LOCALIZATION OF q-ABELIAN GROUPS

Carles Casacuberta

Abstract. Let q be a prime number. We prove that the $(1/q)$-localization of a q-abelian group (i.e. a group in which $(xy)^q = x^q y^q$ for all x, y) is again a q-abelian group. This provides an example of a class of groups which need not be nilpotent and behave well under localization.

First we point out some facts on the structure of q-abelian groups and describe a simple procedure to obtain nontrivial examples.

1. Introduction.

A group G in which $(xy)^n = x^n y^n$ holds for all elements x, y and some fixed integer n has been called *n-abelian*.

This concept was first considered in [12] and extensively analyzed in [7], [2], [3], [1]. Most of the standard notation was introduced by R. Baer in [3].

Many other authors have contributed to the description of the structure and properties of n-abelian groups. One of the most complete general studies is [17]. Useful generalizations and applications have been recently described in [10], [5] and [11].

Our main interest in n-abelian groups is the following: if A is an abelian group and q is a prime number, then the $(1/q)$-localization of A is the natural map $A \to A \otimes \mathbb{Z}[1/q], \quad a \mapsto a \otimes 1$. This localization can be obtained, up to isomorphism, by taking the direct limit of the *telescope*:

$$A \xrightarrow{f} A \xrightarrow{f} A \xrightarrow{f} \dots$$

where $f(a) = qa$.

Now, if G is q-abelian, we can also consider:

$$G \xrightarrow{f} G \xrightarrow{f} G \xrightarrow{f} \dots$$

where $f(x) = x^q$. As expected, the direct limit of this system has unique q^{th} roots and the natural map from G to it is universal with respect to homomorphisms from G to *q-abelian* groups with unique q^{th} roots. This idea has been developed in a slightly more general setting by I. Pop in [13]. We have observed that this procedure, in fact, surprisingly gives the $(1/q)$-localization of G in the category of all groups (that is, the map above mentioned turns out to be universal with respect to homomorphisms from G to *arbitrary* groups having unique q^{th} roots). This is proved in section **5**. Hence, q-abelian groups are $(1/q)$-*localizable by telescoping* ([9]).

Many other concepts and constructions can be transferred from abelian groups to n-abelian groups. This general principle is the starting point of [3], where, among other things, the author points out (without proof) that the elements of finite order in an n-abelian group form a subgroup. We supply a proof of this fact in section **3** and use it to obtain a clearer understanding of the effect of localization on a q-abelian group.

We thank Luise-Charlotte Kappe for her valuable comments and for kindly making known to us the extensive literature on the subject, from the first paper of F. W. Levi (1944) until her own very recent work. We are also indebted to Manuel Castellet, Pere Menal and the referee for many helpful suggestions.

2. Notation and remarks.

We denote $[x, y] = xyx^{-1}y^{-1}$.

Let n be a fixed integer. In this paper we always assume $n \geq 2$.

Definition 1 ([3]). A group G is *n-abelian* if $(xy)^n = x^n y^n$ for all elements x, y of G.

That is, G is n-abelian if and only if the power map $x \mapsto x^n$ is an endomorphism of G.

Clearly n-abelian groups form a *variety*, which we denote n-$\mathcal{A}b$. It contains the following subvarieties:

(i) Abelian groups.
(ii) Groups in which $x^n = 1$ for all x.
(iii) Groups in which $x^n = x$ for all x.

In fact, n-$\mathcal{A}b$ is the *smallest* variety containing (i), (ii) and (iii). This was proved by J. L. Alperin in [1].

The verbal subgroup of a given group G with respect to n-$\mathcal{A}b$ is called the *n-commutator* of G, and is usually denoted $[G, G; n]$. This definition contains the following information ([18, I.2]): $[G, G; n]$ is the subgroup of G generated by all the elements of the form $x^n y^n (xy)^{-n}$ (which is a normal subgroup), $G/[G, G; n]$ is n-abelian and the epimorphism $G \twoheadrightarrow G/[G, G; n]$ is universal among all homomorphisms $G \to K$ with K n-abelian.

It is clear that $[G, G; n]$ is always contained in the commutator subgroup $[G, G]$.

The *free n-abelian groups* are the free objects in n-$\mathcal{A}b$.

It follows again from general considerations ([18, I.3]) that the free n-abelian group on a set S may be described as $F_n = F/[F, F; n]$, where F is the free group on S.

