A homotopy idempotent construction by means of simplicial groups

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Abstract

We obtain a simplicial group model for localization of (not necessarily nilpotent) spaces at sets of primes by applying a suitable functor dimensionwise, as in earlier work of Quillen and Bousfield–Kan. For a set of primes P and any group G, let $G \to L_P G$ be a universal homomorphism from G into a group which is uniquely divisible by primes not in P, and denote also by L_P the prolongation of this functor to simplicial groups. We prove that, if X is any connected simplicial set and J is any free simplicial group which is a model for the loop space ΩX, then the classifying space $\overline{W} L_P J$ is homotopy equivalent to the localization of X at P. Thus, there is a map $X \to \overline{W} L_P J$ which is universal among maps from X into spaces Y for which the semidirect products $\pi_k(Y) \rtimes \pi_1(Y)$ are uniquely divisible by primes not in P. This approach also yields a neat construction of fibrewise localization.

0 Introduction

Certain constructions in homotopy theory are especially suited to the use of simplicial groups as models. One of such constructions is Bousfield–Kan completion. As shown in [3, IV.4], if R is a subring

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of the rationals or $R = \mathbf{Z}/p$, then the R-completion $R_\infty X$ of any reduced simplicial set X is weakly equivalent to the classifying space $\overline{W}(GX)\widehat{\ }_R$, where GX is Kan's loop group [17] of X and $(GX)\widehat{\ }_R$ is the dimensionwise R-lower central series completion of GX, which is defined in [3, IV.2]. This construction was also studied by Wilkerson in [29] for simply connected CW-complexes of finite type, and by Quillen in [26] using profinite completion.

In this article we use a similar pattern to obtain a new model for localization of spaces at primes, following a suggestion of Dror Farjoun. Let us first recall the relevant definitions. A localization of a space X (a CW-complex or a simplicial set) at a set of primes P is a map $X \to X_P$ which is universal in the homotopy category among maps from X into P-local spaces. A space is P-local if each of its connected components is P-local, and a connected space Y is P-local if the self map $\rho_n\colon \Omega Y \to \Omega Y$ induced by a degree n map of S^1 is a weak homotopy equivalence for all positive integers n with no prime factors in P. If the space Y is simply connected, then this condition means precisely that the homotopy groups $\pi_k(Y)$ are uniquely P'-divisible for $k \geq 2$, where P' denotes the complement of P in the set of all primes. In fact, for X simply connected, one has $\pi_k(X_P) \cong \pi_k(X) \otimes \mathbf{Z}_P$ for all k, where $\mathbf{Z}_P$ denotes the integers localized at P. The properties of X_P are well understood when X is simply connected or nilpotent (see [3], [15], [29]), so we do not intend to obtain any new information in this case. For nonnilpotent spaces, however, the homotopy type of X_P can be very different from that of X, much in the same way as with homological localizations. A survey of results in this direction is offered in [4].

Our construction of X_P in the present article goes as follows. Given a connected CW-complex X, we may assume that it has a single 0-cell (since collapsing a maximal tree does not change the homotopy type of X). Pick a free simplicial group J which is a model for the loop space ΩX, with one nondegenerate generator in dimension n for each $(n+1)$-cell of X, as described in [18] or in [29]. If a reduced simplicial set X is given instead of a CW-complex, then let J be Kan's loop group GX, as in [17]. Now, if J_n denotes the group of n-simplices of J, let $J_n \to L_P J_n$ be its localization at P, i.e. a universal homomorphism from J_n into a uniquely P'-divisible group [27]. We prove that, if we denote by $L_P J$ the simplicial group obtained by applying L_P dimensionwise to J, then the classifying space $\overline{W} L_P J$ is a P-localization of X.

Our argument relies on properties of uniquely P'-divisible groups, together with a spectral sequence described by Quillen in [26]. This spectral sequence is useful in our setting thanks to the fact that free groups behave reasonably well under P-localization. Namely, if F is a free group, then the localization homomorphism $F \to L_P F$ induces isomorphisms $H_*(F; A) \cong H_*(L_P F; A)$ for a wide class of coefficient modules, as shown in [6] and recalled in Section 1 below. This is one basic reason why simplicial groups happen to be very suitable as models for P-localization of spaces, like in the case of profinite completion or Bousfield–Kan completion. However, it is an open problem to decide whether ordinary homological localizations [2] fit into the same pattern or not. What is clear is that localizations with respect to generalized homology theories cannot be modeled in principle using dimensionwise constructions on simplicial groups, since such localizations do not preserve connectivity levels in general [23].

