ_M \cap \mathbb{Z}^{d-1}|$.
Moreover, Q_M ordered componentwise is distributive lattice, corresponding to the order on $\mathcal{C}_M$.
And $kQ_M \cap \mathbb{Z}^{d-1} \cong \mathcal{C}_M^k$, too.

then this also closed under componentwise minimum and maximum

Embedded distributive lattices

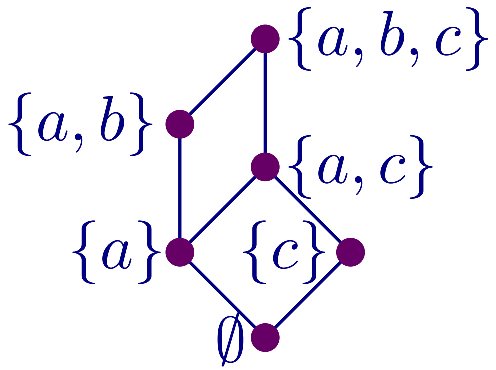
Embedded distributive lattices

poset X | $\mathcal{I}(X)$ distributive lattice



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

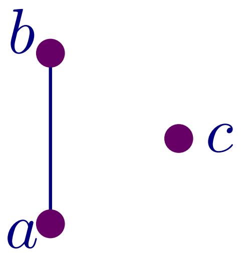
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



Embedded distributive lattices

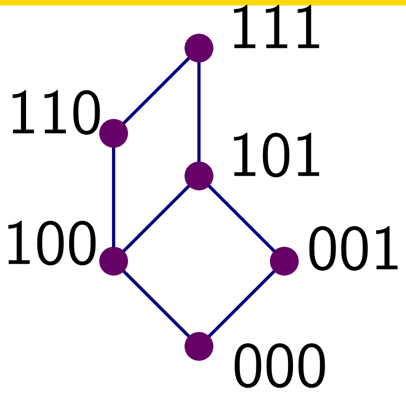
poset X

$\mathcal{I}(X)$ distributive lattice



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

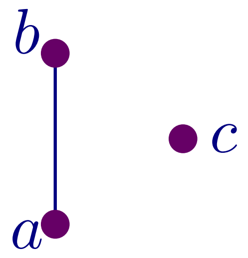
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion



Embedded distributive lattices

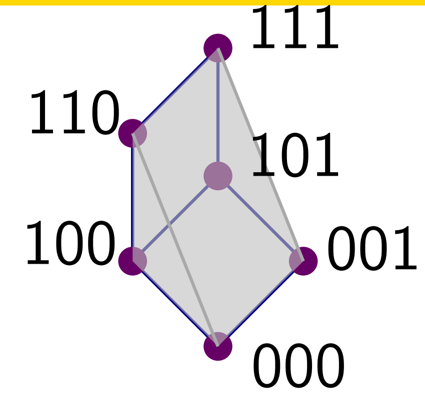
poset X

$\mathcal{I}(X)$ distributive lattice



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

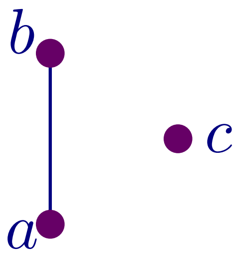
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion



Embedded distributive lattices

chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



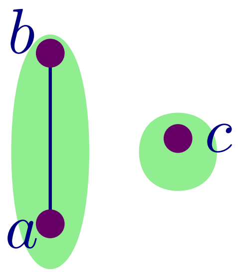
$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

Embedded distributive lattices

chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice

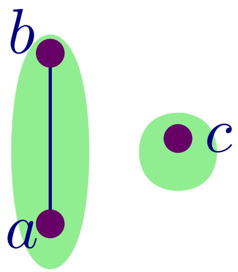


$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

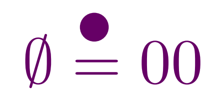
Embedded distributive lattices

chain-partitioned poset X | **embedded $\mathcal{I}(X)$ distributive lattice**



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

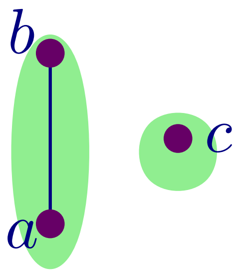
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



Embedded distributive lattices

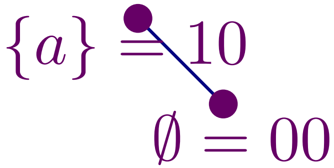
chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

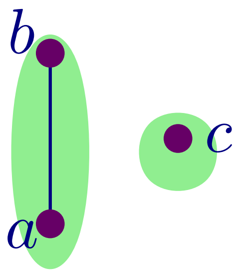
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



Embedded distributive lattices

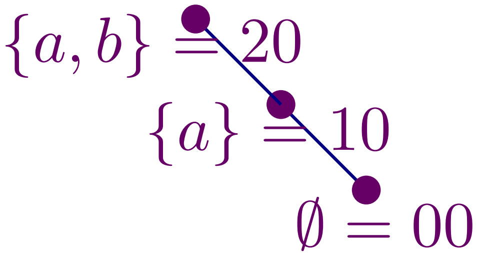
chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



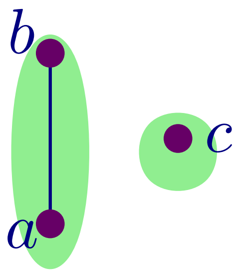
$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



Embedded distributive lattices

chain-partitioned poset X

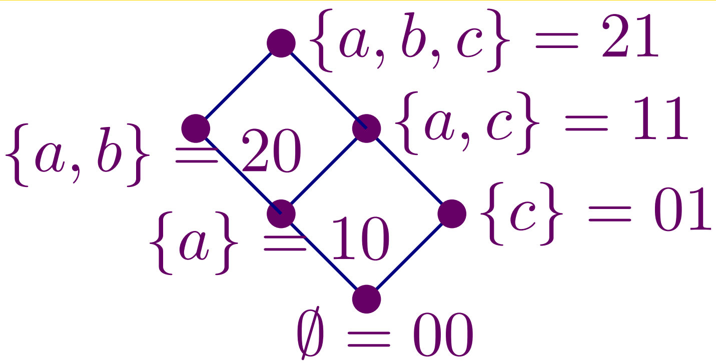


$I \subseteq X$ is an *ideal* if

$y \leq x \in I \implies y \in I$

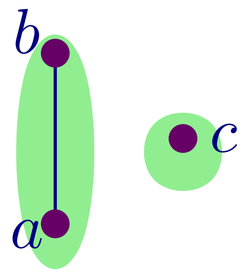
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

embedded $\mathcal{I}(X)$ distributive lattice



Embedded distributive lattices

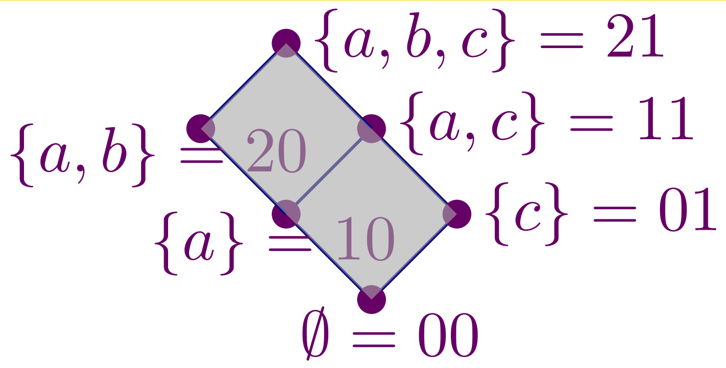
chain-partitioned poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

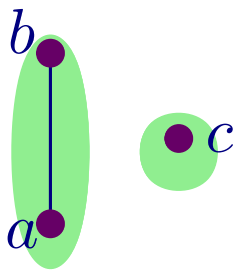
embedded $\mathcal{I}(X)$ distributive lattice