Every n-abelian group is a quotient of some free n-abelian group.

Free n-abelian groups were first considered by O. Grün in [7]. They were an essential tool in [1], and have been recently studied in [19] and [10].

The fact of being a group n-abelian forces it to satisfy several restrictive conditions of a purely combinatorial nature. We shall make use of the following ones, which are contained for example in [1, lemma 1]:

Lemma 1. *If G is n-abelian, then $[x^{n-1}, y^n] = 1$ for all x, y in G.*

Lemma 2. *If G is n-abelian and satisfies:*

$$[G,G] \cap \ker(x \mapsto x^n) = 1,$$

then it is also $(n-1)$-abelian.

Lemma 3. *A torsion-free n-abelian group must be abelian.*

3. The torsion subgroup.

Given an arbitrary group G, we call $T(G)$ the *set* of all elements of G of finite order, and $T_n(G) \subseteq T(G)$ the set of those elements of n-torsion (i.e. such that $x^{n^k} = 1$ for some $k \geq 0$).

If G is n-abelian, then $T_n(G)$ is clearly a normal subgroup of G. We may factor:

$$\overline{G} = G/T_n(G)$$

and then $\overline{G}$ has no elements of n-torsion. It follows from lemma 2 that $\overline{G}$ is also $(n-1)$-abelian.

Lemma 4. *$T(\overline{G})$ is a normal subgroup of $\overline{G}$.*

Proof. Given $x, y \in T(\overline{G})$, pick an integer k such that $x^k = y^k = 1$. Since $\overline{G}$ is n-abelian and $(n-1)$-abelian, we have:

$$x^{n-1}y^{n-1} = (yx)^{n-1} = y^{n-1}x^{n-1}$$

That is, $[x^{n-1}, y^{n-1}] = 1$. Then $(xy)^{k(n-1)} = ((xy)^{n-1})^k = (x^{n-1}y^{n-1})^k = x^{k(n-1)}y^{k(n-1)} = 1$. Hence $xy \in T(\overline{G})$. This means that $T(\overline{G})$ is a subgroup of $\overline{G}$. The normality is obvious. □

From this we immediately obtain:

Proposition 1. *If G is n-abelian, then $T(G)$ is a normal subgroup of G.*

Proof. Given $x, y \in T(G)$, consider their classes $\overline{x}, \overline{y} \in T(\overline{G})$. By lemma 4, $\overline{xy} \in T(\overline{G})$, which means $(xy)^k \in T_n(G)$ for some integer k. Hence $xy \in T(G)$. The normality is again obvious. □

It is interesting to know what this torsion subgroup looks like in the case of a *free* n-abelian group. First recall from lemma 3 that for any n-abelian group G we have $[G,G] \subseteq T(G)$.

Proposition 2. *Let F_n be a free n-abelian group. Then:*

$$T(F_n) = [F_n, F_n].$$

Proof. We only need to prove the inclusion $T(F_n) \subseteq [F_n, F_n]$. This can be derived from a remark in [1]: write $F_n = F/[F, F; n]$ (§2), where F is a free group. Then $[F, F; n] \subseteq [F, F]$ implies $F_n/[F_n, F_n] \cong F/[F, F]$, which is torsion-free. Our claim follows. □

We summarize our conclusions in a structure result which will be useful in section **5**:

Theorem 1. *Let G be an n-abelian group. There is a short exact sequence:*

$$1 \to T(G) \to G \to A \to 1$$

where A is abelian. Moreover, if $T_n(G) = 1$, then the map $x \mapsto x^n$ is an automorphism of $T(G)$.

Proof. The first claim follows from proposition 1 and lemma 3.

The argument to prove the second assertion is standard: given $y \in T(G)$, we can find an integer k such that $(k, n) = 1$ and $y^k = 1$. Write $\lambda k + \mu n = 1$ with appropriate integers λ, μ. This gives:

$$y = y^{\lambda k + \mu n} = y^{\mu n} = (y^{\mu})^n$$

and thus the monomorphism $x \mapsto x^n$ is also an epimorphism in $T(G)$. □

4. New examples of n-abelian groups.

We would like to have at hand explicit examples of noncommutative n-abelian groups. Let us call those examples which are direct products of groups of type (i), (ii) and (iii) in §2 *trivial*.

