ON ORTHOGONAL PAIRS IN CATEGORIES AND LOCALISATION

Carles Casacuberta, Georg Peschke and Markus Pfenniger

In memory of Frank Adams

0 Introduction

Special forms of the following situation are often encountered in the literature: Given a class of morphisms $\mathcal{M}$ in a category $\mathcal{C}$, consider the full subcategory $\mathcal{D}$ of objects $X \in \mathcal{C}$ such that, for each diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ X & & \end{array}$$

with $f \in \mathcal{M}$, there is a unique morphism $h\colon B \to X$ with $hf = g$. The *orthogonal subcategory problem* [13] asks whether $\mathcal{D}$ is reflective in $\mathcal{C}$, i.e., under which conditions the inclusion functor $\mathcal{D} \to \mathcal{C}$ admits a left adjoint $E\colon \mathcal{C} \to \mathcal{D}$; see [17]. Many authors have given conditions on the category $\mathcal{C}$ and the class of morphisms $\mathcal{M}$ ensuring the reflectivity of $\mathcal{D}$, sometimes even providing an explicit construction of the left adjoint $E\colon \mathcal{C} \to \mathcal{D}$; see for example Adams [1], Bousfield [3, 4], Deleanu-Frei-Hilton [9, 10], Heller [15], Yosimura [22], Dror-Farjoun [11], Kelly [12]. The functor E is often referred to as a *localisation functor* of $\mathcal{C}$ at the subcategory $\mathcal{D}$. Most of the known existence results of left adjoints work well when the category $\mathcal{C}$ is cocomplete [12] or complete [19]. Unfortunately, these methods cannot be directly applied to the homotopy category of CW-complexes, as it is neither complete nor cocomplete. This difficulty is often circumvented by resorting to semi-simplicial techniques.

In this paper we offer a construction of localisation functors depending only on the availability of certain weak colimits in the category $\mathcal{C}$. From a technical point of view, the existence of such weak colimits reduces our arguments essentially to the situation in cocomplete categories. From a practical point of view, however, our result is a simple recipe for the explicit construction of localisation functors. It unifies a number of constructions created for specific purposes; cf. [4, 18, 20]. In fact, its scope goes beyond these applications: For example, it can be used to show that there is a whole family of functors extending P-localisation of nilpotent homotopy types to the homotopy category of all CW-complexes. We deal with this issue in [7], where we discuss the geometric significance of these functors as well as their interdependence.

Section 1 of the present paper contains background, followed by the statement and proof of our main result: the affirmative solution of the orthogonal subcategory problem in a wide range of cases. In Section 2 we discuss extensions of a localisation functor in a category $\mathcal{C}$ to localisation functors in supercategories of $\mathcal{C}$. Our results allow us to give, in Section 3, a uniform existence proof for various localisation functors and also to explain their interrelation. The basic features of our project have been outlined in [8].

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1 Orthogonal pairs and localisation functors

We begin by explaining the basic categorical notions we shall use. Our main sources are [1, 3, 4, 13].

A morphism $f\colon A \to B$ and an object X in a category $\mathcal{C}$ are said to be *orthogonal* if the function

$$f^*\colon \mathcal{C}(B, X) \to \mathcal{C}(A, X)$$

is bijective, where $\mathcal{C}(\quad,\quad)$ denotes the set of morphisms between two given objects of $\mathcal{C}$. For a class of morphisms $\mathcal{M}$, we denote by $\mathcal{M}^\perp$ the class of objects orthogonal to each $f \in \mathcal{M}$. Similarly, for a class of objects $\mathcal{O}$, we denote by $\mathcal{O}^\perp$ the class of morphisms orthogonal to each $X \in \mathcal{O}$.

Definition 1.1 An *orthogonal pair* in $\mathcal{C}$ is a pair $(\mathcal{S}, \mathcal{D})$ consisting of a class of morphisms $\mathcal{S}$ and a class of objects $\mathcal{D}$ such that $\mathcal{D}^\perp = \mathcal{S}$ and $\mathcal{S}^\perp = \mathcal{D}$.