Embedded distributive lattices

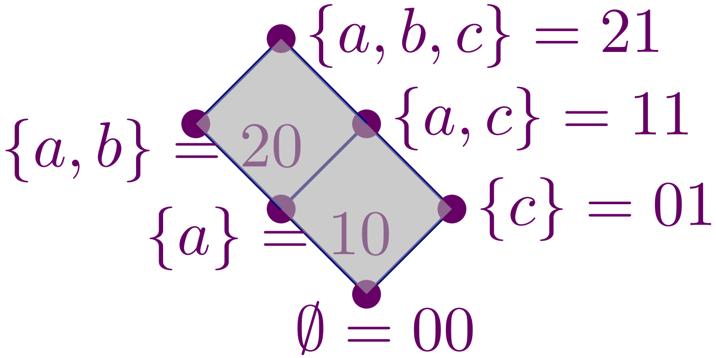
chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

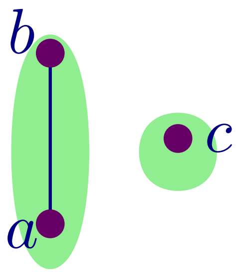


no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Embedded distributive lattices

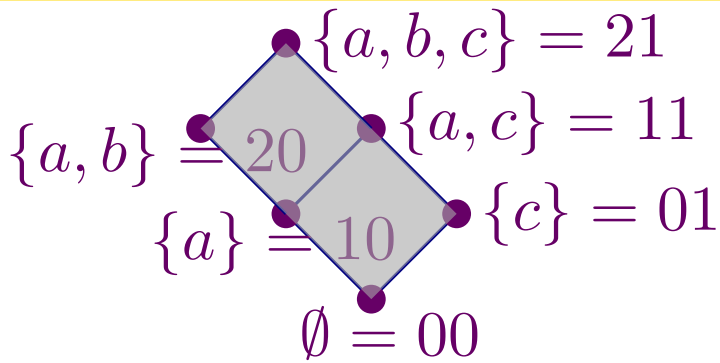
chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



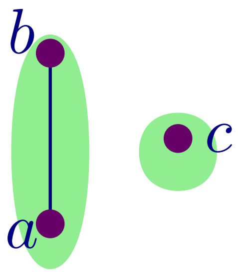
no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:

Embedded distributive lattices

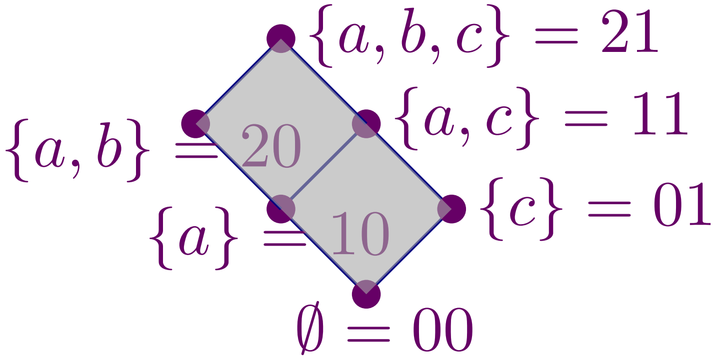
chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



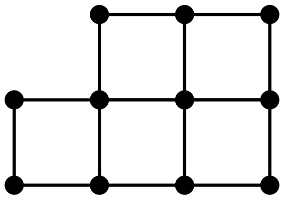
$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:

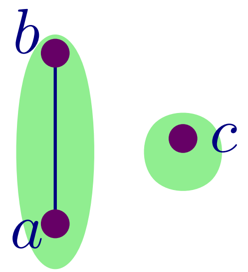


$M[U,L]$

Embedded distributive lattices

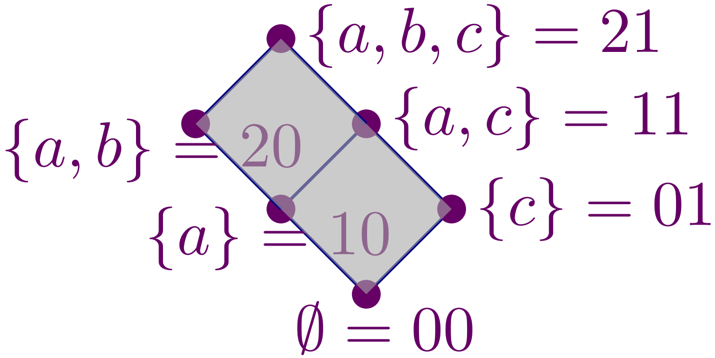
chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice



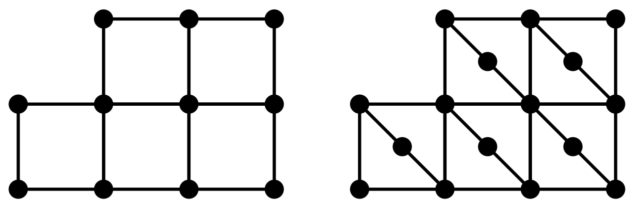
$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X ordered by inclusion



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

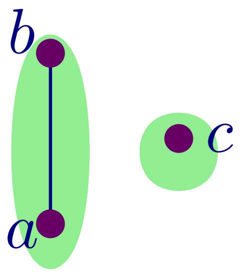
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U,L]$ $k=2$

Embedded distributive lattices

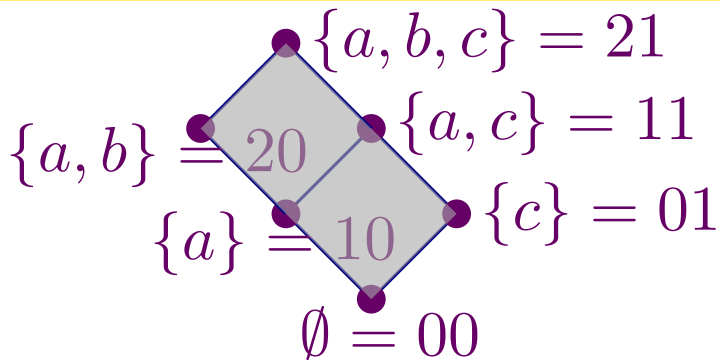
chain-partitioned poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

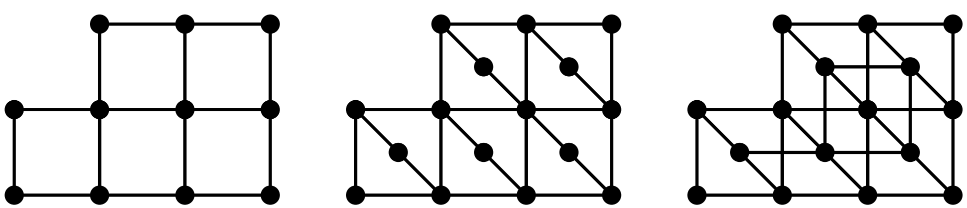
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

embedded $\mathcal{I}(X)$ distributive lattice



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0, 1)$-embedding

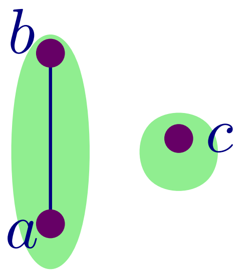
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U, L]$ $\quad$ $k = 2$

Embedded distributive lattices

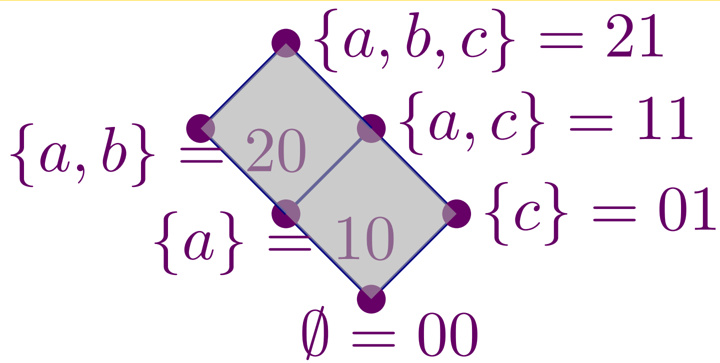
chain-partitioned poset X



embedded $\mathcal{I}(X)$ distributive lattice

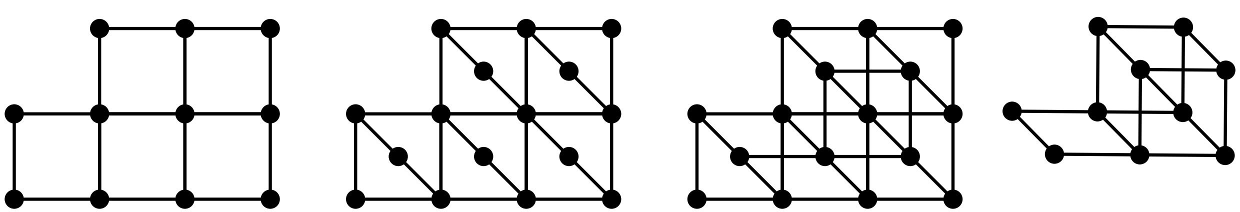
$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