Indeed, as an application of our technique, we find that if the homotopy fibre of a map $X \to Y$ is n-connected for some $n \geq 1$, then the homotopy fibre of the induced map $X_P \to Y_P$ is n-connected too. In practice, this allows one to determine homotopy groups of P-localizations of spaces in a certain range of dimensions, as in [3, IV.5.1] for completions or in [11] for integral homology localizations.

Another useful application of our results is a neat, explicit construction of fibrewise P-localization using simplicial groups. This construction applies to all fibre sequences of connected spaces and does not require any extra assumptions on the fibre, contrary to former approaches such as in [19]. We thank the referee for suggesting that we add this application to the article.

Our results are stated for simplicial sets with only one vertex, as in [3, IV.4] or in many articles on similar topics. This is good enough to yield models for localizations of connected simplicial sets or CW-complexes, but it is dissatisfying in some aspects, e.g. the need to collapse a maximal tree makes our construction functorial only up to homotopy. To remedy this, simplicial groupoids should be used instead of simplicial groups; see [12] or [14] for background about simplicial groupoids. Such a generalization is not carried out in this article, since the necessary localization techniques in the category of groupoids are not yet available.

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1 Uniquely divisible groups

Let P be any set of primes, possibly empty, which will remain fixed throughout. We adopt the usual convention of denoting by P' the complement of P in the set of all primes, and saying that n is a P'-number if n is a positive integer with no prime factors in P. A group G is called P-local or uniquely P'-divisible if the map $x \mapsto x^n$ is bijective in G for every P'-number n. For every group G there is a natural homomorphism, called P-localization,

$$l_G\colon G \to L_P G,$$

which is initial among homomorphisms from G to uniquely P'-divisible groups; see [6] or [27]. (We use the expression $L_P G$ instead of G_P to avoid notational difficulties later in the article; however, we keep denoting by $\mathbf{Z}_P$ the P-localization of the additive group of integers, i.e. the group of rational numbers whose denominator is a P'-number.)

This defines an idempotent functor L_P on groups, in the following sense. A functor L in a category $\mathcal{C}$ is called idempotent if it is equipped with a natural transformation $l\colon \mathrm{Id} \to L$ such that, for every object X, the two morphisms l_{LX} and Ll_X from LX to LLX coincide and are isomorphisms. If L is an idempotent functor, then the objects X such that $l_X\colon X \to LX$ is an isomorphism are called L-local, and the maps $f\colon X \to Y$ such that $Lf\colon LX \to LY$ is an isomorphism are called L-equivalences. Thus, a group homomorphism

$f: G \to K$ will be called a P-equivalence if $L_P f: L_P G \to L_P K$ is an isomorphism.

If F is free, then $l_F: F \to L_P F$ is a monomorphism. The group $L_P F$ was described by Baumslag in [1] using a different terminology. He proved that $L_P F$ is isomorphic to the union of a (possibly transfinite) ascending chain of groups F_i, where $F_0 = F$ and, for each ordinal i, the group F_{i+1} is an amalgamated sum $\mathbf{Z}_P *_U F_i$ for a certain subgroup $\mathbf{Z} \subseteq U \subseteq \mathbf{Z}_P$ and a certain embedding $U \hookrightarrow F_i$. This description shows that F_P has homological dimension 2, and it also shows that the map $l_F: F \to L_P F$ induces homology isomorphisms

$$H_k(F; \mathbf{Z}_P) \cong H_k(L_P F; \mathbf{Z}_P)$$

for all k. In fact, as shown in Corollary 7.3 and Theorem 8.7 of [6], if F is free then the P-localization homomorphism induces isomorphisms

$$H_k(F; A) \cong H_k(L_P F; A) \qquad \text{and} \qquad H^k(L_P F; A) \cong H^k(F; A)$$

for all modules A over $L_P F$ which are P-local in the following sense. An abelian group A with an action of a group G is a P-local module over G if the structure map $\mathbf{Z}G \to \mathrm{End}(A)$ sends the elements of the form $1 + x + x^2 + \cdots + x^{n-1}$ to automorphisms, where $x \in G$ and n is any P'-number. This is equivalent to imposing that the semidirect product $A \rtimes G$ be uniquely P'-divisible; see [6]. If the action of G is trivial, then a P-local module over G is just a $\mathbf{Z}_P$-module.

We shall need a few remarks about uniquely divisible groups, which we collect in this section for convenience. Some of these remarks only rely on the fact that P-localization is an idempotent functor, so we state them in their full generality. The next result follows from Proposition 1.2.1 in [13].

