# Elliptic PDEs 1:
# Why do we study them?

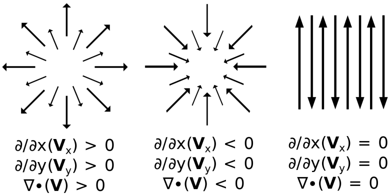
Damià Torres Latorre

Universitat de Barcelona

13th May 2020

# First Examples: The Heat Equation

- Fourier's Law: $q_i = -k\frac{\partial T}{\partial x_i}$.
- Conservation of energy: $\frac{\partial Q}{\partial t} = -\operatorname{Div}\mathbf{q}$.



# First Examples: The Heat Equation (2)

- $\Delta Q = c\rho\Delta T$.
- $\frac{\partial T}{\partial t} = \frac{1}{c\rho}(-\operatorname{Div}\mathbf{q}) = \frac{k}{c\rho}\operatorname{Div}(\nabla T) = \alpha^2\Delta T$.
- If $k$ is not a constant, $\frac{\partial T}{\partial t} = A\operatorname{Div}(B\nabla T)$. $A, B$ can be any functions, we will revisit this formulation later.
- As $t$ increases, the solution $T$ becomes *flatter*.

# Other Physical Applications

- Fick's Law (diffusion): $\varphi_t = D\Delta\varphi$.
- Poisson equation (electrostatics): $\Delta\varphi = \rho/\varepsilon$.

# Elliptic, Parabolic and Hyperbolic PDEs

Take a second order linear PDE, in two variables to make it easy.

$$(a\partial_{xx} + b\partial_{xy} + c\partial yy + d\partial_x + e\partial_y + f)u = 0$$

Now consider the conic section:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

We extend the name of the conic section to the PDE, and we classify:

- $b^2 - 4ac < 0$: Elliptic PDEs.
- $b^2 - 4ac = 0$: Parabolic PDEs.
- $b^2 - 4ac > 0$: Hyperbolic PDEs.

# Lower Order Terms

We can convert a linear second order PDE to a *canonical form* with a change of variables. If we do not, first order terms can be interpreted as transport terms, and the coefficient without derivatives as a reaction term.

$$(\partial_{x_1x_1} + \ldots + \partial_{x_px_p} - \partial_{x_{p+1}x_{p+1}} - \ldots - \partial_{x_qx_q} + \partial_{x_{q+1}} + \ldots + \partial_{x_n} + \lambda)u = 0$$

- No linear terms, all signs equal: elliptic.
- No linear terms, different signs: hyperbolic.
- One linear term, all signs equal: parabolic.
- Otherwise: think about it.

# Stationary Solutions of Parabolic Problems

In physics, the typical parabolic PDE has a time derivative as a first-order term (just as the heat equation). To get the stationary solutions, set $\dot{u} = 0$. Let $\mathcal{L}$ be a positive definite linear second order operator. If the original problem is

$$u_t = \mathcal{L}u$$

The stationary solutions of this problem satisfy

$$\mathcal{L}v = 0$$

# Holomorphic and Harmonic Functions

If $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic, the functions $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, viewed as $u, v : \Omega \to \mathbb{R}$ are harmonic, i.e.,

$$\Delta u = \Delta v = 0$$

Conversely, any harmonic function in $\Omega \subseteq \mathbb{R}^2$ is the real (or imaginary) part of some holomorphic function.

This yields many properties for harmonic functions in two variables, for example:

- Analyticity, power series representation.
- Cauchy integral formula (mean value property).
- Liouville's theorem (in the whole $\mathbb{R}^2$, after some tricks).

## Harmonic Functions

Deriving all the properties from holomorphic functions is good for two variables, but most of them remain when we consider solutions to $\Delta u = 0$ in $\mathbb{R}^n$.

The study of these properties and seeing how can we relax the harmonicity condition to maintain them are one of the most important goals of elliptic PDE theory.

# Regularizing Effect of PDEs

For certain PDEs, the set of solutions is constrained in a way such that if $u$ is a *sufficiently nice* solution, then automatically it is *nicer*, because it is a solution. This is called regularizing effect.

- Solutions of the heat equation are *flattened* as time goes by.
- Harmonic $\mathcal{C}^2$ functions are automatically analytic.

# The Random Walk Expected Value Problem

Let $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ a Jordan curve, i.e., $\Omega$ is the nicest set in all $\mathbb{R}^2$: closed, simply connected, with $\mathcal{C}^1$ boundary.

In the boundary of $\Omega$, there is money, $g : \partial\Omega \to \mathbb{R}$. We start at a point $x_0 \in \Omega$, and we walk randomly until we touch the boundary, then we collect the money there and exit.

How do we choose $x_0$ to maximize the expected benefits?

# The Random Walk Expected Value Problem (2)

Let $u : \Omega \to \mathbb{R}$ be the expected value of the money in each point. It is clear that $u = g$ in $\partial\Omega$.

If $x$ is a point of the interior of $\Omega$, we walk from $x$ to any of the surrounding points. Hence, for small $r > 0$,

$$u(x) = \frac{1}{2\pi r} \int_{\partial B_r(x)} u(y)\mathrm{d}y,$$

$$u(x) = \lim_{r \to 0^+} \frac{1}{2\pi r} \int_{\partial B_r(x)} u(y)\mathrm{d}y$$

# The Random Walk Expected Value Problem (3)

$$\Delta u = c_n \lim_{r \to 0^+} \int_{\partial B_r(x)} (u(y) - u(x)) \mathrm{d}y \text{ in } \mathbb{R}^n.$$

Our expected value function, then, needs to satisfy $\Delta u = 0$.

