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On the existence of group localizations under large-cardinal axioms

Carles Casacuberta and Dirk Scevenels

Abstract. A long-standing open question in categorical group theory asks if every orthogonal pair (consisting of a class of groups and a class of group homomorphisms determining each other by orthogonality in the sense of Freyd–Kelly) is associated with a localization. This is known to be true if one assumes the validity of a suitable large-cardinal axiom (Vopěnka's principle), but so far no proof has been given using the ordinary ZFC axioms of set theory. The answer is affirmative in ZFC if the orthogonal pair is generated by a set of groups or by a set of homomorphisms. In this article we use ideas of Adámek–Rosický and Dugas–Göbel to show that (a) there exist orthogonal pairs which are not generated by any set of groups; (b) the statement that every orthogonal pair is generated by a set of homomorphisms cannot be proved in ZFC, but it follows from Vopěnka's principle.

Existencia de localizaciones de grupos bajo axiomas de cardinales grandes

Resumen. Uno de los problemas abiertos más antiguos de la teoría de grupos categórica es si todo par ortogonal (formado por una clase de grupos y una clase de homomorfismos que se determinan mutuamente por ortogonalidad en el sentido de Freyd–Kelly) se halla asociado a un funtor de localización. Se sabe que esto es cierto si se acepta la validez de un cierto axioma de cardinales grandes (el principio de Vopěnka), pero no se conoce ninguna demostración mediante los axiomas ordinarios (ZFC) de la teoría de conjuntos. También es sabido que la respuesta es afirmativa en ZFC para cualquier par ortogonal generado por un conjunto de grupos o por un conjunto de homomorfismos. En este artículo se usan ideas de Adámek–Rosický y Dugas–Göbel para probar que: (a) existen pares ortogonales que no están generados por ningún conjunto de grupos; (b) la afirmación de que todo par ortogonal está generado por un conjunto de homomorfismos no puede demostrarse en ZFC y sin embargo su veracidad se deduce del principio de Vopěnka.

0. Introduction

An object X and a morphism $\varphi\colon A \to B$ in a category $\mathcal{C}$ are called orthogonal [15] if for every morphism $\alpha\colon A \to X$ there is a unique morphism $\beta\colon B \to X$ such that $\beta \circ \varphi = \alpha$. For a class of morphisms $\mathcal{S}$, we denote by $\mathcal{S}^{\perp}$ the class of objects orthogonal to all morphisms in $\mathcal{S}$. This is called an orthogonality class. Such classes arise in many different contexts, especially in algebra, category theory, geometry and topology.

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One is often interested in deciding if a given orthogonality class $\mathcal{S}^\perp$ is reflective; that is, if the full embedding $\mathcal{S}^\perp \hookrightarrow \mathcal{C}$ has a left adjoint $L\colon \mathcal{C} \to \mathcal{S}^\perp$. (In other words, if for every object X in $\mathcal{C}$ there is a universal morphism $X \to LX$ to an object in $\mathcal{S}^\perp$.) This was called the "orthogonal subcategory problem" by Freyd–Kelly [15]. The corresponding left adjoints are called reflectors or localizations. Their existence and properties have been extensively discussed in the literature; see e.g. [1], [2], [7], [9], [12], [20], [23], [25], [26], [27].

The orthogonal subcategory problem has played an important role in homotopy theory since the decade of 1970, mainly due to the study of homological localizations of spaces and spectra. The approach of Adams [2] led to the introduction of orthogonal pairs in [9] as a useful tool to treat localization problems —in homotopy theory and elsewhere. Precise definitions are recalled below. In this terminology, the orthogonal subcategory problem can be reformulated by asking if every orthogonal pair is associated with a localization.

A fundamental advance was made by Adámek and Rosický in [1] by showing that, in every locally presentable category (as defined by Gabriel–Ulmer [17]), orthogonality classes are reflective if Vopěnka's principle is supposed true. This is a large-cardinal axiom which cannot be proved using ZFC (the Zermelo–Fraenkel axioms with the axiom of choice), yet it is believed to be consistent with ZFC, after more than thirty years of related developments in set theory; see [1, Ch. 6] or [19, § 24].

