

# A note on extensions of nilpotent groups

By

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*Dedicated to the memory of Frank Adams, close friend and inspiring teacher*

## 0 Introduction

In studying the Mislin genus for finitely generated nilpotent groups and its relation to the genus of groups-with-operators (see [1, 2]) we have been led to the following question. Suppose that

$$N \twoheadrightarrow G \xrightarrow{\kappa} Q \tag{0.1}$$

is a short exact sequence of nilpotent groups which splits at every prime  $p$ , that is, such that

$$N_p \twoheadrightarrow G_p \xrightarrow{\kappa_p} Q_p$$

admits a right splitting  $\tau(p) : Q_p \rightarrow G_p$  for each  $p$ . Does it follow that the original sequence (0.1) itself splits? In Theorem 2.1 of [1] we collected together the known results, which we restate here.

**Theorem 0.1** *If  $N \twoheadrightarrow G \xrightarrow{\kappa} Q$  splits at every prime, then it splits provided*

- (a)  *$Q$  is finitely generated and  $N$  is commutative, or*
- (b)  *$Q$  is finitely generated and  $N$  is finite.*

*Moreover, in case (b), given the splitting maps  $\tau(p) : Q_p \rightarrow G_p$ , we may choose the splitting map  $\tau : Q \rightarrow G$  so that  $\tau_p = \tau(p)$ .*

A further advance—though it was not presented as such—was contained in Theorem 1.6 of [1], which effectively gave an example showing that the answer to our question is not always affirmative.

The main purpose of this paper is to generalize both Theorem 0.1 and the counterexample contained in Theorem 1.6 of [1]. Thus we prove (cf. Theorems 2.1 and 2.3, which reproduce part (b) and part (a) of Theorem 0.2 respectively)

**Theorem 0.2** *If  $N \twoheadrightarrow G \xrightarrow{\kappa} Q$  splits at every prime, then it splits provided*

(a)  *$Q$  is finitely generated and  $N_p$  is commutative for almost all primes  $p$ , being torsion for the exceptional values of  $p$ , or*

(b)  *$N$  is torsion and almost torsionfree.*

*Moreover, in case (b), given the splitting maps  $\tau(p) : Q_p \rightarrow G_p$ , we may choose the splitting map  $\tau : Q \rightarrow G$  so that  $\tau_p = \tau(p)$ .*

Recall that  $N$  is *almost torsionfree* if  $TN_p = \{1\}$  for almost all primes  $p$ . Of course a finite (nilpotent) group is almost torsionfree.

This theorem actually presented itself as a byproduct of a study of the Pull-back Theorem [4, Theorem I.3.9] which asserts that (i) for any nilpotent group  $G$  and any partition  $(P_1, P_2)$  of the family of all primes, the square

$$\begin{array}{ccc} G & \rightarrow & G_{P_1} \\ \downarrow & & \downarrow \\ G_{P_2} & \rightarrow & G_0 \end{array}$$

is a pull-back; we also know that (ii) every element of  $G_0$  is expressible as  $x_1x_2$ , where  $x_1$  is the image of an element of  $G_{P_1}$  and  $x_2$  is the image of an element of  $G_{P_2}$ . We need to generalize the pull-back property (i) to any finite partition of the family of primes. In doing this, we also present a different generalization of Theorem I.3.9 of [4], in which we no longer require that  $(P_1, P_2)$  be a partition. We also generalize the supplementary conclusion (ii) in Section 1 below—it is interesting that the hypotheses for the generalizations of the properties (i) and (ii) in fact diverge. The arguments used extend in a natural way to the case that the groups involved are *locally nilpotent*, i.e. such that every finitely generated subgroup is nilpotent; we state the results in Section 1 in this generality. Obviously all the generalizations so far referred to have implications for the homotopy theory of nilpotent spaces, which we plan to pursue in a subsequent paper.