If (E,η) is an idempotent monad [1] in $\mathcal{C}$, then the classes

$$\mathcal{S} = \{f\colon A \to B \mid Ef\colon EA \cong EB\}$$
$$\mathcal{D} = \{X \mid \eta_X\colon X \cong EX\}$$

form an orthogonal pair (note that these could easily be proper classes). The morphisms in $\mathcal{S}$ are then called *E-equivalences* and the objects in $\mathcal{D}$ are said to be *E-local*. Not every orthogonal pair $(\mathcal{S},\mathcal{D})$ arises from an idempotent monad in this way; cf. [19]. If so, we call E the *localisation functor* associated with $(\mathcal{S},\mathcal{D})$. Then the full subcategory of objects in $\mathcal{D}$ is reflective and E is left adjoint to the inclusion $\mathcal{D} \to \mathcal{C}$. The following proposition enables us to detect localisation functors.

Proposition 1.2 *Let $\mathcal{C}$ be a category and $(\mathcal{S},\mathcal{D})$ an orthogonal pair in $\mathcal{C}$. If for each object X there exists a morphism $\eta_X\colon X \to EX$ in $\mathcal{S}$ with EX in $\mathcal{D}$, then*

(i) *η_X is terminal among the morphisms in $\mathcal{S}$ with domain X;*

(ii) *η_X is initial among the morphisms of $\mathcal{C}$ from X to an object of $\mathcal{D}$;*

(iii) *The assignment $X \mapsto EX$ defines a localisation functor on $\mathcal{C}$ associated with $(\mathcal{S},\mathcal{D})$.*

For each class of morphisms $\mathcal{M}$, the pair $(\mathcal{M}^{\perp\perp},\mathcal{M}^{\perp})$ is orthogonal. We say that this pair is *generated* by $\mathcal{M}$ and call $\mathcal{M}^{\perp\perp}$ the *saturation* of $\mathcal{M}$. If $\mathcal{M}^{\perp\perp} = \mathcal{M}$, then $\mathcal{M}$ is said to be *saturated*. This terminology applies to objects as well. Note that if $(\mathcal{S},\mathcal{D})$ is an orthogonal pair then both $\mathcal{S}$ and $\mathcal{D}$ are saturated. The next properties of saturated classes are easily checked and well-known in a slightly more general context [3, 13].

Lemma 1.3 *If a class of morphisms $\mathcal{S}$ is saturated, then*

(i) *$\mathcal{S}$ contains all isomorphisms of $\mathcal{C}$.*

(ii) *If the composition gf of two morphisms is defined and any two of f, g, gf are in $\mathcal{S}$, then the third is also in $\mathcal{S}$.*

(iii) *Whenever the coproduct of a family of morphisms of $\mathcal{S}$ exists, it is in the class $\mathcal{S}$.*

(iv) *If the diagram*

$$\begin{array}{ccc} A & \stackrel{s}{\to} & B \\ \downarrow & & \downarrow \\ C & \stackrel{t}{\to} & D \end{array}$$

is a push-out in which $s \in \mathcal{S}$, then $t \in \mathcal{S}$.

(v) *If α is an ordinal and $F\colon \alpha \to \mathcal{C}$ is a directed system with direct limit T, such that for each $i < \alpha$ the morphism $s_i\colon F(0) \to F(i)$ is in $\mathcal{S}$, then $s_\alpha\colon F(0) \to T$ is in $\mathcal{S}$.*

We call a class of morphisms $\mathcal{S}$ *closed* in $\mathcal{C}$ if it satisfies (i), (ii) and (iii) in Lemma 1.3 above. We restrict attention to closed classes from now on.

We proceed with the statement of our main result. Recall that a *weak colimit* of a diagram is defined by requiring only existence, without insisting on uniqueness, in the defining universal property [17].

Theorem 1.4 *Let $\mathcal{C}$ be a category with coproducts and let $\mathcal{S}$ be a closed class of morphisms in $\mathcal{C}$. Suppose that:*

(C1) *There is a set $\mathcal{S}_0 \subseteq \mathcal{S}$ with $\mathcal{S}_0^{\perp} = \mathcal{S}^{\perp}$.*

(C2) *For every diagram $C \overset{f}{\leftarrow} A \overset{s}{\rightarrow} B$ with $s \in \mathcal{S}$ there exists a weak push-out*

$$\begin{array}{ccc} A & \overset{s}{\rightarrow} & B \\ {\scriptstyle f}\downarrow & & \downarrow \\ C & \overset{t}{\rightarrow} & Z \end{array}$$

with $t \in \mathcal{S}$.

