The derived category of cubic hypersurfaces

One of the main problems in algebraic geometry is the classification problem; classify, up to isomorphism, all the algebraic varieties. The first step towards this goal is to solve the weaker problem; classify up to birational equivalence. The natural beginning to this is to gain an understanding of the simplest varieties: projective spaces. So, one of the problems that have attracted considerable interest in algebraic geometry is to decide whether a smooth projective variety is *rational* (i.e., birationally isomorphic to some projective space).

Already for complex projective varieties to decide whether they are rational is a complicated problem. For surfaces, the Castelnuovo's rationality criterion gives a numerical answer to this problem. In particular, the Castelnuovo's rationality criterion implies that if a surface can be covered by a rational variety, then it is rational. The varieties that can be covered by a rational variety are called *unirational.* The Lüroth problem asks if all *unirational* varieties are *rational* and, by the Castelnuovo's rationality criterion, it has an affirmative answer for complex projective surfaces.

In higher dimensions, Clemens–Griffiths showed that non-singular cubic threefolds are non-rational unirational varieties. Their work used an intermediate Jacobian. Only some months before, Iskovskih–Manin had showed that all non-singular quartic threefolds are irrational, though some of them are unirational. And during the same year, using the torsion subgroup of the third integral cohomology group, Artin–Mumford constructed examples, in all dimensions, of unirational varieties that are not rational. So, the answer to the Lüroth problem in higher dimension is negative.

However, in all these examples, the rationality is preserved in families. Hence, a new question arose: is rationality an open or closed condition? Cubic fourfolds are unirational varieties and some of them are known to be rational. It is expected that they will provide an example where the condition of rationality is neither open nor closed. Unfortunately, no cubic fourfold is known to be irrational.

To attack the problem of the rationality of cubic hypersurfaces the classical approach was based on Hodge theory (see for example the work of Hassett). More recently, following the ideas of Bondal and Orlov, the *bounded derived category of coherent sheaves* has been used to obtain information on the geometry of the variety itself. In particular, it has been conjectured that the derived category should recognize whether a cubic hypersurface is rational or not. The main contributions and conjectures in this direction are due to Kuznetsov, and recently, by Ballard, Favero, and Katzarkov.

In this short summer school we would like to present the derived category new approach to the rationality problem and compare it with the classical Hodge theoretical approach.

Summary of the contents

Summarizing the above discussion, the list of topics covered by this short summer school certainly includes the following:

- Preliminaries about the derived category and semi-orthogonal decompositions;
- Stability conditions on the derived category;
- Cubic threefolds. Hodge theory and derived category.
- Hodge theoretical results for cubic fourfolds;
- Derived category of cubic fourfolds;
- Rationality conjectures in the derived category.