Lemma 1.1 *If L is any idempotent functor in a category $\mathcal{C}$, then the full subcategory of L-local objects is closed under limits.* □

Thus, let $\mathcal{I}$ be any small category and F any diagram indexed by $\mathcal{I}$ in the category of groups. If $F(i)$ is uniquely P'-divisible for every $i \in \mathcal{I}$, where P is any set of primes, then the (inverse) limit of F is also uniquely P'-divisible. Therefore, the class of uniquely

P'-divisible groups is closed under pull-backs, and the kernel of every homomorphism between uniquely P'-divisible groups is uniquely P'-divisible.

Every idempotent functor L in a category $\mathcal{C}$, if regarded as a functor from $\mathcal{C}$ to the full subcategory $\mathcal{D}$ of L-local objects, is left adjoint to the inclusion functor of $\mathcal{D}$ into $\mathcal{C}$. Therefore, L preserves colimits; see [20, V.5]. In particular, L preserves coequalizers, so we have

$$L(\mathrm{coeq}(f, g)) \cong L(\mathrm{coeq}(Lf, Lg))$$

for any two parallel arrows f, g in $\mathcal{C}$, if coequalizers exist in $\mathcal{C}$. This fact will be crucial in Theorem 4.1, so we state it more precisely but omit the standard proof.

Lemma 1.2 *Let L be any idempotent functor in any category. Let $\alpha: B \to C$ be a coequalizer of two morphisms $f, g: A \to B$, and let $\beta: LB \to D$ be a coequalizer of Lf and Lg. Then $LC \cong LD$. Moreover, the unique morphisms $\delta: C \to D$ and $\gamma: D \to LC$ such that $\delta \circ \alpha = \beta \circ l_B$ and $\gamma \circ \beta = L\alpha$ are L-equivalences.* □

A coequalizer $\alpha: B \to C$ of two morphisms $f, g: A \to B$ in an arbitrary category will be called a simplicial coequalizer if there is a morphism $s: B \to A$ such that $f \circ s = g \circ s = \mathrm{id}_B$ (hence, f and g are necessarily epimorphisms). In the category of groups, this implies that C is isomorphic to the quotient of B by the subgroup generated by the elements of the form $f(a)g(a)^{-1}$ with $a \in A$. (The condition that α is a simplicial coequalizer guarantees that this is a normal subgroup, since we may write $bf(a)g(a)^{-1}b^{-1} = f(s(b)as(b)^{-1})\, g(s(b)as(b)^{-1})^{-1}$, for all $a \in A$ and $b \in B$.)

Theorem 1.3 *Let $\alpha: B \to C$ be a simplicial coequalizer of two group homomorphisms $f, g: A \to B$ where A and B are uniquely P'-divisible. Then C is uniquely P'-divisible.*

Proof. The group C is P'-divisible because it is an epimorphic image of a P'-divisible group. Suppose that $\alpha(x)^n = \alpha(y)^n$ with

$x, y \in B$ and n a P'-number. Then $y^{-n}x^n$ is in the kernel of α, so we can write

$$y^{-n}x^n = f(a_1)g(a_1)^{-1} \cdots f(a_k)g(a_k)^{-1} \tag{1.1}$$

for a finite set of elements $\{a_1, \dots, a_k\}$ in A. Now pick $c_0 = s(y^n)$ and $c_i = s(g(a_i))$ for $i = 1, \dots, k$, where s is a common splitting for f and g. Then $f(c_0) = g(c_0) = y^n$ and $f(c_i) = g(a_i) = g(c_i)$ for $i = 1, \dots, k$. Hence, we can rewrite (1.1) as

$$x^n = f(c_0)f(a_1)f(c_1)^{-1} \cdots f(a_k)f(c_k)^{-1} = f(c_0 a_1 c_1^{-1} \cdots a_k c_k^{-1}).$$

Since A is P'-divisible, there is an element $d \in A$ such that

$$d^n = c_0 a_1 c_1^{-1} \cdots a_k c_k^{-1},$$

and the equation $x^n = f(d)^n$ in B tells us that $x = f(d)$. On the other hand, our choices have been made so as to guarantee that $g(d)^n = y^n$, from which we infer that $g(d) = y$. Therefore, $xy^{-1} = f(d)g(d)^{-1}$, and this implies that $\alpha(x) = \alpha(y)$, as desired. □

Corollary 1.4 *Let G be a simplicial group. If G_0 and G_1 are uniquely P'-divisible, then so is $\pi_0(G)$.*

Proof. By definition, the projection $G_0 \to \pi_0(G)$ is a simplicial coequalizer of the faces $d_0, d_1 \colon G_1 \to G_0$, where the common splitting is the degeneracy s_0. □

In fact, we have the following:

Proposition 1.5 *If G is a simplicial group in which G_n is uniquely P'-divisible for every $n \geq 0$, then the homotopy groups $\pi_n(G)$ are P'-divisible for $n \geq 0$.*

Proof. Let NG be the Moore chain complex associated with G; see [7, § 3] or [21, § 17]. Thus, $(NG)_0 = G_0$ and, for $k \geq 1$, $(NG)_k$ is the intersection of the kernels of the faces $d_i \colon G_k \to G_{k-1}$ for $0 \leq i \leq k-1$. The differential $\partial_k \colon (NG)_k \to (NG)_{k-1}$ is the restriction of d_k.