(C3) *There is an ordinal α such that, for every $\beta \leq \alpha$, every directed system $F\colon \beta \to \mathcal{C}$ in which the morphisms $s_i\colon F(0) \to F(i)$ are in $\mathcal{S}$ for $i < \beta$ admits a weak direct limit T satisfying*

(a) *the morphism $s_\beta\colon F(0) \to T$ is in $\mathcal{S}$;*

(b) *for each $s\colon A \to B$ in $\mathcal{S}_0$, every morphism $f\colon A \to T$ factors through $f'\colon A \to F(i)$ for some $i < \alpha$;*

(c) *if two morphisms $g_1, g_2\colon B \to T$ satisfy $g_1 s = g_2 s$ with $s\colon A \to B$ in $\mathcal{S}_0$, then they factor through $g_1', g_2'\colon B \to F(i)$ for some $i < \alpha$, in such a way that $g_1' s = g_2' s$.*

Then the class $\mathcal{S}$ is saturated and the orthogonal pair $(\mathcal{S}, \mathcal{S}^{\perp})$ admits a localisation functor E. Furthermore, for each object X, the localising morphism $\eta_X : X \to EX$ can be constructed by means of a weak direct limit indexed by α.

Proof. For each morphism $s\colon A \to B$ in $\mathcal{S}_0$ fix a weak push-out

$$\begin{array}{ccc} A & \overset{s}{\rightarrow} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t_2} \\ B & \overset{t_1}{\rightarrow} & Z_s \end{array}$$

in which $t_1 \in \mathcal{S}$. Then also $t_2 \in \mathcal{S}$ because $\mathcal{S}$ is closed.

Remark 1.5 With applications in mind, it is worth observing that part (c) of hypothesis (C3) in Theorem 1.4 is satisfied if each map $f\colon Z_s \to T$ factors through $f'\colon Z_s \to F(i)$ for some $i < \alpha$.

Choose next a morphism $u_s \colon Z_s \to B$ rendering commutative the diagram

$$\begin{array}{ccccc} A & \xrightarrow{s} & B & & \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t_2} & \searrow{\scriptstyle \mathrm{id}} & \\ B & \xrightarrow{t_1} & Z_s & & \\ & \searrow{\scriptstyle \mathrm{id}} & & \searrow{\scriptstyle u_s} & \\ & & & & B, \end{array}$$

and note that $u_s \in \mathcal{S}$. Write $\mathcal{D}$ for $\mathcal{S}^{\perp}$. We shall construct, for each object $X \in \mathcal{C}$, a morphism $\eta_X : X \to EX$ with $EX \in \mathcal{D}$ and $\eta_X \in \mathcal{S}$. Set $X_0 = X$. Given $i < \alpha$, assume that X_i has been constructed, together with a morphism $X \to X_i$ belonging to $\mathcal{S}$. Define a morphism $\sigma_i \colon X_i \to X_{i+1}$ as follows: For each $s \colon A \to B$ in the set $\mathcal{S}_0$, consider all morphisms $\varphi \colon A \to X_i$ and $\psi : Z_s \to X_i$ for which no factorisation through $s : A \to B$, resp. $u_s : Z_s \to B$, exists (if there are no such morphisms, then $X_i \in \mathcal{D}$ and we may set $EX = X_i$). Choose a weak push-out

$$\begin{array}{ccc} \coprod_{s \in \mathcal{S}_0}((\coprod_{\varphi} A) \amalg (\coprod_{\psi} Z_s)) & \xrightarrow{\phi} & \coprod_{s \in \mathcal{S}_0}((\coprod_{\varphi} B) \amalg (\coprod_{\psi} B)) \\ {\scriptstyle f}\downarrow & & \downarrow \\ X_i & \xrightarrow{\sigma_i} & X_{i+1} \end{array}$$

