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# ON FINITE GROUPS ACTING ON ACYCLIC COMPLEXES OF DIMENSION TWO

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*Abstract*

We conjecture that every finite group $G$ acting on a contractible $CW$-complex $X$ of dimension 2 has at least one fixed point. We prove this in the case where $G$ is solvable, and under this additional hypothesis, the result holds for $X$ acyclic.

*Dedicat a la memòria d'en Pere Menal*

## 0. Introduction

Let $G$ be a group and $A$ an abelian group. Dicks and Dunwoody ([**4**, Chapter IV]) proved that for each element $\zeta$ of $H^1(G; AG)$ there exists a $G$-tree $T$ with finite edge stabilizers, with the property that for each subgroup $H$ of $G$, the restriction of $\zeta$ to $H$ is zero if and only if $H$ fixes a point of $T$. It is natural to look for analogous geometric explanations of elements of higher cohomology groups; thus, for example, one can ask if for each element $\zeta$ of $H^2(G; AG)$ there exists a contractible 2-dimensional $CW$-complex $X$ admitting an action of $G$ with finite stabilizers for 2-cells, with the property that for each subgroup $H$ of $G$, the restriction of $\zeta$ to $H$ is zero if and only if $H$ acts trivially on $X$ in some sense, perhaps leaving invariant a subtree of the 1-skeleton of $X$. The restriction of $\zeta$ to any finite subgroup of $G$ is zero, but if a finite group leaves a subtree invariant then it fixes a point. With this motivation, we optimistically conjecture that every finite group $G$ acting on a contractible 2-dimensional $CW$-complex $X$ has at least one fixed point.

In this note we prove this conjecture in the case where $G$ is solvable. Our argument is based on a classical result of P.A. Smith ([**8**], [**9**]), stating that every action of a finite $p$-group on a finite dimensional $\mathbf{Z}/p$-acyclic $CW$-complex has a $\mathbf{Z}/p$-acyclic fixed-point set (see [**2**, Chapter III] and further developments e.g. in [**1**], [**3**], [**7**]).

In our context, the hypothesis that $X$ has no cells above dimension 2 is essential. It is known that any finite nilpotent group whose order is not a prime power acts on some contractible 3-dimensional $CW$-complex without fixed points ([**1**]).

On the other hand, we shall prove that for a finite solvable group $G$ acting on a 2-dimensional $CW$-complex $X$, in order to ensure the existence of a fixed point it suffices to assume that $X$ is *acyclic*. For $X$ acyclic, however, the condition that $G$ be solvable cannot be removed, because the alternating group $A_5$ acts on the 2-skeleton of the Poincaré sphere –which is acyclic– without fixed points ([**6**]). Recall that the 1-skeleton of the Poincaré sphere is the complete graph on 5 vertices, and the 2-skeleton is obtained by adding 6 pentagonal faces so as to extend the natural action of $A_5$ on the set of vertices. The fundamental domain of the action is a triangle with angles $\pi/2$, $\pi/5$, $3\pi/10$, and the 60 copies of this fundamental domain triangulate the 2-skeleton, from which it follows that there are no fixed points. The fundamental group of this space is isomorphic to $SL_2(\mathbf{F}_5)$.

Since $X$ being contractible is equivalent to $X$ being simply-connected and acyclic, the question that remains open is: If we add the condition that $X$ be simply-connected, can we delete the condition that $G$ be solvable?

## 1. Statement and proof of the result

Let $G$ be a finite group acting on a $CW$-complex $X$ of dimension 2, and denote by $X^G$ the set of fixed points under the action of $G$. We shall assume that the action is *cellular* ([**5**]); that is, each translation of an open cell is an open cell, and, if a cell is invariant, then it is pointwise fixed. Thus $X^G$ is a subcomplex of $X$. For a subcomplex $Y \subseteq X$, we denote by $C_n(X,Y)$ the group of relative cellular $n$-chains of the pair $(X,Y)$.

