Cálculo efectivo de sistemas espectrales y su relación con la homología persistente multiparamétrica

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Trabajo conjunto con A. Guidolin, J. Divasón y F. Vaccarino

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- We use a previous work where we developed a set of algorithms and programs for computing spectral sequences.

Chain complexes, homology

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  A reduction $\rho : \hat{C} \Rrightarrow C_*$ is given by three maps $f : \hat{C}_* \to C_*$, $g : C_* \to \hat{C}_*$ and $h : \hat{C}_* \to \hat{C}_{*+1}$ satisfying some properties, which in particular imply that $H_*(\hat{C}_*) \cong H_*(C_*)$.

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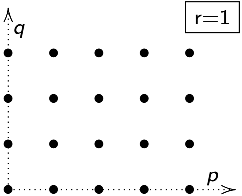

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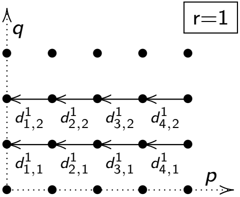

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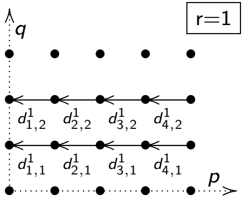

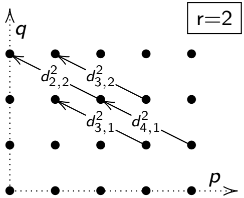

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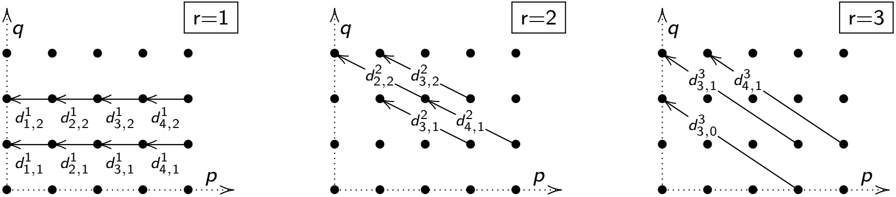

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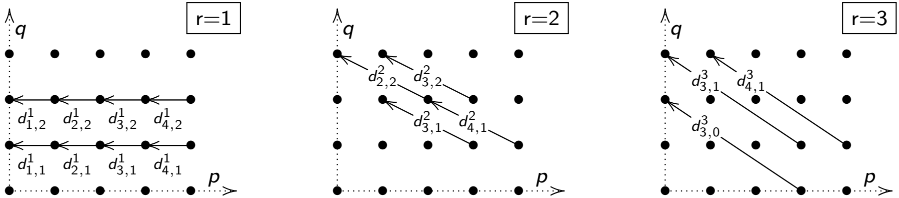

Since $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ for each $r \geq 1$, one can define the **final groups** of the spectral sequence as $E^\infty_{p,q} = \bigcap_{r\geq 1} E^r_{p,q}$.

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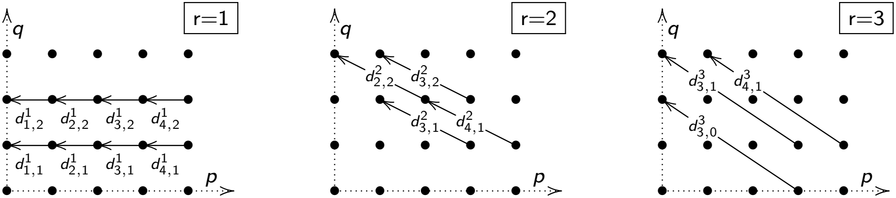

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- The **Adams spectral sequence** converges to the homotopy groups of a simplicial set.