The answer to the orthogonal subcategory problem is affirmative in ZFC if the orthogonal pair is generated by a set of objects and the category is complete and well powered. A proof of this fact can be found in [8]. (This was previously shown by Pfenniger in unpublished work; another predecessor was [21].) On the other hand, it has long been known that the answer is affirmative if the category is cocomplete and the orthogonal pair is generated by a set of morphisms, provided that some additional conditions are satisfied. The formulation of these additional conditions varies depending on the authors, but the construction of the reflector is essentially the same everywhere and dates back to the work of Gabriel–Ulmer [17]; see also [4], [20].

If the given category has products, then a set of objects and their product generate the same orthogonal pair, and similarly for morphisms and coproducts. Thus one may ask, without loss of generality, if every orthogonal pair is generated by either a single object or a single morphism in categories with products or coproducts, respectively. Localizations associated with orthogonal pairs generated by single morphisms (here called φ-localizations) have been the subject of much research in the past decade and have led to interesting developments in group theory and in homotopy theory; see [5], [6], [7], [10], [13], [22], [24], among others.

In a different direction, Dugas and Göbel considered in [14] the problem whether non-hereditary torsion theories of abelian groups are singly generated or singly cogenerated. This is very much related to the above queries, and their results provide the clue to discuss generators for orthogonal pairs in the category of groups, as we do in this article. We prove the following facts, which summarize the current status of knowledge.

- There are orthogonal pairs in the category of groups which are not generated by any set of groups.

- It is impossible to prove (in ZFC) that every orthogonal pair in the category of groups is generated by a set of homomorphisms. Indeed, the assumption that there are no measurable cardinals is consistent with ZFC, and under this assumption we are able to exhibit a counterexample.

- Vopěnka's principle implies that every orthogonal pair in the category of groups is generated by a set of homomorphisms. Therefore, if we assume the validity of Vopěnka's principle, then every orthogonal pair in the category of groups is associated with a localization, and every localization is φ-localization for some homomorphism φ.

Strictly speaking, the problem whether every orthogonal pair in the category of groups is associated with a localization remains open, since a positive answer may be given in ZFC. However, it is unreasonable to try to find a counterexample, since the existence of a counterexample in ZFC would imply the inconsistency of Vopěnka's principle and other large-cardinal axioms.

This article is a complement to [11], which answers in a similar way the question whether every homotopical localization is f-localization for some map f of simplicial sets. Here we develop the group-theoretical counterpart, by providing a detailed review of former results from various sources and supplementing it with Theorem 5 and Theorem 6 below, which extend analogous results from [14] to non necessarily abelian groups.

1. Orthogonal pairs

Let $\mathcal{C}$ be any category. Assume given a functor $E\colon \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\eta\colon \mathrm{Id} \to E$. This is called a *pointed endofunctor* or a *coaugmented functor*. Assume, in addition, that $\eta_{EX} = E\eta_X$ for all X (that is, E is *well pointed*), and $\eta_{EX}\colon EX \to EEX$ is an isomorphism for every X. A functor E with these properties is called *idempotent*. The pair (E, η) is more commonly called an *idempotent monad* or an *idempotent triple*. This terminology suggests that there is a third element implicit; indeed, one obtains a natural transformation $\mu\colon EE \to E$ by taking the inverse of η_{EX} for every X, and then μ and η share the properties of a multiplication and a unit. An idempotent functor is also called a *localization*.

As in [2] or [9], we shall focus our attention on the objects isomorphic to EX for some X, which are called *E-local objects*, and the morphisms $\varphi\colon X \to Y$ such that $E\varphi\colon EX \to EY$ is an isomorphism, which are called *E-equivalences*. It follows from the definition that $\eta_X\colon X \to EX$ is an E-equivalence for all X. It is in fact a terminal E-equivalence out of X, and it is also an initial morphism from X to an E-local object.

For a class of objects $\mathcal{O}$, we denote by $\mathcal{O}^{\perp}$ the class of morphisms orthogonal to all objects of $\mathcal{O}$. Similarly, for a class of morphisms $\mathcal{M}$, we denote by $\mathcal{M}^{\perp}$ the class of objects orthogonal to all morphisms of $\mathcal{M}$. By a slight abuse of terminology, $\mathcal{M}^{\perp}$ also denotes the full subcategory of $\mathcal{C}$ over these objects.