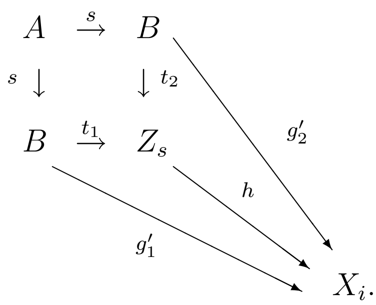
with $\sigma_i \in \mathcal{S}$, in which f is the coproduct morphism and ϕ is the corresponding coproduct of copies of $s : A \to B$ and $u_s : Z_s \to B$ (which is therefore a morphism in $\mathcal{S}$). Iterate this procedure until reaching the ordinal α. If $\beta \leq \alpha$ is a limit ordinal, define X_β by choosing a weak direct limit of the system $\{X_i, \, i < \beta\}$, according to (C3). Set $EX = X_\alpha$. The construction guarantees that the composite morphism $\eta_X \colon X \to EX$ is in $\mathcal{S}$. We claim that $EX \in \mathcal{D}$. Since $\mathcal{D} = \mathcal{S}_0^{\perp}$, it suffices to check that EX is orthogonal to each morphism in $\mathcal{S}_0$. Take a diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ {\scriptstyle f}\downarrow & & \\ EX & & \end{array}$$

with $s \in \mathcal{S}_0$. Then f factors through $f' \colon A \to X_i$ for some $i < \alpha$ and hence, either f' factors through $s \colon A \to B$, or there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ X_i & \xrightarrow{\sigma_i} & X_{i+1} \end{array}$$

which provides a morphism $g\colon B \to EX$ such that $gs = f$. Now suppose that there are two maps $g_1, g_2\colon B \to EX$ with $g_1 s = g_2 s = f$. Then we can choose an object X_i with $i < \alpha$, and morphisms $g_1', g_2'\colon B \to X_i$ such that $g_1' s = g_2' s$. Using the weak push-out property of Z_s, we obtain a morphism $h\colon Z_s \to X_i$ rendering commutative the diagram



Then, either h factors through $u_s : Z_s \to B$ and $g_1' = g_2'$, or there is a commutative diagram

$$\begin{array}{ccc} Z_s & \stackrel{u_s}{\to} & B \\ h\downarrow & & \downarrow k \\ X_i & \stackrel{\sigma_i}{\to} & X_{i+1} \end{array}$$

which yields

$$\sigma_i g_1' = \sigma_i h t_1 = k u_s t_1 = k = k u_s t_2 = \sigma_i h t_2 = \sigma_i g_2'$$

and hence $g_1 = g_2$. This shows that $EX \in \mathcal{D}$.

To complete the proof it remains to show that $\mathcal{S}^{\perp\perp} = \mathcal{S}$. The inclusion $\mathcal{S} \subseteq \mathcal{S}^{\perp\perp}$ is trivial. For the converse, let $f : A \to B$ be orthogonal to all objects in $\mathcal{D}$. Since $\eta_A : A \to EA$ is in $\mathcal{S}$ and $EB \in \mathcal{D}$, there is a unique morphism Ef rendering commutative the diagram

$$\begin{array}{ccc} A & \stackrel{f}{\to} & B \\ \eta_A\downarrow & & \downarrow \eta_B \\ EA & \stackrel{Ef}{\to} & EB. \end{array}$$

But $\eta_B f$ is orthogonal to EA and this provides a morphism $g\colon EB \to EA$ which is two-sided inverse to Ef. Hence Ef is an isomorphism and $f \in \mathcal{S}$ because $\mathcal{S}$ is closed. $\square$

Given an orthogonal pair $(\mathcal{S}, \mathcal{D})$, the class $\mathcal{S}$ is saturated and, a fortiori, closed. Therefore

Corollary 1.6 *Let $\mathcal{C}$ be a category with coproducts and $(\mathcal{S}, \mathcal{D})$ an orthogonal pair in $\mathcal{C}$. Suppose that some set $\mathcal{S}_0 \subseteq \mathcal{S}$ generates the pair $(\mathcal{S}, \mathcal{D})$ and that the class $\mathcal{S}$ satisfies conditions* (C2) *and* (C3) *in Theorem* 1.4. *Then the pair $(\mathcal{S}, \mathcal{D})$ admits a localisation functor E.*

Moreover, if the category $\mathcal{C}$ is cocomplete, then it follows from Lemma 1.3 that for each orthogonal pair $(\mathcal{S}, \mathcal{D})$ condition (C2) and part (a) of condition (C3) are automatically satisfied. This leads to Corollary 1.7 below. An object X has been called *presentable* [14] or *s-definite* [3] if, for some sufficiently large ordinal α, the functor $\mathcal{C}(X, \quad)$ preserves direct limits of directed systems $F\colon \alpha \to \mathcal{C}$. For example, all groups are presentable [3]. For finitely presented groups it suffices to take α to be the first infinite ordinal.