Since $(NG)_k$ is a pull-back of a diagram where all groups are uniquely P'-divisible by assumption, $(NG)_k$ itself is uniquely P'-divisible, for $k \geq 0$, by Lemma 1.1. For the same reason, $\ker \partial_k$ is uniquely P'-divisible for $k \geq 1$. So is also $\operatorname{im} \partial_{k+1}$, since P'-roots exist in $\operatorname{im} \partial_{k+1}$ as it is an epimorphic image of $(NG)_{k+1}$, and they are unique because $\operatorname{im} \partial_{k+1}$ is contained in $(NG)_k$. Now Lemma 1.6 below implies that, for $n \geq 1$, the abelian group

$$\pi_n(G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$$

is uniquely P'-divisible. The case $n = 0$ has been stated in Corollary 1.4. □

In general, a group which is P'-divisible and P'-torsion-free need not be uniquely P'-divisible; a counterexample is given after Theorem 39.6 in [1]. However, if a group A is nilpotent and P'-divisible, then A is uniquely P'-divisible if and only if A is P'-torsion-free; see Theorem 13.6 in [1] or Corollary I.2.3 in [15]. This fact yields the following result, as in Corollary 13.7 in [1].

Lemma 1.6 *Let $A = G/N$, where G and N are uniquely P'-divisible groups and A is nilpotent. Then A is uniquely P'-divisible.* □

2 Prolongation of functors

Every functor L from a category $\mathcal{C}$ to a category $\mathcal{D}$ can be extended to a functor from the category of simplicial objects over $\mathcal{C}$ to the category of simplicial objects over $\mathcal{D}$, by applying L at each dimension, and to the face and degeneracy maps. Such an extension is called a prolongation of the functor L. If we view a simplicial object over $\mathcal{C}$ as a functor $X\colon \Delta^{\mathrm{op}} \to \mathcal{C}$, where Δ^{op} is the opposite of the category whose objects are the ordered sets $[n] = \{0, 1, \ldots, n\}$ and whose morphisms $[n] \to [m]$ are order-preserving maps, then the prolongation of L assigns to every X the composite functor

$$\Delta^{\mathrm{op}} \xrightarrow{X} \mathcal{C} \xrightarrow{L} \mathcal{D}.$$

We shall be especially interested in the prolongation of endofunctors from the category of groups to the category of simplicial groups. Recall from [25, II.3.7] that the category of simplicial groups admits a simplicial model category structure where, for a simplicial group G and a simplicial set X, one defines $G \otimes X$ as the simplicial group where $(G \otimes X)_n$ is a free product of copies of G_n indexed by X_n. In the terminology of [17, § 4], a loop homotopy between two homomorphisms $f, g\colon G \to K$ of simplicial groups is a homomorphism $H\colon G \otimes \Delta[1] \to K$ such that $H_0 = f$ and $H_1 = g$. (Every loop homotopy is also a homotopy between the underlying maps of simplicial sets, but not conversely.) The following result is commonplace.

Lemma 2.1 *Let L be any functor in the category of simplicial groups which is a prolongation of a functor from groups. If two homomorphisms f and g are loop homotopic, then so are Lf and Lg.*

Proof. We show, more generally, that for every simplicial set X and every simplicial group G, there is a homomorphism

$$LG \otimes X \to L(G \otimes X)$$

which is natural in G and X and is the identity when X is a point. This implies our claim, by choosing $X = \Delta[1]$ and considering the composite

$$LG \otimes \Delta[1] \longrightarrow L(G \otimes \Delta[1]) \xrightarrow{LH} LK,$$

where H is a loop homotopy between f and g, as in [8].