$\mathcal{I}(X)$ set of ideals of X ordered by inclusion



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:

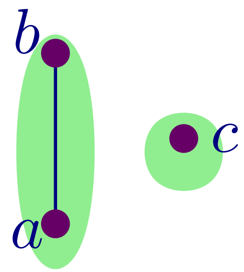


$M[U,L]$ $k = 2$

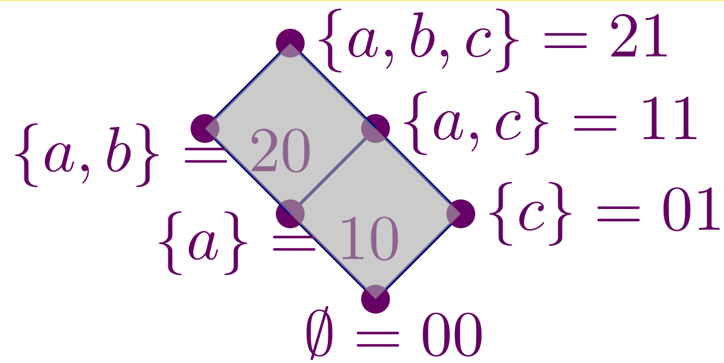
Embedded distributive lattices

chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice

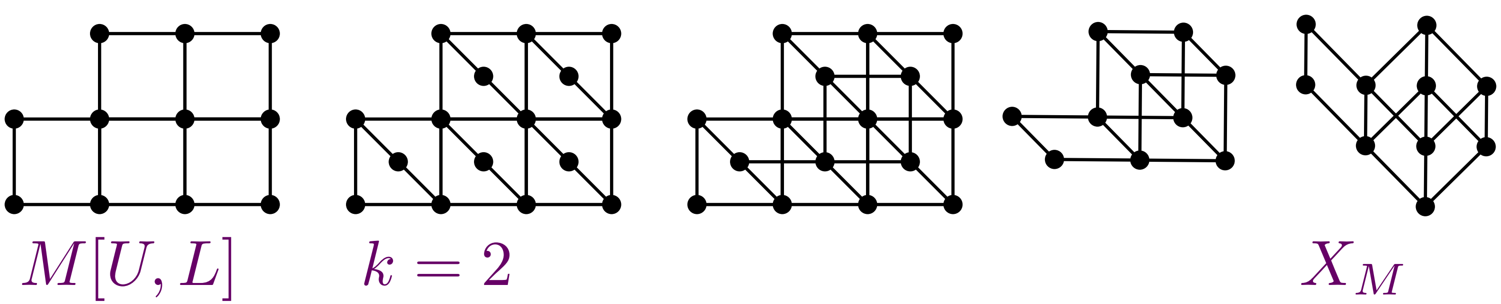


$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

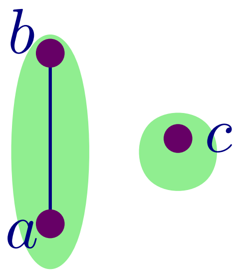
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



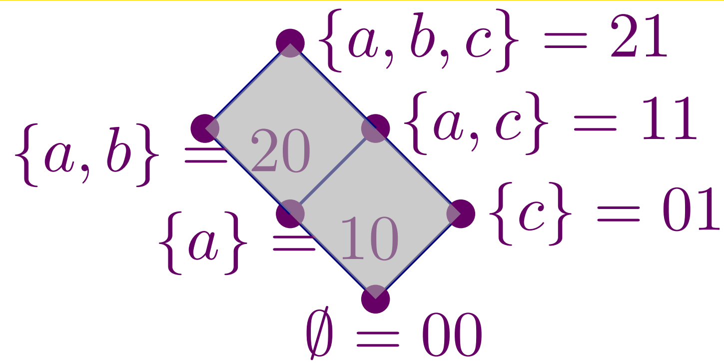
Embedded distributive lattices

chain-partitioned poset X

embedded $\mathcal{I}(X)$ distributive lattice

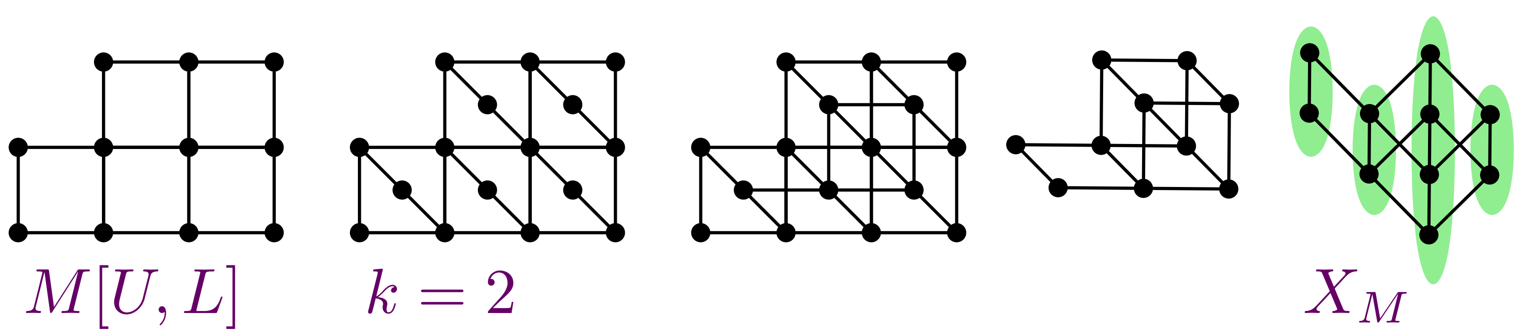


$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion



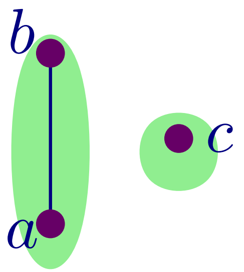
no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



Embedded distributive lattices

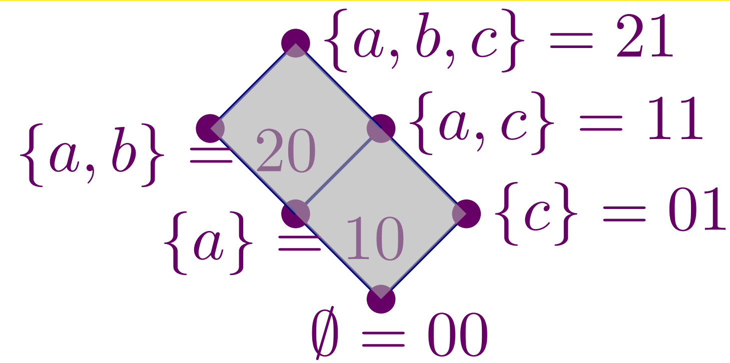
chain-partitioned poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