An *orthogonal pair* $(\mathcal{S}, \mathcal{D})$ consists of a class $\mathcal{S}$ of morphisms and a class $\mathcal{D}$ of objects such that $\mathcal{S}^{\perp} = \mathcal{D}$ and $\mathcal{D}^{\perp} = \mathcal{S}$. If $(\mathcal{S}, \mathcal{D})$ is any orthogonal pair, then $\mathcal{D}$ is closed under limits and $\mathcal{S}$ is closed under colimits. If $\mathcal{M}$ is any class of morphisms, then $(\mathcal{M}^{\perp\perp}, \mathcal{M}^{\perp})$ is an orthogonal pair, which is said to be *generated* by $\mathcal{M}$. Similarly, if $\mathcal{O}$ is any class of objects, then $(\mathcal{O}^{\perp}, \mathcal{O}^{\perp\perp})$ is the orthogonal pair generated by $\mathcal{O}$. Every localization functor E gives rise to an orthogonal pair, by letting $\mathcal{S}$ be the class of E-equivalences and $\mathcal{D}$ the class of E-local objects. A class $\mathcal{D}$ of objects is called *reflective* [23, IV.3] if it is part of an orthogonal pair $(\mathcal{S}, \mathcal{D})$ which is associated with a localization. Any two localization functors associated with the same orthogonal pair are naturally isomorphic. In fact, if E_1 and E_2 are localizations onto two classes $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, and we assume that $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then there is a unique natural transformation $E_2 \to E_1$ compatible with the units $\eta_1\colon \mathrm{Id} \to E_1$ and $\eta_2\colon \mathrm{Id} \to E_2$.

From now on we shall work exclusively in the category of groups, although many of our statements continue to hold in a much broader context.

Theorem 1 *If an orthogonal pair $(\mathcal{S}, \mathcal{D})$ in the category of groups is generated by either*

(a) *a set of groups, or*

(b) *a set of homomorphisms, or*

(c) *the union of a set of homomorphisms and a proper class of epimorphisms,*

then it is associated with a localization.

Proof. A proof in the case when $(\mathcal{S},\mathcal{D})$ is generated by a set $\mathcal{D}_0$ of groups is given in [8, Corollary 2.4]. The argument uses that the category of groups is complete and well powered. We repeat it here in a concise form. For a group G, take the inverse limit of the functor from the comma category $G \downarrow \mathcal{D}_0$ into the category of groups, sending a homomorphism $G \to D$ to the group D. This inverse limit $\widehat{G}$ exists because it is indexed by a set, and comes with a natural homomorphism $\eta\colon G \to \widehat{G}$, which is called pro-$\mathcal{D}_0$-completion (profinite completion is a special case). This yields in fact a monad, but not necessarily an idempotent one. Now look at the two homomorphisms $\eta_{\widehat{G}}$ and $\widehat{\eta_G}$ from $\widehat{G}$ to its own pro-$\mathcal{D}_0$-completion, and let E_1G be their equalizer. This also comes with a natural homomorphism $\eta_1\colon G \to E_1G$, which defines again a monad. Let E_2G be the equalizer of $E_1\eta_1$ and η_1E_1, and repeat the same step until the sequence

$$\cdots \hookrightarrow E_{\alpha+1}G \hookrightarrow E_\alpha G \hookrightarrow \cdots \hookrightarrow E_2G \hookrightarrow E_1G \hookrightarrow \widehat{G}$$

stabilizes at some (possibly transfinite) ordinal, by cardinality reasons. Call EG this inverse limit. Then E is idempotent and the class of E-local objects is precisely $\mathcal{D}$.

Now suppose that $(\mathcal{S},\mathcal{D})$ is generated by a set $\mathcal{S}_0$ of homomorphisms. Then the associated localization functor is constructed by the "orthogonal reflection construction", as described in [1, 1.37], by repeatedly taking suitable push-outs and coequalizers indexed by means of the generating set $\mathcal{S}_0$, possibly transfinitely, until the sequence

$$G \to L_1G \to L_2G \to \cdots \to L_\alpha G \to L_{\alpha+1}G \to \cdots$$

stabilizes by cardinality reasons (the length will depend on the cardinalitites of the domains and codomains of the homomorphisms in $\mathcal{S}_0$). This construction works because that the category of groups is locally presentable.

A different argument, based on Freyd's solution-set condition, is given in [15, Theorem 4.1.3]. That argument holds in categories which are complete, cocomplete, bounded, co-well-powered, and equipped with a proper factorization system. It is more general, as it allows the orthogonal pair to be generated by the union of a set of morphisms and a proper class of epimorphisms, hence proving (c).