Thus, for every $n \geq 0$, we apply L to the natural inclusions of G_n into $G \otimes X$, yielding homomorphisms $LG_n \to L(G \otimes X)$, which add up together into a homomorphism $LG \otimes X \to L(G \otimes X)$ with the desired properties. □

A homotopy functor is a functor which carries weak equivalences into weak equivalences. Functors defined by prolongation in the category of simplicial groups are far from being homotopy functors in general (for instance, the assumption that B be free cannot be deleted from Theorem 5.1 below). However, the technique of prolongation can be used to construct homotopy functors in the category of reduced simplicial sets by the following method, which was exploited by Bousfield and Kan in [3, IV.4]. Given a functor L in the category of groups, we may assign to each reduced simplicial set X the reduced simplicial set $\overline{W}LGX$, where G is Kan's loop group functor and $\overline{W}$ is the classifying space functor (see [17, § 10]). A weak equivalence $f\colon X \to Y$ yields a weak equivalence $Gf\colon GX \to GY$ of simplicial groups, which is then a loop homotopy equivalence, since GX and GY are free; see Proposition 6.5 in [17]. By Lemma 2.1, the induced map $LGf\colon LGX \to LGY$ is a loop homotopy equivalence, so $\overline{W}LGf$ is a homotopy equivalence. This shows that $\overline{W}LG$ is indeed a homotopy functor.

Well-known instances of this construction yield alternative descriptions of the Dold–Thom infinite symmetric product [9],

$$SP^\infty X \simeq \overline{W}(GX)_{\mathrm{ab}},$$

and the Bousfield–Kan R-completion functor

$$R_\infty X \simeq \overline{W}\widehat{(GX)}_R,$$

where R is a subring of the rationals or $\mathbf{Z}/p$, and the corresponding functor on groups is R-lower central series completion [3, IV.2].

In the rest of the article, we describe another homotopy functor which is obtained analogously, by choosing L to be localization at a set of primes P in the category of groups, that is, the functor which assigns to every group a uniquely P'-divisible group in a universal way. An interesting feature of the resulting functor on reduced simplicial sets is that it is homotopy idempotent, contrary to the two examples above.

3 Localization of spaces

For a set of primes P, a simplicial set X (or a CW-complex) is called P-local if it is local in the sense of Dror Farjoun [10] with respect to all degree q maps of the circle S^1 for $q \in P'$. That is, X is P-local if and only if X is fibrant and the maps

$$\rho_q\colon \mathrm{map}(S^1, X) \to \mathrm{map}(S^1, X)$$

induced by degree q maps $S^1 \to S^1$ are weak equivalences for $q \in P'$.

As explained in [6], a connected space X is P-local if and only if the fundamental group $\pi_1(X)$ and each of the semidirect products $\pi_k(X) \rtimes \pi_1(X)$, $k \geq 2$, are uniquely P'-divisible. Using the terminology given in Section 1, X is P-local if and only if $\pi_1(X)$ is a P-local group and, for $k \geq 2$, $\pi_k(X)$ is P-local as a module over $\pi_1(X)$.

A map $f\colon X \to Y$ is called a P-equivalence if, for every map $g\colon X \to Z$ where Z is P-local, there is a map $h\colon Y \to Z$, unique up to homotopy, such that $h \circ f \simeq g$. A recognition principle for P-equivalences was given in Theorem 3.2 of [6], as follows:

Theorem 3.1 *A map $f\colon X \to Y$ of connected spaces is a P-equivalence if and only if*

- $f_*\colon \pi_1(X) \to \pi_1(Y)$ *is a P-equivalence of groups, and*
- $f^*\colon H^k(Y; A) \cong H^k(X; A)$ *for all k and every P-local module A over $\pi_1(Y)$.* □

According to Theorem 3.3 in [6], for every space X there is a map $l_X\colon X \to X_P$ where X_P is P-local and l_X is a P-equivalence. Such a map will be called a P-localization of X. Up to homotopy, this coincides with Dror Farjoun localization [10] with respect to a set of degree q maps of S^1 with $q \in P'$. In the subcategory of simply connected (or nilpotent) spaces, it coincides, up to homotopy, with the classical localization at sets of primes; see [3], [15].

4 Proof of the main result

As in the previous sections, we fix a set of primes P and denote by L_P the P-localization functor in the category of groups. For a reduced simplicial set X, we consider the map $\eta_X\colon X \to \overline{W}L_PGX$ which is adjoint to the homomorphism $GX \to L_PGX$ given by prolongation of L_P to simplicial groups; that is, η_X is the composite of the natural weak equivalence $X \to \overline{W}GX$ with the map induced by the homomorphism $GX \to L_PGX$.

Our main result is that η_X is homotopy equivalent to the P-localization $l_X\colon X \to X_P$ described in Section 3. We devote the rest of this section to giving a proof of this claim.