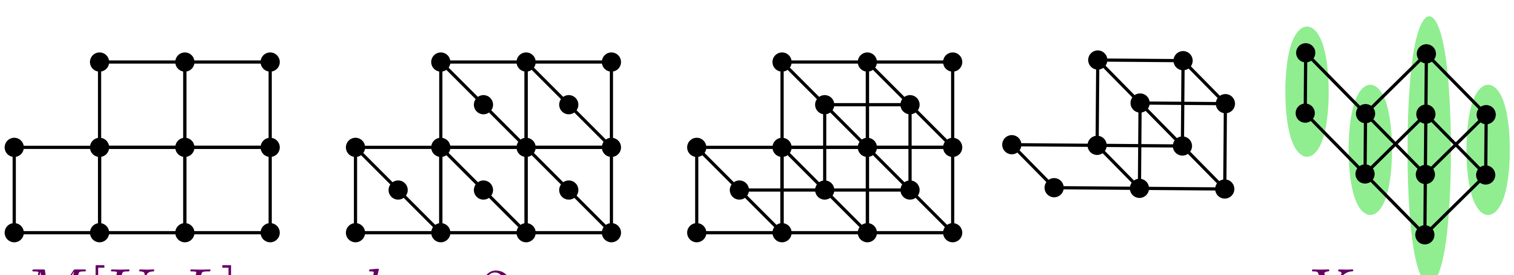
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

embedded $\mathcal{I}(X)$ distributive lattice

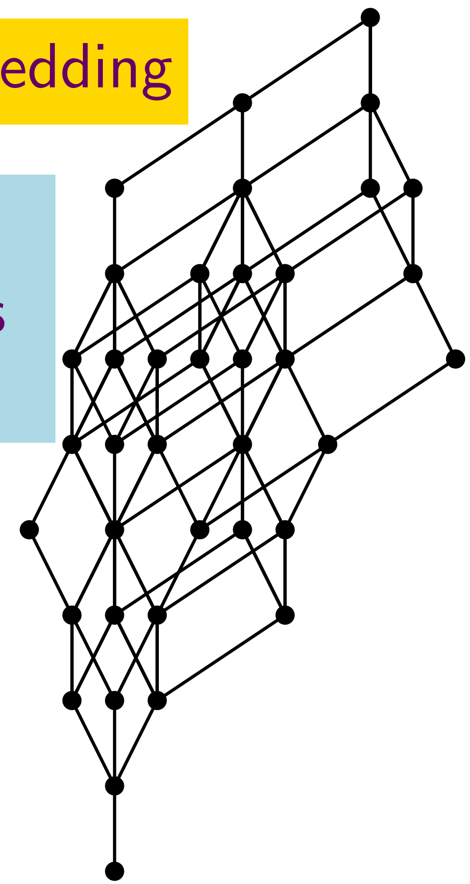


no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:

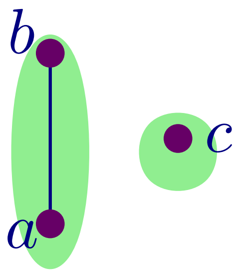


$M[U,L]$ $k=2$ X_M



Embedded distributive lattices

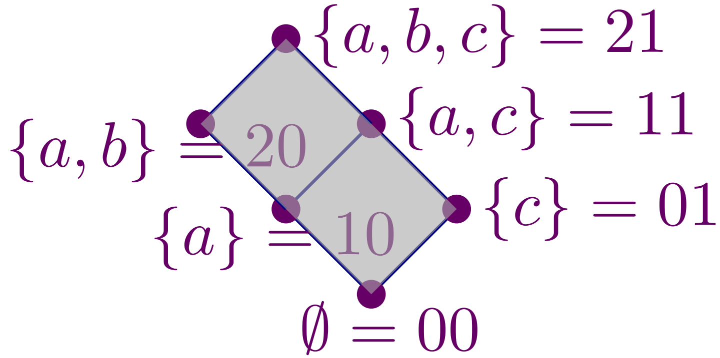
chain-partitioned poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

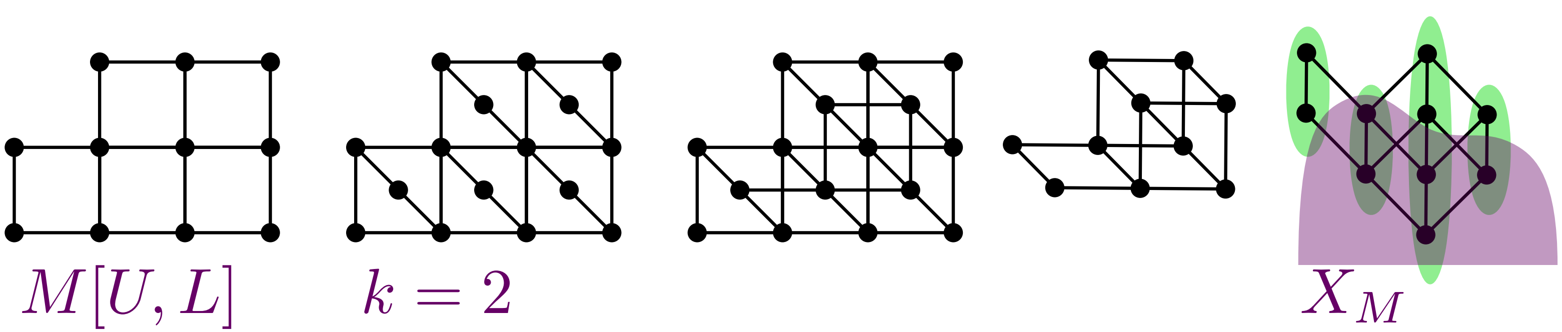
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

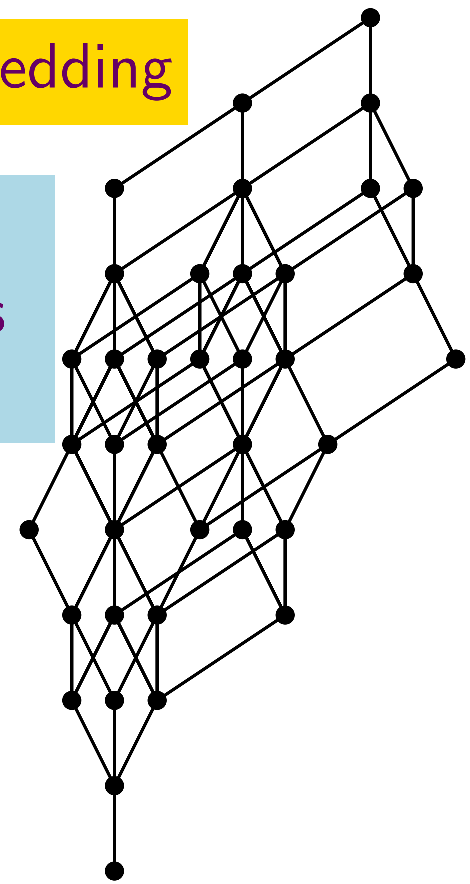
embedded $\mathcal{I}(X)$ distributive lattice



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

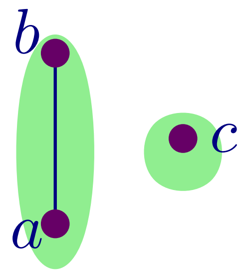
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:





Embedded distributive lattices

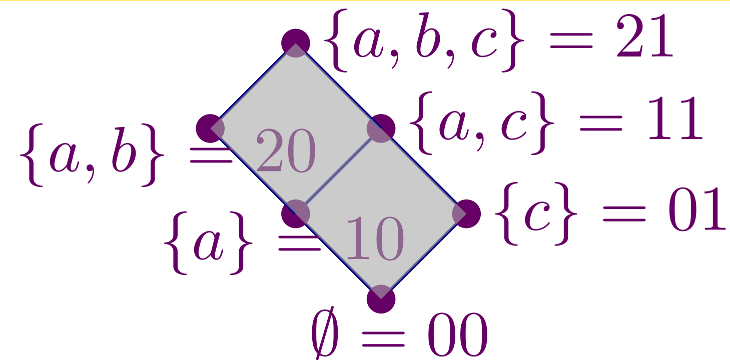
chain-partitioned poset X



$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$

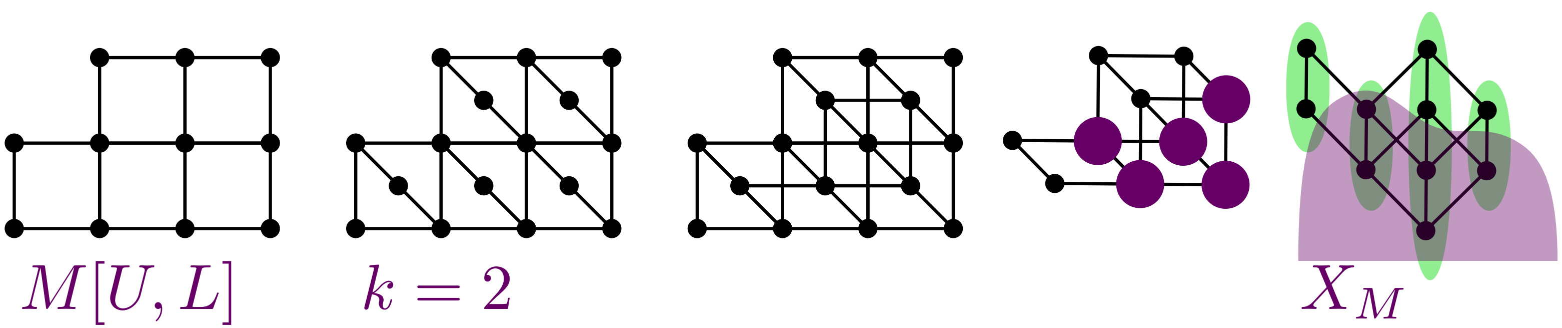
$\mathcal{I}(X)$ set of ideals of X
ordered by inclusion