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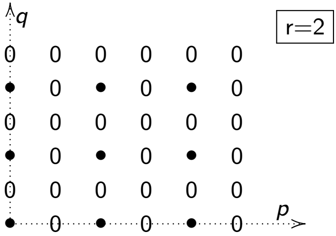

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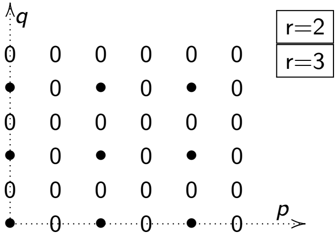

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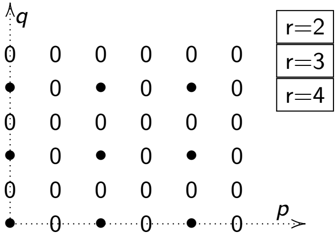

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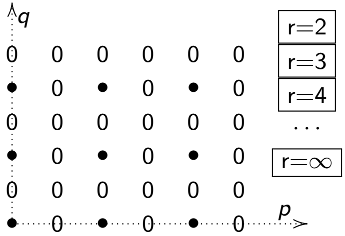

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They are not algorithms producing the desired H_*

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It holds:

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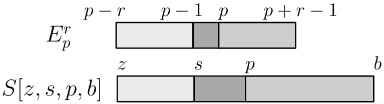

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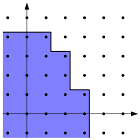

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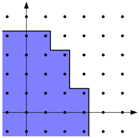
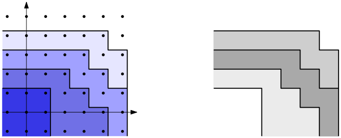


We denote $D(\mathbb{Z}^m)$ the collection of all downsets of $\mathbb{Z}^m$, which is a poset with respect to the inclusion $\subseteq$.



Motivating example

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Theorem (Serre, 1951)

Let $G \hookrightarrow E \to B$ be a **fibration** and suppose the base B is 1-reduced. There is a spectral sequence converging to $H_*(E)$ whose second page is given by $E^2_{p,q} = H_p(B; H_q(G))$.

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Theorem (Matschke, 2013)

Consider a **tower of fibrations**

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and suppose the base B is 1-reduced. There exists a $D(\mathbb{Z}^2)$-spectral system converging to $H_*(E)$ whose second page is given by

$$S^*_n(P;2) = H_{p_2}(B; H_{p_1}(M; H_{n-p_1-p_2}(G))), \quad P = (p_1, p_2) \in \mathbb{Z}^2.$$

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The result is a **basis-divisors** description of the group, that is:

- a list of combinations $(c_1, \ldots, c_{\alpha+k})$
- a list of torsion coefficients $(b_1, \ldots, b_k, 0, \overset{\alpha}{\ldots}, 0)$.

Algorithms

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To compute the differential map
$d : S_2 \equiv S[z_2, s_2, p_2, b_2] \to S_1 \equiv S[z_1, s_1, p_1, b_1]$ applied to an element
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- We reduce them considering the corresponding divisors.

Algorithms

If a I-filtered chain complex C_* is not of finite type, we use the effective homology method and we consider a pair of reductions $C_* \Lleftarrow \hat{C}_* \Rrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered over I).

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Theorem

Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ be a reduction between the I-filtered chain complexes (C_*, F) and (D_*, F'), and suppose that f and g are compatible with the filtrations. Then, given four indices $z \leq s \leq p \leq b$ in I, the map f induces an isomorphism $f^{z,s,p,b} : S_n[z, s, p, b] \to S'_n[z, s, p, b]$ whenever the homotopy $h : (C_*, F) \to (C_{*+1}, F)$ satisfies the conditions

$$h(F_z) \subseteq F_s \qquad \text{and} \qquad h(F_p) \subseteq F_b.$$

Discrete Morse Theory for algorithmic efficiency

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Let $F = (F_i)_{i \in I}$ be an I-filtration of C_*, and let $V = \{(\sigma_j; \tau_j)\}_{j \in J}$ be an admissible discrete vector field on C_* such that, for all $j \in J$, the cells σ_j and τ_j appear together in the filtration. Then there exists a reduction $\rho =: C_* \Rrightarrow C_*^c$, where C_*^c is the **critical** chain complex (generated by the cells which do not appear in the vector field), which is compatible with the filtrations.