Section 2 closes with a broad generalization of the counterexample implicit in Theorem 1.6 of [1]. We study the group  $\text{Ext}(Q, N)$  when  $Q$  and  $N$  are commutative, and establish conditions under which  $\text{Ext}(Q, N)$  is uncountable although  $\text{Ext}(Q_p, N_p) = 0$  for all  $p$ . A more detailed study of this situation is currently being undertaken by Robert Militello.

# 1 On the pull-back and quasi-push-out properties

We start with Theorem I.3.9 of [4] which asserts that if  $G$  is a nilpotent group and  $(P_1, P_2)$  is a partition of the set of all primes, then the square

$$\begin{array}{ccc} G & \xrightarrow{e_{P_1}} & G_{P_1} \\ e_{P_2} \downarrow & & \downarrow r_{P_1} \\ G_{P_2} & \xrightarrow{r_{P_2}} & G_0 \end{array} \quad (1.1)$$

is a pull-back. Moreover, the Remark following Theorem I.3.9 asserts that every element of  $G_0$  is expressible as  $(r_{P_1}x_1)(r_{P_2}x_2)$ ,  $x_i \in G_{P_i}$ . If  $G$  is commutative this last is equivalent to the square being a push-out (in the category of commutative groups). We will describe this property, in general, by saying that (1.1) is a *quasi-push-out*. We will generalize these two properties in two directions.

We say that  $(P_1, P_2, \dots, P_k)$  is a (finite) *presentation* of the family  $P$  of primes if  $P = \cup_i P_i$ . It is a *thin* presentation if  $\cap_i P_i = \emptyset$ . Thus a partition coincides with a thin presentation in the case  $k = 2$ . However, observe the divergence of the conditions stated in Theorems 1.1 and 1.2 below.

**Theorem 1.1** *Let  $G$  be a locally nilpotent group and let  $(P_1, P_2, \dots, P_k)$  be a finite partition of the family  $P$  of primes. Then  $G_P$  is the pull-back of  $G_{P_1}, G_{P_2}, \dots, G_{P_k}$  over  $G_0$ .*

**Theorem 1.2** *Let  $G$  be a locally nilpotent group and let  $(P_1, P_2, \dots, P_k)$  be a thin presentation of the family  $P$  of primes. Then  $G_0$  is the quasi-push-out of  $G_{P_1}, G_{P_2}, \dots, G_{P_k}$ , in the sense that every element of  $G_0$  can be expressed as  $\prod_{i=1}^k r_{P_i}x_i$ ,  $x_i \in G_{P_i}$ .*

Before giving the proofs, we explain the second direction of our generalization. In the next theorem, we no longer insist that  $(P_1, P_2)$  be a partition.

**Theorem 1.3** *Let  $P_1, P_2$  be any families of primes and let  $P = P_1 \cup P_2$ ,  $P_0 = P_1 \cap P_2$ . Let  $G$  be a locally nilpotent group. Then the square*

$$\begin{array}{ccc} G_P & \xrightarrow{e_{P_1}^P} & G_{P_1} \\ e_{P_2}^P \downarrow & & \downarrow e_{P_0}^{P_1} \\ G_{P_2} & \xrightarrow{e_{P_0}^{P_2}} & G_{P_0}, \end{array}$$

where all arrows are localization maps, is a pull-back and a quasi-push-out.

The strategy of proof is based on the following lemmas, as in [4]. We use the notation  $G \in \mathcal{P}$  to indicate that  $G$  has the pull-back and quasi-push-out properties asserted by Theorem 1.3.