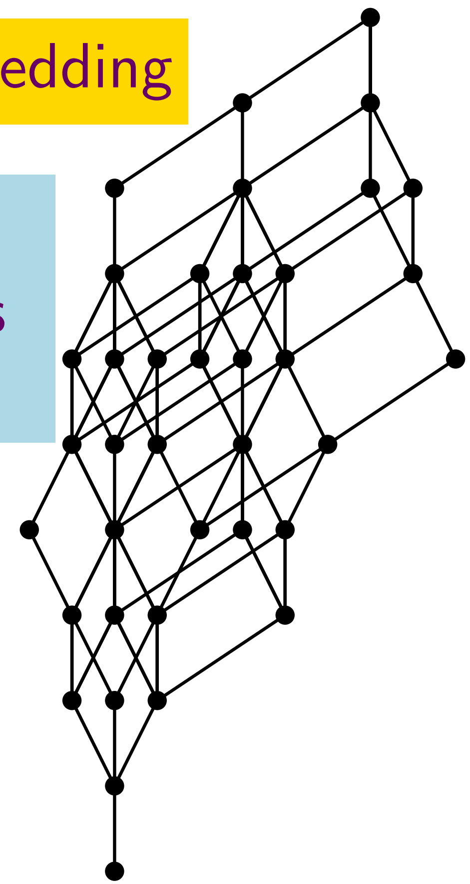
embedded $\mathcal{I}(X)$ distributive lattice



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

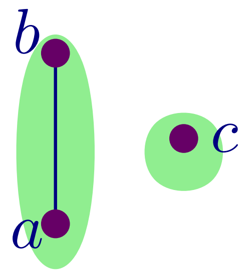
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:





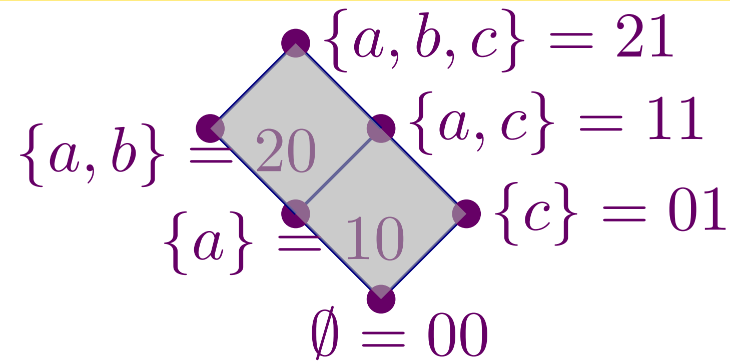
Embedded distributive lattices

chain-partitioned poset X



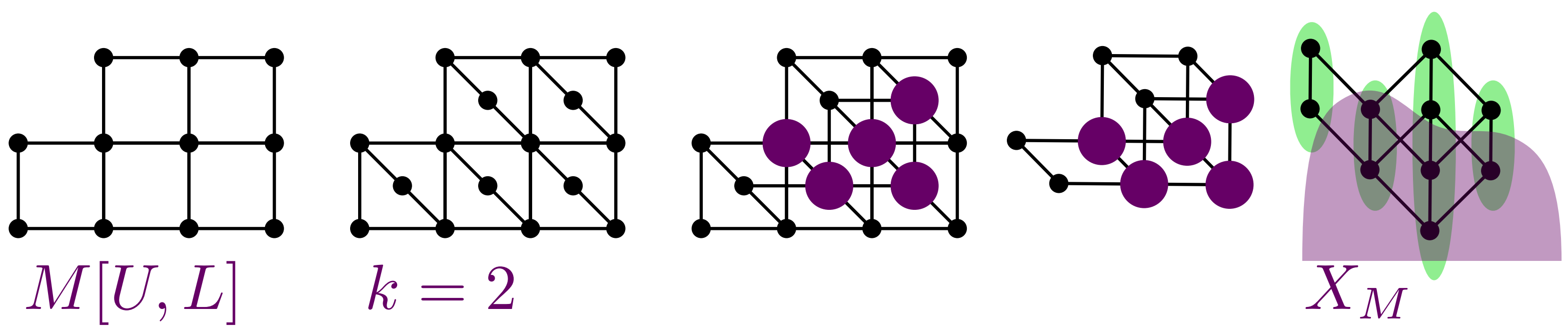
$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

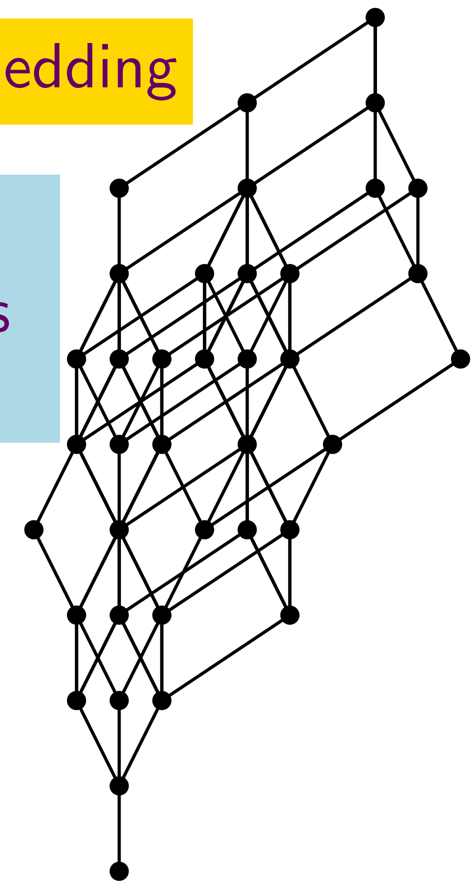
embedded $\mathcal{I}(X)$ distributive lattice



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

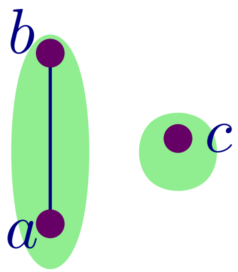
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:





Embedded distributive lattices

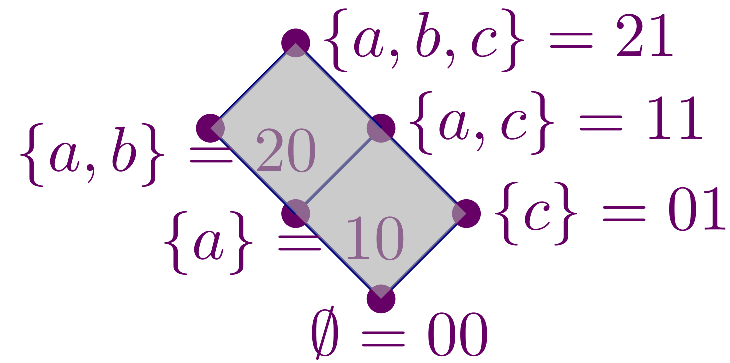
chain-partitioned poset X | **embedded $\mathcal{I}(X)$ distributive lattice**