In view of theorem 1, it seems natural to start analyzing the structure of those groups in which $x \mapsto x^n$ is an automorphism, in our attempt to find nontrivial examples.

We have found the following decompositions:

Proposition 3. *Let T be a group in which $x \mapsto x^n$ is an automorphism. There are short exact sequences:*

(a) $1 \to Z(T) \to T \to Q \to 1$

where $Z(T)$ is the center of T and the map $x \mapsto x^n$ is the identity map in Q.

(b) $1 \to N \to T \to A \to 1$

where A is abelian, the map $x \mapsto x^n$ is the identity map in N, and N is maximal with respect to that property.

Proof.

(a) By lemma 1, $x^{n-1} \in Z(T)$ for all $x \in T$. Hence $(\overline{x})^{n-1} = \overline{1}$ for all $\overline{x} \in Q = T/Z(T)$.

(b) Let $N = \{x \in T \mid x^n = x\}$. It is clear that N is a normal subgroup of T. Now, given $x, y \in T$, we have:

$$[x,y]^{n-1} = (xyx^{-1}y^{-1})^{n-1} = x^{n-1}y^{n-1}x^{-(n-1)}y^{-(n-1)} = [x^{n-1}, y^{n-1}] = 1$$

because T is $(n-1)$-abelian by lemma 2 and the $(n-1)^{\text{th}}$ powers lie in the center of T.

It follows that $[x,y]^n = [x,y]$ and hence $[T,T] \subseteq N$. This implies that $A = T/N$ is abelian. $\square$

Example 1. Let $n = 5$ and $T = \mathcal{Q}_8 \times \mathbb{Z}/8$, where $\mathcal{Q}_8$ denotes the quaternion group of order 8. T is a *trivial* example, in our sense, of a group in which $x \mapsto x^n$ is an automorphism.

One obtains:

$$\begin{array}{ll} Z(T) \cong \mathbb{Z}/2 \times \mathbb{Z}/8 & N \cong \mathcal{Q}_8 \times \mathbb{Z}/4 \\ Q \cong \mathbb{Z}/2 \times \mathbb{Z}/2 & A \cong \mathbb{Z}/2. \end{array}$$

Thus, in spite of the symmetry, the extensions (a) and (b) in proposition 3 need not be split.

It is not difficult to show a nontrivial example:

Example 2. Let $G = \langle s, t \mid s^3 = t^2 = (st)^2 \rangle$. This group has order 12 and is indecomposable. One readily checks that the map $x \mapsto x^7$ is a homomorphism (and hence an automorphism), which is not the identity because $t^7 = t^{-1}$.

Following the notation of proposition 3, we find that $Q \cong \Sigma_3$ and $N = \langle s \rangle \cong \mathbb{Z}/6$.

In fact, this example turns out to be a particular case of a more general situation. It gives the idea of a method which produces many nontrivial examples on n-abelian groups, as we next describe.

This method is essentially based on the following obvious remark:

Lemma 5. *If G_1, G_2 are n-abelian groups, Q is an arbitrary group, and $f_1\colon G_1 \to Q$, $f_2\colon G_2 \to Q$ are homomorphisms, then the pull-back $B = \{(x,y) \in G_1 \times G_2 \mid f_1(x) = f_2(y)\}$ is n-abelian.*

Now take an arbitrary group homomorphism $f\colon G \to Q$ and let Q act on a given group N through $\omega\colon Q \to \mathrm{Aut}(N)$. Then G acts on N through ωf.

In this situation we have a well-defined homomorphism: $\varphi\colon N \rtimes G \to N \rtimes Q$, $\varphi(a,x) = (a, f(x))$, rendering the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & N \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\ & & \Vert & & \varphi\big\downarrow & & f\big\downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & N \rtimes Q & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

It is plain that the right square is a pull-back square. The argument is the following:

Write $B = \{(x,(a,y)) \in G \times (N \rtimes Q) \mid f(x) = y\}$. Then φ and the projection $N \rtimes G \twoheadrightarrow G$ induce a homomorphism $\theta\colon N \rtimes G \to B$, namely $\theta(a,x) = (x,\varphi(a,x)) = (x,(a,f(x)))$. Clearly θ is mono and epi.

We immediately obtain:

Theorem 2. *Let $f\colon G \to Q$ be an arbitrary group homomorphism, and let Q act on a given group N.*

(a) If G and $N \rtimes Q$ are n-abelian, then $N \rtimes G$ is also n-abelian.