**Lemma 1.4** *Let  $G' \twoheadrightarrow G \xrightarrow{\kappa} G''$  be a central extension of nilpotent groups. Then  $G \in \mathcal{P}$  if  $G', G'' \in \mathcal{P}$ .*

PROOF. We first check the pull-back property for  $G$ . Note that it suffices to prove that if  $x_i \in G_{P_i}$  with  $e_{P_0}^{P_1}x_1 = e_{P_0}^{P_2}x_2 = x_0$  then there exists  $x \in G_P$  with  $e_{P_i}^P x = x_i$ , for the uniqueness of  $x$  follows from the fact that the kernel of  $e_{P_i}^P: G_P \rightarrow G_{P_i}$  consists of elements which are  $(P \setminus P_i)$ -torsion. Thus, with the given elements  $x_i$ , we have  $e_{P_0}^{P_1}\kappa_{P_1}x_1 = e_{P_0}^{P_2}\kappa_{P_2}x_2 = \kappa_{P_0}x_0$ . Since  $G'' \in \mathcal{P}$ , there exists an element  $x'' \in G''_P$  with  $e_{P_i}^P x'' = \kappa_{P_i}x_i$ ,  $i = 1, 2$ . Let  $x'' = \kappa_P \bar{x}$ . Then  $\kappa_{P_i} e_{P_i}^P \bar{x} = e_{P_i}^P x'' = \kappa_{P_i} x_i$ , so that  $x_i = (e_{P_i}^P \bar{x})x'_i$  with  $x'_i \in G'_{P_i}$ . Moreover  $x_0 = e_{P_0}^{P_i} x_i = (e_{P_0}^{P_i} \bar{x})(e_{P_0}^{P_i} x'_i)$ , so that  $e_{P_0}^{P_1} x'_1 = e_{P_0}^{P_2} x'_2$ . Since  $G' \in \mathcal{P}$ , there exists  $x' \in G'_P$  with  $e_{P_i}^P x' = x'_i$ ,  $i = 1, 2$ . Thus  $e_{P_i}^P (\bar{x}x') = (e_{P_i}^P \bar{x})x'_i = x_i$ , and the first part of the lemma is proved (note that we did not use here the assumption that the extension is central). Now we turn to the quasi-push-out property. Let  $x_0 \in G_{P_0}$ . Then  $\kappa_{P_0}x_0 = (e_{P_0}^{P_1}x''_1)(e_{P_0}^{P_2}x''_2)$ ,  $x''_i \in G''_{P_i}$ , since  $G'' \in \mathcal{P}$ . Let  $x''_i = \kappa_{P_i}x_i$ ,  $x_i \in G_{P_i}$ , so that  $\kappa_{P_0}x_0 = \kappa_{P_0}(e_{P_0}^{P_1}x_1)(e_{P_0}^{P_2}x_2)$ . Thus  $x_0 = (e_{P_0}^{P_1}x_1)(e_{P_0}^{P_2}x_2)x'_0$  with  $x'_0 \in G'_0$ . Since  $G' \in \mathcal{P}$ ,  $x'_0 = (e_{P_0}^{P_1}x'_1)(e_{P_0}^{P_2}x'_2)$ ,  $x'_i \in G'_{P_i}$ . Since  $G'$  is central in  $G$ , we conclude finally that

$$x_0 = (e_{P_0}^{P_1}(x_1x'_1))(e_{P_0}^{P_2}(x_2x'_2)). \quad \square$$

**Lemma 1.5** *Let  $\{G^\alpha, \varphi^{\alpha\beta}\}$  be a directed system of nilpotent groups such that  $G^\alpha \in \mathcal{P}$  for all  $\alpha$ . Then  $G = \varinjlim \{G^\alpha, \varphi^{\alpha\beta}\}$  belongs to  $\mathcal{P}$ .*