$I \subseteq X$ is an *ideal* if

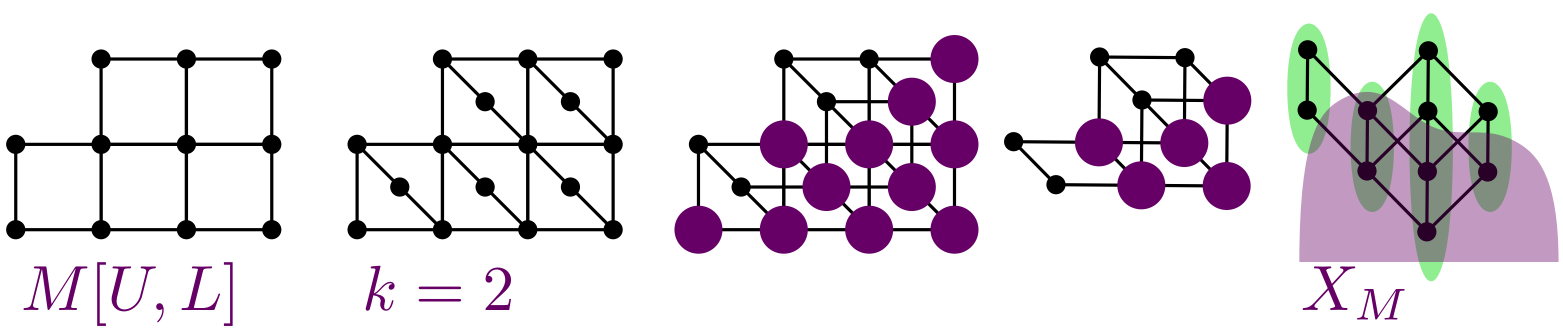
$y \leq x \in I \implies y \in I$

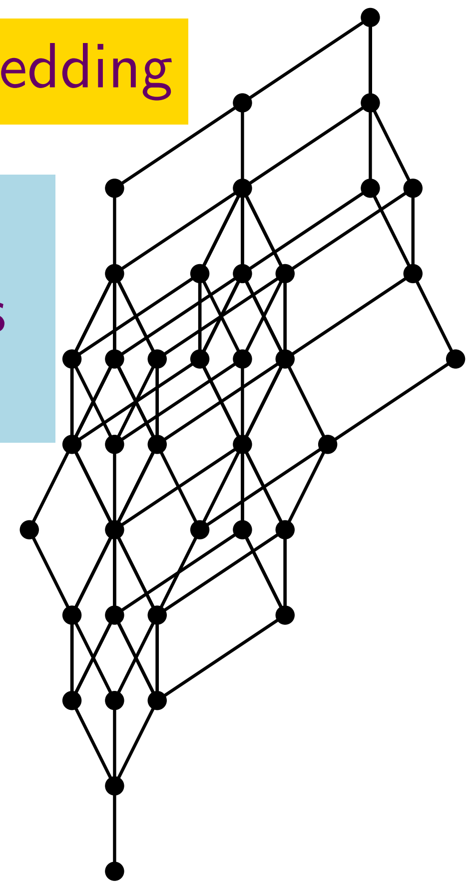
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

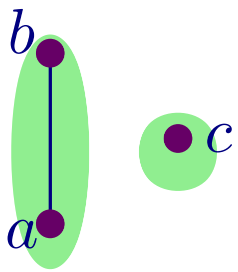
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:





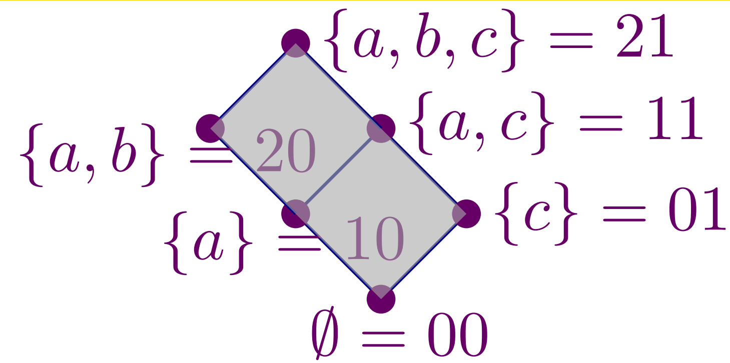
Embedded distributive lattices

chain-partitioned poset X

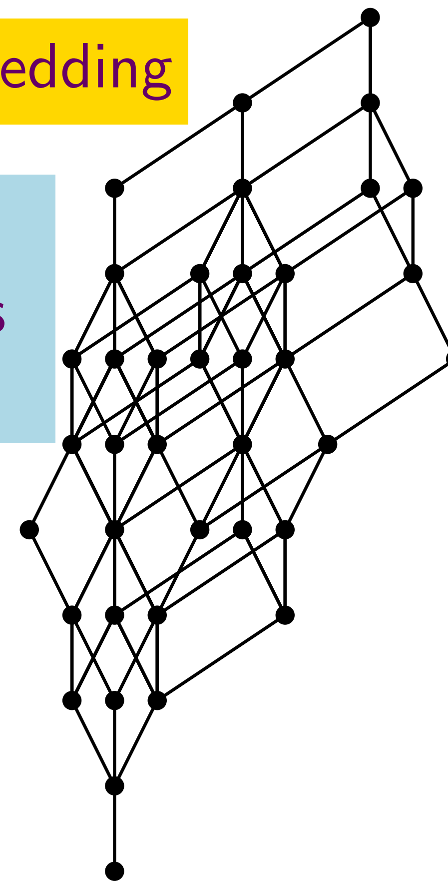


$I \subseteq X$ is an *ideal* if
$y \leq x \in I \implies y \in I$
$\mathcal{I}(X)$ set of ideals of X ordered by inclusion

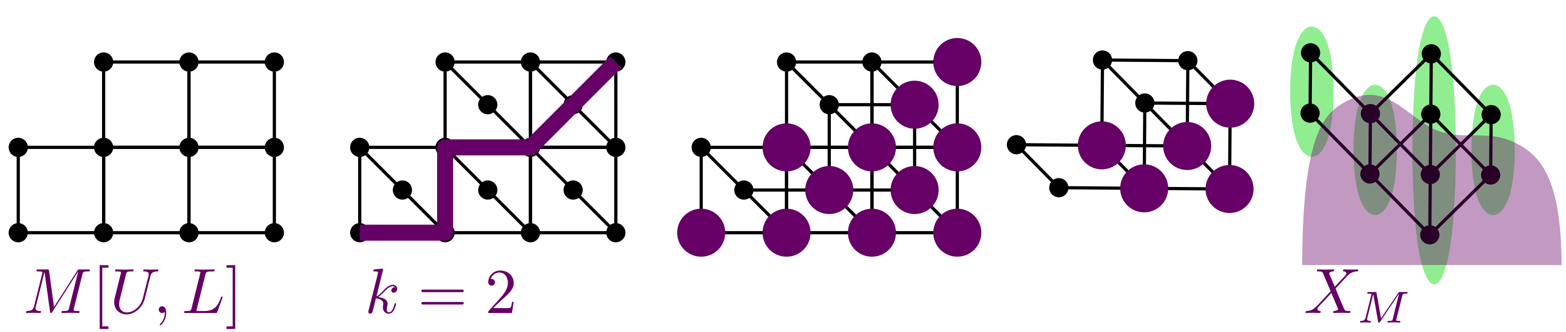
embedded $\mathcal{I}(X)$ distributive lattice



no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding



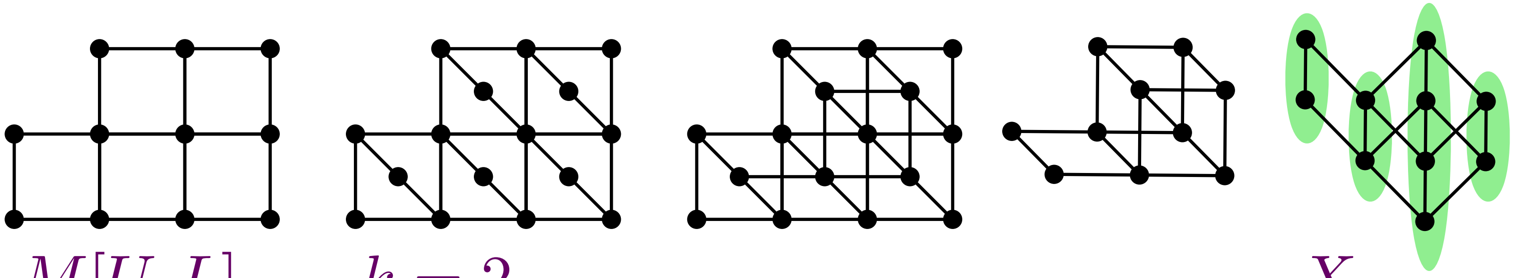
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