Corollary 1.7 [3] *Let $\mathcal{C}$ be a cocomplete category. Let $(\mathcal{S}, \mathcal{D})$ be the orthogonal pair generated by an arbitrary set $\mathcal{S}_0$ of morphisms of $\mathcal{C}$. Suppose that the domains and codomains of morphisms in $\mathcal{S}_0$ are presentable. Then $(\mathcal{S}, \mathcal{D})$ admits a localisation functor.*

Since any colimit of presentable objects is again presentable, the following definition together with the results of [19] imply Corollary 1.9 below.

Definition 1.8 A set $\{E_\alpha\}$ of objects of a category $\mathcal{C}$ is a *cogenerator set* of $\mathcal{C}$ if any morphism $f\colon X \to Y$ of $\mathcal{C}$ inducing bijections $f_*\colon \mathcal{C}(E_\alpha, X) \cong \mathcal{C}(E_\alpha, Y)$ for each α, is an isomorphism.

Corollary 1.9 *Let $\mathcal{C}$ be a cocomplete category. Suppose that $\mathcal{C}$ has a cogenerator set whose elements are presentable. Then any orthogonal pair generated by an arbitrary set of morphisms of $\mathcal{C}$ admits a localisation functor.*

2 Extending localisation functors

Let E be a localisation functor on the subcategory $\mathcal{C}'$ of $\mathcal{C}$. We wish to discuss extensions of E over $\mathcal{C}$. Familiar examples include the extension of P-localisation of abelian groups to nilpotent groups and further to all groups. Two problems arise here: existence —for which we often refer to Theorem 1.4— and uniqueness. An appropriate setting for discussing the latter is obtained by partially ordering the collection of all orthogonal pairs in $\mathcal{C}$ as follows: For two given orthogonal pairs $(\mathcal{S}_1, \mathcal{D}_1)$, $(\mathcal{S}_2, \mathcal{D}_2)$ in $\mathcal{C}$ we write $(\mathcal{S}_1, \mathcal{D}_1) \geq (\mathcal{S}_2, \mathcal{D}_2)$ if $\mathcal{D}_1 \supseteq \mathcal{D}_2$ (or, equivalently, if $\mathcal{S}_1 \subseteq \mathcal{S}_2$).

Remark 2.1 If E_1, E_2 are localisation functors associated to $(\mathcal{S}_1, \mathcal{D}_1)$ and $(\mathcal{S}_2, \mathcal{D}_2)$ respectively, and if $(\mathcal{S}_1, \mathcal{D}_1) \geq (\mathcal{S}_2, \mathcal{D}_2)$, then there is a natural transformation of functors $E_1 \to E_2$. In fact, the restriction of E_2 to $\mathcal{D}_1$ is left adjoint to the inclusion $\mathcal{D}_2 \to \mathcal{D}_1$.

An orthogonal pair $(\mathcal{S},\mathcal{D})$ of $\mathcal{C}$ is said to *extend* the orthogonal pair $(\mathcal{S}',\mathcal{D}')$ of the subcategory $\mathcal{C}'$ if both $\mathcal{S}'\subseteq\mathcal{S}$ and $\mathcal{D}'\subseteq\mathcal{D}$. The collection of all extensions of $(\mathcal{S}',\mathcal{D}')$ is partially ordered. Moreover we have

Proposition 2.2 *Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$ and $(\mathcal{S}',\mathcal{D}')$ an orthogonal pair in $\mathcal{C}'$. If $(\mathcal{S},\mathcal{D})$ is an extension of $(\mathcal{S}',\mathcal{D}')$ to $\mathcal{C}$, then*

$$((\mathcal{S}')^{\perp\perp},(\mathcal{S}')^{\perp})\geq(\mathcal{S},\mathcal{D})\geq((\mathcal{D}')^{\perp},(\mathcal{D}')^{\perp\perp}),$$

where orthogonality is meant in $\mathcal{C}$.