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Corollary

Under the same hypotheses, the generalized spectral sequences associated with the I-filtrations of C_* and C_*^c are isomorphic.

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A. Guidolin, A. R. *Effective Computation of Generalized Spectral Sequences.* Proceedings ISSAC 2018, 183-190.

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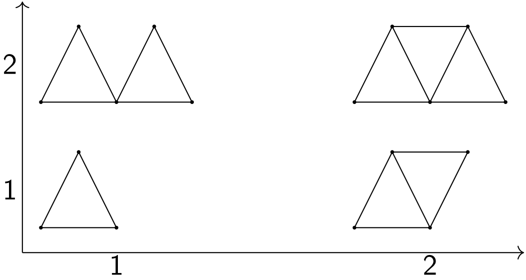
Example



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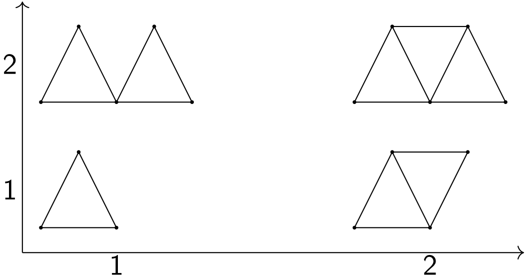


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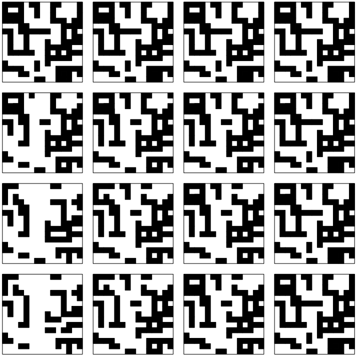

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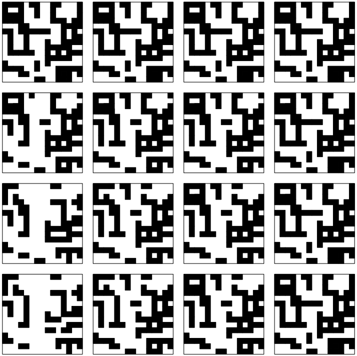

Associated simplicial complex: 203 vertices, 408 edges and 208 triangles.

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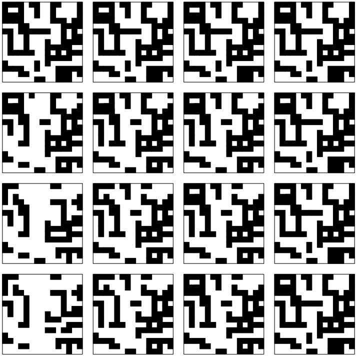

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Reduced chain complex: 21 vertices, 23 edges and 5 triangles.

Generalized Serre spectral sequence: example

First stages of the Postnikov tower for computing the homotopy groups of the sphere S^3, given by the following tower of fibrations:

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Multi-parameter persistence and persistence of I-filtrations

Multi-parameter filtrations (or $\mathbb{Z}^m$-filtrations) of simplicial complexes:

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Associated invariant: **rank invariant**

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Similarly, for an I-filtration $(F_i)_{i\in I}$, we define the **rank invariant** as the collection of integers

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A. Guidolin, J. Divasón, A. R., F. Vaccarino. *Computing invariants for multipersistence via spectral systems and effective homology*. Journal of Symbolic Computation 104, 724–753, 2021.

Relation between multi-parameter persistence and spectral systems

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A **partially ordered abelian group** $(I, +, \leq)$ is an abelian group $(I, +)$ endowed with a partial order $\leq$ that is **translation invariant**: for all $p, t, t' \in I$, if $t \leq t'$ then $p + t \leq p + t'$.