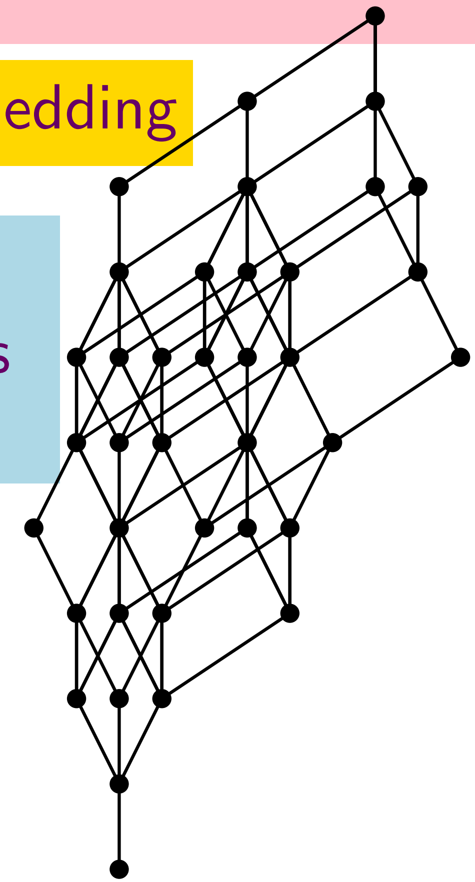
Back to order

no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



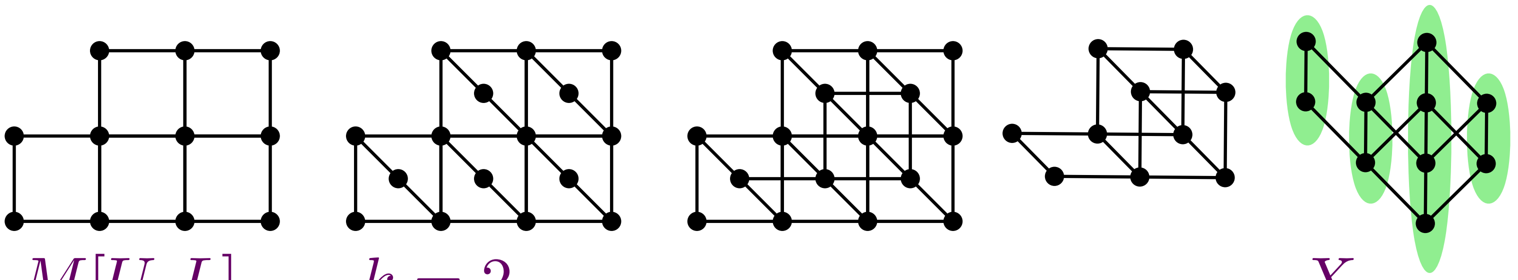
$M[U,L]$ $k = 2$ X_M



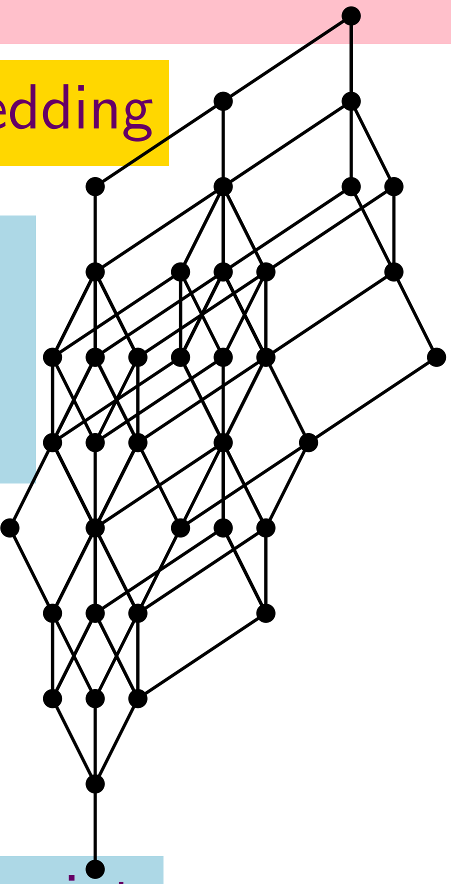
Back to order

no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U,L]$ $\quad$ $k=2$ $\quad$ X_M

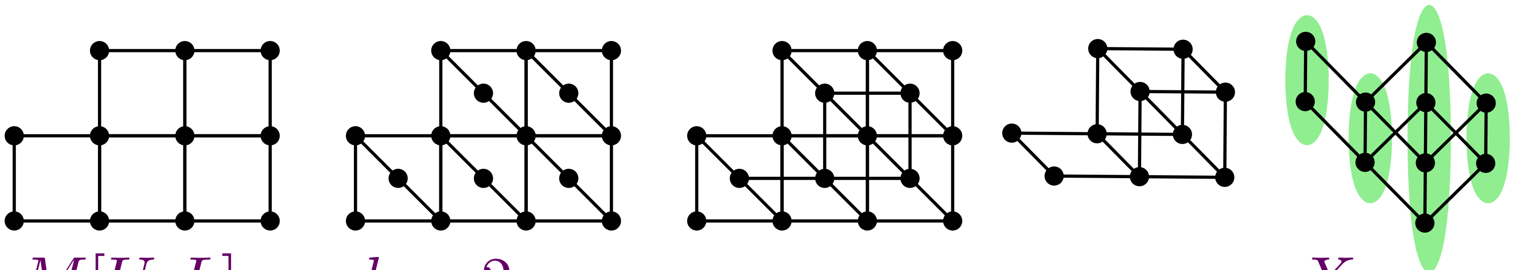


In particular, Q_M is an order polytope if M has no interior points

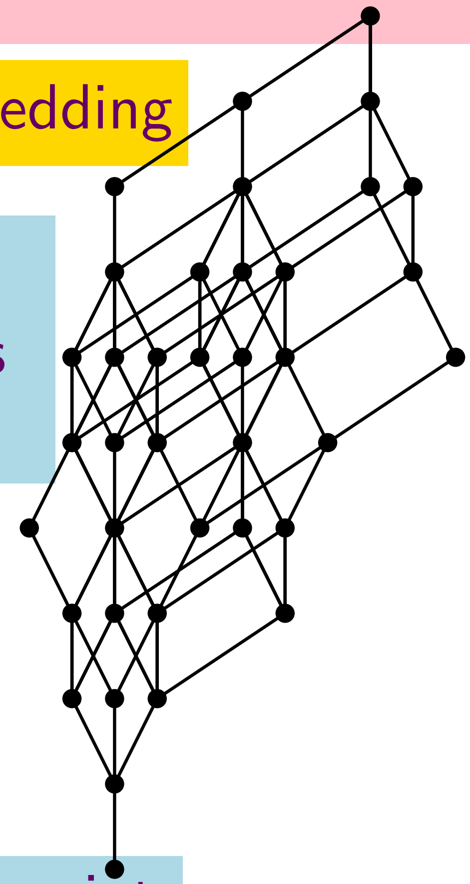
Back to order

no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U,L]$ $k=2$ X_M



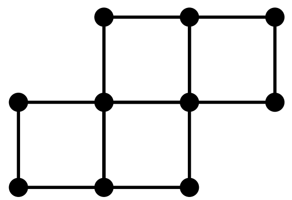
In particular, Q_M is an order polytope if M has no interior points

M is a *snake*

Back to order

no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U,L]$

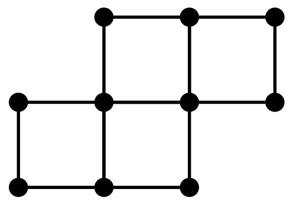
In particular, Q_M is an order polytope if M has no interior points