In this situation, we call the orthogonal pair in $\mathcal{C}$ generated by the class $\mathcal{S}'$ the *maximal extension* of $(\mathcal{S}',\mathcal{D}')$, and the one generated by $\mathcal{D}'$ the *minimal extension*. A convenient tool for recognising such extremal extensions is given in the next proposition.

Proposition 2.3 *Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$, $(\mathcal{S}',\mathcal{D}')$ an orthogonal pair in $\mathcal{C}'$, and $(\mathcal{S},\mathcal{D})$ an extension of $(\mathcal{S}',\mathcal{D}')$ to $\mathcal{C}$. Then*

(a) *$(\mathcal{S},\mathcal{D})$ is the maximal extension of $(\mathcal{S}',\mathcal{D}')$ if and only if there is a subclass $\mathcal{S}_0\subseteq\mathcal{S}'$ such that $\mathcal{S}_0^{\perp}\subseteq\mathcal{D}$.*

(b) *$(\mathcal{S},\mathcal{D})$ is the minimal extension of $(\mathcal{S}',\mathcal{D}')$ if and only if there is a subclass $\mathcal{D}_0\subseteq\mathcal{D}'$ such that $\mathcal{D}_0^{\perp}\subseteq\mathcal{S}$.*

Of course $(\mathcal{S}',\mathcal{D}')$ admits a unique extension to $\mathcal{C}$ if and only if the minimal and the maximal extensions coincide.

Example 2.4 Let $\mathcal{C}$ be the category of finite groups and $\mathcal{C}'$ the subcategory of finite nilpotent groups. Fix a prime p and consider the orthogonal pair $(\mathcal{S}',\mathcal{D}')$ in $\mathcal{C}'$ associated to p-localisation [16]. The class $\mathcal{D}'$ consists of all p-groups, and the orthogonal pair $(\mathcal{S},\mathcal{D})=((\mathcal{D}')^{\perp},\mathcal{D}')$ in $\mathcal{C}$ is both the maximal and the minimal extension of $(\mathcal{S}',\mathcal{D}')$ to $\mathcal{C}$. The pair $(\mathcal{S},\mathcal{D})$ admits a localisation functor —namely, mapping each finite group G onto its maximal p-quotient—, which is therefore the unique extension to all finite groups of the p-localisation of finite nilpotent groups.

3 Applications of the basic existence result

Examples 3.1, 3.2 and 3.3 below discuss well-known functors, each of whose constructions may be viewed as particular cases of Theorem 1.4. Examples 3.4 to 3.7 are new.

Example 3.1 Let $\mathcal{H}_1$ be the pointed homotopy category of simply-connected CW-complexes, and P a set of primes. The P-localisation functor described by Sullivan [21] is associated to the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by the set

$$\mathcal{S}_0 = \{\rho_n^k \colon S^k \to S^k \mid \deg \rho_n^k = n,\ k \geq 2,\ n \in P'\},$$

where P' denotes the set of primes not in P. Objects in $\mathcal{D}$ are simply-connected CW-complexes whose homotopy groups are $\mathbf{Z}_P$-modules. Morphisms in $\mathcal{S}$ are $H_*(\quad;\mathbf{Z}_P)$-equivalences. The hypotheses of Corollary 1.6 are fulfilled by taking α to be the first infinite ordinal and using homotopy colimits.

Example 3.2 Let $\mathcal{H}$ denote the pointed homotopy category of connected CW-complexes and h_* an additive homology theory. Take $\mathcal{S}$ to be the class of morphisms $f \colon X \to Y$ inducing an isomorphism $f_* \colon h_*(X) \cong h_*(Y)$. We know from [4] that $\mathcal{S}$ satisfies the hypotheses of Theorem 1.4: Choose α to be the smallest infinite ordinal whose cardinality is bigger than the cardinality of $h_*(\mathrm{pt})$; the collection of all CW-inclusions $\varphi \colon A \to B$ with $h_*(\varphi) = 0$ and $\mathrm{card}(B) < \mathrm{card}(\alpha)$ represents a set $\mathcal{S}_0$ with $\mathcal{S}_0^{\perp} = \mathcal{S}^{\perp}$.