PROOF. Recall first that localization commutes with direct limits over directed sets. Now let  $x_i \in G_{P_i}$ ,  $i = 1, 2$ , with  $e_{P_0}^{P_1}x_1 = e_{P_0}^{P_2}x_2 = x_0$ . Then there exists  $\alpha$  and  $x_i^\alpha \in G_{P_i}^\alpha$  such that  $\varphi_{P_i}^\alpha x_i^\alpha = x_i$ ,  $i = 1, 2$ , where  $\varphi^\alpha$  is the canonical homomorphism  $\varphi^\alpha: G^\alpha \rightarrow G$ . Moreover, since  $\varphi_{P_0}^\alpha e_{P_0}^{P_1}x_1^\alpha = \varphi_{P_0}^\alpha e_{P_0}^{P_2}x_2^\alpha$ , we may find  $\beta$  and  $x_i^\beta \in G_{P_i}^\beta$  such that  $\varphi_{P_i}^\beta x_i^\beta = x_i$  and  $e_{P_0}^{P_1}x_1^\beta = e_{P_0}^{P_2}x_2^\beta$ . Since  $G^\beta \in \mathcal{P}$ , there exists  $x^\beta \in G_P^\beta$  with  $e_{P_i}^P x^\beta = x_i^\beta$ , whence  $e_{P_i}^P \varphi_{P_i}^\beta x^\beta = x_i$ ,  $i = 1, 2$ , establishing the pull-back property for  $G$ . The quasi-push-out property is obvious.  $\square$

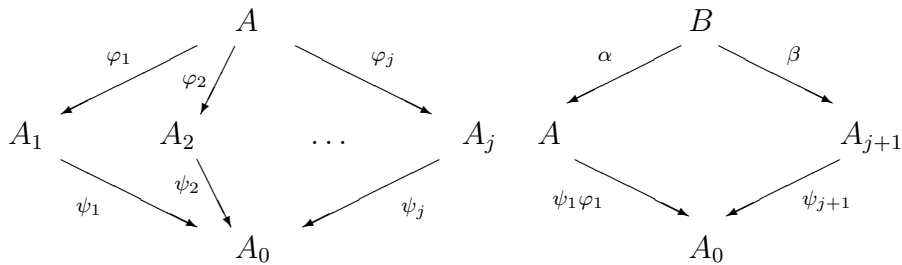
**Lemma 1.6** *The groups  $\mathbf{Z}$ ,  $\mathbf{Z}/p^n$  belong to  $\mathcal{P}$ .*

PROOF. If  $G = \mathbf{Z}$ , we must show that if the reduced fractions (with positive denominators)  $m_1/n_1 \in \mathbf{Z}_{P_1}$ ,  $m_2/n_2 \in \mathbf{Z}_{P_2}$  are equal as rational numbers, then  $m_1 = m_2 = m$ ,  $n_1 = n_2 = n$ , and  $n$  is a  $P'$ -number. This, however, is clear since  $P'_1 \cap P'_2 = P'$ . We must also consider the reduced fraction  $m/n$  where  $n$  is a  $P'_0$ -number. Write  $n$  as  $uvw$ , where  $u$  is a  $(P_1 \setminus P_0)$ -number,  $v$  is a  $(P_2 \setminus P_0)$ -number, and  $w$  is a  $P'$ -number. Since  $u, v$  are mutually prime,  $1 = au + bv$  for some  $a, b \in \mathbf{Z}$ . Thus  $m/n = (ma/vw) + (mb/uw)$ , and  $vw$  is a  $P'_1$ -number while  $uw$  is a  $P'_2$ -number. This shows that  $\mathbf{Z}$  belongs to  $\mathcal{P}$ . If  $G = \mathbf{Z}/p^n$ , both conclusions are obvious if  $p \notin P$  or if  $p \in P_0$ . If  $p \in P_1 \setminus P_0$ , then  $G_{P_2} = G_{P_0} = \{0\}$  and  $e_{P_1}^P = \text{id}$ , so, again, both conclusions are clear.  $\square$