M is a *snake*

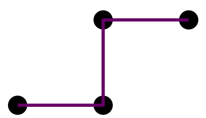
Back to order

no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U, L]$



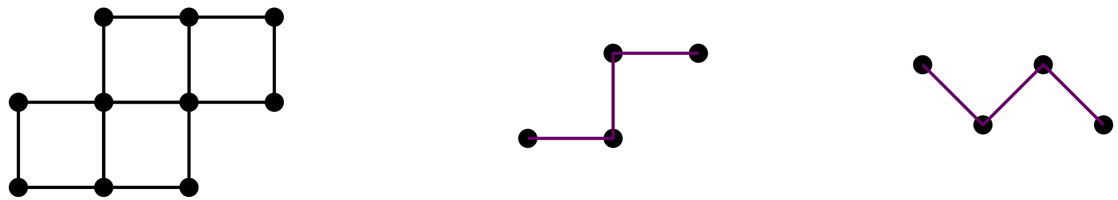
In particular, Q_M is an order polytope if M has no interior points

M is a *snake*

Back to order

no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U,L]$ X

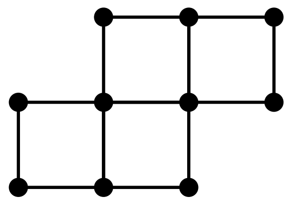
In particular, Q_M is an order polytope if M has no interior points

M is a *snake*

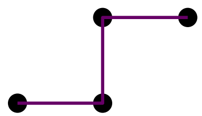
Back to order

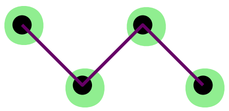
no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



$M[U, L]$





X

In particular, Q_M is an order polytope if M has no interior points

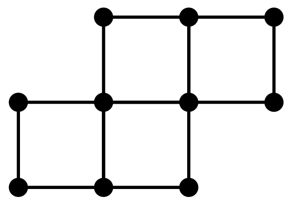
M is a *snake*

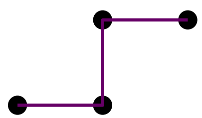
Back to order

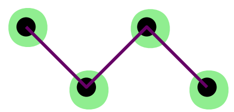
no chain-partition ≅ singleton chain-partition ≅ $(0,1)$-embedding

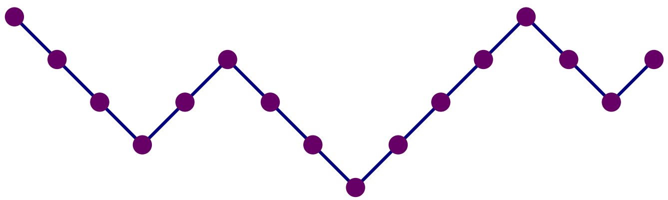
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:

zig-zag-posets


$M[U,L]$




X



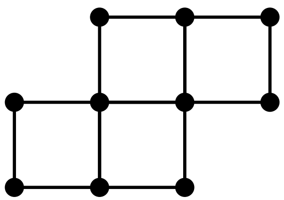
In particular, Q_M is an order polytope if M has no interior points

M is a *snake*

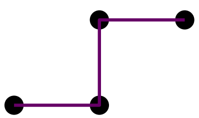
Back to order

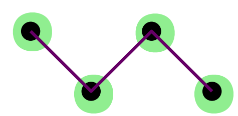
no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:



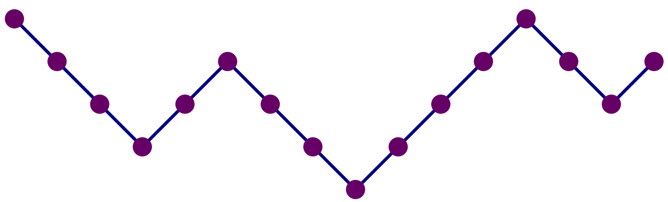
$M[U,L]$





X

zig-zag-posets



In particular, Q_M is an order polytope if M has no interior points

M is a *snake*

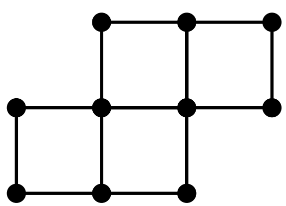
If M snake, then h^* of L_{P_M} equals ω of zig-zag poset

Back to order

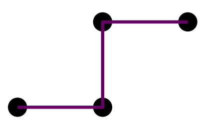
no chain-partition $\cong$ singleton chain-partition $\cong$ $(0,1)$-embedding

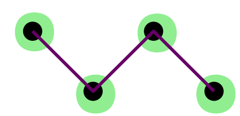
Thm (KMR):
the embedded distributive lattice $kQ_M \cap \mathbb{Z}^{d-1}$ corresponds to a chain-partitioned poset in the following way:

zig-zag-posets

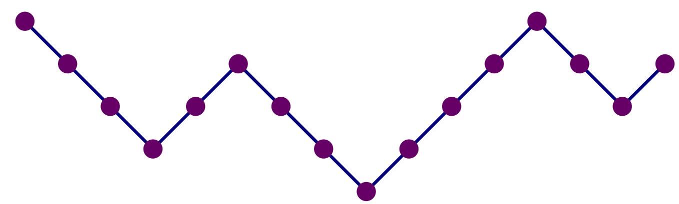


$M[U,L]$





X

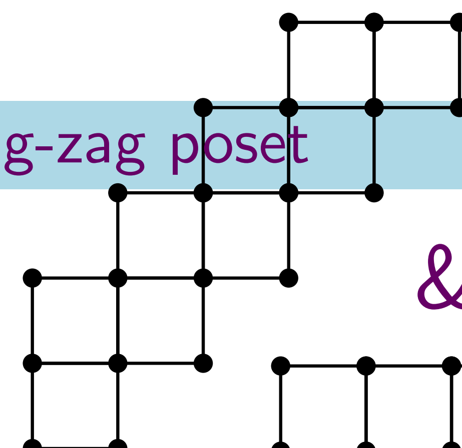


In particular, Q_M is an order polytope if M has no interior points

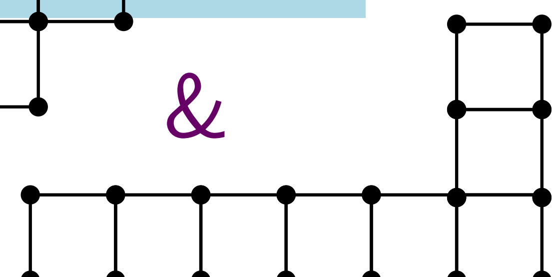
M is a *snake*

If M snake, then h^* of L_{P_M} equals ω of zig-zag poset

ω unimodal, for unions of chains and graded posets $\implies$ h^* unimodal for:



&



What we did:

- P_M as generalized lattice paths (formula for L_{P_M} for some snakes),
- *Ehrhart-equivalent* distributive polytope Q_M,
- unimodality of h^* for some snakes

What we did:

- P_M as generalized lattice paths (formula for L_{P_M} for some snakes),
- *Ehrhart-equivalent* distributive polytope Q_M,
- unimodality of h^* for some snakes

What we would like to do next:

- unimodality of h^* for more lattice path matroids
- generalize the order polynomial and ω to chain-partitioned posets,
- determine which matroid polytopes are order polytopes,
- another conjecture of de Loera et. al:
  - *coefficients of L_{P_M} are non-negative*

  true for uniform matroids [Ferroni: arXiv Monday]

  false for order polytopes...find counterexample?

What we did:

- P_M as generalized lattice paths (formula for L_{P_M} for some snakes),
- *Ehrhart-equivalent* distributive polytope Q_M,
- unimodality of h^* for some snakes

What we would like to do next:

- unimodality of h^* for more lattice path matroids
- generalize the order polynomial and ω to chain-partitioned posets,
- determine which matroid polytopes are order polytopes,
- another conjecture of de Loera et. al:
  - *coefficients of* L_{P_M} *are non-negative*

  true for uniform matroids [Ferroni: arXiv Monday]

  false for order polytopes...find counterexample?