(b) Suppose that the map $x \mapsto x^n$ is the identity map in $N \rtimes Q$. Then it is an automorphism of $N \rtimes G$ if and only if it is an automorphism of G, and it is the identity map in $N \rtimes G$ if and only if it is the identity map in G.

Proof. Assertion (a) follows from lemma 5.

To prove (b), use the isomorphism θ above described:

$\theta((a,x)^n) = (\theta(a,x))^n = (x,(a,f(x)))^n = (x^n,(a,f(x))^n) = (x^n,(a,f(x))) = \theta(a,x^n)$.

Hence $(a,x)^n = (a,x^n)$ and our claim follows. $\square$

Corollary 1. *Let G be an abelian group acting on a finite group N through $\omega\colon G \to \mathrm{Aut}(N)$. Let r be the order of $N \rtimes (G/\ker\omega)$. Then $N \rtimes G$ is n-abelian at least for $n \equiv 0,1$ mod r.*

Proof. Apply theorem 2(a) to $f\colon G \twoheadrightarrow G/\ker\omega$. $\square$

Example 3. A whole family of examples which arise in this way are the *dicyclic* groups:

$$G_m = \langle s,t \mid s^m = t^2 = (st)^2\rangle$$

with m odd and $n = 2m+1$.

Changing $z = st^2$ one obtains:

$$G_m = \langle z,t \mid z^m = t^4 = 1, \quad tzt^{-1} = z^{-1}\rangle.$$

Hence $G_m \cong \mathbb{Z}/m \rtimes \mathbb{Z}/4$, where the action $\omega\colon \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/m)$ is given by the relation $tzt^{-1} = z^{-1}$. Then $\ker\omega = \langle t^2\rangle$ and $\mathbb{Z}/m \rtimes ((\mathbb{Z}/4)/\ker\omega) \cong \mathbb{Z}/m \rtimes \mathbb{Z}/2$ is the dihedral group of order $2m$.

By theorem 2(b) the map $x \mapsto x^{2m+1}$ is an automorphism of G_m which is not the identity, because it is a nontrivial automorphism of $\mathbb{Z}/4$ (being m odd).

Observe that example 2 is just the case $m = 3$.

Note. Given an arbitrary group G, plainly the set of those integers n such that G is n-abelian is multiplicatively closed. It is called the *exponent semigroup* of G. The study of this set was the motivation for [12] and the subject of several recent papers (see [19] for some references, and also [10], [5]).

Finally, we would like to know when the examples produced by theorem 2 are nontrivial in our sense. The next proposition guarantees it under some rather general assumptions.

If a group G acts on a group N, let us denote N^G the subgroup of invariant elements under the action.

Proposition 4. *Let G be an indecomposable group acting on an abelian group $A \neq 0$. Suppose that $A^G = 0$. Then $A \rtimes G$ cannot be properly decomposed as a direct product $B \times K$ with B abelian.*

The proof is an easy consequence of the following fact: if a group G acts on an abelian group A through $\omega\colon G \to \mathrm{Aut}(A)$, then:

$$(a,x) \in Z(A \rtimes G) \Longleftrightarrow \begin{cases} x \in Z(G) \cap \ker \omega \\ a \in A^G. \end{cases}$$

It follows from proposition 4 that the groups G_m in example 3 are nontrivial in our sense.

5. Localization of q-abelian groups.

5.1. Definitions and remarks.

Let P be a fixed set of primes.

We recall the following concept from [8]: a group K is P-*local* if the map $x \mapsto x^p$ is bijective in K for each prime p *not* in P.

Given a group G, a homomorphism $l\colon G \to G_P$ is a P-*localization* if G_P is P-local and l is universal (initial) among all homomorphisms $f\colon G \to K$ in which K is P-local.

It is well-known that each group admits a P-localization (see for example [15]). Then it is plainly unique up to isomorphism and functorial.

However, it is not easy to handle the P-localization of an arbitrary group. For example, as far as we know, there is no good description of $\ker l$ in general.

In this section we fix a prime q and take P to be the set of all primes $p \neq q$. Hence a P-*local group* (or $(1/q)$-*local group*) will be a group in which each element has a unique q^{th} root.