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Theorem

Let $(I, +, \leq)$ be a partially ordered abelian group, and let $(F_i)_{i \in I}$ be an I-filtration of chain complexes. Then, for any $v, w \in I$ such that $v, w \geq 0$ there is an exact sequence

$$\cdots \to S_n[-\infty, -\infty, p - v, p - v + w] \xrightarrow{\ell} S_n[-\infty, -\infty, p, p + w] \xrightarrow{\ell}$$
$$\xrightarrow{\ell} S_n[p - v - w, p - v, p, p + w] \xrightarrow{k}$$
$$\xrightarrow{k} S_{n-1}[-\infty, -\infty, p - v - w, p - v] \xrightarrow{\ell} S_{n-1}[-\infty, -\infty, p - w, p] \to \cdots ,$$

which yields the relation

$$\dim_{\mathbb{F}} S_n[p - v - w, p - v, p, p + w] = \beta_n^{p, p+w} - \beta_n^{p-v, p+w}$$
$$+ \beta_{n-1}^{p-v-w, p-v} - \beta_{n-1}^{p-v-w, p}.$$

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Consider the partially ordered abelian group $(\mathbb{Z}^m, +, \leq)$ and a $\mathbb{Z}^m$-filtration $(F_P)_{P \in \mathbb{Z}^m}$. The previous theorem gives a relation between the rank invariant of multi-parameter persistence and the dimension of the terms of the spectral system over $\mathbb{Z}^m$.

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Our results generalize those obtained by Basu and Parida for spectral sequences and persistent homology (defined from $\mathbb{Z}$-filtrations)

S. Basu, L. Parida . *Spectral Sequences, Exact Couples and Persistent Homology of filtrations*. Expositiones Mathematicae 35 (1), 119–132, 2017.

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We compute these groups by using our previous programs for computing spectral systems.

Generalizing the rank invariant in the finite case

We generalize existing programs for computing multi-parameter persistence, which are valid for specific situations. We consider integer coefficients and a general poset I.

We define the quotient group:

$$H_n^{v,w} := \frac{F_v C_n \cap \operatorname{Ker} d_n}{F_v C_n \cap d(F_w C_{n+1})},$$

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We compute these groups by using our previous programs for computing spectral systems. We obtain the rank and also the generators and the torsion coefficients.

Computation of a new descriptor

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We propose a new descriptor which tries to express the notions of **birth** and **death**.

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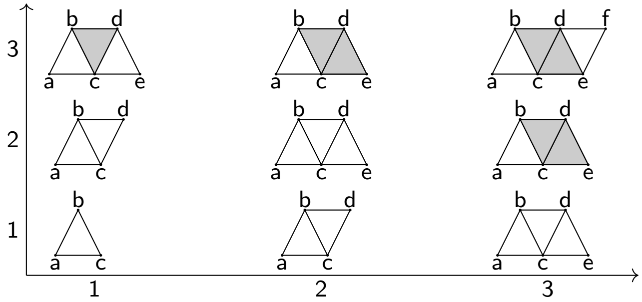
For regular persistent homology, defined from $\mathbb{Z}$-filtrations:

$$M_n^{i,j} := \frac{F_i C_n \cap d(F_j C_{n+1}) + F_{i-1} C_n}{F_i C_n \cap d(F_{j-1} C_{n+1}) + F_{i-1} C_n}$$

Multipersistence

Example:



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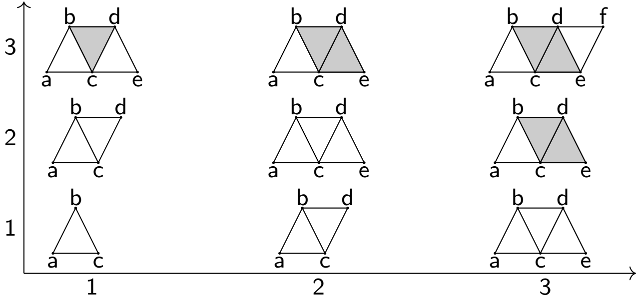