A simpler proof of the fact that the orthogonal complement $\mathcal{D}$ of any class of group epimorphisms is reflective is given in [10, Proposition 2.1]. Namely, for a group G, let TG be the intersection of all kernels of epimorphisms from G onto groups in $\mathcal{D}$, and let $EG = G/TG$. Then E is a localization onto the class $\mathcal{D}$. ■

In the special case when $(\mathcal{S},\mathcal{D})$ is generated by a class of epimorphisms of the form $A_\alpha \to 1$, where A_α ranges through a class $\mathcal{A}$ of groups, the corresponding localization is called $\mathcal{A}$-*reduction*. Thus, a group R is $\mathcal{A}$-*reduced* if and only if the set $\mathrm{Hom}(A_\alpha, R)$ has only the trivial element for every A_α in $\mathcal{A}$. For example, a group is torsion-free if and only if it is $\mathcal{A}$-reduced when $\mathcal{A}$ is the set of all finite cyclic groups. In this example, the $\mathcal{A}$-reduction functor takes every group onto its largest torsion-free quotient. Whenever the class $\mathcal{A}$ is a set, we may consider the free product G of all its members and call G-*reduction* the corresponding functor; that is, a group R is G-reduced if and only if $\mathrm{Hom}(G, R)$ has only the trivial element.

More generally, if an orthogonal pair $(\mathcal{S},\mathcal{D})$ is generated by a set of homomorphisms, we may take their free product φ. The associated localization is denoted by L_φ and called φ-*localization*, as first done in [6]. Groups G such that $L_\varphi G$ is the trivial group are called φ-*acyclic*. In Section 3 we will need the following fact.

Theorem 2 *If φ is any group homomorphism, then there is a group G such that the φ-acyclic groups are precisely the groups annihilated by G-reduction.*

Proof. Let κ be any infinite cardinal such that each φ-acyclic group is a directed colimit of φ-acyclic groups of cardinality smaller than κ. The existence of such a cardinal is proved in

[24, Theorem 7], using [5, Lemma 3.2]. Then the free product G of a set of representatives of isomorphism classes of φ-acyclic groups of cardinality smaller than κ has the desired property.

The referee pointed out the following alternative argument. Terminology and details can be found in [1]. Choose a regular cardinal λ such that the domain and the codomain of φ are λ-presentable. Then the orthogonal complement of φ is closed under λ-directed colimits, and hence the functor L_φ from the category of groups to itself is accessible (as it preserves λ-directed colimits). Now the full subcategory of φ-acyclic groups is the pseudo-equalizer of L_φ and the constant functor with value the trivial group. It follows from the Limit Theorem of Makkai–Paré (see [1, Theorem 2.77]) that this is an accessible category, and this implies that there exists a set of φ-acyclic groups such that every φ-acyclic group is a directed colimit of these. Take G to be their free product. ■

A group with the property stated in Theorem 2 is called a universal φ-acyclic group (although it need not be unique). It is a discrete analogue of the universal acyclic spaces defined by Bousfield in [5, Theorem 4.4].

2. Abelian groups and measurable cardinals

An uncountable cardinal λ is *measurable* if it admits a nontrivial, two-valued, λ-additive measure; that is, if a function μ can be defined on any set X of cardinality λ assigning to each subset of X a value 0 or 1, in such a way that $\mu(X) = 1$, $\mu(x) = 0$ for all $x \in X$, and $\mu(\cup_i A_i) = \Sigma_i \mu(A_i)$ if the subsets A_i are pairwise disjoint and the set of indices i has cardinality smaller than λ. It is well known that the existence of measurable cardinals cannot be proved in ZFC, since every measurable cardinal is (strongly) inaccessible; see [1, A.10] or [18, 5.27].

There is an important occurrence of measurable cardinals in infinite abelian group theory, which is explained in Fuchs' book [16, § 94] (steming from Łoś). We next recall it briefly. For each cardinal κ, let $\mathbf{Z}^\kappa$ be the cartesian product of κ copies of the additive group of integers, that is, the abelian group of all functions $f\colon \kappa \to \mathbf{Z}$. Denote by $\mathbf{Z}^{<\kappa}$ the subgroup of those functions $f \in \mathbf{Z}^\kappa$ whose support (i.e., the set of indices $i \in \kappa$ for which $f(i) \neq 0$) has cardinality smaller than κ.