Given a nonzero abelian group $A$, a space $X$ is said to be $A$-*acyclic* if $\widetilde{H}_k(X;A) = 0$ for all $k$, where $\widetilde{H}$ denotes reduced homology. Recall that the condition $\widetilde{H}_{-1}(X;A) = 0$ is equivalent to the augmentation homomorphism $C_0(X) \otimes A \to A$ being surjective, and hence equivalent to $X$ being nonempty.

We prove

**Theorem 1.1.** *Let $G$ be a finite solvable group acting on a $CW$-complex $X$ of dimension 2. If $\widetilde{H}_*(X;\mathbf{Z})$ is finite, and the orders of $G$, $H_1(X;\mathbf{Z})$ are coprime, then the natural map $\widetilde{H}_*(X^G;\mathbf{Z}) \to \widetilde{H}_*(X;\mathbf{Z})$ is injective.*

*Proof:* Under our assumptions, the graded group $\widetilde{H}_*(X;\mathbf{Z})$ is necessarily concentrated in degree 1, since it is free abelian in all other degrees. Moreover, $H_1(X;\mathbf{Z}/p)=0$ (and hence $X$ is $\mathbf{Z}/p$-acyclic) for every prime $p$ dividing the order of $G$.

Since $G$ is solvable, we can find a series of subgroups

$$\{1\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k = G \tag{1.1}$$

in which each $G_{i-1}$ is normal in $G_i$ and $G_i/G_{i-1} \cong \mathbf{Z}/p_i$, where $p_i$ is a prime. Then the action of $G$ on $X$ induces an action of $G_i/G_{i-1}$ on $X^{G_{i-1}}$ and

$$X^{G_i} = \left(X^{G_{i-1}}\right)^{G_i/G_{i-1}} . \tag{1.2}$$

We prove inductively that the map $\widetilde{H}_*(X^{G_i};\mathbf{Z}) \to \widetilde{H}_*(X;\mathbf{Z})$ is a monomorphism for all $i=0,\ldots,k$. This is trivial for $G_0$. Thus suppose that it has been established for $G_{i-1}$. Then $X^{G_{i-1}}$ is $\mathbf{Z}/p$-acyclic for every prime $p$ dividing the order of $G$. Since the order of $G_i/G_{i-1}$ is a prime $p_i$, applying Smith's Theorem ([**9**]) to the action of $G_i/G_{i-1}$ on $X^{G_{i-1}}$ we obtain, by (1.2), that $X^{G_i}$ is $\mathbf{Z}/p_i$-acyclic. This tells us in particular that $X^{G_i}$ is nonempty and connected. Further, for every abelian group $A$ we have an exact sequence

$$\begin{aligned} 0 \longrightarrow H_2(X^{G_i};A) \longrightarrow H_2(X;A) \longrightarrow H_2(X,X^{G_i};A) \longrightarrow \\ \longrightarrow H_1(X^{G_i};A) \longrightarrow H_1(X;A) \longrightarrow H_1(X,X^{G_i};A) \longrightarrow 0, \end{aligned} \tag{1.3}$$

from which we infer that $H_2(X,X^{G_i};\mathbf{Z}/p_i)=0$. But, since $X$ has no cells above dimension 2, the group $H_2(X,X^{G_i};\mathbf{Z})$ embeds in the free abelian group $C_2(X,X^{G_i})$ and hence it is free abelian itself. This forces $H_2(X,X^{G_i};\mathbf{Z})=0$, showing that $H_1(X^{G_i};\mathbf{Z})$ embeds in $H_1(X;\mathbf{Z})$. ■

**Corollary 1.2.** *Every action of a finite solvable group $G$ on a $\mathbf{Z}$-acyclic CW-complex $X$ of dimension 2 has at least one fixed point.*

*Proof:* It follows from Theorem 1.1 that the fixed-point set $X^G$ is $\mathbf{Z}$-acyclic, so in particular it is nonempty. ■

**Note.** Robert Oliver has kindly informed us that an, as yet unpublished, paper by Yoav Segev contains a different proof of Corollary 1.2, with the additional assumption that $X$ be finite.

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