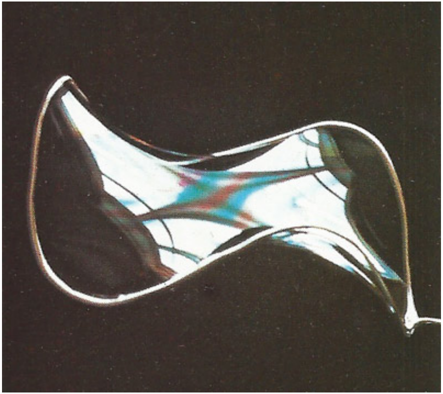
If we want to compute the expected time of arrival at the boundary, we set $T = 0$ at $\partial\Omega$, and we use the fact that:

$$T(x) = T(r, x) + \frac{1}{2\pi r} \int_{\partial B_r(x)} T(y) \mathrm{d}y$$

$$\Delta T = -c(x)$$

# Minimal Surfaces

Consider a membrane with a fixed boundary, think of a soap bubble in a wire. It will have the least tensioned possible shape.



# Minimal Surfaces (2)

Let $v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be the surface, wieved as a graph, let $g : \partial\Omega \to \mathbb{R}$ be the wire.

The tension of a surface can be approximated by $c|\nabla v|^2$, where $c$ depends on phyisics.

$$\min E(v) = \int_\Omega |\nabla v|^2, \quad v|_{\partial\Omega} = g$$

# Minimal Surfaces (3)

A necessary condition for a minimizer $v$ is, for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$,

$$E(v) \leq E(v + t\varphi) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} E(v + t\varphi) = 0 \text{ at } t = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla v + t\nabla\varphi|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (2t\nabla v \cdot \nabla\varphi + t^2|\nabla\varphi|^2) =$$

$$= 2\int_\Omega \nabla v \cdot \nabla\varphi = -2\int_\Omega \Delta v \varphi$$

$$\Delta v = 0$$

# The Obstacle Problem

Minimize the *energy* of a surface $v : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ above an obstacle $\varphi$.

$$\min \int_\Omega |\nabla v|^2, \quad v \geq \varphi.$$

This is equivalent to a nonlinear elliptic PDE problem:

$$\begin{cases} v \geq \varphi \text{ in } \Omega, \\ \Delta v \leq 0 \text{ in } \Omega, \\ \Delta v = 0 \text{ in the set } v > \varphi. \end{cases} \tag{1}$$

We can change $\Delta$ for any elliptic operator $\mathcal{L}$, and we retrieve a more general obstacle problem, which shares some of the properties and known results.

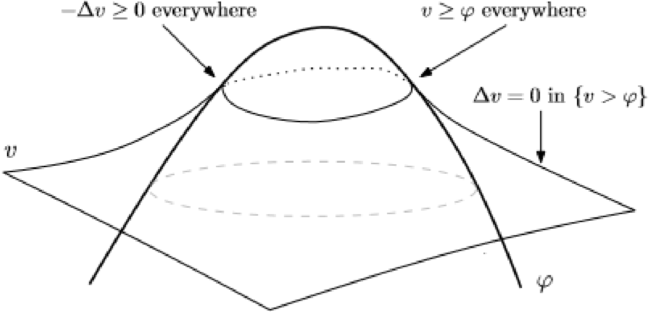
# The Obstacle Problem (2)

- Let $\Omega$ be a bounded Lipschitz domain (i.e., the boundary is locally the graph of a Lipschitz function, with the Lipschitz constants globally bounded).
- Let $\varphi \in \mathcal{C}^\infty$.
- We must add a boundary condition to equation 1, $v|_{\partial\Omega} = g \in L^2(\partial\Omega)$.

Then, if exists any function $w \in H^1(\Omega)$ satisfying $w \geq \varphi$, $w|_{\partial\Omega} = g$, there exists a unique solution $v \in H^1(\Omega)$ for the obstacle problem.

Recall $H^1(\Omega) = \{w : \Omega \to \mathbb{R}, w \in L^2(\Omega), \nabla w \in L^2(\Omega)\}$.

# The Free Boundary




The free boundary is defined as $\Gamma = \partial\{u > 0\} \cap \Omega$, where $u = v - \varphi$, that is, the boundary of the contact set.

# Optimal Stopping

Recall what we did with the Random Walk. Now, we have $\Omega$ with a benefit in all points, $\varphi : \Omega \to \mathbb{R}$, but we can stop the random walk at will and get the money.
Our expected value satisfies $u \geq \varphi$ because we can stop if we want to.

$$u(x) \geq \frac{1}{2\pi r} \int_{\partial B_r(x)} u(y) \mathrm{d}y \Rightarrow \Delta u \leq 0$$

If $u > \varphi$, we do not stop here. Hence, we walk. And when we walk, $\Delta u = 0$ in the set $\{u > \varphi\}$.
Observe that these conditions are exactly the same as in the obstacle problem.

# Elliptic PDEs 1: Why do we study them?

Damià Torres Latorre

Universitat de Barcelona

13th May 2020

# Bonus: The Truth

*"... we study elliptic PDE because we are such nerds"*

- maybe not Alessio Figalli, 2019