We are going to show that q-abelian groups can be P-*localized by telescoping* in the sense of [9]. This allows us to describe several good properties of their P-localization, completely analogous to those of the abelian case.

Our construction is based on the following argument (which is valid for an arbitrary set P):

Recall that if K is a P-local group and H is a subgroup of K, then the *isolator* of H in K, denoted $I(K,H)$, is the smallest P-local subgroup of K containing H. If H is already P-local, then it is said to be *isolated* in K.

It often occurs that we have a P-localization functor in some variety $\mathcal{V}$ of groups (that is, given G in $\mathcal{V}$, we have a homomorphism $\lambda\colon G \to G_*$ in $\mathcal{V}$ which is universal among all $f\colon G \to K$ in $\mathcal{V}$ with K P-local). This is the case, for example, with $\mathcal{V} = \mathcal{N}il_c$ = nilpotent groups of class c or less ([8]).

A necessary and sufficient condition in order to know that this functor is the restriction of the P-localization in the whole category of all groups can be given as follows:

Proposition 5. *Let $\mathcal{V}$ be a variety of groups. Suppose that each group G in $\mathcal{V}$ has a P-localization $\lambda\colon G \to G_*$ in $\mathcal{V}$.*

The following statements are equivalent:

(a) G_P lies in $\mathcal{V}$ for each group G in $\mathcal{V}$.

(b) If K is a P-local group and a given subgroup $H \subseteq K$ is in $\mathcal{V}$, then $I(K,H)$ is also in $\mathcal{V}$.

(c) For each G in $\mathcal{V}$ there is a unique isomorphism rendering the following diagram commutative:

$$\begin{array}{ccc} G & \xrightarrow{l} & G_P \\ \downarrow & & \\ G_* & & \end{array}$$

This criterion has been more or less explicitly used in [6] and [16], where the case $\mathcal{V} = \mathcal{N}il_c$ was considered and affirmatively answered. The case $c = 1$ ($\mathcal{V} = \mathcal{A}b$) is particularly simple ([6, theorem 1.1.14]). Thus for each abelian group A we may write $A_P = A \otimes \mathbb{Z}_P$ without ambiguity. In our case $P = \{p \neq q\}$, $\quad \mathbb{Z}_P = \mathbb{Z}[1/q]$ is the smallest subring of $\mathbb{Q}$ containing $1/q$.

In this section we prove:

(a) There is a P-localization functor (with "good" properties) in the variety q-$\mathcal{A}b$ of all q-abelian groups.

(b) This functor agrees with the P-localization in the category of all groups.

5.2. Main result.

Let G be a given q-abelian group. For each $i = 1, 2, 3, \ldots$, let G_i be a copy of G, and denote x_i the element $x \in G$ viewed in G_i.

Define $f_i\colon G_i \to G_{i+1}$ by $f_i(x_i) = (x_{i+1})^q$.

Consider:

$$G_* = \varinjlim (G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \ldots)$$

and the homomorphism $\lambda\colon G \to G_*$ given by $\lambda(x) = x_1$.

Each element of G_* corresponds to some $x_i \in G_i$, which is identified with $x_{i+k}^{q^k} \in G_{i+k}$ for all $k \geq 0$. Given two elements of G_*, we may assume, without restriction, that they lie in some common G_i.

Recall that a group homomorphism $f\colon K \to L$ is called *P-injective* if $\ker f$ is a q-torsion subgroup, *P-surjective* if for any $y \in L$ there exists some integer r such that $y^{q^r} \in \operatorname{im} f$, and *$P$-bijective* if it is both P-injective and P-surjective.

Plainly each f_i in our construction is P-bijective. In this situation, the directed system is called a *telescope*.

Theorem 3. *$\lambda\colon G \to G_*$ is a P-localization in the variety q-$\mathcal{A}b$ of q-abelian groups.*

Proof. This is a particular case of [13]. The argument is essentially a reproduction of the procedure in the abelian case. We give it to make it clear why we must stay within q-$\mathcal{A}b$:

(a) *G_* is P-local.*
Given $x_i \in G_i$, then $x_{i+1} \in G_{i+1}$ is clearly a q^{th} root for it. Suppose now that x_i^q and y_i^q are identified in G_*. Then $x_{i+k}^{q^{k+1}} = y_{i+k}^{q^{k+1}}$ for some k. That is, $x^{q^{k+1}} = y^{q^{k+1}}$ as elements of G. But then $x_{i+k+1}^{q^{k+1}} = y_{i+k+1}^{q^{k+1}}$, which tells us that x_i and y_i are identified in G_*.