It is an instructive exercise to prove that

$$\mathrm{Hom}(\mathbf{Z}^{\aleph_0}/\mathbf{Z}^{<\aleph_0},\, \mathbf{Z}) = 0;$$

see e.g. [16, Lemma 94.1]. (Here, of course, $\mathbf{Z}^{<\aleph_0}$ is the direct sum of countably many copies of $\mathbf{Z}$.) More generally, the following holds.

Theorem 3 *If a cardinal λ is measurable, then* $\mathrm{Hom}(\mathbf{Z}^\lambda/\mathbf{Z}^{<\lambda},\, \mathbf{Z}) \neq 0$. *Conversely, if we assume that* $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa},\, \mathbf{Z}) \neq 0$ *for some cardinal κ, then the smallest cardinal with this property is measurable.*

Proof. If λ is measurable, then a nonzero homomorphism $\varphi\colon \mathbf{Z}^\lambda \to \mathbf{Z}$ with $\varphi(\mathbf{Z}^{<\lambda}) = 0$ is defined by assigning to each function $f\colon \lambda \to \mathbf{Z}$ the unique integer z such that $f^{-1}(z)$ has measure 1; cf. [14, p. 83] or [16, p. 161].

Conversely, if $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa},\, \mathbf{Z}) \neq 0$ for some cardinal κ, then it follows from [16, Theorem 94.4] that κ admits a nontrivial, two-valued, countably additive measure. Then, by [3, Theorem 6.1.11] or [18, Lemma 27.1], the least such cardinal κ is measurable. ■

This tells us that the question whether there exists some cardinal κ such that $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa},\, \mathbf{Z})$ is nonzero cannot be answered positively in ZFC, since it is impossible to prove in ZFC that measurable cardinals exist. On the other hand, a negative answer in ZFC would imply that measurable cardinals do not exist, and such a development is not to be expected.

In the next section we show that the question whether every orthogonal pair is generated by a set of homomorphisms in the category of groups is undecidable in a similar manner.

3. On the existence of generators

Vopěnka's principle states that *for any proper class of models of the same language, there is one that is elementarily embeddable into another*; see [18, p. 414] or [19, p. 335]. Among other things, this implies the existence of arbitrarily large measurable cardinals. Hence, Vopěnka's principle cannot be proved in ZFC. One of the various equivalent formulations of this principle [1, Ch. 6] says that *no locally presentable category contains a rigid proper class of objects.* (A class of objects is called *rigid* if it admits no other morphisms than identities.) Thus, according to Vopěnka's principle, given a proper class of objects A_i in any locally presentable category, there is a nonidentity morphism $A_i \to A_j$ for some indices i and j.

As explained in [1, Corollary 6.24], this statement has the following consequence: in a locally presentable category, every full subcategory which is closed under limits is reflective, and it is in fact a small-orthogonality class (i.e., the orthogonal complement of a set of morphisms). Since in every orthogonal pair $(\mathcal{S}, \mathcal{D})$ the class $\mathcal{D}$ is closed under limits, the following holds.

Theorem 4 *Suppose that Vopěnka's principle is true. Then every orthogonal pair $(\mathcal{S}, \mathcal{D})$ in a locally presentable category is generated by a set of morphisms.* ■

The category of groups is locally presentable, since the isomorphism classes of finitely presented groups form a set and every group is a directed colimit of finitely presented groups. Therefore, Vopěnka's principle implies that every orthogonal pair in the category of groups is generated by a set of homomorphisms —in fact, by a single homomorphism. In other words,

Corollary 1 *If Vopěnka's principle holds, then for every idempotent functor L in the category of groups there is a homomorphism φ such that L is naturally isomorphic to L_φ.* ■

Next we show that it is impossible to prove this statement using the ordinary ZFC axioms of set theory.

Theorem 5 *Let $\mathcal{A}$ be the class of groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ for all cardinals κ. If the orthogonal pair $(\mathcal{S}, \mathcal{D})$ associated with $\mathcal{A}$-reduction is generated by a set of homomorphisms, then there exists a measurable cardinal.*

Proof. Suppose that there is a set of homomorphisms generating $(\mathcal{S}, \mathcal{D})$. Then their free product φ is a homomorphism such that φ-localization coincides with $\mathcal{A}$-reduction. Let G be a universal φ-acyclic group, as defined at the end of Section 1. Then G-reduction and $\mathcal{A}$-reduction annihilate the same groups and hence coincide. Since the groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ are annihilated by $\mathcal{A}$-reduction, we may infer that $\mathrm{Hom}(G, \mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}) \neq 0$ for all κ.