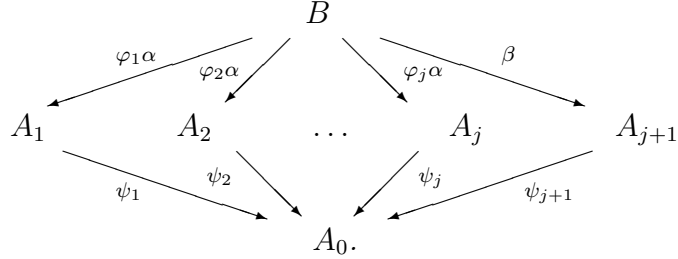
PROOF OF THEOREM 1.3. Starting from Lemma 1.6 and repeatedly using Lemma 1.4, we conclude that  $G \in \mathcal{P}$  if  $G$  is finitely generated commutative. Lemma 1.5 then allows us to drop the condition of finite generation. Assume next  $G$  nilpotent and proceed by induction on the nilpotency class of  $G$ . For if  $\text{nil } G = c$ , so that  $\Gamma^c G = \{1\}$ , we set  $\Gamma = \Gamma^{c-1} G$  and we have the central extension  $\Gamma \twoheadrightarrow G \twoheadrightarrow G/\Gamma$ , with  $\text{nil } G/\Gamma = c - 1$ . Thus Lemma 1.4 allows us to infer that  $G \in \mathcal{P}$ . Finally, the proof in the case of  $G$  locally nilpotent is achieved again using Lemma 1.5.  $\square$

The above argument applies *mutatis mutandi* to provide proofs of Theorems 1.1 and 1.2. However, it is more economic now to infer Theorems 1.1 and 1.2 from Theorem 1.3 as follows.

**Lemma 1.7** *If the diagrams*



*are pull-backs in an arbitrary category, so is the diagram*



PROOF. This only requires a routine checking of the characteristic universal property.  $\square$

PROOF OF THEOREM 1.1. We argue by induction on  $k$ . By Theorem 1.3,  $G_{P_1 \cup P_2}$  is the pull-back of  $G_{P_1}$  and  $G_{P_2}$  over  $G_0$ , since  $P_1 \cap P_2 = \emptyset$ . Assume inductively that  $G_{P_1 \cup \dots \cup P_i}$  is the pull-back of  $G_{P_1}, \dots, G_{P_i}$  over  $G_0$ . Then, again by Theorem 1.3,  $G_{P_1 \cup \dots \cup P_{i+1}}$  is the pull-back of  $G_{P_1 \cup \dots \cup P_i}$  and  $G_{P_{i+1}}$  over  $G_0$ . Therefore, by Lemma 1.7,  $G_{P_1 \cup \dots \cup P_{i+1}}$  is the pull-back of  $G_{P_1}, \dots, G_{P_{i+1}}$  over  $G_0$ .  $\square$

**Theorem 1.8** *Let  $G$  be a locally nilpotent group and let  $(P_1, P_2, \dots, P_k)$  be any family of sets of primes with intersection  $P_0$ . Then  $G_{P_0}$  is the quasi-push-out of  $G_{P_1}, G_{P_2}, \dots, G_{P_k}$ .*

PROOF. We argue again by induction on  $k$ . Set  $Q = P_2 \cap \dots \cap P_k$ . Then, given  $x \in G_{P_0}$ , by Theorem 1.3 we may write

$$x = (e_{P_0}^{P_1} x_1)(e_{P_0}^Q y), \quad x_1 \in G_{P_1}, \quad y \in G_Q.$$

But, by the inductive hypothesis, we have

$$y = \prod_{i=2}^k e_Q^{P_i} x_i, \quad x_i \in G_{P_i},$$

and hence

$$e_{P_0}^Q y = \prod_{i=2}^k e_{P_0}^{P_i} x_i,$$

which proves the theorem.  $\square$

Of course, Theorem 1.2 is a special case of Theorem 1.8. It is easy to see that Theorem 1.8 in fact remains true if we assume the family  $(P_1, P_2, \dots)$  to be infinite, provided we understand that a product  $\prod_i e_{P_0}^{P_i} x_i$  has only finitely many nontrivial factors.