(b) *G_* is q-abelian.*
$(x_iy_i)^q = x_i^q y_i^q$ because G_i is q-abelian.

(c) *λ is universal.*
Let $\varphi\colon G \to K$ be a homomorphism where K is q-abelian and P-local. Define $\varphi_1\colon G_1 \to K$ by $\varphi_1(x_1) = \varphi(x)$ and inductively $\varphi_i\colon G_i \to K$ by $(\varphi_i(x_i))^q = \varphi_{i-1}(x_{i-1})$. This has sense because K is P-local. Notice that each φ_i is a group homomorphism because K is q-abelian: $(\varphi_i(x_i)\varphi_i(y_i))^q = (\varphi_i(x_i))^q(\varphi_i(y_i))^q = \varphi_{i-1}(x_{i-1})\varphi_{i-1}(y_{i-1}) = \varphi_{i-1}(x_{i-1}y_{i-1}) = (\varphi_i(x_iy_i))^q$.

We have $\varphi_i f_{i-1}(x_{i-1}) = \varphi_i(x_i^q) = \varphi_{i-1}(x_{i-1})$ for each i and hence we obtain a homomorphism $\varphi_*\colon G_* \to K$ such that $\varphi_*(\lambda(x)) = \varphi_1(x_1) = \varphi(x)$. Finally, if $\psi\colon G_* \to K$ also satisfies $\psi\lambda = \varphi$, then $\psi = \varphi_*$ because we lifted φ in the only possible way. □

Now our aim is to prove that, in fact, $\lambda\colon G \to G_*$ coincides with $l\colon G \to G_P$. In view of proposition 5, we only need to prove the following:

Theorem 4. *If K is any P-local group and a subgroup $H \subseteq K$ is q-abelian, then the isolator $I(K,H)$ of H in K is also q-abelian.*

The proof essentially uses only lemma 1 and the following observation:

Lemma 6. *If K is a P-local group and x, y are elements of K, then $[x, y^q] = 1$ implies $[x,y] = 1$.*

Proof. The expression $(xyx^{-1})^q = xy^qx^{-1} = y^q$ gives $xyx^{-1} = y$ because K has unique q^{th} roots. □

Proof of theorem 4. Let $R(H) = \{x \in K \mid x^{q^r} \in H \quad \text{for some} \quad r \geq 0\}$.

Write $R(H) = \bigcup_{r=0}^{\infty} L_r$, where $L_r = \{x \in K \mid x^{q^r} \in H\}$.

We know that $L_0 = H$ is q-abelian. We are going to prove inductively that each L_r is a q-abelian subgroup of K.

Assume that L_{r-1} is a q-abelian subgroup. Let x, y be arbitrary elements of L_r. Then x^q and y^q lie in L_{r-1}.

Observe that $[x^{q(q-1)}, y^{q^2}] = 1$ by lemma 1. Then lemma 6 gives $[x^{q-1}, y] = 1$. Obviously $[x, y^{q-1}] = 1$ by symmetry. Since L_{r-1} is q-abelian, we have:

$$x^{q^2} y^{q^2} = (x^q y^q)^q = (x x^{q-1} y y^{q-1})^q = x^{q(q-1)} (xy)^q y^{q(q-1)}.$$

Thus $x^q y^q = (xy)^q$. This shows that xy also lies in L_r, and therefore L_r is a subgroup. Moreover, it is a q-abelian subgroup.

Now $R(H)$ is an isolated q-abelian subgroup of K containing H. Then $I(K, H) \subseteq R(H)$ and our claim follows. (In fact, $I(K, H) = R(H)$). □

Corollary 2. *We have an isomorphism:*

$$\begin{array}{ccc} G & \xrightarrow{l} & G_P \\ \downarrow & & \\ G_* & & \end{array}$$

Corollary 3. *If G is q-abelian, then its P-localization $l\colon G \to G_P$ is P-bijective.*

Proof. Plainly λ is P-bijective. □

5.3. Consequences.

Many of the pleasant properties of the classical localization of nilpotent groups only depend on the fact of l being P-bijective (see [8]). These features therefore hold for q-abelian groups as well. We list some of them:

Corollary 4. $\ker l = T_q(G)$.