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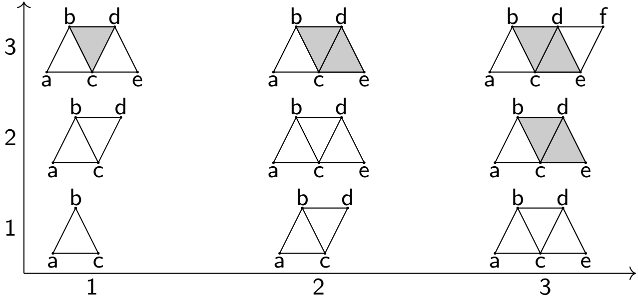


The class corresponding to the (boundary of) the triangle *bcd* is born at both positions $(1,2)$ and $(2,1)$ and the (boundary of) the triangle *cde* is born at positions $(1,3),(2,2)$ and $(3,1)$.

Computation of a new descriptor

Computation of a new descriptor

Definition

Let (F_P) be a $\mathbb{Z}^m$-filtration and consider the canonically associated $D(\mathbb{Z}^m)$-filtration $(F_p = \sum_{P \in p} F_P)$. For each $p \leq b$ in $D(\mathbb{Z}^m)$ we define

$$M_n^{p,b} = \frac{\hat{F}_p C_n \cap d(\hat{F}_b C_{n+1})}{A_{p,n} + B_{b,n}}$$

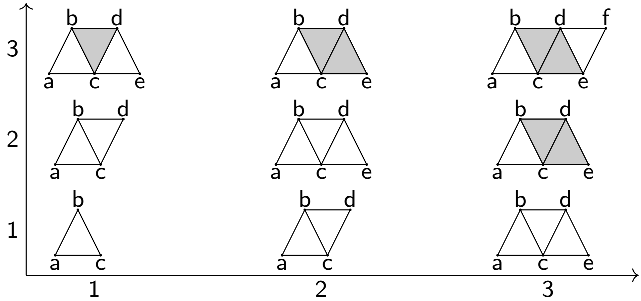
where

$$\begin{aligned}
\hat{F}_p C_n =& \{\sigma | \sigma \in F_{P_j} C_n \text{ for all } 1 \leq j \leq k\} = \textstyle\bigcap_j F_{P_j} C_n \\
A_{p,n} =& \textstyle\sum_Q (\hat{F}_p C_n \cap d(\hat{F}_b C_{n+1}) \cap F_Q C_n) + \sum_X (\hat{F}_p C_n \cap d(\hat{F}_b C_{n+1}) \cap F_X C_n) \\
B_{p,n} =& \textstyle\sum_R (\hat{F}_p C_n \cap d(\hat{F}_b C_{n+1}) \cap d(F_R C_{n+1})) \\
& + \textstyle\sum_Y (\hat{F}_p C_n \cap d(\hat{F}_b C_{n+1}) \cap d(F_Y C_{n+1}))
\end{aligned}$$

with $Q \in \mathbb{Z}^m$ not comparable with the points P_j defining the downset p, $X \in p \setminus \{P_1, \ldots, P_k\}$, $R \in \mathbb{Z}^m$ not comparable with the points B_j defining the downset b and $Y \in b \setminus \{B_1, \ldots, B_r\}$.

Multi-parameter persistence

Example:



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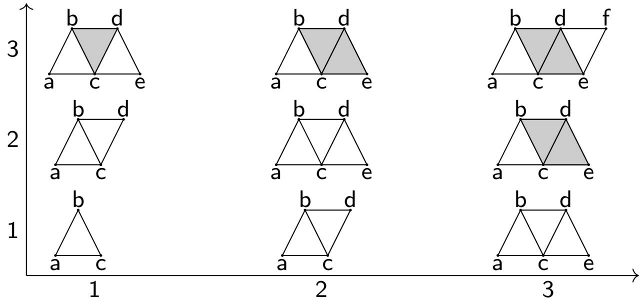


```
> (multiprst-m-group K (list '(1 3) '(2 2) '(3 1))
             (list '(2 3) '(3 2)) 1)
Multipersistence group M[((1 3) (2 2) (3 1)),((2 3) (3 2))]_{1}
Component Z
> (multiprst-m-gnrts K (list '(1 3) '(2 2) '(3 1))
             (list '(2 3) '(3 2)) 1)
({CMBN 1}<1 * CD><-1 * CE><1 * DE>)
```