Notice, however, that there is no useful analog of Theorem 1.8 in the case of the pull-back property, for we would have to assume a presentation  $(P_1, \dots, P_k)$  of  $P$  such that  $P_i \cap P_j$ ,  $i \neq j$ , is independent of  $(i, j)$ . This only seems sensible in the case of a partition.

## 2 The splitting problem

We consider a short exact sequence of nilpotent groups

$$N \twoheadrightarrow G \xrightarrow{\kappa} Q, \quad (2.1)$$

which we suppose splits at every prime  $p$ . Indeed, let  $\tau(p): Q_p \rightarrow G_p$  be a splitting at the prime  $p$ , so that  $\kappa_p \tau(p) = \text{id}_{Q_p}$ .

**Theorem 2.1** *If  $N$  is torsion but almost torsionfree, then (2.1) splits. Moreover, there is a splitting  $\tau: Q \rightarrow G$  such that  $\tau_p = \tau(p)$ .*

PROOF. Let  $P = \{p_1, p_2, \dots, p_k\}$  be the (finite) set of primes at which  $N$  has torsion. We consider the partition  $(p_1, p_2, \dots, p_k, P')$  of the family of all primes. For each  $i = 1, 2, \dots, k$  we write  $\tau_i = \tau(p_i): Q_{p_i} \rightarrow G_{p_i}$ . Now  $N_{P'} = \{1\}$  so that  $\kappa_{P'}: G_{P'} \cong Q_{P'}$ . Set  $\tau' = (\kappa_{P'})^{-1}: Q_{P'} \rightarrow G_{P'}$ . Note that each of  $\tau_1, \tau_2, \dots, \tau_k, \tau'$  rationalizes to a section of  $\kappa_0: G_0 \rightarrow Q_0$ ; but  $\kappa_0$  is an isomorphism so that the only section of  $\kappa_0$  is  $(\kappa_0)^{-1}$ . Thus the splitting maps  $\tau_1, \tau_2, \dots, \tau_k, \tau'$  agree over  $Q_0$ . We may therefore invoke Theorem 1.1 to infer the existence of  $\tau: Q \rightarrow G$  such that

$$\tau_{p_i} = \tau_i, \quad i = 1, 2, \dots, k; \quad \tau_{P'} = \tau'.$$

Then  $e_{p_i} \kappa \tau = \kappa_{p_i} \tau_i e_{p_i} = e_{p_i}$  and  $e_{P'} \kappa \tau = \kappa_{P'} \tau' e_{P'} = e_{P'}$ . Thus by the uniqueness assertion implicit in a pull-back,  $\kappa \tau = \text{id}_Q$  and  $\tau$  is a splitting of (2.1). By construction,  $\tau_{p_i} = \tau(p_i)$ ,  $i = 1, 2, \dots, k$ ; and, if  $p \in P'$ , then  $\tau_p$  is the unique section of the isomorphism  $\kappa_p$  so that  $\tau_p = \tau(p)$ . This completes the proof of the theorem.  $\square$

This result generalizes part (b) of Theorem 0.1. Furthermore, we may use it, together with the following observation, to yield a generalization of part (a).

**Lemma 2.2** *Let  $N \twoheadrightarrow G \xrightarrow{\kappa} Q$  be an extension of nilpotent groups and  $\Gamma \subseteq N$  a subgroup which is normal in  $G$ . Assume given a splitting  $s: Q \rightarrow G/\Gamma$  of the extension*

$$N/\Gamma \twoheadrightarrow G/\Gamma \xrightarrow{\varepsilon} Q$$

and a homomorphism  $t: G/\Gamma \rightarrow G$  such that  $\varepsilon \pi t = \varepsilon$  in the diagram

$$\begin{array}{ccccc} \Gamma & \twoheadrightarrow & G & \xrightarrow{\pi} & G/\Gamma \\ \downarrow \iota & & = \downarrow & & \varepsilon \downarrow \\ N & \twoheadrightarrow & G & \xrightarrow{\kappa} & Q. \end{array}$$

Then the composite  $ts$  is a section of  $\kappa$ .