Theorem 4.1 *For any simplicial group G and any set of primes P there is a natural isomorphism $\pi_0(L_P G) \cong L_P \pi_0(G)$.*

Proof. From Lemma 1.2 it follows that the natural homomorphism $\pi_0(G) \to \pi_0(L_P G)$ is a P-equivalence of groups, and Corollary 1.4 tells us that the group $\pi_0(L_P G)$ is uniquely P'-divisible. $\square$

Theorem 4.2 *For every reduced simplicial set X and every set of primes P, the natural map $\eta_X\colon X \to \overline{W} L_P GX$ is a P-equivalence.*

Proof. First, the induced homomorphism of fundamental groups,

$$\pi_1(X) \to \pi_1(\overline{W} L_P GX),$$

is a P-equivalence if and only if the homomorphism

$$\pi_0(GX) \to \pi_0(L_P GX)$$

induced by the localization map $GX \to L_P GX$ is a P-equivalence, which is the case by Lemma 1.2.

Secondly, we have to prove that the homomorphisms

$$H^k(\overline{W} L_P GX; A) \to H^k(X; A)$$

induced by η_X are isomorphisms for every P-local module A over $\pi_0(L_P GX)$; see Section 3. For this, we use a first-quadrant spectral sequence due to Quillen [26], which, for a simplicial group G and a module A over $\pi_0(G)$, takes the form

$$E_2^{r,s} = \pi^r \mathbf{H}^s(G; A) \ \Rightarrow H^{r+s}(\overline{W} G; A),$$

where $\mathbf{H}^s(G; A)$ is regarded as a cosimplicial abelian group with $H^s(G_n; A)$ in dimension n.

In our situation, $(GX)_n$ is a free group for every n, and therefore, by Corollary 7.3 and Theorem 8.7 in [6], the P-localization homomorphism $(GX)_n \to L_P(GX)_n$ induces isomorphisms

$$H^s(L_P(GX)_n; A) \cong H^s((GX)_n; A)$$

for $s \geq 0$ and every P-local module A over $\pi_0(L_P GX)$. This is in fact an isomorphism of cosimplicial abelian groups, and Quillen's spectral sequence yields an isomorphism

$$H^k(\overline{W} L_P GX; A) \cong H^k(\overline{W} GX; A),$$

for $k \geq 0$, as desired. □

Our next aim is to prove that, for every X, the space $\overline{W} L_P GX$ is P-local.

If X is a simplicial set and G is a simplicial group, then the (unpointed) function space $\mathrm{map}(X, G)$, whose n-simplices are the simplicial maps $X \times \Delta[n] \to G$, admits a natural structure of a simplicial group where multiplication of two maps is defined pointwise. Therefore, the following result is straightforward.

Lemma 4.3 *Let G be a simplicial group which is uniquely P'-divisible at every dimension. Then, for every simplicial set X, the simplicial group $\mathrm{map}(X, G)$ is uniquely P'-divisible at every dimension.* □

Lemma 4.4 *Let G be any simplicial group. Then*

$$\pi_n(G) \rtimes \pi_0(G) \cong \pi_0(\mathrm{map}(S^n, G)) \qquad \textit{for } n \geq 0.$$

Proof. Recall that $\pi_0(G)$ acts on $\pi_n(G)$ by

$$[g] \cdot [x] = [s_0^n(g)\; x\; s_0^n(g)^{-1}],$$

where $g \in G_0$, $x \in (NG)_n$, and s_0 denotes the corresponding degeneracy of G. Thus, the map

$$\phi \colon \pi_0(\mathrm{map}(S^n, G)) \to \pi_n(G) \rtimes \pi_0(G),$$

given by $\phi([f]) = ([f(e_n)\, s_0^n(f(e_0))^{-1}],\ [f(e_0)])$, is a group homomorphism (compare with Theorem 1.7 in [24]); here e_0 and e_n are the nondegenerate simplices of S^n. This homomorphism has a two-sided inverse given by $\phi^{-1}([x], [g]) = [f]$, where $f(e_0) = g$ and $f(e_n) = x\, s_0^n(g)$. $\square$

Theorem 4.5 *For any simplicial group G and any set of primes P, the simplicial set $\overline{W}L_PG$ is P-local.*

PROOF. By Lemma 4.4, we have

$$\pi_n(\overline{W}L_PG) \rtimes \pi_1(\overline{W}L_PG) \cong \pi_0(\mathrm{map}(S^{n-1},\ L_PG)),$$

and, by Lemma 4.3, the simplicial group $\mathrm{map}(S^{n-1},\ L_PG)$ is uniquely P'-divisible at every dimension, so we may use Corollary 1.4 to complete the argument. $\square$

Now, Theorem 4.2 and Theorem 4.5 yield together our main result:

Theorem 4.6 *If X is any reduced simplicial set and P is any set of primes, then the space $\overline{W}L_PGX$ is homotopy equivalent to the P-localization X_P.* $\square$

5 Applications

In practice, one often works with connected CW-complexes instead of reduced simplicial sets. The following extension of Theorem 4.6 yields manageable models for P-localizations of connected CW-complexes, by using the free simplicial groups considered by Kan in [18]. Given a connected CW-complex X, replace it, if necessary, by a homotopy equivalent CW-complex with a single 0-cell. Then let J be a free simplicial group with a nondegenerate generator in dimension n for every $(n+1)$-cell of X, as described in [18]. Thus, J is loop homotopy equivalent to the free simplicial group obtained by applying Kan's loop group functor G to the reduced singular complex of X (which is much larger than J in general). By the next result, we may use J to calculate X_P as well.

Theorem 5.1 *Let X be any reduced simplicial set and P any set of primes. If J is any free simplicial group which is weakly equivalent to GX, then $\overline{W}L_PJ \simeq X_P$.*

Proof. Since J is a free simplicial group, there is a loop homotopy equivalence $h\colon J \to GX$, by Proposition 6.5 in [17]. By Lemma 2.1, L_Ph is a loop homotopy equivalence, from which it follows that the induced map $\overline{W}L_PJ \to \overline{W}L_PGX$ is a homotopy equivalence. □

The assumption that J be free in Theorem 5.1 can be weakened, by imposing only that the homomorphisms

$$H_i(J_n; A) \to H_i(L_PJ_n; A)$$

induced by the P-localization $J_n \to L_PJ_n$ be isomorphisms for all i, n, and every P-local module A over $\pi_0(L_PJ)$. This is seen by looking carefully at the proof of Theorem 4.2. For example, J could be nilpotent at every dimension (by Theorem 4.3 in [6]); compare also with Corollary 3.5 in [26].

As a consequence of our description of the P-localization functor, we gain a good control of its "low dimensional behaviour", as in [3, IV.5]. Indeed, if two CW-complexes or simplicial sets X and Y have isomorphic n-skeleta, then their P-localizations X_P and Y_P have isomorphic n-skeleta too, since our construction of P-localization is carried out dimensionwise. We state this result in the same form as in [3, IV.5.1].

Theorem 5.2 *Fix an integer $n \geq 0$ and a set of primes P. If $f\colon X \to Y$ is a map of spaces such that the induced homomorphism $\pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k \leq n$ and an epimorphism for $k = n+1$, then $\pi_k(X_P) \to \pi_k(Y_P)$ is also an isomorphism for $k \leq n$ and an epimorphism for $k = n+1$.* □

In other words, if the homotopy fibre of $f\colon X \to Y$ is n-connected for some $n \geq 1$, then the homotopy fibre of $f_P\colon X_P \to Y_P$ is n-connected as well. A consequence of Theorem 5.2 is that the homotopy groups $\pi_k(X_P)$ of the P-localization of a space X can easily be calculated for $k \leq n$ if either the $(n+1)$-skeleton or the nth Postnikov section of X is a nilpotent space; cf. [15]. That is, if $\pi_1(X)$ is a nilpotent group acting nilpotently on $\pi_k(X)$ for $2 \leq k \leq n$, then

$$\pi_k(X_P) \cong \pi_k(X) \otimes \mathbf{Z}_P \qquad \text{for } k \leq n.$$

A similar result was proved for integral homology localizations in Theorem 1.1 of [11].

Our description of P-localization of spaces using simplicial groups admits a relative version, namely fibrewise P-localization of fibre sequences. Fibrewise localizations or completions were first discussed by Sullivan in Theorem 4.2 of [28] and Bousfield–Kan in [3, IV.5.7], and later developed by May in [22]. Fibrewise localization at sets of primes was described by Llerena in [19] under the assumption that the fibre be nilpotent. More recently, fibrewise localizations have been considered in greater generality by Dror Farjoun in [10, 1.F].

Our construction of fibrewise P-localization applies to any fibre sequence $F \to X \to B$ in the category of reduced simplicial sets. Thus, we are implicitly assuming that the map $f\colon X \to B$ is surjective and the induced homomorphism $\pi_1(X) \to \pi_1(B)$ is an epimorphism. By [25, II.3.10], the induced epimorphism $Gf\colon GX \to GB$ is a fibration of simplicial groups. Its fibre is the simplicial group K where $K_n = \ker(Gf)_n$. Both the natural homomorphism $GF \to K$ and its adjoint map $F \to \overline{W}K$ are weak equivalences; cf. [25, II.3.11].