Corollary 5. *Let G be q-abelian and $f\colon G \to K$ be a given homomorphism. Then f is a P-localization if and only if K is P-local and f is P-bijective.*

Corollary 6. *P-localization is an exact functor in q-$\mathcal{A}b$, and it preserves central extensions.*

Now we can use our structure result (theorem 1) together with corollary 6 to describe, with more clarity, the effect of P-localization on a q-abelian group.

Given G q-abelian, consider its q-torsion subgroup $T_q(G)$. We have a group extension:

(1) $$1 \to T_q(G) \to G \to Q \to 1$$

where Q has no q-torsion. The projection $G \twoheadrightarrow Q$ induces then an isomorphism $G_P \cong Q_P$.

Using theorem 1, write:

(2) $$1 \to T(Q) \to Q \to A \to 1$$

where A is abelian and $x \mapsto x^q$ is an automorphism of $T(Q)$. That is, $T(Q)$ is just P-local. Hence we obtain an extension:

(3) $$1 \to T(Q) \to Q_P \to A_P \to 1$$

where $A_P = A \otimes \mathbb{Z}[1/q]$. Moreover, whenever (2) splits, then (3) also splits.

We illustrate this with an example:

Example 4. Let $\mathbb{Z}$ act nontrivially on $\mathbb{Z}/3$. Take $q = 7$ and $P = \{p \neq 7\}$.

Since $\mathrm{Aut}(\mathbb{Z}/3) \cong \mathbb{Z}/2$ is P-local, the homomorphism $\omega\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3)$ induces a homomorphism $\omega' : \mathbb{Z}[1/7] \to \mathrm{Aut}(\mathbb{Z}/3)$ such that $\omega' l = \omega$, where $l\colon \mathbb{Z} \hookrightarrow \mathbb{Z}[1/7]$ is the P-localization.

The group $\mathbb{Z}/3 \rtimes \mathbb{Z}$ is 7-abelian by corollary 1, and nonnilpotent. The sequence:

$$1 \to \mathbb{Z}/3 \to \mathbb{Z}/3 \rtimes \mathbb{Z} \to \mathbb{Z} \to 1$$

is just (2). We immediately infer that the embedding $\mathbb{Z}/3 \rtimes \mathbb{Z} \hookrightarrow \mathbb{Z}/3 \rtimes \mathbb{Z}[1/7]$ P-localizes.

Observe that $(a, b)^7 = (a, 7b)$ in $\mathbb{Z}/3 \rtimes \mathbb{Z}[1/7]$ (as in the proof of theorem 2). Thus we might also have used corollary 5.

5.4. A comment on the behaviour of the homology.

If G is a nilpotent group and P is an arbitrary set of primes, then it is well-known ([8]) that the P-localization $l\colon G \to G_P$ induces a P-localization $l_*\colon H_k(G) \to H_k(G_P)$ for each $k \geq 1$. H_k denotes homology with integral coefficients.

For arbitrary groups this is far from being true.

When trying to check it for q-abelian groups, a serious difficulty arises: there exist examples of finitely generated infinite groups in which $x^q = 1$ for each element x. Such groups are obviously q-abelian and their P-localization is trivial. However, there is no reason, a priori, to suspect that their integral homology groups are q-torsion groups (although we are not able to show any explicit counterexample).

We shall avoid this difficulty by only restricting our attention to q-abelian groups for which the torsion subgroup is finite, as in example 4.

Then $H_k(T_q(G); \mathbb{Z}_P) = 0$ for $k \geq 1$ and hence the Lyndon-Hochschild-Serre spectral sequence associated with the extension (1) gives an isomorphism:

$$H_k(G; \mathbb{Z}_P) \cong H_k(Q; \mathbb{Z}_P)$$

for all $k \geq 0$, induced by the projection $G \twoheadrightarrow Q$. Since this projection also induces an isomorphism $G_P \cong Q_P$, we only need to check that $H_k(Q; \mathbb{Z}_P)$ is isomorphic to $H_k(Q_P)$ for each $k \geq 1$.

Now consider the spectral sequences associated with (2) and (3). We have:

$$E^2_{r,s} = H_r(A; H_s(T(Q); \mathbb{Z}_P))$$
$$\tilde{E}^2_{r,s} = H_r(A_P; H_s(T(Q); \mathbb{Z}_P))$$

and a morphism $l^*_{*,*}\colon \{E^*_{*,*}\} \to \{\tilde{E}^*_{*,*}\}$ induced by the localization map.