PROOF.  $\kappa ts = \varepsilon \pi ts = \varepsilon s = \text{id}_Q$ .  $\square$

**Theorem 2.3** *Assume that  $Q$  is finitely generated,  $N_p$  is commutative for all primes  $p$  except for a finite number  $p_1, p_2, \dots, p_k$ , and  $N_{p_i}$  is torsion for  $i = 1, 2, \dots, k$ . Then (2.1) splits.*

PROOF. Write  $P = \{p_1, p_2, \dots, p_k\}$ , and set  $\Gamma = N_{p_1} N_{p_2} \cdots N_{p_k}$ . Then  $\Gamma$  is a direct factor in  $TN$  and certainly normal in  $G$ . If  $\pi$  denotes the projection of  $G$  onto  $G/\Gamma$ , then for each prime  $p$  the homomorphism  $\pi_p \tau(p)$  splits the extension

$$N/\Gamma \twoheadrightarrow G/\Gamma \xrightarrow{\varepsilon} Q, \quad (2.2)$$

at the prime  $p$ , since  $\varepsilon_p \pi_p \tau(p) = \kappa_p \tau(p) = \text{id}_{Q_p}$ . But  $N/\Gamma$  is commutative because  $(N/\Gamma)_p$  is commutative for all  $p$ . Hence, (2.2) splits by part (a) of Theorem 0.1. We next check that

$$\Gamma \twoheadrightarrow G \xrightarrow{\pi} G/\Gamma \quad (2.3)$$

also splits, so that the theorem will follow from Lemma 2.2. Since  $\Gamma$  is torsion and almost torsionfree, by Theorem 2.1 it suffices to check that (2.3) splits at every prime  $p$ . If  $p \in P$  then the  $p$ -localization of (2.3) coincides with the  $p$ -localization of (2.1) and hence splits. If  $p \in P'$ , then  $\Gamma_p = \{1\}$  and hence  $\pi_p$  also admits an obvious section. This completes the argument.  $\square$

It remains to show that *some* restriction on the sequence (2.1) is necessary in order to deduce from the splitting of its localizations that it itself splits.

We confine ourselves to the situation in which both  $N$  and  $Q$  are commutative and will further assume that  $Q$  is a group of *pseudo-integers* [3], i.e. such that  $Q_p \cong C_p$  for all primes  $p$ , where  $C$  is cyclic infinite.

**Lemma 2.4** *Let  $Q$  be a group of pseudo-integers. Then  $H_n(Q) = 0$  for  $n \geq 2$ .*

PROOF. If  $n \geq 2$  we have

$$H_n(Q)_p \cong H_n(Q_p) \cong H_n(C_p) \cong H_n(C)_p = 0$$

for all  $p$ . It follows that  $H_n(Q) = 0$ .  $\square$



**Proposition 2.5** *Let  $N$  be commutative and  $Q$  a group of pseudo-integers. Then, for every central extension  $N \twoheadrightarrow G \twoheadrightarrow Q$ ,  $G$  is commutative.*

PROOF. Since  $\text{Hom}(H_2(Q), N) = 0$ , we have  $H^2(Q; N) = \text{Ext}(Q, N)$ .  $\square$

We will thus concentrate on abelian extensions

$$N \twoheadrightarrow G \twoheadrightarrow Q$$

with  $Q$  a group of pseudo-integers.