In the case $h_* = H_*(\quad;\mathbf{Z}_P)$, the corresponding orthogonal pair $(\mathcal{S}, \mathcal{D})$ extends the pair $(\mathcal{S}', \mathcal{D}')$ associated with P-localisation of nilpotent spaces (see [4]). It is indeed the *minimal* extension of $(\mathcal{S}', \mathcal{D}')$, because the spaces $K(\mathbf{Z}_P, n)$, $n \geq 1$, belong to $\mathcal{D}'$ (cf. Proposition 2.3).

Example 3.3 Let $\mathcal{G}$ be the category of groups and P a set of primes. The P-localisation functor described by Ribenboim [20] is associated to the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by the set

$$\mathcal{S}_0 = \{\rho_n \colon \mathbf{Z} \to \mathbf{Z} \mid \rho_n(1) = n,\ n \in P'\}.$$

Groups in $\mathcal{D}$ are those in which P'-roots exist and are unique. Such groups have been studied for several decades (see [2, 20] and the references there). The hypotheses of Theorem 1.4 are readily checked (use Corollary 1.7). We may choose α to be the first infinite ordinal. We denote by $l \colon G \to G_P$ the P-localisation homomorphism.

If $(\mathcal{S}', \mathcal{D}')$ is the orthogonal pair corresponding to P-localisation of nilpotent groups, then, since $\mathcal{S}_0 \subset \mathcal{S}'$, Proposition 2.3 implies that $(\mathcal{S}, \mathcal{D})$ is the *maximal* extension of $(\mathcal{S}', \mathcal{D}')$. In particular, for each group G there is a natural homomorphism from G_P to the Bousfield $H\mathbf{Z}_P$-localisation of G (cf. [5]).

Example 3.4 Example 3.3 can be generalised to the category $\mathcal{C}$ of π-groups for a fixed group π; that is, objects are groups with a π-action and morphisms are action-preserving group homomorphisms. Let $F(\xi)$ be the free π-group on one generator (it can be explicitly described as the free group on the symbols ξ^x, $x \in \pi$, with the obvious left π-action; cf. [18]). Define a π-homomorphism $\rho_{n,x}\colon F(\xi) \to F(\xi)$ for each $x \in \pi$, $n \in \mathbf{Z}$, by the rule

$$\rho_{n,x}(\xi) = \xi(x \cdot \xi)(x^2 \cdot \xi) \dots (x^{n-1} \cdot \xi)$$

and consider the set of morphisms

$$\mathcal{S}_0 = \{\rho_{n,x}\colon F(\xi) \to F(\xi) \mid \ x \in \pi, \ n \in P'\}.$$

By Corollary 1.7, the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by $\mathcal{S}_0$ admits a localisation functor. It again suffices to take the first infinite ordinal as α in the construction. Example 3.3 is the special case $\pi = \{1\}$.

We extend the term *P-local* to the π-groups in $\mathcal{D}$ and the term *P-equivalences* to the morphisms in $\mathcal{S}$. They are particularly relevant to the next example.

Example 3.5 This example is extracted from [7]. Let $\mathcal{H}$ be the pointed homotopy category of connected CW-complexes and P a set of primes. We consider the class $\mathcal{D}$ of those spaces X in $\mathcal{H}$ for which the power map $\rho_n\colon \Omega X \to \Omega X$, $\rho_n(\omega) = \omega^n$ is a homotopy equivalence for all $n \in P'$. Then there exists a set of morphisms $\mathcal{S}_0$ such that $\mathcal{S}_0^{\perp} = \mathcal{D}$, namely

$$\mathcal{S}_0 = \{\rho_n^k\colon S^1 \wedge (S^k \cup \mathrm{pt}) \to S^1 \wedge (S^k \cup \mathrm{pt}) \mid \ k \geq 0, \ n \in P'\},$$

where $\rho_n^k = \rho_n \wedge \mathrm{id}$, $\rho_n : S^1 \to S^1$ denotes the standard map of degree n, and pt denotes a disjoint basepoint. Morphisms in $\mathcal{S} = \mathcal{D}^{\perp}$ turn out to be those $f : X \to Y$ for which $f_* : \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups and $f_*\colon H_*(X; A) \to H_*(Y; A)$ is an isomorphism for each abelian $\pi_1(Y)_P$-group A which is P-local in the sense of Example 3.4. The conditions of Corollary 1.6 are satisfied. One can take α to be the first infinite ordinal. Spaces in $\mathcal{D}$ will be called *P-local* and maps in $\mathcal{S}$ will be called *P-equivalences*. We denote the P-localisation map by $l\colon X \to X_P$. The pair $(\mathcal{S}, \mathcal{D})$ extends the pair $(\mathcal{S}', \mathcal{D}')$ corresponding to P-localisation of nilpotent spaces.