Multipersistence

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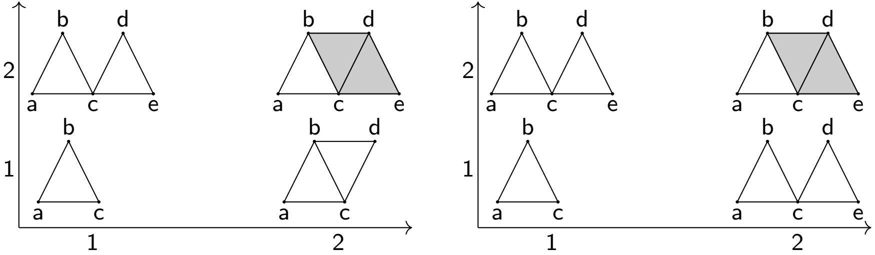
Our new descriptor distinguishes different filtrations:



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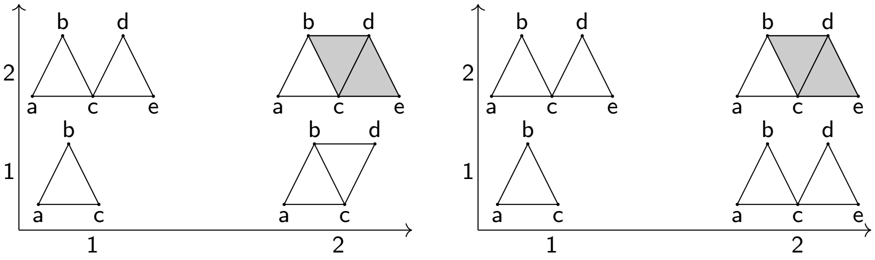


```
> (multiprst-m-group K1 (list '(1 2) '(2 1)) (list '(2 2)) 1)
Multipersistence group M[((1 2) (2 1)),((2 2))]_{1}
NIL
> (multiprst-m-group K2 (list '(1 2) '(2 1)) (list '(2 2)) 1)
Multipersistence group M[((1 2) (2 1)),((2 2))]_{1}
Component Z
```

Effective homology for infinitely generated spaces

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If a I-filtered chain complex C_* is not of finite type, we use the effective homology method and we consider a pair of reductions $C_* \twoheadleftarrow \hat{C}_* \twoheadrightarrow D_*$ from the initial chain complex C_* to another one D_* of finite type (also filtered over I).

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Theorem

Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction between the I-filtered chain complexes (C_*, F) and (D_*, F'), and suppose that the maps f, g and h are compatible with the filtrations. Then, $H_n^{p,b}(D_*) \cong H_n^{p,b}(C_*)$ for all $p \leq b$ in I and $M_n^{p,b}(D_*) \cong M_n^{p,b}(C_*)$ for every $p < b$.

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This result allows us to apply our programs to compute multi-parameter persistence of filtered complexes of infinite type.

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(Joint work with D. Miguel, A. Guidolin, and J. Rubio)

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- J. Cuevas-Rozo, M. Marco-Buzunáriz, A. R. *Computing with Kenzo from Sage*. MEGA 2019, software presentation.
- J. Cuevas-Rozo, J. Divasón, M. Marco-Buzunáriz, A. R. *A Kenzo interface for algebraic topology computations in SageMath*. ISSAC 2019, Best software demo award.
- J. Cuevas-Rozo, J. Divasón, M. Marco-Buzunáriz, A. R. *Integration of the Kenzo system within SageMath for new Algebraic Topology Computations*. Mathematics 9(7), 722, 2021.

¡Muchas gracias!