Let κ be a regular cardinal that is bigger than the cardinality of G. (Recall that a cardinal λ is *regular* if it is infinite and cannot be expressed as a sum of cardinals $\Sigma_{i<\alpha}\lambda_i$ where $\alpha < \lambda$ and $\lambda_i < \lambda$ for all i. The first infinite cardinal $\aleph_0$ is regular and so is every successor cardinal.) Let $\beta\colon G \to \mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ be a nonzero homomorphism. As in the Wald–Łoś Lemma [14, Lemma 2.6], β can be lifted to a nonzero homomorphism $\alpha\colon G \to \mathbf{Z}^\kappa$, as follows. For each element $g \in G$, pick a representative $\phi(g) \in \mathbf{Z}^\kappa$ of the image $\beta(g)$. Thus, for each pair of elements g and h of G, the element $\phi(g) + \phi(h) - \phi(gh)$ lies in $\mathbf{Z}^{<\kappa}$. Let S be the union of the supports of the elements $\phi(g) + \phi(h) - \phi(gh)$ for all pairs of elements g and h of G. The assumption that $\mathrm{card}(G) < \kappa$ ensures that $\mathrm{card}(S) < \kappa$ as well, since κ is regular. Thus, if we define $\alpha(g)$ by setting to zero all the components in S of the element $\phi(g)$, then $\alpha(g)$ and $\phi(g)$ map onto the same element in $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$, and $\alpha\colon G \to \mathbf{Z}^\kappa$ is a homomorphism.

Now, composition with a suitable projection yields a nonzero homomorphism $G \to \mathbf{Z}$ and this implies that $\mathbf{Z}$ is not G-reduced. Hence, $\mathbf{Z}$ is not $\mathcal{A}$-reduced, and this implies the existence of a measurable cardinal, by Theorem 3. ■

Since the statement that all cardinals are nonmeasurable is consistent with ZFC, we conclude that it is impossible to prove in ZFC that every orthogonal pair is generated by a set of homomorphisms. In contrast, we next exhibit an orthogonal pair in ZFC which is not generated by any set of groups.

Theorem 6 *Let $\mathcal{B}$ be the class of groups $\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}$ for all nonmeasurable cardinals κ. Then the orthogonal pair $(\mathcal{S},\mathcal{D})$ associated with $\mathcal{B}$-reduction is not generated by any set of groups.*

PROOF. Suppose that $(\mathcal{S},\mathcal{D})$ is generated by a set of groups, and let G be their product; that is, $\mathcal{S}=\{G\}^\perp$. Let C be the product of all the abelian subgroups of G. Then, since G is $\mathcal{B}$-reduced, we have $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}, C)=0$ for all nonmeasurable cardinals κ. Therefore, the group C is strongly cotorsion-free, in the sense of [14]. By [14, Theorem 4.2], there is a nonzero slender group A such that $\mathrm{Hom}(A,C)=0$. (A torsion-free abelian group A is *slender* if every homomorphism $f\colon \mathbf{Z}^{\aleph_0}\to A$ vanishes on almost all components; see [16, § 94].) Since $\mathrm{Hom}(A,G)$ is trivial, the homomorphism $A\to 0$ is in $\mathcal{S}$. On the other hand, since A is slender, Theorem 94.4 in [16] implies that $\mathrm{Hom}(\mathbf{Z}^\kappa/\mathbf{Z}^{<\kappa}, A)=0$ for all nonmeasurable cardinals κ; therefore, A is in $\mathcal{D}$. This implies that $A=0$, so we have arrived at a contradiction that proves our claim. ∎

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Carles Casacuberta
Departament d'Àlgebra i Geometria
Universitat de Barcelona
Gran Via de les Corts Catalanes, 585
E-08007 Barcelona, Spain
casac@mat.ub.es

Dirk Scevenels
Departement Wiskunde
Katholieke Universiteit Leuven
Celestijnenlaan 200 B
B-3001 Heverlee, Belgium
dirk.scevenels@yucom.be