Since the orthogonal pair corresponding to $H_*(\quad; \mathbf{Z}_P)$-localisation is minimal among those pairs extending P-localisation of nilpotent spaces (see Example 3.2), for each space X there is a natural map from X_P to the $H_*(\quad; \mathbf{Z}_P)$-localisation of X.

Example 3.6 Let $\mathcal{H}$ denote the pointed homotopy category of connected CW-complexes and P a set of primes. Consider the orthogonal pair $(\mathcal{S}, \mathcal{D})$ generated by the set

$$\mathcal{S}_0 = \{\rho_n^k \colon S^k \to S^k \mid \deg \rho_n^k = n,\ k \geq 1,\ n \in P'\}.$$

The class $\mathcal{D}$ consists of spaces whose homotopy groups are P-local, and one finds, with the same methods as in [7, 9], that $\mathcal{S}$ consists of morphisms $f : X \to Y$ such that $f_* : \pi_1(X) \to \pi_1(Y)$ is a P-equivalence of groups and $f^* : H^k(Y; A) \to H^k(X; A)$ is an isomorphism for $k \geq 2$ and every $\mathbf{Z}_P[\pi_1(Y)_P]$-module A. This class $\mathcal{S}$ is not closed under homotopy colimits, because the natural map from S^1 to $K(\mathbf{Z}_P, 1)$, which is the homotopy colimit of a certain direct system of maps $\rho_n^1, n \in P'$, fails to induce an isomorphism in H^2 with coefficients in the group ring $\mathbf{Z}_P[\mathbf{Z}_P]$, and hence does not belong to $\mathcal{S}$. Thus, Corollary 1.6 does not apply in this case. In fact, the orthogonal pair $(\mathcal{S}, \mathcal{D})$ does not admit a localisation functor [7].

On the other hand, if we delete from $\mathcal{S}_0$ the maps ρ_n^1, $n \in P'$, then the resulting class $\mathcal{D}$ consists of spaces whose higher homotopy groups are P-local, and $\mathcal{S}$ consists of morphisms $f \colon X \to Y$ inducing an isomorphism of fundamental groups and such that $f^* \colon H^k(Y; A) \to H^k(X; A)$ is an isomorphism for all k and every $\mathbf{Z}_P[\pi_1(Y)]$-module A. This orthogonal pair $(\mathcal{S}, \mathcal{D})$ is the maximal extension to $\mathcal{H}$ of the pair described in Example 3.1. Now Corollary 1.6 provides a localisation functor associated to $(\mathcal{S}, \mathcal{D})$. This functor induces an isomorphism of fundamental groups and P-localises the higher homotopy groups, i.e., corresponds to fibrewise localisation with respect to the universal covering fibration $\tilde{X} \to X \to K(\pi_1(X), 1)$.

Example 3.7 Fix a group G and let $\mathcal{H}(G)$ be the category whose objects are maps $X \to K(G, 1)$ in $\mathcal{H}$ and whose morphisms are homotopy commutative triangles. Given an abelian G-group A, let $\mathcal{S}(A)$ be the class of morphisms f such that $f_* \colon H_*(X; A) \to H_*(Y; A)$ is an isomorphism. Then $\mathcal{S}(A)$ satisfies the conditions of Theorem 1.4. Example 3.2 corresponds to the particular case $G = \{1\}$. In [7] we show that several idempotent functors on $\mathcal{H}$ extending P-localisation of nilpotent spaces can be obtained by splicing localisation functors with respect to twisted homology in a suitable way. In fact, Example 3.5 can be alternatively obtained as a special case of this procedure.

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Carles CASACUBERTA
Centre de Recerca Matemàtica
Institut d'Estudis Catalans
Apartat 50
E–08193 Bellaterra, Barcelona
Spain

Georg PESCHKE
Department of Mathematics
University of Alberta
Edmonton T6G 2G1
Canada

Markus PFENNIGER
School of Mathematics
University of Wales
Dean Street
Bangor, LL57 1UT
United Kingdom