We claim that $l^2_{r,s}$ is an isomorphism for all $r, s \geq 0$. This is obvious for $s = 0$. When $s \geq 1$, observe that $H_s(T(Q); \mathbb{Z}_P)$ is a finite P-local group. Then our claim is deduced from the following result, which is contained in [14]:

Lemma 7. *Let A be an abelian group, and suppose that A_P acts on a finite P-local abelian group C. Then $l\colon A \to A_P$ induces an isomorphism $H_k(A; C) \cong H_k(A_P; C)$ for each $k \geq 0$.*

It follows that $l^{\infty}_{r,s}$ is also an isomorphism for all $r, s \geq 0$, and hence l induces:

$$H_k(Q;\mathbb{Z}_P) \cong H_k(Q_P;\mathbb{Z}_P)$$

for all $k \geq 0$.

Finally observe that both A_P and $T(Q)$ have P-local homology groups. Thus Q_P has also P-local homology groups, and $H_k(Q_P;\mathbb{Z}_P)$ is naturally isomorphic to $H_k(Q_P)$ for each $k \geq 1$.

Summarizing, we have obtained:

Theorem 5. *Let G be a q-abelian group such that $T(G)$ is finite. Then $l\colon G \to G_P$ induces a P-localization of the integral homology groups.*

References

[[1]] [redacted] L. Alperin, *A classification of n-abelian groups*, Canad. J. Math. **21** (1969), 1238–1244.

[[2]] [redacted] Baer, *Endlichkeitskriterien für Kommutatorgruppen*, Math. Ann. **124** (1952), 161–177.

[[3]] [redacted] Baer, *Factorization of n-soluble and n-nilpotent groups*, Proc. Amer. Math. Soc. **4** (1953), 15–26.

[[4]] [redacted] Baumslag, *Some aspects of groups with unique roots*, Acta Math. **104** (1960), 217–303.

[[5]] [redacted] Brandl and L. C. Kappe, *On n-Bell groups*, (to appear in Comm. Algebra).

[[6]] [redacted] García Rodicio, *Métodos homológicos en grupos P-locales*, Universidad de Santiago de Compostela, 1986, Alxebra **43**.

[[7]] [redacted] Grün, *Über eine charakteristische Untergruppe*, Beiträge zur Gruppentheorie IV, Math. Nachr. **3** (1949/50), 77–94.

[[8]] [redacted] Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland Math. Studies **15**, Amsterdam, 1975.

[[9]] [redacted] Kahn and H. Scheerer, *Localisation des groupes et des espaces nilpotents par des télescopes*, C. R. Acad. Sci. Paris **281** (1975), 419–422.

[[10]] [redacted]C. Kappe, *On n-Levi groups*, Arch. Math. **47** (1986), 198–210.

[[11]] [redacted]C. Kappe and R. F. Morse, *Groups with 3-abelian normal closures*, (to appear in Arch. Math.).

[[12]] [redacted]W. Levi, *Notes on group theory I and VII*, J. Indian Math. Soc. **8** (1944), 1–7, and **9** (1945), 37–42.

[[13]] [redacted]op, *A generalization of simple spaces*, Rev. Roumaine Math. Pures Appl., tome **XXV**, n. 6 (1980), 925–932.

[[14]] [redacted]Reynol, *P-localization of some classes of groups*, preprint, IME Universidade de São Paulo, 1987.

[[15]] [redacted]Ribenboim, *Torsion et localisation de groupes arbitraires*, Lecture Notes in Math. **740** (1979), 444–456.

[[16]] [redacted]Ribenboim, *Equations in groups, with special emphasis on localization and torsion, I and II*, preprint, 1986.

[[17]] [redacted]Ronse and R. Teitler, *Groupes dans lesquels l'élévation à une puissance entière est un endomorphisme*, Acad. Roy. Belg. Bull. Cl. Sci. (5), tome **LXII** (1976), 539–564.

[[18]] [redacted]Stammbach, *Homology in Group Theory*, Lecture Notes in Math. **359** (1973).

[[19]] [redacted]Tamura, *Free E-m groups and free E-m semigroups*, Proc. Amer. Math. Soc. **84**, n. 3 (1982), 318–324.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain