L'Enseignement Mathématique (2) 54 (2008), 61–63

21

POSTNIKOV PIECES OF FINITE DIMENSION

by Carles CASACUBERTA and Manuel CASTELLET

The authors do not know the answer to the following question:

QUESTION 21.1. *Can the higher homotopy groups of a finite CW-complex X be nonzero vector spaces over the field $\mathbf{Q}$ of rationals?*

Of course, if X is nilpotent, then the answer is negative, since the homotopy groups of a nilpotent finite CW-complex are finitely generated (see [2]). On the other hand, if X is not nilpotent, then the higher homotopy groups of X need not be finitely generated, not even as modules over the integral group ring $\mathbf{Z}[\pi_1(X)]$ of the fundamental group of X (Stallings gave a counterexample in [5]). Hence, the question does make sense.

It seems very unlikely that there exists a finite CW-complex X such that, for example, $\pi_2(X) \cong \mathbf{Q}$. However, we have been unable to discard this possibility so far. What is relevant here is that infinitely many primes are asked to be invertible in $\pi_2(X)$. It is well known that one can attach a 3-cell to $S^1 \vee S^2$ so as to render any *finite* number of primes invertible in the second homotopy group.

Let us now explain why Question 21.1 is relevant and give a reason why we would be pleased if the answer were negative.

One might ask for examples of CW-complexes of finite dimension that are *Postnikov pieces*, i.e., that have only a finite number of nonzero homotopy groups. Many such examples come to mind: a wedge of circles, a torus, a Klein bottle, or in fact any compact surface of higher genus, and products of these. *Note that each of these examples is a $K(G,1)$*; that is, an aspherical space. Rational spheres and rational complex projective spaces are Postnikov pieces that are not aspherical and admit models which are finite-dimensional, yet surely not finite.

The following theorem appeared in [1]:

THEOREM 21.2. *If X is a CW-complex of finite dimension with only a finite number of nonzero Postnikov invariants, then X is a Postnikov piece and its homotopy groups $\pi_n(X)$ are $\mathbf{Q}$-vector spaces for $n \geq 2$.*

The proof uses homotopical localization with respect to the map from $B\mathbf{Z}/p$ to a point, where p is any prime, together with Miller's proof of the Sullivan conjecture. Alternatively, it follows from a result in [3], which relies on a similar line of argument.

Thus, it is not surprising that Question 21.1 has implications around Serre's classical theorem about the higher homotopy groups of 1-connected finite CW-complexes (see Théorème 10 on p. 217 of [4]). Thus, let us be optimistic and state the following:

CONJECTURE 21.3. *If X is a finite CW-complex that is a Postnikov piece, then X is a $K(G,1)$.*

By Theorem 21.2, if X is a finite CW-complex that is a Postnikov piece, then the higher homotopy groups of X are $\mathbf{Q}$-vector spaces. Hence, the fate of this conjecture will depend on our ability to find finite CW-complexes whose higher homotopy groups are nonzero $\mathbf{Q}$-vector spaces.

It would be very exciting to generalize Serre's old theorem along these lines, if the conjecture were true. So far we can only report that the conjecture holds when the dimension of X is less than or equal to 3.

REFERENCES

[1] CASACUBERTA, C. On Postnikov pieces of finite dimension. *Collect. Math. 49* (1998), 257–263.

[2] HILTON, P., G. MISLIN and J. ROITBERG. *Localization of Nilpotent Groups and Spaces.* North-Holland Math. Studies, vol. 15, North-Holland, 1975.

[3] MCGIBBON, C. A. and J. A. NEISENDORFER. On the homotopy groups of a finite dimensional space. *Comment. Math. Helv. 59* (1984), 253–257.

[4] SERRE, J.-P. Cohomologie modulo 2 des complexes d'Eilenberg–Mac Lane. *Comment. Math. Helv. 27* (1953), 198–232.

[5] STALLINGS, J. A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math. 85* (1963), 541–543.

Carles Casacuberta

Facultat de Matemàtiques
Universitat de Barcelona
Gran Via de les Corts Catalanes, 585
E-08007 Barcelona
Spain
e-mail: carles.casacuberta@ub.edu

Manuel Castellet

Centre de Recerca Matemàtica
Apartat 50
E-08193 Bellaterra
Barcelona
Spain
e-mail: manuel@castellet.cat

Achevé de composer le 13 mai 2008 à 23 : 08