Lemma 5.3 *The kernel K of the epimorphism $Gf\colon GX \to GB$ is a free simplicial group.*

Proof. Since $(GX)_n$ is free for all n, the subgroup K_n is also free for all n. Thus, in order to prove that K is a free simplicial group, we have to exhibit bases for each K_n which are stable under the degeneracies of K. We shall not distinguish notationally between the degeneracies of distinct simplicial sets. Recall however that the degeneracy $s_i\colon (GX)_{n-1} \to (GX)_n$ is the homomorphism spanned by $s_{i+1}\colon X_n \to X_{n+1}$, for $i = 0, \ldots, n-1$. Since $(GB)_n$ is free on the set $B_{n+1} - s_0(B_n)$, we may define inductively homomorphisms

$$\sigma_n\colon (GB)_n \to (GX)_n$$

for each n, by imposing that $\sigma_n \circ s_i = s_i \circ \sigma_{n-1}$ for $i = 0, \ldots, n-1$, where $n \geq 1$, and that $(Gf)_n \circ \sigma_n$ be the identity for all n. (Note that such a σ does not necessarily commute with the face operators.) Then the image of σ_n is a Schreier system of coset representatives of the kernel K_n in $(GX)_n$ (cf. [7, § 4] or [16, § 18]). Hence, as in Corollary 4.7 in [7], the group K_n is freely generated by the elements

$$\sigma_n(w)\, x\, \sigma_n((Gf)_n(x))^{-1}\, \sigma_n(w)^{-1},$$

with $w \in (GB)_n$ and $x \in X_{n+1} - s_0(X_n) - \sigma_n(B_{n+1} - s_0(B_n))$. By our choice of σ_n, these elements form a basis which is closed under the degeneracies of GX, hence of K. $\square$

Now, for each n, we apply the relative P-localization functor of [5] to the group extension

$$K_n \rightarrowtail (GX)_n \twoheadrightarrow (GB)_n,$$

yielding a commutative diagram

$$\begin{array}{ccccc} K_n & \rightarrowtail & (GX)_n & \twoheadrightarrow & (GB)_n \\ \downarrow & & {\scriptstyle \varepsilon_n}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ L_P K_n & \rightarrowtail & E_n & \twoheadrightarrow & (GB)_n \end{array} \tag{5.1}$$

which is universal in the category of group extensions among morphisms from the upper extension into extensions with P-local kernel; cf. Theorem 1.4 in [5]. This allows us to endow the sequence of groups E_n with the structure of a simplicial group, which we denote by E. Thus, we obtain a fibration $E \to GB$ of simplicial groups with fibre $L_P K$, and a commutative diagram of fibre sequences

$$\begin{array}{ccccc} F & \to & X & \to & B \\ {\scriptstyle l}\downarrow & & {\scriptstyle e}\downarrow & & {\scriptstyle \simeq}\downarrow \\ \overline{W} L_P K & \to & \overline{W} E & \to & \overline{W} GB, \end{array} \tag{5.2}$$

where the left vertical map l is the composite $F \to \overline{W} K \to \overline{W} L_P K$, and the map e is adjoint to $\varepsilon\colon GX \to E$. We call (5.2) the fibrewise P-localization of the given fibration. This terminology is justified by the following theorem.

Theorem 5.4 *The map $l\colon F \to \overline{W}L_P K$ is a P-localization and the map $e\colon X \to \overline{W}E$ is a P-equivalence of reduced simplicial sets.*

Proof. The first claim follows from Theorem 5.1, since K is a free simplicial group (by Lemma 5.3) which is weakly equivalent to GF. To prove the second claim, we use the description of P-equivalences given in Theorem 3.1. By Proposition 1.3 in [5], the homomorphism ε_n in (5.1) is a P-equivalence for all n. From the fact that ε_0 and ε_1 are P-equivalences it follows, by Theorem 4.1, that the homomorphism $\pi_0(GX) \to \pi_0(E)$ induced by ε_0 is a P-equivalence of groups. Next, let A be any P-local module over $\pi_0(E)$. Then (5.2) induces a morphism of first-quadrant spectral sequences for cohomology with coefficients in A. (A suitable reference for such spectral sequences with twisted coefficients is [25, II.6.17] for homology or Proposition 2.1 in [26] for cohomology.) Since we already proved that the left vertical arrow l in (5.2) is a P-localization, we see that $l^*\colon H^k(\overline{W}L_P K; A) \to H^k(F; A)$ is an isomorphism for all k. This implies that $e^*\colon H^k(\overline{W}E; A) \cong H^k(X; A)$ for all k as well. □

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