**Proposition 2.6** *Every such abelian extension splits at every prime.*

PROOF.  $N_p \twoheadrightarrow G_p \twoheadrightarrow Q_p$  is an extension of  $\mathbf{Z}_p$ -modules with  $Q_p$  free.  $\square$

We will now study  $\text{Ext}(Q, N)$ , writing our abelian groups additively. Then, if we enumerate the primes,  $Q$  is characterized by a set of non-negative integers  $(n_1, n_2, \dots)$ , in the sense that  $Q = \langle 1/p_1^{n_1}, 1/p_2^{n_2}, \dots \rangle$ . Thus there is an exact sequence

$$\mathbf{Z} \twoheadrightarrow Q \twoheadrightarrow \bigoplus_i \mathbf{Z}/p_i^{n_i},$$

giving rise to an exact sequence

$$\text{Hom}(\mathbf{Z}, N) \rightarrow \text{Ext}\left(\bigoplus_i \mathbf{Z}/p_i^{n_i}, N\right) \rightarrow \text{Ext}(Q, N) \rightarrow 0. \quad (2.4)$$

Now  $\text{Ext}(\bigoplus_i \mathbf{Z}/p_i^{n_i}, N) = \prod_i \text{Ext}(\mathbf{Z}/p_i^{n_i}, N) = \prod_i N/p_i^{n_i}N$ , so that (2.4) becomes

$$N \xrightarrow{\theta} \prod_i N/p_i^{n_i}N \twoheadrightarrow \text{Ext}(Q, N).$$

Moreover it is easy to see that the  $i$ th component of  $\theta$  is just the standard projection of  $N$  onto  $N/p_i^{n_i}N$ . We conclude that

$$\text{Ext}(Q, N) \cong \text{coker } \theta;$$

and it only remains to describe conditions under which  $\text{coker } \theta \neq \{0\}$ . Obviously a sufficient condition is that  $N$  be countable and  $\prod_i N/p_i^{n_i}N$  uncountable. Now  $\prod_i N/p_i^{n_i}N$  is uncountable if  $N/p_i^{n_i}N \neq \{0\}$  for infinitely many  $i$ . This will occur if there are infinitely many  $i$  such that  $n_i > 0$  and  $N$  is not  $p_i$ -divisible. Thus we infer

**Theorem 2.7** *Let  $N$  be a countable abelian group and let*

$$Q = \langle 1/p_i^{n_i}, i = 1, 2, \dots \rangle$$

*be a group of pseudo-integers. Then  $\text{Ext}(Q, N)$  is uncountable, provided there exist infinitely many  $i$  such that  $n_i > 0$  and  $N$  is not  $p_i$ -divisible.  $\square$*

Notice that we may take as suitable groups  $N$

(i) any countable abelian group having a  $\mathbf{Z}$ -summand;

(ii) any countable abelian torsion group such that  $N_p$  is non-zero and reduced for infinitely many primes  $p$ .

In case (i) any group of pseudo-integers not isomorphic to  $\mathbf{Z}$  would be a suitable  $Q$ ; in case (ii)  $Q$  would need to be chosen more carefully. Of course, case (i) immediately allows us to generalize to the following result.

**Corollary 2.8** *Let  $N$  be an abelian group having a  $\mathbf{Z}$ -summand and let  $Q$  be a group of pseudo-integers,  $Q \not\cong \mathbf{Z}$ . Then  $\text{Ext}(Q, N)$  is uncountable.*

In fact, we know that the conclusion of Corollary 2.8 holds so long as  $Q$  is  $\mathbf{Z}^k$ -like but  $Q \not\cong \mathbf{Z}^k$ , with the same assumption on  $N$ .

## References

- [1] C. CASACUBERTA and P. HILTON, On special nilpotent groups, *Bull. Soc. Math. Belg. Sér. B* **49** (1989), no. 3, 261–273.
- [2] P. HILTON, On the genus of groups with operators, *Topology Appl.* **25** (1987), 113–120.
- [3] P. HILTON, On groups of pseudo-integers, *Acta Math. Sinica (N. S.)* **4** (1988), no. 2, 189–192.
- [4] P. HILTON, G. MISLIN and J. ROITBERG, *Localization of Nilpotent Groups and Spaces*, North-Holland Math. Studies **15** (1975).

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